§ 22 Ljapunov's Theorem for ~~immediately~~ eventually norm continuous semigroups

(22.1) Def. A C_0-sg is T eventually (schließlich) norm-continuous if $\exists \tau > 0$
$\| T(\tau + t) - T(\tau) \| \longrightarrow 0$ as $t \downarrow 0$

Consequence: $T : [\tau, \infty) \to \mathcal{L}(X)$ is norm continuous. (exercise)

T is immediately norm-continuous if $T : (0, \infty) \to \mathcal{L}(X)$ is norm-continuous

(22.2) Examples. 1. T hol. $\Rightarrow$ T immediately norm continuous.

2. $\tau > 0$, $T(\tau)$ compact $\overset{\text{Lemma}}{\Longrightarrow}$ $T : [\tau, \infty) \to \mathcal{L}(X)$ continuous.

Lemma. $S_n \to S$ strongly in $\mathcal{L}(X)$
$K \in \mathcal{L}(X)$ compact $\Rightarrow$ $S_n K \to SK$ in norm.

$\downarrow$
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(22.3) <u>Theorem.</u> Let T be eventually norm continuous and $1 \in \sigma_{ap}(T(1))$. Then $\exists k \in \mathbb{Z}$ s.t. $2\pi i k \in \sigma_{ap}(A)$.

<u>Proof.</u> $\exists x_n$ $\|x_n\| = 1$ $\|T(1)x_n - x_n\| \to 0$. $u(t) := (T(t+\tau)x_n)_{n \in \mathbb{N}}$

$u : [0,1] \to \ell^\infty(X)$ continuous

[Pf. $\|u(t) - u(t_0)\|_\infty = \sup_n \|T(t+\tau)x_n - T(t_0+\tau)x_n\|$

$\leq \|T(t+\tau) - T(t_0+\tau)\| \to 0$ $t \to t_0$].

$q : \ell^\infty(X) \to \ell^\infty(X)/c_0(X) =: \hat{X}$

$q \circ u : [0,1] \to \hat{X}$ continuous. 1)

$\Rightarrow$ $\exists k \in \mathbb{Z}$

$$\int_0^1 e^{-2\pi i k t}\, q(u(t))\, dt \neq 0$$

$$\| $$

$$q\left(\left(\int_0^1 e^{-2\pi i k t}\, T(t+\tau)x_n\, dt\right)_{n \in \mathbb{N}}\right)$$

1) $q\circ n \neq 0$. In fact, let

$m \in [\tau, \tau+1) \cap \mathbb{N}_0$. Then

$T(m)x_n - x_n =$

$\sum_{k=0}^{m-1} T(k)\,(T(1)x_n - x_n) \longrightarrow 0$

Thus ~~$q(m$~~ $n(m-\tau)$ ~~x~~ $-(x_n)_{n\in\mathbb{N}} =$

$(T(m)x_n)_{n\in\mathbb{N}} - (x_n)_{n\in\mathbb{N}} \in c_0$.

Thus $q(n(m-\tau)) = q((x_n)_{n\in\mathbb{N}}) \neq 0$.

$$\Rightarrow \quad \left(\int_0^1 e^{-2\pi i k t}\, T(t+\tau) x_n \, dt \right)_{n\in\mathbb{N}} \notin c_0(X)$$

$$y_n := \int_0^1 e^{-2\pi i k t}\, T(t+\tau) x_n \, dt$$

$\Rightarrow \quad \exists\, \delta > 0 \quad \exists$ ss $(y_{n_k})_{k\in\mathbb{N}}$ s.t.

$$\| y_{n_k} \| \geq \delta > 0$$

$$(A - 2\pi i k)\, y_{n_k} = T(1+\tau) x_{n_k} - T(\tau) x_{n_k} \longrightarrow 0$$

$\Rightarrow \quad 2\pi i k \in \sigma_{ap}(A) \qquad \square$

<u>Rk</u> <u>Quotients.</u>

E Banach $\quad F$ closed subspace

$E/_F$ Banach for

$$\| [x] \| = \operatorname{dist}(x, F)$$

$q : E \to E/_F$ contraction.

(22.3) <u>Theorem (Ljapunov).</u>

Let T be eventually norm continuous. If $\operatorname{Re}\lambda < 0$ $\forall \lambda \in \sigma(A)$

then

$$\|T(t)\| \le M e^{-\varepsilon t}.$$

<u>Proof.</u> $r(T(1)) = e^{\omega(A)}$

claim $r(T(1)) < 1$

Assume $r(T(1)) \geq 1$

Then $\exists\, \varrho \geq 1,\ \theta \in \mathbb{R}$ s.t. $\varrho = r(T(1))$

$\Rightarrow \quad \varrho e^{i\theta} \in \sigma_{ap}(T(1))$

~~Define~~

<u>1st case:</u> $\varrho = 1,\ \theta = 0.$

$\Rightarrow 1 \in \sigma_{ap}(T(1)) \Rightarrow \exists k \quad 2\pi i k \in \sigma_{ap}(A)$

ϟ

<u>2nd case</u> $S(t) = e^{(-i\theta - \log \varrho)t}\, T(t)$

is a C_0-sg. Its generator: $A - i\theta - \log\varrho$

~~r(T(t))~~ $r(S(1)) = r\left(e^{-i\theta} e^{-\log\varrho}\, T(1)\right) = 1$

1st case $\Rightarrow$ $\exists k \in \mathbb{Z}$ $\quad 2\pi i k \in \sigma_{ap}(A - i\theta - \log\varrho)$

$\Rightarrow$ $2\pi i k + i\theta + \log\varrho \in \sigma_{ap}(A)$

But $\operatorname{Re}(2\pi i k + i\theta + \log\varrho) = \log\varrho \geq 0$ $\lightning$

<u>Exercise</u> Let T be an eventually norm continuous C_0-sg with generator A.

Then

a) $\sigma_{ap}(T(t)) \setminus \{0\} = e^{t\sigma_{ap}(A)}$

b) $\sigma(T(t)) \setminus \{0\} = e^{t\sigma(A)}$.

§ 23 Dathko's Theorem.

(23.1) Bochner Integral

(Ω, Σ) measurable space, X Banach space.

$f: \Omega \to X$ simple function $:\Leftrightarrow$

$$f = \sum_{j=1}^{n} x_j 1_{A_j} \qquad A_j \in \Sigma, \; x_j \in X.$$

f measurable $:\Leftrightarrow$ $\exists f_n : \Omega \to X$ simple functions, $f_n(\omega) \to f(\omega)$ $\forall \omega \in \Omega$.

(23.1) Pettis' Theorem X separable

~~Equ.~~ $f : \Omega \to X$

(i) f ~~sepa~~ measurable

(ii) $x' \circ f$ measurable $\forall x' \in X$

(iii) $\exists W \subset X'$ separating $x' \circ f$ meas. $\forall x' \in W$.

Consequence: $\|f\|$ measurable.

(Ω, Σ, μ)

$1 \leq p < \infty$

$L^p(\Omega, X) := \{ f : \Omega \to X$ measurable :

$$\int \|f(\omega)\|^p \, d\mu(\omega) < \infty \}$$

Banach space for $\|f\|_p$

<u>Rk.</u> $f : \Omega \to X$ measurable $\Rightarrow$

$\exists X_0 \subset X$ separable closed subspace

$f(\omega) \in X_0$ a.e.

<u>Theorem.</u> $p = 1$ $\quad f \in L^1(\Omega; X) \Rightarrow$

$\exists!\, y \in X$ s.t.

$$\langle x', y \rangle = \int_\Omega \langle x', f(\omega) \rangle \, d\mu(\omega) \quad \forall x' \in X'.$$

$$y =: \int_\Omega f(\omega) \, d\mu(\omega)$$

Thus $\left\langle \int_\Omega f(\omega) \, d\mu(\omega), x' \right\rangle =$

$$\int \langle f(\omega), x' \rangle \, d\mu(\omega) \quad \forall x' \in X'$$

Consequence:

$$\left\| \int_\Omega f(\omega)\, d\mu(\omega) \right\| \le \int_\Omega \|f(\omega)\|\, d\mu(\omega)$$

$\forall f \in L^1(\Omega, X)$.

(23.2) <u>Theorem (Datko).</u> Let T be a C_0-sg with generator A, $1 \le p < \infty$. Equ.:

(i) $\omega(A) < 0$

(ii) $\int_0^\infty \|T(t)x\|^p\, dt < \infty \quad \forall x \in X$.

<u>Proof.</u> (ii) $\Rightarrow$ (i) $S: X \longrightarrow L^p(0,\infty; X)$

$x \longmapsto T(\cdot)x$

is linear & continuous.

Pf. $x_n \longrightarrow x$, $T(\cdot)x \longrightarrow f$ in L^p

$$\Rightarrow \int_0^\infty \|T(t)x_n - f(t)\|^p\, dt \longrightarrow 0 \quad (n \to \infty).$$

$\Rightarrow \exists$ ss $\|T(t)x_{n_k} - f(t)\| \longrightarrow 0$ a.e.

But $\quad T(t)x_{n_k} \longrightarrow T(t)x \qquad \forall t \in (0,\infty)$

$\Rightarrow \quad f(t) = T(t)x \quad$ a.e.

i.e. $\quad f = T(\cdot)x \quad$ in $L^p(0,\infty; X)$

Consequence : $\exists c > 0$ such that

$$\int_0^\infty \| T(t)x \|^p \, dt \leq c \, \|x\|^p \qquad \forall x \in X.$$

Suppose that $\omega(A) \geq 0$. $\Rightarrow$

$r(T(1)) \geq 1. \quad \Rightarrow \quad$ ~~$\exists |\lambda| \geq 1$~~

~~$\| T(1)x$~~ $\quad \lambda \in \sigma(T(1)), \quad |\lambda| = r(T(1)).$

$\Rightarrow \quad \lambda \in \sigma_{ap}(T(1)) \quad \Rightarrow \quad \exists x_k \in X, \; \|x_k\| = 1$

$$\| T(1)x_k - \lambda x_k \| \longrightarrow 0$$

$$\Rightarrow \quad \| T(n)x_k - \lambda^n x_k \| \longrightarrow 0 \qquad \forall n \in \mathbb{N}$$

[in fact $\quad T(n) - \lambda^n =$

$$\lambda^n \left[\left(\frac{T(1)}{\lambda} \right)^n - I \right] = -\lambda^n \sum_{k=1}^{n-1} \left(\frac{T(1)}{\lambda} \right)^k \left(I - \frac{T(1)}{\lambda} \right)$$

$$= \lambda^{n-1} \sum_{k=1}^{n-1} \left(\frac{T(1)}{\lambda} \right)^k \left(T(1) - \lambda I \right) . \Big]$$

$\Rightarrow \quad \exists$ ss

$$\| T(n) x_{k_\ell} - \lambda^n x_{k_\ell} \| \le \tfrac{1}{2} \quad \forall n \in \mathbb{N} \quad \ell \ge n$$

[$n = 1$ choose x_{k_1} $\quad \| T(1) x_{k_1} - \lambda x_{k_1} \| \le \frac{1}{2}$

$n = 2$ choose $k_2 > k_1$ $\quad \| T(2) x_{k_2} - \lambda^2 x_{k_2} \| \le \frac{1}{2}$

$n = 3$ choose $k_3 > k_2$ $\quad \| T(3) x_{k_3} - \lambda^3 x_{k_3} \| \le \frac{1}{2}$]

...

$$\Rightarrow \quad \| T(n) x_{k_\ell} \| = \| \lambda^n x_{k_\ell} - (\lambda^n x_{k_\ell} - T(n) x_{k_\ell}) \|$$

$$\ge 1 - \tfrac{1}{2} = \tfrac{1}{2} \quad \forall \ell \ge n.$$

$$C_1 := \sup_{s \in [0,1]} \| T(s) \|$$

Let $m \in \mathbb{N}$, $0 \le t \le m$. $\exists! \, n \in \mathbb{N}_0$
$t \in (n-1, n]$ Thus $n \le m$

For $\ell \ge m$

$$\tfrac{1}{2} \le \| T(n) x_{k_\ell} \| = \| T(n-t) T(t) x_{k_\ell} \|$$

$$\le C_1 \| T(t) x_{k_\ell} \| .$$

Thus $\quad \| T(t) x_{k_\ell} \| \ge \dfrac{1}{2 C_1} \quad \forall \, t \in [0, m],$
$\ell \ge m.$

$$C = C\|x_{k_\ell}\|^p \geq \int_0^\infty \|T(t)x_{k_\ell}\|^p \, dt$$

$$\geq \int_0^m \|T(t)x_{k_\ell}\|^p \, dt$$

$$\geq \frac{1}{(2C_1)^p} \, m \qquad \forall \ell \geq m \qquad \lightning.$$

Chapter 4 The non-homogeneous equation

§ 24 The non-homogeneous problem: classical & mild solutions

A closed operator, $J \subset \mathbb{R}$ interval

$f \in L^1(J, X)$

$$(E) \qquad \dot{u}(t) = Au(t) + f(t) \qquad \text{on } J.$$

(24.1) Definition (classical solution)

Assume that $f \in C(J; X)$.

u classical solution $:\Longleftrightarrow$

$u \in C^1(J, X)$, $u(t) \in D(A)$ $\forall t \in J$

and (E) holds.

(24.2) Definition (mild solution)

$f \in L^1(J; X)$

u mild solution $:\Longleftrightarrow$

$u \in C(J, X)$, $\int_s^t u(r)\,dr \in D(A)$

& $u(t) - u(s) = A \int_s^t u(r)\,dr + \int_s^t f(r)\,dr$

$\forall s, t \in J$.

~~Rk~~

(24.3) **Proposition** Let $f \in C(J, X)$.

u classical solution $\Longleftrightarrow$

u mild solution & $u \in C^1(J, X)$

Proof "$\Rightarrow$" u classical solution

$\Rightarrow$ $Au = \dot{u} - f \in C(J, X)$

$\Rightarrow$ $u \in C(D(A))$ graph norm

$A \in \mathcal{L}(D(A), X)$ $\Rightarrow$

$\int_s^t u(r)\,dr \in D(A)$ & $A \int_s^t u(r)\,dr$

$$\int_s^t Au(r)\,dr = \int_s^t \dot{u}(r) - f(r)\,dr$$

$$= u(t) - u(s) - \int_s^t f(r)\,dr$$

$\Rightarrow$ u mild solution.

"$\Leftarrow$" Let u be a mild sol. & $u \in C^1(J; X)$. Let $t \in J$

$$u(t+h) - u(t) = \int_t^{t+h} \dot{u}(r)\,dr$$

$$= A\int_t^{t+h} u(r)\,dr + \int_t^{t+h} f(r)\,dr$$

$$\frac{1}{h}\int_t^{t+h} u(r)\,dr \longrightarrow u(t)$$

$$\frac{1}{h}\int_t^{t+h} f(r)\,ds \longrightarrow f(t)$$

$$\frac{1}{h}\left(u(t+h) - u(t)\right) \longrightarrow \dot{u}(t)$$

A closed $\Rightarrow$ $u(t) \in D(A)$ &

$\dot{u}(t) = Au(t) + f(t)$. □

Thus, if u is a mild solution, it is a classical solution as soon as $u \in C^1$.

(24.4) <u>Proposition</u>. Let $J = [0, \tau]$,

$f \in L^1(0, \tau; X)$, $x \in X$.

Let A be the generator of

a C_0-sg T. ~~$u \in C([0,\tau]; X)$.~~

Then $\exists!$ mild sol. u of

$$\begin{cases} \dot u = Au + f \\ u(0) = x \end{cases}$$

Namely $u(t) = T(t)x + (T * f)(t)$

$$(T * f)(t) = \int_0^t T(t-s) f(s)\, ds$$

<u>Proof</u>. 1. <u>Uniqueness</u> u mild solution

for $x = 0$, $f = 0$ $\Rightarrow$

$\int_0^t u(r)\, dr \in D(A)$ & $A \int_0^t u(r)\, dr = u(t) - \cancel{x}$

Let $v(t) = \int_0^t u(r)\, dr$ $\Rightarrow$

$v \in C^1([0,\tau]; X)$, $v(t) \in D(A)$ &, $v(0) = 0$

& $\dot v(t) = u(t) = Av(t)$. Thus

v is a classical solution of

$$\begin{cases} v'(t) = Av(t) \\ v(0) = 0 \end{cases}$$

We know that $v \equiv 0$. Hence $u \equiv 0$.

2. $f \equiv 0$ $\quad u(t) = T(t)x$

$$\int_s^t T(r)x\,dr \in D(A) \quad \&$$

(AWI) $$A\int_s^t T(r)x\,dr = T(t)x - T(s)x$$

Thus $T(\cdot)x$ is the mild solution.

3. $x = 0$, $f \in L^1(0,T;X)$.

$u = T * f$.

a) $u \in C([0,T];X)$

$t_n \to t_0$ $\quad u(t_n) - u(t_0) =$

$$\int_0^{t_n} T(t_n - s) f(s)\,ds - \int_0^{t_0} T(t_0 - s) f(s)\,ds =$$

$$\int_0^{t_0} \big(T(t_n - s) - T(t_0 - s)\big) f(s)\, ds + \int_{t_0}^{t_n} T(t_n - s) f(s)\, ds$$

$$\Rightarrow \quad \|u(t_n) - u(t_0)\| \leq$$

$$\int_0^{t_0} \|(T(t_n - s) - T(t_0 - s)) f(s)\|\, ds + \int_{t_0}^{t_n} M e^{\omega(t_n - s)} \|f(s)\|\, ds$$

$$\longrightarrow 0 \qquad \text{(dominated convergence).}$$

b) $$\int_0^t u(r)\, dr =$$

$$\int_0^t \int_0^r T(r-s) f(s)\, ds\, dr =$$

$$\int_0^t \int_s^t T(r-s) f(s)\, dr\, ds =$$

$$\int_0^t \underbrace{\int_0^{t-s} T(r) f(s)\, dr}_{\in D(A)}\, ds$$

$$A \int_0^{t-s} T(r) f(s)\, dr = T(t-s) f(s) - f(s)$$

$$\overset{L}{\Longrightarrow} \int_0^t u(r)\, dr \in D(A) \ \&$$

$$A \int_0^t u(r)\, dr = \int_0^t T(t-s) f(s)\, ds - \int_0^t f(s)\, ds$$

$$= u(t) - u(0) - \int_0^t f(s)\, ds \qquad \square$$

(24.5) **Lemma** A a.t.g., $\rho(A) \neq \emptyset$.

Let $f \in L^1(J; X)$, $f(t) \in D(A)$ $\forall t \in J$.

Assume $Af \in L^1(J, X)$. Then

$$\int_J f(t)\,dt \in D(A) \ \& \ A \int_J f(t)\,dt = \int_J Af(t)\,dt.$$

Pf in the case where $\rho(A) \neq \emptyset$.

1st case: $0 \in \rho(A)$.

$Af \in L^1(J, X) \Rightarrow A^{-1}Af \in L^1(J, D(A))$

since $A^{-1} \in \mathcal{L}(X, D(A))$

(24.6) $\Rightarrow \int_J f(t)\,dt \in D(A)$

$A \in \mathcal{L}(D(A), X) \Rightarrow$

$$A \int_J f(t)\,dt = \int_J Af(t)\,dt.$$

2nd case: $\lambda \in \rho(A)$.

$(\lambda - A)f \in L^1(J, X) \Rightarrow$

$\int_J f(t)\,dt \in \cancel{X}\, D(A)$ & $(\lambda - A)\int_J f(t)\,dt$

$= \int_J (\lambda - A) f(t)\,dt \longrightarrow$ Beh.

(24.6) Lemma $f \in L^1(\Omega, \Sigma, \mu; X)$

$S \in \mathcal{L}(X, Y) \Rightarrow Sf \in L^1(\Omega, \Sigma; \mu, X)$

$$S \int_\Omega f(\omega)\, d\mu(\omega) = \int_\Omega (Sf)(\omega)\, d\mu(\omega).$$

Pf. $f_n : \Omega \to X$ simple $f_n(\omega) \to f(\omega)$

$\Rightarrow Sf_n : \Omega \to Y$ " $Sf_n(\omega) \to Sf(\omega)$

$\Rightarrow Sf$ measurable

$$\int_\Omega \| Sf(\omega) \| \, d\mu(\omega) \le \|S\| \int_\Omega \|f(\omega)\| d\mu(\omega)$$

$\Rightarrow Sf \in L^1(\Omega, \Sigma; \mu; Y)$

Let ~~b~~

$$\int_\Omega Sf \, d\mu \overset{!}{=} S \int_\Omega f \, d\mu$$

Let $y' \in Y'$. Then

$$\langle S \int_\Omega f\, d\mu, y' \rangle = \langle \int_\Omega f\, d\mu, S' y' \rangle$$

$$\overset{\S 23.1}{=} \int_\Omega \langle S' y', f(\omega) \rangle \, d\mu(\omega)$$

$$= \int_\Omega \langle y', Sf(\omega) \rangle \, d\mu(\omega) \overset{23.1}{=} \langle y', \int Sf(\omega) d\mu(\omega) \rangle$$

$\square$

28.6.17