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(Schmanzahl)
Primzahlen

§ 30 Interpolation

$M \subset \mathbb{C}$ $\qquad C^b(M) := \{ f : M \to \mathbb{C} : \text{bounded continuous} \}$

$\mathrm{Hol}(\Omega) := \{ f : \Omega \to \mathbb{C} \text{ hol.} \}$

$\Omega \subset \mathbb{C}$ open.

(30.1) **Maximum Principle:** $\Omega \subset \mathbb{C}$ open bded, $f \in \mathrm{Hol}(\Omega) \cap C(\bar{\Omega})$

$$\Rightarrow \quad \|f\|_{\bar{\Omega}} = \|f\|_{\partial\Omega}$$

$$\|f\|_M = \sup_{z \in M} |f(z)| \qquad M \subset \mathbb{C}.$$

(30.2) **Theorem (maximum principle on S)**

$S := \{ z \in \mathbb{C} : 0 < \mathrm{Re}\, z < 1 \}$

$h \in C^b(\bar{S}) \cap \mathcal{H}(S) \quad \Rightarrow$

$$\|h\|_{\bar{S}} = \|h\|_{\partial S}$$

<u>Pf:</u> $S_k = \{z \in S : |\operatorname{Im} z| \le k\}$

$\psi_n(z) = \dfrac{n}{z+n}$ $\quad \psi_n \in \mathcal{H}ol(S) \cap C^0(\bar{S})$

$$\|\psi_n h\|_{\bar{S}_k} = \|\psi_n h\|_{\partial S_k}$$

$$\|\psi_n\|_{\{|\operatorname{Im} z| \ge k\}} \longrightarrow 0 \quad k \to \infty$$

Let $n \in \mathbb{N}$. $\quad \exists k \in \mathbb{N}$

$$\|\psi_n h\|_{\bar{S}} = \|\psi_n h\|_{S_k}$$

$$\|\psi_n h\|_{\partial S} = \|\psi_n h\|_{\partial S_k}$$

$$\Rightarrow \quad \|\psi_n h\|_{\bar{S}} = \|\psi_n h\|_{\partial S} \le \|h\|_{\partial S}$$

$\forall n \in \mathbb{N}$

$$n \to \infty \Rightarrow \quad \|h\|_{\bar{S}} \le \|h\|_{\partial S}. \quad \square$$

(30.3) <u>Corollary</u> (3-lines Theorem).

Let $h \in C^b(\bar{S}) \cap \mathcal{H}ol(S)$

Then $\|h\|_{\{Re z = \tau\}} \le \|h\|^{1-\tau}_{\{Re z = 0\}} \cdot \|h\|^{\tau}_{\{Re z = 1\}}$

<u>Proof.</u> Let $b_j > \|h\|_{\{Re z = j\}}$ $j = 0, 1$.

$$H(z) := \left(\frac{b_0}{b_1}\right)^z h(z)$$

$\overset{30.2}{\Rightarrow}$ $\left(\frac{b_0}{b_1}\right)^{\tau} \|h\|_{\{Re z = \tau\}} \le$

$$\max\left\{\left(\frac{b_0}{b_1}\right)^0 b_0 , \left(\frac{b_0}{b_1}\right)^1 b_1\right\} = b_0$$

$\Rightarrow$ $\|h\|_{\{Re z = \tau\}} \le b_1^{\tau} b_0^{1-\tau}$. $\square$

§ 31 The Stein interpolation theorem

(Ω, Σ, μ) σ-finite.

$\Sigma_c := \{ B \in \Sigma : \mu(B) < \infty \}$

$S(\Sigma_c) := \text{lin}\{ 1_A : A \in \Sigma_c \}$

simple functions.

$L^1_{loc}(\Omega) = \{ u \cancel{\in L^1} : \Omega \to \mathbb{C}$ meas.

$u 1_A \in L^1(\mu) \quad \forall A \in \Sigma_c \}$

Rk. $u \in L^1_{loc}(\Omega),\ v \in S(\Sigma_c) \Rightarrow uv \in L^1(\Omega, \mu)$

$p_0, p_1, q_0, q_1 \in [1, \infty]$, $M_0, M_1 \geq 0$

$\tau \in \cancel{\mathbb{Z}}\ [0,1]$

$$\frac{1}{p_\tau} = \frac{1-\tau}{p_0} + \frac{\tau}{p_1} \qquad \frac{1}{q_\tau} = \frac{1-\tau}{q_0} + \frac{\tau}{q_1}$$

$$M_\tau = M_0^{1-\tau} M_1^\tau$$

$$\Phi : \overline{S} \longrightarrow L(S(\Sigma_c), L^1_{loc})$$

$L(E,F) = \{ T : E \to F \text{ linear} \}$

(a) $\| \Phi(j + is) u \|_{q_j} \leq M_j \|u\|_{p_j}$

$j = 0, 1,\ s \in \mathbb{R},\ u \in S(\Sigma_c)$

(ii) $A, B \in \Sigma_c \Rightarrow$

$$z \in \overline{S} \mapsto \int_\Omega (\Phi(z) 1_A) 1_B \, d\mu$$

$\in \mathcal{H}ol(S) \cap C^b(\overline{S})$.

(31.1) Theorem (Stein)

$$\| \Phi(\tau + is) u \|_{q_\tau} \leq M_\tau \|u\|_{p_\tau}$$

$\forall u \in S(\Sigma_c),\ s \in \mathbb{R},\ \tau \in [0,1]$

(31.2) Corollary (Riesz - Thorin)

$B \in L(S(\Sigma_c), L^1_{loc}(\Omega))$

$\|Bu\|_{q_j} \leq M_j \|u\|_{p_j}$