(1) § 14 Hille's proof

(14.1) Proof of HY

Let A be d.d., $(0,\infty) \subseteq \varrho(A)$, $\|\lambda R(\lambda, A)\| \leq 1 \ \forall \lambda > 0$.

Define

$$V_n(t) := (I - \tfrac{t}{n} A)^{-n} = (\tfrac{n}{t})^n R(\tfrac{n}{t}, A)^n \quad \forall t > 0 \ \forall n \in \mathbb{N}$$

and $V_n(0) := I \quad \forall n \in \mathbb{N}$.

Then $\|V_n(t)\| \leq 1 \quad \forall n \in \mathbb{N}, t \geq 0$.

Moreover: $\forall n \in \mathbb{N} \ \forall x \in X$: $V_n(t)x = (\tfrac{n}{t})^n R(\tfrac{n}{t}, A)^n x \xrightarrow{t \downarrow 0} x$ (*)

since $\lambda R(\lambda, A) x \xrightarrow{\lambda \to \infty} x$ by (3.6).

By (3.3) (Neumann series representation of $R(\lambda, A)$)

$(0, \infty) \longrightarrow \mathcal{L}(X)$, $\lambda \longmapsto R(\lambda, A)^n$

is differentiable with

$$\frac{d}{d\lambda} R(\lambda, A)^n = (-1) n \cdot R(\lambda, A)^{n+1}$$

$$\Rightarrow \frac{d}{d\lambda} \lambda^n R(\lambda, A)^n = n \lambda^{n-1} R(\lambda, A)^n - n \cdot \lambda^n \cdot R(\lambda, A)^{n+1}$$

$$= ((\lambda - A) - \lambda) n \cdot \lambda^{n-1} R(\lambda, A)^{n+1}$$

$$= -A n \lambda^{n-1} R(\lambda, A)^{n+1} \quad \forall n \in \mathbb{N}, \lambda > 0$$

$$\left.\begin{aligned} &\Rightarrow V_n \text{ is differentiable on } (0,\infty) \text{ with} \\ &\frac{d}{dt} V_n(t) = -A n \cdot (\tfrac{n}{t})^{n-1} R(\tfrac{n}{t}, A)^{n+1} \cdot (-\tfrac{n}{t^2}) \\ &= A (\tfrac{n}{t})^{n+1} R(\tfrac{n}{t}, A)^{n+1} \end{aligned}\right\} (**)$$

(*) + (**) $\Rightarrow$ V_n is strongly continuous $\forall n \in \mathbb{N}$.

Moreover:

$$V_n(t)x - V_m(t)x = \lim_{\varepsilon \downarrow 0} \int_\varepsilon^{t-\varepsilon} \frac{d}{ds} V_m(t-s) V_n(s) x \, ds$$

$$= \lim_{\varepsilon \downarrow 0} \int_\varepsilon^{t-\varepsilon} \underbrace{[-V_m'(t-s) V_n(s) x + V_m(t-s) V_n'(s) x]}_{(***)} ds$$

with

$$(***) = -A (I - \tfrac{t-s}{m} A)^{-m-1} (I - \tfrac{s}{n} A)^{-n} x$$
$$+ (I - \tfrac{t-s}{m} A)^{-m} A (I - \tfrac{s}{n} A)^{-n-1} x$$

$$= (I - \tfrac{t-s}{m} A)^{-m-1} [(I - \tfrac{s}{n}A) - (I - \tfrac{t-s}{m}A)] (I - \tfrac{s}{n}A)^{-n-1} Ax$$

$$= (\tfrac{t-s}{m} - \tfrac{s}{n}) (I - \tfrac{t-s}{m}A)^{-m-1} (I - \tfrac{s}{n}A)^{-n-1} A^2 x$$

$\forall t > 0,\ x \in D(A^2),\ n, m \in \mathbb{N}$.

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$$\Rightarrow \|V_n(t)x - V_m(t)x\| \le \int_0^t \frac{s}{n} + \frac{t-s}{m}\, ds \cdot \|A^2 x\|$$

$$= \frac{t^2}{2}\left(\frac{1}{n} + \frac{1}{m}\right)\|A^2x\| \le \frac{T^2}{2}\left(\frac{1}{n} + \frac{1}{m}\right)\|A^2 x\|$$

$\forall t \in (0, T],\ x \in D(A^2),\ n, m \in \mathbb{N},\ T > 0$.

$D(A^2)$ dicht in X

$\Rightarrow$ $(V_n(\cdot)x)_{n\in\mathbb{N}}$ is Cauchy sequence in $C([0,T], X)$ $\forall x \in X\ \forall T > 0$.

$\Rightarrow \exists\ T(t)x := \lim_{n\to\infty} V_n(t)x \quad \forall t \ge 0,\ \forall x \in X$

and the limit is uniform on $[0,T]$ $\forall T > 0$.

In particular: $t \mapsto T(t)x$ is continuous $\forall x \in X$. and $\|T(t)\| \le 1\ \forall t \ge 0$

Moreover:

(a) $T(0) = I$

$$\left(\tfrac{n}{t}\right)^n R(\tfrac{n}{t}, A)^n Ax = A \left(\tfrac{n}{t}\right)^n R(\tfrac{n}{t}, A)^n x \qquad \forall x \in D(A)$$

$\downarrow n \to \infty$

$T(t)Ax$

A closed

$\Rightarrow$ (b) $T(t)x \in D(A)$ and $T(t)Ax = AT(t)x \quad \forall t \ge 0$

$$(\ast\ast) \Rightarrow V_n(t)x - x = \int_0^t \left(\tfrac{n}{s}\right)^{n+1} R(\tfrac{n}{s}, A)^{n+1} Ax\, ds$$

$$= \int_0^t \left(\tfrac{n}{s}\right) R(\tfrac{n}{s}, A)\, V_n(s) Ax\, ds$$

$\downarrow n \to \infty$

$$T(t)x - x = \int_0^t T(s)Ax\, ds \qquad \forall x \in D(A),\ t > 0.$$

$$\Rightarrow (c)\ \frac{d}{dt} T(t)x = AT(t)x \qquad \forall t > 0,\ x \in D(A)$$

③

(14.2) Lemma

Let A be d.d. + closed and
$T:[0,\infty)\to\mathcal{L}(X)$ strongly cont. with $\|T(t)\|\le 1$ $\forall t\ge 0$
satisfying (a), (b), (c).

$\Rightarrow$ T is contractive C_0-sgr. with generator A.

Proof:

Given $x\in D(A)$ $u(t):=T(t)x$ $\forall t\ge 0$ is the unique solution of

$$(CP)\begin{cases} u\in C^1([0,\infty),X),\ u(t)\in D(A)\ \forall t\ge 0\\ \dot u(t)=Au(t)\quad \forall t>0\\ u(0)=x.\end{cases}$$

(uniqueness is proved as in (2.3)).

Now let $s>0$ and $v(t):=T(t+s)x$ for $x\in D(A)$.

$\Rightarrow$ v is solution of (CP) with $v(0)=T(s)x$.

$\Rightarrow$ $v(t)=T(t)T(s)x$.

$\underset{D(A)\text{ dense}}{\Rightarrow}$ $T(t)T(s)=T(t+s)$ $\forall t,s\ge 0$.

Let B the generator of T.

$\overset{(c)}{\Rightarrow}$ $A\subseteq B$

Since $D(A)$ is dense and invariant, it is a core for B (see (12.11)).

$\Rightarrow$ $\overline{D(A)}^{\|\cdot\|_B}=D(B)$ $\overset{A\text{ closed}}{\Rightarrow}$ $A=B$. □

(14.3) Corollary (of (14.1))

Let A be the generator of a contractive C_0-sgr. T.

$$\Rightarrow\quad T(t)x=\lim_{n\to\infty}\frac{n}{t}R\left(\frac{n}{t},A\right)x$$

uniformly on $[0,J]$ $\forall x\in X$ $\forall J>0$.