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Chapter 2 Holomorphic Semigroups

§16 Holomorphic functions.

$\Omega \subset \mathbb{C}$ open, X complex Banach space.

(16.1) **Definition:** $f : \Omega \to X$

holomorphic $:\Leftrightarrow$

$$f'(z) := \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists for all $z \in \Omega$.

Consequence: a) $x' \circ f : \Omega \to \mathbb{C}$ hol.

$\forall x' \in X'$ & $(x' \circ f)' = x' \circ f'$.

b) $\sup\limits_{z \in K} \|f(z)\| < \infty \quad \forall K \subset \Omega$

compact.

(16.2) <u>Theorem</u> Let $f : \Omega \to X$ be [locally] bounded. Let $\Phi \subset X'$ be separating. If

$$\varphi \circ f : \Omega \to \mathbb{C}$$

is hol. $\forall \varphi \in \Phi$, then f is holomorphic.

<u>Pf.</u> Green Book Appendix

(16.3) <u>Corollary.</u> Let $T : \Omega \to \mathcal{L}(X, Y)$ such that for all $x \in X$, $y' \in Y'$

$\langle T(\cdot)x, y' \rangle$ is holomorph.

Then T is holomorphic.

<u>Pf.</u> $\Phi = \{ \varphi_{x,y'} : x \in X, y' \in Y' \}$

$\varphi_{x,y'}(S) = \langle Sx, y' \rangle \qquad (S \in \mathcal{L}(X, Y))$

□

(16.4) <u>Lemma</u> Let $g_n : \Omega \to \mathbb{C}$ be hol., $|g_n(z)| \le M$, $\forall n \in \mathbb{N}, z \in \Omega$,

$$g(z) = \lim_{n\to\infty} g_n(z) \quad \forall z \in \Omega.$$

Then g is holomorphic.

<u>Proof.</u> Let $\overline{B}(z_0, r) \subset \Omega \Rightarrow$

$$g_n(z) = \frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{g_n(w)}{w-z}\, dw$$

$\forall z \in B(z_0, r)$ $\quad n \to \infty \Rightarrow$

$$g(z) = \frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{g(w)}{w-z}\, dw$$

$\Rightarrow g$ hol. on $B(z_0, r)$

W.l.g. $z_0 = 0$ $\quad \dfrac{1}{w-z} = \dfrac{1}{w} \dfrac{1}{1-\frac{z}{w}}$

$= \sum_{n=0}^{\infty} \dfrac{z^n}{w^{n+1}}$. $\Rightarrow g(z) =$

$$\sum_{n=0}^{\infty} \underbrace{\frac{1}{2\pi i} \int_{|w|=r} \frac{g(w)}{w^{n+1}}\, dw}_{a_n} \, z^n \qquad \square$$

(16.5) <u>Proposition</u>. Let $T_n : \Omega \to \mathcal{L}(X, Y)$ be holomorphic, $\|T_n(z)\| \leq M$ $\forall z \in \Omega, n \in \mathbb{N}$. Assume that

$$T(z)x := \lim_{n \to \infty} T_n(z)x$$

exists $\forall x \in X$.

Then

$$T : \Omega \to \mathcal{L}(X, Y)$$

is holomorphic.

<u>Proof.</u> a) $T(z) \in \mathcal{L}(X, Y)$ ~~by the~~ clear.

(~~Banach Steinhaus Thm~~)

b) Let $y' \in Y'$, $x \in X$ (16.4) $\Rightarrow$

$\langle T(.)x, y'\rangle : \Omega \to \mathbb{C}$ holomorphic

(16.3) $\Rightarrow$ T is hol. $\square$

(16.6) <u>Uniqueness Theorem.</u> Ω connected

$f, g : \Omega \to X$ hol.

$\exists\ z_k \in \Omega$, $\lim z_k = z_0 \in \Omega$,

$z_k \neq z_0\ \forall k$, $f(z_k) = g(z_k)\ \forall k$

$\Rightarrow f(z) = g(z)\ \forall z \in \Omega$

§ 17 Holomorphic semigroups

$\theta \in [0, \pi)$

$\Sigma_\theta = \{ r e^{i\alpha} : r > 0, |\alpha| < \theta \}$.

(17.1) Definition A C_0-sg T is holomorphic

if $\exists\, \theta \in (0, \pi/2)$ & $\tilde{T} : \Sigma_\theta \to \mathcal{L}(X)$,

a hol extension of T s.t.

$\sup_{\substack{z \in \Sigma_\theta \\ |z| \le 1}} \|\tilde{T}(z)\| < \infty$.

Consequences.

a) $\tilde{T}(z_1 + z_2) = \tilde{T}(z_1)\tilde{T}(z_2)$

Pf. 1st case. $z_2 = t \in (0, \infty)$.

Then $\tilde{T}(z_1 + t) = \tilde{T}(z_1)\tilde{T}(z_2)$ if $z_1 \in (0, \infty)$

uniqueness thm $\Rightarrow \forall z_1 \in \Sigma_\theta$.

<u>2nd case</u> Let $z_1 \in \Sigma_\theta$ Then

$\tilde{T}(z_1 + z_2) = \tilde{T}(z_1)\tilde{T}(z_2)$ if $z_2 \in (0,\infty)$

by case 1. Uniqueness Theorem

$\Rightarrow \forall z_2 \in \Sigma_\theta$.

b) $\exists M, \omega$ $\quad \|\tilde{T}(z)\| \le M e^{(\mathrm{Re}\, z)\omega}$

$z \in \Sigma_\theta$.

<u>Proof</u> $M := \sup\limits_{\substack{z \in \Sigma_\theta \\ |z| \le 1}} \|\tilde{T}(z)\|$

Let $z = re^{i\alpha} \in \Sigma_\theta$. $\exists!\, n \in \mathbb{N}$,

$r = [n, n+1)$. $\|\tilde{T}(z)\| = \|\tilde{T}((r-n)e^{i\theta})$.

$\tilde{T}(ne^{i\theta})\| \le M \|(\tilde{T}(e^{i\theta}))^n\| \le M M^n$

$\le M M^r = M e^{\omega' r} = M e^{|z|\omega'}$ $\quad \omega' = \ln M$

~~But for $|\mathrm{Im}\, z| \le$~~ But $|z| = r$

$= |r\cos\alpha| \frac{1}{|\cos\alpha|} \le \mathrm{Re}\, z \frac{1}{\cos\theta}$

$\square$

$\omega = \frac{\omega'}{\cos\theta}$.

c) $$\lim_{\substack{z \in \Sigma_\theta \\ z \to 0}} \tilde{T}(z)x = x \qquad \forall x \in X$$

<u>Pf</u> 1. $x = T(t)y \qquad t > 0.$

Then $$\tilde{T}(z)x = \tilde{T}(z+t)y \underset{z \to 0}{\to} \tilde{T}(t)y = x$$

2. $\{T(t)y : t > 0,\ y \in X\}$ dense in X

Equicontinuity Lemma $\Rightarrow$ claim. □

d) $\tilde{T}(z)X \subset D(A)$ and

$$\frac{d}{dz}\tilde{T}(z) = A\tilde{T}(z)$$

<u>Pf.</u> Let $z \in \Sigma_\theta$. Then

$$\tilde{T}(z)' = \lim_{t \downarrow 0} \frac{\tilde{T}(z+t) - \tilde{T}(z)}{t}$$

$$= \lim_{t \downarrow 0} \frac{T(t)\tilde{T}(z) - \tilde{T}(z)}{t}$$ □

e.g.) Let $z \in \Sigma_\theta$. Then $T_z(t) = \tilde{T}(tz)$

defines a C_0-sg. Generator: $z \cdot A$

<u>Proof.</u> $B :=$ generator of T_z.

1. Let ~~y~~ $x = \tilde{T}(w) y$, ~~$s > 0$~~ $w \in \Sigma_\theta$

$$\frac{T_z(t)x - x}{t} = z \frac{\tilde{T}(w + tz)y - T(w)y}{tz}$$

$$\longrightarrow z A T(w) y \quad (t \downarrow 0)$$

by f)

Thus $x \in D(B)$ & $Bx = zAx$.

2. $D = \operatorname{span}\{\tilde{T}(w)y : $ ~~$s > 0$~~ $w \in \Sigma_\theta\}$

is dense in X, invariant by T_z,

and $D \subset D(B)$. Thus D is a

core of B.

But $D \subset D(A)$ & $ABx = zAx$

$(x \in D)$. Thus $B = \overline{zA|_D} = zA$. $\square$

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