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§ 26 Periodic solutions on Hilbert spaces

Let X be a complex Banach space.

(26.1) Proposition

Let T be a C_0-sgr. on X with generator A, $1 \in \varrho(T(1))$.

$\Rightarrow$ $2\pi i \mathbb{Z} \subseteq \varrho(A)$ and

$$\sup_{k \in \mathbb{Z}} \|R(2\pi i k, A)\| < \infty.$$

Proof:

We obtain

$$(A - 2\pi i) \int_0^1 e^{-2\pi i k s} T(s) x \, ds = T(1)x - x \quad \forall x \in X$$

and

$$\int_0^1 e^{-2\pi i k s} T(s) (A - 2\pi i) x \, ds = T(1)x - x \quad \forall x \in D(A).$$

for each $k \in \mathbb{Z}$.

$\Rightarrow$ $A - 2\pi i k$ is invertible with

$$(A - 2\pi i k)^{-1} = S_k (T(1) - I)^{-1}$$

where $S_k \in \mathcal{L}(X)$,

$$S_k x := \int_0^1 e^{-2\pi i k s} T(s) x \, ds \quad \forall x \in X, k \in \mathbb{Z}.$$

~~Moreover~~:

$\Rightarrow$ $2\pi i \mathbb{Z} \subseteq \varrho(A)$ and

$$\sup_{k \in \mathbb{Z}} \|R(2\pi i k, A)\| \leq \sup_{s \in [0,1]} \|T(s)\| \cdot \|R(1, T(1))\| < \infty.$$

$\square$

Let now H be a separable, complex Hilbert space.

(26.2) Reminder

$u : (0,1) \longrightarrow H$ is measurable

$\Leftrightarrow$ $(u(\cdot) | x)$ is measurable $\forall x \in H$

and this implies: $\|u(\cdot)\|$ is measurable

[$\|u(t)\| = \sup_{n\in\mathbb{N}} |(u(t) \mid x_n)| \quad \forall t \in (0,1)$
where $\{x_n : n \in \mathbb{N}\}$ is dense in H.]

(26.3) Remark 2

As in the scalar-valued case $L^2((0,1);H)$ becomes a Hilbert space by setting

$$(u \mid v) := \int_0^1 (u(t) \mid v(t))_H \, dt$$

and we define the Fourier coefficients

$$\hat{u}(k) := \int_0^1 e^{-2\pi i k t} u(t)\, dt$$

for $k \in \mathbb{Z}$, $u \in L^2((0,1);H)$.

(26.4) Theorem (Plancherel)

The mapping

$$L^2((0,1);H) \longrightarrow \ell^2(H)$$
$$u \longmapsto (\hat{u}(k))_{k\in\mathbb{Z}}$$

is a unitary (i.e., linear, bij., isometric) with inverse

$$\ell^2(H) \longrightarrow L^2((0,1);H)$$
$$(a_k)_{k\in\mathbb{Z}} \longmapsto \sum_{k=-\infty}^{\infty} e^{2\pi i k \cdot} a_k .$$

We are now ready to prove.

(26.5) Theorem

For a generator A of a C_0-sgr. T on H the following are equivalent:

(a) $2\pi i \mathbb{Z} \subseteq \varrho(A)$ and $\sup_{k\in\mathbb{Z}} \|R(2\pi i k, A)\| < \infty$.

(b) $\forall f \in L^2((0,1);H)$: $\exists!$ mild periodic solution of

(E) $\quad \dot{u}(t) = Au(t) + f(t) \quad \forall t \in (0,1)$

(c) $1 \in \varrho(T(1))$.

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<u>proof</u>:

(26.1) $\Rightarrow$ „(c) $\Rightarrow$ (a)".

(25.4) $\Rightarrow$ „(b) $\Rightarrow$ (c)".

Assume (a) and let $f \in L^2((0,1);H)$.

$$\overset{(26.4)}{\Rightarrow} f_N := \sum_{|k| \le N} e^{2\pi i k \cdot} \hat{f}(k) \xrightarrow[N \to \infty]{L^2} f$$

By (26.4) there is a unique $u \in L^2((0,1);H)$ with $\hat{u}(k) = R(2\pi i k, A)\hat{f}(k) \quad \forall k \in \mathbb{Z}$.

We show: u is periodic mild solution of (E).

We set $u_N := \sum_{|k| \le N} e^{2\pi i k \cdot} \hat{u}(k) \in C^1([0,1];H)$

Then $u_N(0) = u_N(1)$ and

$$\dot{u}_N(t) = \sum_{|k| \le N} 2\pi i k \, e^{2\pi i k t} \hat{u}(k)$$

$$= \sum_{|k| \le N} e^{2\pi i k t} \cdot \left(2\pi i k \cdot R(2\pi i k, A)\hat{f}(k)\right)$$

$$= \sum_{|k| \le N} e^{2\pi i k t} \hat{f}(k) + A \sum_{|k| \le N} e^{2\pi i k t} R(2\pi i k, A)\hat{f}(k)$$

$$= f_N(t) + A u_N(t) \quad \forall t \in [0,1]$$

$\Rightarrow$ u_N is classical periodic solution of (E) with respect to f_N for each $N \in \mathbb{N}$.

$$\Rightarrow (*): u_N(t) = T(t)u_N(0) + T * f_N(t) \quad \forall t \in [0,1]$$

We check that $u_N \longrightarrow u$ uniformly.

We obtain

$$\|T * f_N(t) - T * f(t)\| \le \int_0^t \|T(t-s)(f_N(s) - f(s))\| \, ds$$

$$\le M e^{\omega t} \int_0^t \|f_N(s) - f(s)\| \, ds$$

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$$\leq M e^{\omega t}\left(\int_0^t \|f_\nu(s) - f(s)\|^2 ds\right)^{\frac{1}{2}} t^{\frac{1}{2}}$$

$$\xrightarrow[\nu\to\infty]{} 0 \quad \text{uniformly for } t \in [0,1].$$

We also obtain with (*)

$$\int_0^1 T(t-s)\,u_\nu(s)ds = \int_0^1 T(t-s)\Big(T(s)u_\nu(0) + \int_0^s T(s-r)f_\nu(r)dr\Big)ds$$

$$= T(t)\,u_\nu(0) + \int_0^1\int_0^s T(t-r)\,f_\nu(r)\,dr\,ds \quad \forall t\in[0,1]$$

$$\Rightarrow \quad T(t)u_\nu(0) = \int_0^1 T(t-s)u_\nu(s)ds - \int_0^1\int_r^1 T(t-r)f_\nu(r)\,ds\,dr$$

$$= \int_0^1 T(t-s)u_\nu(s)ds - \int_0^1 (1-r)\,T(t-r)\,f_\nu(r)dr$$

$$= \int_0^1 T(t-s)u_\nu(s)ds - (1-t)\int_0^1 (t-r)T(t-r)f_\nu(r)dr \quad \forall t\geq 0,\ \nu\in\mathbb{N}$$

As above we obtain

$$\int_0^1 T(t-s)\,u_\nu(s)ds \xrightarrow{\nu\to\infty} \int_0^1 T(t-s)\,u(s)ds$$

and

$$(1-t)\int_0^1 (t-r)\,T(t-r)\,f_\nu(r)dr \xrightarrow{\nu\to\infty} (1-t)\int_0^1 (t-r)T(t-r)f(r)dr$$

uniformly for $t\in[0,1]$.

$\overset{(*)}{\Rightarrow}$ u_ν converges uniformly on $[0,1]$ and u is necessarily the limit.

$\Rightarrow$ $u \in C([0,1];X)$ and $u(1) = u(0)$.

Since

$$\begin{array}{ccc} u_\nu(t) - u_\nu(0) & = A\int_0^t u_\nu(s)ds & + \int_0^t f_\nu(s)\,ds \\ \downarrow \ \nu\to\infty \ \downarrow & & \downarrow \\ u(t) - u(0) & & \int_0^t f(s)ds \end{array}$$

and $\int_0^t u_\nu(s)ds \longrightarrow \int_0^t u(s)ds$, the closedness of A implies

$$\int_0^t u(s)\,ds \in D(A) \text{ and}$$

$$u(t) - u(0) = A\int_0^t u(s)\,ds + \int_0^t f(s)\,ds \quad \forall t \geq 0.$$

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$\Rightarrow$ u is periodic mild solution ∎

Uniqueness follows from the spectral mapping theorem for the point sp. + (25.3)

(26.6) <u>Theorem</u> (Gearhard – Prüss)

For a C_0-sgr. T on H with generator A the following are equivalent.

(a) $\omega(A) < 0$

(b) $\{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda > 0\} \subseteq \varrho(A)$ and $\sup_{\mathrm{Re}\,\lambda > 0} \|R(\lambda, A)\| < \infty$.

(26.7) <u>Remark</u>

(a) $\Rightarrow \mathbb{C}_+ := \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda \geq 0\} \subseteq \varrho(A)$ and $\sup_{\mathrm{Re}\,\lambda \geq 0} \|R(\lambda, A)\| < \infty$.

<u>Proof</u> (of Theorem (26.6)):

"(b) $\Rightarrow$ (a)":

Since $r(T(1)) = e^{\omega(A)}$ it suffices to show that $r(T(1)) < 1$.

Let $\lambda \in \mathbb{C}$, $|\lambda| \geq 1$, $\lambda = r e^{i\theta}$, $r \geq 1$.

We write $r = \log r_0$ with $r_0 \geq 0$.

$\Rightarrow \mu = e^{r_0 + i\theta}$ and

$$\mu - T(1) = \mu\,(I - e^{-r_0 - i\theta}T(1)) = \mu\,(I - S(1))$$

for $S(t) := e^{(-r_0 - i\theta)t} T(t)$, $t \geq 0$.

Now S is C_0-sgr. with generator $A - r_0 - i\theta$. By (26.7) we have $2\pi i \mathbb{Z} \subseteq \varrho(A - r_0 - i\theta)$ and the resolvent

$$R(2\pi i k, A - r_0 - i\theta) = R(2\pi i k + r_0 + i\theta, A)$$

satisfies

$$\sup_{k \in \mathbb{Z}} \| R(2\pi i k, A - r_0 - iA)\| < \infty.$$

$(26.3) \Rightarrow 1 \in \varrho(S(1)) \Rightarrow \mu \in \varrho(T(1))$

"(a) $\Rightarrow$ (b)":

Let $\omega(A) < 0 \Rightarrow \exists\, M \geq 1, \varepsilon > 0$:
$\|T(t)\| \leq M e^{-\varepsilon t}$ $\forall t \geq 0$.

$$\Rightarrow \|R(\lambda, A)x\| \leq \int_0^\infty e^{-\mathrm{Re}\lambda t} \|T(t)\| \, dt \cdot \|x\|$$
$$\leq \int_0^\infty e^{-(\mathrm{Re}\lambda + \varepsilon)t} \cdot M \, dt \cdot \|x\|$$
$$= \frac{M}{\mathrm{Re}\lambda + \varepsilon} \|x\| \leq \frac{M}{\varepsilon} \|x\|$$

$$\Rightarrow \|R(\lambda, A)\| \leq \frac{M}{\varepsilon} \quad \forall \lambda \in \mathbb{C}, \ \mathrm{Re}\lambda > 0.$$ $\square$