§ 31 Interpolation of semigroups.

(Ω, Σ, μ). $p_1 \in [1, \infty)$, $\theta \in (0, \frac{\pi}{2}]$ $\mathbb{K} = \mathbb{C}$

T bound hol C_0-sg of angle θ,

$$M_1 := \sup_{z \in \Sigma_\theta} \| T(z) \|_{\mathcal{L}(L^{p_1})} < \infty .$$

$p_0 \in [1, \infty]$, $p_0 \neq p_1$.

$$\| T(t) u \|_{p_0} \leq M_0 \| u \|_{p_0} \qquad u \in L^{p_1} \cap L^{p_0}$$

$\tau \in (0,1)$

$$M_\tau := M_0^{1-\tau} M_1^{\tau} , \qquad \theta_\tau = \tau \theta ,$$

$$\frac{1}{p_\tau} = \frac{1-\tau}{p_0} + \frac{\tau}{p_1}$$

(31.1) <u>Theorem.</u> For $z \in \Sigma_{\theta_\tau}$ $\exists !$ $T_{p_\tau}(z) \in \mathcal{L}(L^{p_\tau})$

consistent with $T(z)$. Moreover,

$$\| T_\tau(z) \| \leq M_\tau \qquad (z \in \Sigma_{\theta_\tau})$$

and

$$(T_\tau(z))_{z \in \Sigma_{\theta_\tau}}$$

is a hol. C_0-sg of angle θ_τ.

<u>Proof.</u> Let $0 < \theta' < \theta$

$$\psi : \bar{S} \longrightarrow \overline{\Sigma_{\theta'}} \setminus \{0\}$$

$$z \longmapsto e^{i\theta' \cdot z}$$

is bijective, holomorphic on S.

$$[\, z = is + t6 \qquad e^{i\theta' \cdot z} = e^{-\theta' s}\, e^{i\theta' 6} \quad]$$

$$\phi := T \circ \psi : \bar{S} \longrightarrow L(\mathcal{S}, L^1_{loc}(\Omega))$$

$$z \mapsto \int \phi(z) 1_B \, 1_C = \int_\Omega T(e^{i\theta' z}) 1_A \, 1_C \, d\mu$$

is holomorph and $\in C(\bar{S})$.

Stein $\Rightarrow$

$$\| T(\psi(\tau + is)) u \|_{p_\tau} \leq M_\tau \|u\|_{p_\tau}.$$

$$\psi(\tau + is) = e^{i\theta'(\tau + is)} = e^{-\theta' s} e^{i\theta' \tau} \in \Sigma_{\theta_\tau}$$

$$w \in \Sigma_{\theta_\tau} \Rightarrow \exists \theta' \; \exists s \quad \psi(\tau + is) = w$$

$$\Rightarrow \quad \| T(w) u \|_{p_\tau} \leq M_\tau \|u\|_{p_\tau} \qquad u \in \mathcal{S}$$

$$w \in \Sigma_{\theta_\tau}.$$

$\Rightarrow \quad \exists \; T_{p_\tau}(w) \in \mathcal{L}(L^{p_\tau})$ consistent with $T(w)$

$w \mapsto \int T(w)(1_A) 1_B$ holomorphic $\Rightarrow$

$T_{p_\tau} : \Sigma_{\theta_\tau} \to \mathcal{L}(L^{p_\tau})$ holomorphic.

Strong continuity.

$$\| T_{p_\tau}(t) f - f \|_{L^{p_\tau}} \leq \underbrace{\| T(t) f - f \|_{p_0}^{1-\tau}}_{\to 0} \; \underbrace{\| T(t) f - f \|_{p_1}^{\tau}}_{\text{bded}}.$$

$$\frac{1}{p_\tau} = \frac{1-\tau}{p_0} + \frac{\tau}{p_1} \qquad t \downarrow 0$$

$\Rightarrow \quad T_{p_\tau}(t) f \to f$ in L^{p_τ} $\forall f \in \mathcal{S}$

$\mathcal{S}$ dense $\Rightarrow$ $T_{p_\tau}(t) f \to f$ $\quad t \downarrow 0$

$\forall f \in L^{p_\tau}$. $\quad \square$

§ 32 Invariance and interpolation.

$S \in \mathcal{L}(L^2(\mu))$

S substochastic $:\iff$ $S \geq 0$ &

$\|Sf\|_1 \leq \|f\|_1$

S submarkovian $:\iff$ $f \leq 1 \Rightarrow Sf \leq 1$

$\iff$ $S \geq 0$ &

$\|Sf\|_\infty \leq \|f\|_\infty$

(32.1) Lemma. S submarkovian $\iff$
S^* substochastic.

$H = L^2(\mu)$ $\quad a$ a densely defined closed

accretive form. $A \sim a$

$-A$ generates a C_0 - sg. T