Reminder:

A form $a : D(a) \times D(a) \longrightarrow \mathbb{C}$ is sectorial if $\exists\, \theta \in [0, \frac{\pi}{2}) : a(u) \in \Sigma_\theta \ \forall u \in D(a)$.

($\Leftrightarrow \exists\, c > 0 : |(\mathrm{Im}\, a)(u)| \leq c \cdot (\mathrm{Re}\, a)(u) \ \forall u \in D(a)$)

(19.7) Definition

Let H be a complex Hilbert space. A sectorial form $a : D(a) \times D(a) \longrightarrow \mathbb{C}$ is a sectorial form on H if $D(a)$ is a subspace of H.

(19.8) Consequence

(i) $(u|v)_a := (\mathrm{Re}\, a)(u,v) + (u|v)_H$ for $u, v \in D(a)$ defines a scalar product on $D(a)$.

(ii) $\|u\|_a := (u|u)_a = ((\mathrm{Re}\, a)(u) + \|u\|_H^2)^{\frac{1}{2}}$ for $u \in D(a)$ defines a norm on $D(a)$.

(iii) a is continuous with respect to $\|\cdot\|_a$, i.e., $\exists\, M \geq 0 : |a(u,v)| \leq M \cdot \|u\|_a \cdot \|v\|_a$ $\forall u, v \in D(a)$.

Proof: (i), (ii) are obvious

Since a is accretive, $\mathrm{Re}\, a$ is symmetric and $\exists\, c > 0$:

$$|a(u)|^2 = (\mathrm{Re}\, a(u))^2 + (\mathrm{Im}\, a(u))^2$$
$$\leq (\mathrm{Re}\, a(u))^2 + c^2 (\mathrm{Re}\, a(u))^2 = (1+c^2)\, \mathrm{Re}\, a(u)^2$$

the Cauchy-Schwartz inequality (19.6) implies

$$|a(u,v)| \leq \underbrace{2\sqrt{1+c^2}}_{=: M} (\mathrm{Re}\, a(u))^{\frac{1}{2}} (\mathrm{Re}\, a(v))^{\frac{1}{2}}$$
$$\leq M \cdot \|u\|_a \cdot \|v\|_a \quad \forall u, v \in D(a) \ \square$$

(19.9) Definition

A sectorial form a on a Hilbert space H is...

(a) ... closed if $(D(a), \|\cdot\|_a)$ is complete.

(b) ... densely defined if $D(a)$ is dense in H.

(19.10) Definition

Let a be a d.d., closed, sectorial form on a Hilbert space H. We define the operator A associated with a by

$$D(A) := \{x \in D(a) : \exists y \in H : a(x,v) = (y|v)_H \ \forall v \in D(a)\}$$

$Ax := y$ for $y \in H$ with " " ", $x \in D(A)$.

Notation: $A \sim a$

(19.11) Remark

A is well-defined since $\overline{D(a)}^H = H$.

(19.12) Theorem

Let a be a d.d., closed, sectorial form on a Hilbert space H, $A \sim a$.

$\Rightarrow$ A is m-sectorial

($\Leftrightarrow$ $-A$ generates a contractive hol. sgr.)

For the proof we need the Lax-Milgram-Theorem.

(19.13) Definition

Let V be a complex Hilbert space.

An anti-linear mapping $L : V \longrightarrow \mathbb{C}$ is continuous if $\exists c \geq 0 : |Lv| \leq c\|v\| \ \forall v \in V$.

We set $V^* := \{L : V \longrightarrow \mathbb{C} : L$ antilinear $+$ continuos$\}$

(19.14) **Example**

Let $y \in V$ and define $L_y v := (y|v)_V \; \forall v \in V$.

$\Rightarrow L \in V^*$

(19.15) **Lax-Milgram-Theorem**

Let V be a complex Hilbert space, $a: V \times V \longrightarrow \mathbb{C}$ a sesquilinear form which is continuous and coercive (i.e. $\exists \alpha > 0: \operatorname{Re} a(u) \geq \alpha \|u\|_V^2 \; \forall u \in V$).

Then: $\exists B \in \mathcal{L}(V)$ invertible with

$$a(u,v) = (Bu|v) \quad \forall u,v \in V.$$

In particular: $\forall L \in V^* \, \exists w \in V$ with

$$Lv = a(w,v) \quad \forall v \in V.$$

Proof:

Let $u \in V$. By Riesz-Fréchet there is a unique $Bu \in V$ with

$$a(u,v) = (Bu|v) \quad \forall v \in V.$$

and $B: V \longrightarrow V$ is clearly linear + injective.

* B is continuous: $\exists M \geq 0$ such that

$$\|Bu\|^2 = (Bu|Bu) = a(u, Bu) \leq M \cdot \|u\| \cdot \|Bu\| \quad \forall u \in V.$$

$\Rightarrow \|Bu\| \leq M \cdot \|u\| \quad \forall u \in V.$

* ~~B~~ rg B is closed:

$\alpha \|u\|^2 \leq \operatorname{Re} a(u) = \operatorname{Re} (Bu|u) \leq \|Bu\| \cdot \|u\|$

$\Rightarrow \alpha \|u\| \geq \|Bu\| \quad \forall u \in V$

This shows that B is closed.

* rg B is dense:

Take $v \in rg(B)^{\perp}$. $\Rightarrow 0 = (Bu|v) \; \forall u \in V$

$\Rightarrow a(v) = (Bv|v) = 0 \Rightarrow v = 0$

Now take $L \in V^*$.

$\overset{\text{Riesz-Fréchet}}{\Longrightarrow}$ $\exists u \in V$ with

$Lv = (u|v)_V \quad \forall v \in V.$

$\Rightarrow Lv = (BB^{-1}u|v)_V = a(\underbrace{B^{-1}u}_{=:w}, v) \quad \forall v \in V.$ $\square$

<u>Proof (of (19.12))</u>:

Let $x \in D(A)$. $\Rightarrow (Ax|x) = a(x) \in \Sigma_\theta$ for some $\theta \in [0, \frac{\pi}{2})$.

$\Rightarrow A$ sectorial.

We now show: $I + A$ is surjective.

Let $y \in H$ and define

~~$L_y v := (y|v)_H \quad \forall v \in D(a)$~~

$L_y v := (y|v)_H \quad \forall v \in D(a)$

$\Rightarrow L_y$ is antilinear and continuous since

$|L_y v| \le \|y\|_H \cdot \|v\|_H \le \|y\|_H \cdot \|v\|_a$

$\forall v \in D(a)$.

$\tilde{a}(u,v) := a(u,v) + (u|v)_H \quad \forall u, v \in D(a)$

defines a continuous + coercive form on $(D(a), \|\cdot\|_a)$.

$\overset{LM}{\underset{(19.15)}{\Longrightarrow}}$ $\exists x \in D(a)$:

$(y|v)_H = L_y v = \tilde{a}(x,v) = a(x,v) + (x|v)_H \quad \forall v \in D(a).$

$\Rightarrow \quad a(x,v) = (y - x|v)_H \quad \forall v \in D(a).$

$\Rightarrow x \in D(A)$ and $Ax = y - x$,

i.e. $y \in rg(I + A)$ $\square$

(19.16) <u>Definition</u>

Let a be a closed, sectorial form on a Hilbert space H. A subspace W of $D(a)$ is a <u>form core</u> if W is dense in $(D(a), \|\cdot\|_a)$.

(19.17) <u>Proposition</u>

Let a be a d.d., closed, sectorial form on a Hilbert space H, $A \sim a$.

$\Longrightarrow$ $D(A)$ is a form core for a.

<u>Proof</u>:

By Lax-Milgram we find $B \in \mathcal{L}(D(a))$ invert. with

$$a(u,v) + (u|v)_H = (Bu|v)_a \quad \forall v \in D(a).$$

It suffices to show that $B\,D(A)$ is dense in $D(a)$.

Let $w \in [B\,D(A)]^{\perp}$.

$\Rightarrow$ $0 = (Bu|w)_a = a(u,w) + (u|w)_H$

$= (Au+u|w)_H \quad \forall u \in D(A)$.

$\overset{I+A\text{ surj.}}{\Longrightarrow}$ $w = 0$. $\square$

We now prove the converse of (19.13).

(19.18) <u>Theorem</u>

Let A be m-sectorial on a Hilbert space H.

$\Rightarrow$ $\exists!$ d.d., closed, sectorial form a on H with $A \sim a$.

<u>Proof</u>:

"<u>Existence</u>":

Define $a: D(A) \times D(A) \longrightarrow \mathbb{C}$ by

$$a(u,v) := (Au|v)_H \quad \forall u,v \in D(A).$$

$\overset{A\text{ sectorial}}{\Longrightarrow}$ a sectorial

By (19.8)(iii) a is continuous on $(D(A), \|\cdot\|_a)$. $\Rightarrow$ $\exists M > 0: |a(u,v)| \leq M \|u\|_a \cdot \|v\|_a$ $\forall u,v \in D(A)$.

Now take the completion

$$D(\bar{a}) := (D(A), \|\cdot\|_a)^{\sim}.$$

By the universal property of the completion there is a unique $j \in \mathcal{L}(D(\bar{a}), H)$ with $j(u) = u \quad \forall u \in D(A)$:

$$\begin{tikzcd} D(\bar{a}) \arrow[r, "j"] & H \\ D(a) \arrow[u, hook] \arrow[ur, "u \mapsto u"'] \arrow[r, phantom, "\circlearrowleft"] & \end{tikzcd}$$

We show: j is injective.

Let $u \in D(\bar{a})$, $j(u) = 0$.

$\Rightarrow \exists\, u_n \in D(A)$ with $u_n \longrightarrow u$

$\Rightarrow (u_n)_{n\in\mathbb{N}}$ is Cauchy in $D(A)$ and

$u_n \longrightarrow 0$ in H.

Choose $\tilde{M} > 0$: $\|u_n\|_a \leq \tilde{M} \quad \forall n \in \mathbb{N}$.

Now let $\varepsilon > 0$ and take $N \in \mathbb{N}$ with

$$\|u_n - u_m\|_a \leq \frac{\varepsilon}{M \cdot \tilde{M}} \quad \forall n, m \geq N.$$

We obtain

$$\begin{aligned} |a(u_n)| &\leq |a(u_n, u_n - u_m)| + |a(u_n, u_m)| \\ &\leq M \|u_n\| \cdot \|u_n - u_m\| + |(Au_n | u_m)_H| \\ &\leq M \cdot \tilde{M} \cdot \frac{\varepsilon}{M \cdot \tilde{M}} + \|Au_n\|_H \cdot \underbrace{\|u_m\|_H}_{\downarrow\, 0} \quad \forall n, m \geq N \end{aligned}$$

$\overset{m \to \infty}{\Rightarrow} |a(u_n)| \leq \varepsilon \quad \forall n \geq N.$

$\Rightarrow a(u_n) \xrightarrow{n \to \infty} 0$

$\Rightarrow \|u_n\|_a^2 = \operatorname{Re} a(u_n) + \|u_n\|_H^2 \xrightarrow{n \to \infty} 0$

$\Rightarrow u = 0 \quad \Rightarrow j$ is injective.

We identify $D(\bar{a})$ with the subspace $j(D(\bar{a})) \subseteq H$.

There is a unique cont. extension

$$\overline{a} : D(\overline{a}) \times D(\overline{a}) \longrightarrow \mathbb{C}$$

of a to $D(\overline{a}) \times D(\overline{a})$.

Choose $\theta \in [0, \frac{\pi}{2})$ with. $a(u) \in \overline{\Sigma_\theta}$ $\forall u \in D(A)$.

For $u = \lim\limits_{n\to\infty} \underset{D(A)}{\overset{\in}{u_n}} \in D(\overline{a})$ we obtain

$$\overline{a}(u) = \lim_{n\to\infty} a(u_n) \in \overline{\Sigma_\theta}$$

$\Rightarrow$ $\overline{a}$ is sectorial.

Let $B \sim \overline{a}$. Take $u \in D(A)$.

Since $\overline{a}(u,v) = (Au|v)_H$ $\forall v \in D(A)$

we obtain by density of $D(A)$ in $D(\overline{a})$:

$$\overline{a}(u,v) = (Au|v)_H \quad \forall v \in D(\overline{a})$$

$\Rightarrow$ $u \in D(B)$ and $Bu = Au$.

$\Rightarrow$ $A \subseteq B$ $\underset{-A,\,-B \text{ generators}}{\Longrightarrow}$ $A = B$.

"Uniqueness":

Let b be a closed, d.d., sectorial form on H with $A \sim b$.

$\Rightarrow$ $a(u,v) = (Au|v)_H = b(u,v)$ $\forall u,v \in D(A)$. $(*)$

In particular we have

$$\|u\|_a = \|u\|_b \quad \forall u \in D(A).$$

By definition of $\overline{a}$, $D(A)$ is dense in $(D(\overline{a}), \|\cdot\|_a)$. By (19.17), $D(A)$ is dense in $(D(b), \|\cdot\|_b)$.

$\overset{(*)}{\Longrightarrow}$ $a = b$ $\square$