Let $-\infty < a < b < \infty$.

1

(24.7) Definition

* $f \in L^1((a,b);X)$ is weakly differentiable if $\exists\, f' \in L^1((a,b);X)$ with
$$-\int_a^b \varphi'(t) f(t)\,dt = \int_a^b \varphi(t) f'(t)\,dt$$
for each $\varphi \in C_c^\infty((a,b))$.
In this case f' is the weak derivative of f.

* We set
$$W^{1,1}((a,b);X) := \{ f \in L^1((a,b);X) : f \text{ weakly diff.} \}$$

(24.8) Remark

By setting
$$\|u\|_{W^{1,1}} := \int_a^b \|u(t)\|\,dt + \int_a^b \|u'(t)\|\,dt \quad \text{for}$$
$u \in W^{1,1}((a,b);X)$ we obtain a complete norm $\|\cdot\|_{W^{1,1}}$ on $W^{1,1}((a,b);X)$.

(24.9) Theorem

(i) Each $u \in W^{1,1}((a,b);X)$ can be identified with a unique continuous representative $u \in C([a,b];X)$.
and
$$u(t) = u(s) + \int_s^t u'(r)\,dr \quad \forall\, a \le s < t \le b.$$

(ii) For $v \in L^1((a,b);X)$ and $x \in X$ we obtain $u \in W^{1,1}((a,b);X)$ by setting
$$u(t) := x + \int_a^t v(s)\,ds \quad \forall\, t \in (a,b)$$
and $u' = v$.

(iii) If $u \in W^{1,1}((a,b);X)$ then $\frac{d}{dt}u(t)$ exists almost everywhere and $u'(t) = \frac{d}{dt}u(t)$ a.e.

Now let A be generator of C_0-sgr. T on Bspace X.

(24.10) **Theorem**

Let $x \in D(A)$, $f \in W^{1,1}((0,J); X)$, $J \geq 0$ and u the corresponding mild solution of E with $u(0)=x$

(i.e. $u(t) = T(t)x + T * f(t)$ $\forall t \geq 0$.

$\Rightarrow$ $u \in C^1([0,J]; X)$, i.e. u is a classical solution

(24.11) **Remark** We now in case of $f=0$:

$T(t)x$ is the classical solution of

$$\begin{cases} \dot{u}(t) = Au(t), \ t \geq 0, \\ u(0) = x. \end{cases}$$

if $x \in D(A)$.

Proof (of Theorem 24.10)

We may assume $x=0$.

$$\Rightarrow u(t) = \int_0^t T(t-s) f(s)\,ds$$

$$= \underbrace{\int_0^t T(t-s) f(0)\,ds}_{1} + \overbrace{\int_0^t T(t-s) \int_0^s f'(r)\,dr\,ds}^{2} \quad \forall t \geq 0$$

with

$$1 \underset{r=t-s}{=} \underline{\int_0^t T(r) f(0)\,dr} \in D(A)$$

and

$$2 \underset{\text{Fubini}}{=} \int_0^t \int_r^t T(t-s) f'(r)\,ds\,dr$$

$$\underset{w=t-s}{=} \int_0^t \int_0^{t-r} T(w) f'(r)\,ds\,dr$$

By Lemma (24.5) we obtain
$u(t) \in D(A)$ $\forall t \in [0,T]$ and

$$Au(t) = T(t)f(0) - f(0) + \int_0^t A\left(\int_0^{t-r} T(w) f'(r)\,dw\right)dr$$
$$= T(t)f(0) - f(0) + \int_0^t T(t-r)f'(r)\,dr - \int_0^t f'(r)\,dr \quad \forall t \in [0,T]$$

$\Rightarrow Au \in C([0,T];X)$.

~~The~~ Lemma (24.12) now proves the claim. $\square$

(24.12) Lemma

Let $f \in C([0,T;X)$ and u a mild solution of (E) on $[0,T]$, $u(0)=0$.
If $u(t) \in D(A)$ and $Au(t) \in C([0,T];X)$ $\forall t \in [0,T]$, then u is classical solution.

Proof:
By (24.5) we obtain

$$u(t) = A\int_0^t u(r)\,dr + \int_0^t f(r)\,dr$$
$$= \int_0^t Au(r)\,dr + \int_0^t f(r)\,dr \quad \forall t \in [0,T]$$

$\Rightarrow$ u is differentiable with
$u'(t) = Au(t) + f(t)$ $\forall t \in [0,T]$. $\square$

Exercise

Let $x \in D(A)$, $f \in L^1(0,T,X)$ and u be the corresponding mild solution of (E).
If $f(t) \in D(A)$ $\forall t \in [0,T]$ and $Af \in L^1((0,T);X)$, then u is a classical solution.

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(24.13) <u>Theorem</u>

For a closed operator A the following are equivalent

(a) $\forall x \in X$: $\exists!$ mild solution of

$$\dot{u}(t) = Au(t) \quad \forall t \geq 0,$$
$$u(0) = x.$$

(b) A generates a C_0-sgr.

In that case: the mild solution for $x \in X$ in (a) is $T(\cdot)x$ where T is the C_0-sgr. generated by A.

<u>proof</u>: Exercise.

§25. Periodic solutions

<u>Setting</u>:

Let A be the generator of a C_0-sgr. T
5 on a Banach space X.
For $f \in L^1((0,1);X)$ we consider

(E) $\quad u(t) = Au(t) + f(t) \quad \forall t \in [0,1]$.

The mild solutions of (E) are given by

$$u(t) = T(t)x + T * f\,(t) \qquad \forall t \in [0,1]$$

where $x \in X$.

(25.1) <u>Definition</u>

A mild solution u of (E) is <u>periodic</u> if $u(0) = u(1)$.

(25.2) <u>Remark</u>

(E) has a periodic mild solution
$\Leftrightarrow \exists x \in X: \quad x - T(1)x = (T * f)(1)$.

(25.3) <u>Proposition</u> („uniqueness").

The following are equivalent:

(a) $\forall f \in L^1((0,1);X)$: $\exists$ at most one periodic mild solution of (E).

(b) The homogenous problem
$$u'(t) = Au(t) \quad \forall t \in [0,1],$$
has ~~[illegible]~~ exactly one ~~solution~~ periodic mild solution.
(namely $u = 0$).

(c) $1 \notin \sigma_p(T(1))$.

Proof:

"(a) ⇒ (b)": obvious. ($f=0$).

"(b) ⇒ (c)": Take $x \in X$ with $x - T(1)x = 0$.

The mild solution u with initial value x for $f=0$ (i.e., $u(t) = T(t)x \ \forall t \in [0,1]$) is periodic.

$\overset{}{\underset{(b)}{\Longrightarrow}} \ u = 0 \Rightarrow x = 0.$

"(c) ⇒ (a)": Take two periodic solutions u, v of (E) for some $f \in L^1((0,1);X)$.

Then $x := u(0) - v(0)$ satisfies

$$x - T(1)x = 0.$$

$\underset{(c)}{\Longrightarrow} \ u(0) = v(0) \Rightarrow u = v.$ □

(25.4) <u>Theorem</u>

The following are equivalent.

(a) $\forall f \in L^1((0,1);X)$: $\exists!$ periodic mild sol. of (E).

(b) $\forall f \in C([0,1];X)$: $\exists!$ " " " " "

(c) $1 \in \varrho(T(1))$.

Proof: "(a) ⇒ (b)": ✓

"(b) ⇒ (c)": By (25.3) we know that

$$I - T(1) \text{ is injective.}$$

Now let $x \in X$ and $f(t) := T(t)x \ \forall t \in [0,1]$.

$\Rightarrow \exists$ periodic mild solution for $(E)_f$

$\Rightarrow \exists y \in Y$: $\quad y - T(1)y = T * f(1)$

Since

$$T * f(t) = \int_0^t T(t-s)T(s)x\,ds = \int_0^t T(t)x\,ds = tT(t)x$$

we obtain $\quad y - T(1)y = T(1)x.$

$\Rightarrow \quad (x+y) - T(1)(x+y) = x$

$\Rightarrow \quad (I - T(1))$ surjective

$\Rightarrow \quad 1 \in \varrho(T(1))$.

„(c) $\Rightarrow$ (a)": Let $f \in L^1((0,1);X)$.

and $\quad x := (I - T(1))^{-1} (T * f)(1)$

$\Rightarrow \quad x - T(1)x = (T * f)(1)$.

$\Rightarrow \quad (E)$ has a periodic mild sol.

$\Rightarrow \quad (a)$. $\square$

(25.3): uniqueness

(25.5) Proposition

Let $x \in X$, with $\cancel{T(1)x}$, $x - T(1)x = T * f(1)$

$f: \mathbb{R}_+ \longrightarrow X$ 1-periodic with

$f|_{(0,1)} \in L^1((0,1);X)$.

Set $u(t) := T(t)x + (T * f)(t) \quad \forall t \geq 0$.

~~u is periodic~~

$\Rightarrow \quad u(t+1) = u(t) \quad \forall t \geq 0$.

Proof: Define $w(t) := u(t-n)$ for $t \in [n, n+1)$.

$\Rightarrow \quad w \in C([0,\infty);X)$ and $w(t+1) = w(t) \; \forall t \in [0,\infty)$.

Claim: w is mild solution of

$$w'(t) = Aw(t) + f(t) \quad \forall t \geq 0.$$

Let $t \geq 0$, $t \in [n, n+1)$.

$$\Rightarrow \int_0^t w(s)\,ds = \sum_{k=1}^{n} \int_{k-1}^{k} w(s)\,ds + \int_n^t w(s)\,ds$$

$$= \sum_{k=1}^{n} \int_0^1 w(s+k-1)\,ds + \int_0^{t-n} w(s+n)\,ds$$

$$= \sum_{k=1}^{n} \int_0^1 u(s)\,ds + \int_0^{t-n} u(s)\,ds. \in D(A)$$

and

$$A\int_0^t w(s)\,ds = \overbrace{n\cdot(u(1)-u(0))}^{=0} - \sum_{k=1}^{n}\int_0^1 f(s)\,ds$$

$$+\, u(t-n) - u(0) - \int_0^{t-n} f(s)\,ds$$

$$= w(t) - w(0) - \int_0^t f(s)\,ds$$

This proves the claim.

$(24.*) \Rightarrow \quad w(t) = T(t)\,u(0) + T*f\,(t) \quad \forall t \geq 0,$

i.e. $w = u$.

$\square$