§ 31 Interpolation of semigroups.

(Ω, Σ, μ). $p_1 \in [1, \infty)$, $\theta \in (0, \frac{\pi}{2}]$ $\mathbb{K} = \mathbb{C}$

T bound hol C_0-sg of angle θ,

$M_1 := \sup_{z \in \Sigma_\theta} \| T(z) \|_{\mathcal{L}(L^{p_1})} < \infty.$

$p_0 \in [1, \infty]$, $p_0 \neq p_1$.

$$\| T(t) u \|_{p_0} \leq M_0 \|u\|_{p_0} \qquad u \in L^{p_1} \cap L^{p_0}$$

$\tau \in (0,1)$

$$M_\tau := M_0^{1-\tau} M_1^\tau, \quad \theta_\tau = \tau \theta,$$

$$\frac{1}{p_\tau} = \frac{1-\tau}{p_0} + \frac{\tau}{p_1}$$

(31.1) Theorem. For $z \in \Sigma_{\theta_\tau}$ $\exists!$ $T_{p_\tau}(z) \in \mathcal{L}(L^{p_\tau})$ consistent with $T(z)$. Moreover,

$$\| T_\tau(z) \| \leq M_\tau \qquad (z \in \Sigma_{\theta_\tau})$$

and

$$(T_\tau(z))_{z \in \Sigma_{\theta_\tau}}$$

is a hol. C_0-sg of angle θ_τ.

<u>Proof.</u> Let $0 < \theta' < \theta$

$$\psi : \bar{S} \longrightarrow \overline{\Sigma_{\theta'}} \setminus \{0\}$$
$$z \longmapsto e^{i\theta' \cdot z}$$

is bijective, holomorphic on S.

$[z = is + \sigma \qquad e^{i\theta' \cdot z} = e^{-\theta' s} e^{i\theta' \sigma} \quad]$

$$\Phi := T \circ \psi : \bar{S} \to L(\mathcal{S}, L^1_{loc}(\Omega))$$

$$z \mapsto \int \Phi(z) 1_B \, 1_C = \int_\Omega T(e^{i\theta' z}) 1_A \, 1_C \, d\mu$$

is holomorph and $\in C(\bar{S})$.

Stein $\Rightarrow$

$$\| T(\psi(\tau + is)) u \|_{p_\tau} \le M_\tau \|u\|_{p_\tau}.$$

$$\psi(\tau + is) = e^{i\theta'(\tau + is)} = e^{-\theta' s} e^{i\theta' \tau} \in \Sigma_{\theta_\tau}$$

$$w \in \Sigma_{\theta_\tau} \Rightarrow \exists \theta' \; \exists s \quad \psi(\tau + is) = w$$

$$\Rightarrow \quad \| T(w)u \|_{p_\tau} \leq M_\tau \|u\|_{p_\tau} \qquad u \in \mathcal{S}$$

$$w \in \Sigma_{\theta_\tau}.$$

$\Rightarrow \quad \exists\ T_{p_\tau}(w) \in \mathcal{L}(L^{p_\tau})$ consistent with

$T(w)$

$w \mapsto \int T(w)(1_A)\, 1_B$ holomorphic $\Rightarrow$

$T_{p_\tau} : \Sigma_{\theta_\tau} \to \mathcal{L}(L^{p_\tau})$ holomorphic.

Strong continuity.

$$\| T_{p_\tau}(t)f - f \|_{L^{p_\tau}} \leq \underbrace{\| T(t)f - f \|_{p_0}^{1-\tau}}_{\downarrow\, 0} \ \underbrace{\| T(t)f - f \|_{p_1}^{\tau}}_{\text{bded}}.$$

$$\frac{1}{p_\tau} = \frac{1-\tau}{p_0} + \frac{\tau}{p_1} \qquad t \downarrow 0$$

$\Rightarrow \quad T_{p_\tau}(t)f \to f$ in L^{p_τ} $\forall f \in \mathcal{S}$

$\mathcal{S}$ dense $\Rightarrow$ $T_{p_\tau}(t)f \to f$ $\quad t \downarrow 0$

$\forall f \in L^{p_\tau}$. $\square$

§ 32 Invariance and interpolation.

$S \in \mathcal{L}(L^2(\mu))$

S substochastic $:\Leftrightarrow$ $S \geq 0$ &

$\|Sf\|_1 \leq \|f\|_1$

S submarkovian $:\Leftrightarrow$ $f \leq 1 \Rightarrow Sf \leq 1$

$\Leftrightarrow$ $S \geq 0$ &

$\|Sf\|_\infty \leq \|f\|_\infty$

(32.1) Lemma. S submarkovian $\Leftrightarrow$ S^* substochastic.

$H = L^2(\mu)$ a a densely defined closed accretive form. $A \sim a$

$-A$ generates a C_0-sg. T

$\Leftrightarrow \quad a(\mathrm{Re}\, u, \mathrm{Im}\, u) \in \mathbb{R} \quad \forall u \in V$

$\Leftrightarrow \quad a(u,v) \in \mathbb{R} \quad \forall u,v \in V_{\mathbb{R}} := V \cap L^2(\Omega, \mathbb{R})$.

(b) We may assume that $\mathbb{K} = \mathbb{R}$

$C = \{ u \in L^2(\Omega; \mathbb{R}) : u \leq 1 \}$

$Pu = u \wedge 1$

$u - Pu = u - u \wedge 1 = (u-1)^+ \quad \square$

~~Thus ...~~

(32.4) Theorem. T C_0-sg on L^2

a) Assume T is substochastic & submarkovian.

Then $\forall p \in [1, \infty)$ $\exists$ C_0-sg T_p

s.t. a) $T_2 = T$

b) $T_p(t) f = T_q(t) f \quad \forall f \in L^p \cap L^q$

consistency.

b) T_p is holomorphic for $1 < p < \infty$

$\| T_p(t) f \|_p \leq \| f \|_p \qquad 1 \leq p \leq \infty.$

Pf. $\| T(t) f \|_\infty \leq \| f \|_\infty$

$\| T(t) f \|_1 \leq \| f \|_1$

Riesz Thorin $\Rightarrow$ $\| T(t) f \|_p \leq \| f \|_p$.

& Continuity at 0: Let $t_n \downarrow 0$

$u \in L^p_{\mathbb{R}}(\Omega) \cap L^2(\Omega)$ Then $T(t_n)u \to u$

in L^2. Hence $T(t_n)u \to u$ a.e. after ss.

Lemma 32.5 $\Rightarrow$ $T(t_n) u \to u$ in L^p. $\square$

(32.5) Lemma. Let $1 \leq p < \infty$ and

let $f_n, f \in L^p(\Omega)$ $f_n \to f$ a.e.

$\overline{\lim} \| f_n \|_p \leq \| f \|_p$ $\Rightarrow$ $f_n \to f$ in L^p.

Pf. $\tilde{f}_n := (\operatorname{sgn} f_n) (|f| \wedge |f_n|)$

$\tilde{f}_n(x) \to f(x)$ a.e.

[1st. case $f(x) \neq 0$ & $f_n(x) \to f(x)$

$\Rightarrow$ $\exists n_0$ $\forall n \geq n_0$ $|f_n(x)| \neq 0$ $\Rightarrow$

$$\operatorname{sgn} f_n(x) = \frac{f_n(x)}{|f_n(x)|} \longrightarrow \frac{f(x)}{|f(x)|}$$

$|f(x)| \wedge |f_n(x)| \longrightarrow |f(x)|$

hence $\tilde{f}_n(x) \longrightarrow f(x)$ ~~to~~ if the $f(x) \neq 0$

2nd case $f(x) = 0 \Rightarrow \tilde{f}_n(x) \to 0$.

Thus the claim follows from Lebesgue's Theorem.

$$|f_n| = |\tilde{f}_n| + |f_n - \tilde{f}_n|$$

$$\cancel{|f| \wedge |f_n|}$$

[Pf. a) $|f(x)| < |f_n(x)| \Rightarrow$

$$|\tilde{f}_n| + |f_n - \tilde{f}_n| = \underbrace{|f| \wedge |f_n|}_{= |f|} + \left| f_n - \frac{f_n}{|f_n|} |f| \right|$$

$$= |f| + |f_n| \left| 1 - \frac{|f|}{|f_n|} \right|$$

$$= |f| + \cancel{|f_n|} \, \big| |f_n| - |f| \big|$$

$$= |f| + |f_n| - |f| = |f_n|.$$

b) $|f(x)| \geq |f_n(x)| \Rightarrow$

$$|\tilde{f}_n| + |f_n - \tilde{f}_n| = |f_n| + \left| f_n - \frac{f_n}{|f_n|} |f_n| \right|$$

$$= |f_n| + \cancel{|f_n| \left| 1 - \frac{1}{|f_n|} \right|} \quad .]$$

$$\Rightarrow \quad |f_n|^p \geq |\tilde{f}_n|^p + |f_n - \tilde{f}_n|^p$$

$$\Big[\; c = a + b \quad \Rightarrow \quad c^p = (a+b)^p \geq a^p + b^p$$

$$\cancel{p(\log(a+b)) \geq \frac{1}{n}\log(a^p+b^p)}$$

$p \geq 1$

$$(a+b)^p = (a+b)(a+b)^{p-1}$$

$$= a(a+b)^{p-1} + b(a+b)^{p-1}$$

$$\geq a a^{p-1} + b b^{p-1} = a^p + b^p. \; \Big]$$

$$\Rightarrow \quad \|f_n - \tilde{f}_n\|_p^p \leq \int(|f_n|^p - |\tilde{f}_n|^p)$$

$$= \|f_n\|_p^p - \|\tilde{f}_n\|_p^p \longrightarrow 0$$

$$\Rightarrow \quad f_n = (f_n - \tilde{f}_n) + \tilde{f}_n \longrightarrow f \text{ in } L^p(\Omega). \; \square$$

§ 33 Elliptic operators.

$\Omega \subset \mathbb{R}^d$ open, $\cancel{\mathbb{K} = \mathbb{R}}$.

(33.1) **Lemma.** $u \in H^1_0(\Omega)$ real $\Rightarrow$

$u \wedge 1$, $(u-1)^+ \in H^1_0(\Omega)$ &

$D_j(u \wedge 1) = D_j u \, 1_{\{u<1\}}$

$D_j(u-1)^+ = D_j u \, 1_{\{u>1\}}$.

$a_{ij} \in L^\infty(\Omega)$ real valued, $0 < \alpha \leq 1$

$$\sum_{i,j=1}^d a_{ij}(x)\, \xi_i \overline{\xi_j} \geq \alpha |\xi|^2$$

$\forall \xi \in \mathbb{C}^d \quad \forall x \in \Omega$.

$H = L^2(\Omega)$

$D(a) = H^1_0(\Omega)$

$$a(u,v) = \int_\Omega \sum_{i,j=1}^d a_{ij}(x)\, \partial_i u(x)\, \overline{\partial_j v(x)}\, dx.$$

(33.2) <u>Lemma.</u> a is closed, accretive, dd.

<u>Proof.</u> $\mathrm{Re}\, a(u) \geq \alpha \int_\Omega |\nabla u|^2 \, dx$

$\Rightarrow$ accretive. &

$$\|u\|_a^2 = \mathrm{Re}\, a(u) + \|u\|_H^2 \geq \alpha \|u\|_{H^1}^2$$

Moreover,

$$\mathrm{Re}\, a(u) \leq c \int_\Omega \sum_{i,j=1}^d |\partial_i u|\,|\partial_j u| \, dx$$

$$= c \int_\Omega \Big(\sum_{j=1}^d |\partial_j u| \Big)^2 dx$$

$$\overset{\cancel{CS}}{\leq} c\, 2^d \int_\Omega \sum_{j=1}^d |\partial_j u|^2 \, dx$$

Hence norms are equivalent $\Rightarrow$

closed. □

Rk $\Big(\sum_{j=1}^d a_j\Big)^2 \leq 2^d \sum_{j=1}^d a_j^2$

[$d \Rightarrow d+1$ $\Big(\sum_{j=1}^{d+1} a_j\Big)^2 \leq 2 \Big(\sum^d a_j\Big)^2 + 2 a_{d+1}^2$

$(a+b)^2 \leq 2(a^2+b^2)$ $\leq 2\, 2^d \sum_{j=1}^d a_j^2 + 2 a_{d+1}^2$ □

(33.3) <u>Theorem.</u> There exist consistent C_0-sg T_p on L^p ~~consist~~ such that $T_2 = T$.

Moreover, $\|T_p(t)\| \leq 1$ $\quad 1 \leq p < \infty$,

Finally T_p is hol. for $1 < p < \infty$.

<u>Rk.</u> Even T_1 is holomorphic (more difficult)

ENDE