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§ 18 Holomorphic contraction semigroups

X

(18.1) <u>Definition.</u> Let $\theta \in (0, \frac{\pi}{2}]$.

a) An operator A is <u>θ-m-diss.</u> if $\Sigma_\theta \subset \rho(A)$ & $\|\lambda R(\lambda, A)\| \leq 1 \ \forall \lambda \in \Sigma_\theta$

b) A C_0-sg T is a <u>θ-contractive</u> if it has a contractive hol extension $\tilde{T} : \Sigma_\theta \to \mathcal{L}(X)$

c) A <u>hol. contraction sg</u> is a C_0-sg T having a contractive hol. extension to some sector Σ_θ, where $0 < \theta \leq \frac{\pi}{2}$.

(18.2) <u>Theorem.</u> Let A be a dd operator $\theta \in (0, \pi/2)$. Equ:

(i) A is θ-m-diss.

(ii) A generates a θ-contractive C_0-sg

(iii) $e^{\pm i\theta}A$ generates a contractive C_0-sg.

(iv) $e^{\pm i\theta}A$ is diss. & $wI - A$ surj.
for some $w \in \Sigma_{\theta + \frac{\pi}{2}}$.

~~Pf.~~

(7.1) Recall: $\Lambda \subset \mathbb{C}$ open, connected,
$d : \Lambda \to (0, \infty)$ continuous
A an operator.
(a) $\| \lambda x - Ax \| \geq d(\lambda) \|x\|$
(b) $\exists \lambda_0 \in \Lambda$ $(\lambda_0 - A)$ surj.
$\Rightarrow \Lambda \subset \rho(A)$.

Recall: A m-diss $\Rightarrow$ $\mathbb{C}_+ \subset \rho(A)$.

Pf of (18.2). (i) $\Rightarrow$ (ii) Let $z \in \Sigma_\theta$.
$\lambda > 0$. $\lambda(\lambda - zA)^{-1} = \frac{\lambda}{z}\left(\frac{\lambda}{z} - A\right)^{-1}$
$\Rightarrow$ zA m-diss. $\Rightarrow$ zA generates
a C_0-sg T_z, $\| T_z(t) \| \leq 1$

Hille $\Rightarrow$ $T_z(1) = \lim\limits_{n\to\infty} \left(I - \frac{zA}{n}\right)^{-n}$ strongly.

Since $z \mapsto \left(I - \frac{zA}{n}\right)^{-n}$ is holomorphic

also $z \mapsto T_z(1)$ is holomorphic.

Thus $z \mapsto \tilde{T}(z) := T_z(1) : \Sigma_\theta \to \mathcal{L}(X)$

is holomorphic. If $t > 0$, then

$\tilde{T}(t) = T_t(1) \cancel{= T(t)} = s\text{-}\lim\limits_{n\to\infty} \left(I - \frac{t}{n}A\right)^{-n} = T(t).$

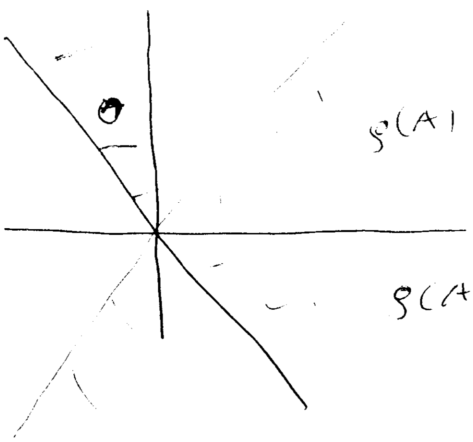
This proves (ii).

(ii) $\Rightarrow$ (iii) (17.1) e).

(iii) $\Rightarrow$ (iv) $\operatorname{Re}\lambda > 0 \Rightarrow \lambda \in \rho(e^{\pm i\theta}A) = e^{\pm i\theta}\rho(A)$

$\Rightarrow e^{\mp i\theta}\lambda \in \rho(A)$

$\Rightarrow \Sigma_{\theta + \frac{\pi}{2}} \subset \rho(A).$



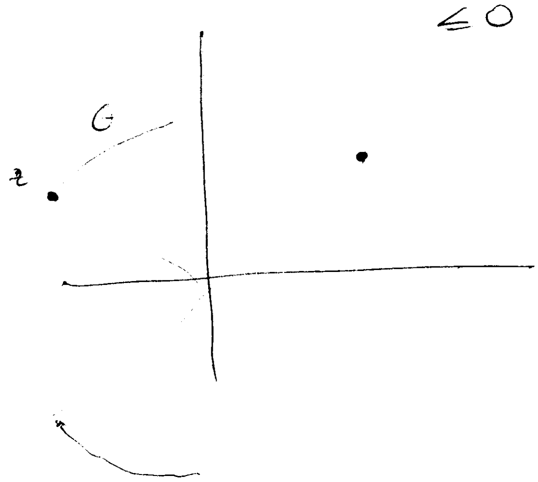
$e^{i\alpha}A$ is diss. for $|\alpha| \le \theta$. In fact,

let $|\alpha| \le \theta$, $x \in D(A)$, $x' \in J(x)$. Then

$Re\langle e^{\pm i\theta}Ax, x'\rangle \le 0$ (since $e^{\pm i\theta}A$ diss)

$\Rightarrow$ $Re\langle e^{\pm i\theta} Ax, x'\rangle \le 0$ $\Rightarrow$ $Re\, e^{i\alpha}\langle Ax, x'\rangle$

≤ 0. $\Rightarrow$



$\|\lambda \cancel{e^{i\alpha}} x - e^{i\alpha}Ax\| \ge \lambda\|x\|$

$\lambda > 0$

$\|\lambda e^{-i\alpha} x - Ax\| \ge \lambda\|x\|$

$\Rightarrow$ $\|\mu x - Ax\| \ge |\mu|\,\|x\|$

$\mu \in \Sigma_\theta$

$1-A$ surj $\Rightarrow$ $\Sigma_\theta \subset \rho(A)$ &

$\|\mu R(\mu, A)\| \le 1$ $\forall \mu \in \Sigma_\theta$.

i.e. A is θ-m-dissipative. □

Exercise: Show that $\tilde{T} : \Sigma_\theta \to \mathcal{L}(X)$

has a unique strongly continuous extension to Σ_θ. Moreover, $(\tilde{T}(te^{\pm i\theta}))_{t\geq 0}$ is a C_0-sg and $e^{\pm i\theta}A$ its generator.

§ 19 Sectorial operators and forms.

H complex Hilbert space, $\theta \in [0, \pi/2)$

(19.1) Theorem. A an operator on H

Equ.

(i) $-A$ generates a θ-contractive C_0-sg

(ii) a) $(Ax|x) \in \Sigma_\theta$ $\forall x \in D(A)$ &

b) $I + A$ is surjective

Proof. This is (18.2) since

$e^{\pm i\theta}(-A)$ diss $\iff$ $\mathrm{Re}\,(e^{\pm i\theta}Ax|x) \geq 0$ $\forall x \in D(A)$ $\iff$ $(Ax|x) \in \Sigma_\theta$ $\forall x \in D(A)$. □

Rk. a) $\iff$ $W(A) \subset \Sigma_\theta$.

$W(A) \subset \Sigma_0 \iff A$ symmetric.

(19.2) <u>Definition</u>. A an operator on H.

a) Let $\theta \in [0, \pi/2)$. A is θ <u>sectorial</u> if $(Ax \mid x) \in \Sigma_\theta \quad \forall x \in D(A)$;

A is <u>m-θ-sectorial</u> if in addition $(I + A)$ is surjective

b) A is <u>sectorial</u> if $\exists\, \theta \in [0, \pi/2)$ such that A is θ sectorial

c) A <u>m-sectorial</u> $:\Longleftrightarrow$ $\exists\, \theta \in [0, \pi/2)$ s.t. A is m-θ sectorial, i.e. θ-sectorial and $(I+A)$ surjective.

(19.3) ~~Lax Milgram~~ Propositi[on]
<u>Replacing (19.1)</u>

$-A$ generates a ~~holom~~ θ'-contractive C_0-sg $\Longleftrightarrow$ A is d.d. & m-θ-sectorial

$\theta = \frac{\pi}{2} - \theta'$ or

<u>Lumer Phillips complex:</u>
<u>Corollary</u>: A m-sectorial $\Longleftrightarrow$ $-A$ generates a contractive hol. sg

(1.9.4) Forms.

Let $a : D(a) \times D(a) \to \mathbb{C}$ be sesquilinear, $D(a)$ a vector space.

a symmetric $:\Leftrightarrow$ $a(u,v) = \overline{a(v,u)}$

$\Rightarrow$ $a(u) \in \mathbb{R}$ $\forall u \in D(a)$.

$a^*(u,v) = \overline{a(v,u)}$ is also ~~sec~~ sesquilinear.

$$h(u,v) := \frac{a + a^*}{2}$$

$$k(u,v) := \frac{a - a^*}{2i}$$

are symmetric

$a = h + ik$ $\quad$ $\operatorname{Re} a := h$, $\operatorname{Im} a := k$

a <u>sectorial</u> $:\Leftrightarrow$ $\exists\, \theta \in [0, \pi/2)$

$a(u) \in \Sigma_\theta$ $\forall u \in D(a)$

$\Leftrightarrow$ $\exists c > 0$ $\quad \underline{|k(u)|} \leq c\, h(u)$ $\quad u \in D(a)$
$\cancel{h(u)}$

a <u>accretive</u> $:\Leftrightarrow$ $\operatorname{Re} a(u) \geq 0$ $\forall u \in D(a)$

(19.5) Polarization:

$$a : D(a) \times D(a) \to \mathbb{C} \quad \text{form}$$

1. $a(u,v) = \frac{1}{4}\left[a(u+v) - a(u-v) + i\,a(u+iv) - i\,a(u-iv)\right]$

2. a sym. $\Rightarrow$ $a(u+iv),\ a(u-iv) \in \mathbb{R}$ $\Rightarrow$

$$\mathrm{Re}\, a(u,v) = \frac{1}{4}\left[a(u+v) - a(u-v)\right]$$

(19.6) Cauchy-Schwarz. $a, h : V \times V \to \mathbb{C}$ forms. h sym. & accretive. &

$$|a(u)| \leq M\, h(u) \qquad (u \in V)$$

a) a sym. $\Rightarrow$

$$|a(u,v)| \leq M\, h(u)^{1/2}\, h(v)^{1/2}$$

Pf. a) a sym.

$$|a(u,v)| = \mathrm{Re}\, e^{i\theta} a(u,v)$$

$$= \mathrm{Re}\, a(\tilde{u}, v) \qquad \tilde{u} = e^{i\theta} u$$

$$= \frac{1}{4} \left[a(\tilde{u}+v) - a(\tilde{u}-v) \right]$$

$$\leq \frac{1}{4} M \left[h(\tilde{u}+v) + h(\tilde{u}-v) \right]$$

$$= \frac{1}{4} M \left[2h(\tilde{u}) + 2h(v) \right]$$

$$= \frac{M}{2} \left[h(u) + h(v) \right]$$

$$u \mapsto \sqrt{\varepsilon}\, u \qquad v \mapsto \frac{1}{\sqrt{\varepsilon}} v$$

$$|a(u,v)| = \frac{M}{2} \left[\varepsilon h(u) + \frac{1}{\varepsilon} h(v) \right]$$

$$\varepsilon = \frac{h(v)^{1/2}}{h(u)^{1/2}} \qquad = \frac{M}{2}\, 2\, h(v)^{1/2} h(u)^{1/2}.$$

b) a arbitrary $\qquad a = a_1 + i a_2 \qquad a_j$ sym.

~~$|a(u,v)|$~~ $\quad |a_1(u)| \leq |a(u)| \leq M\, h(u)$

$|a_2(u)| \leq M h(u)$

a) $\Rightarrow$ ~~$|a(u)| \leq ?$~~

$$|a(u,v)| \leq |a_1(u,v)| + |a_2(u,v)|$$

$$\leq 2M\, h(u)^{1/2} h(v)^{1/2} \qquad \text{by a)}. \quad \square$$

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