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Mapping theorems for Sobolev spaces of vector-valued functions

by

Wolfgang Arendt and Marcel Kreuter (Ulm)

Abstract. We consider Sobolev spaces with values in Banach spaces, with emphasis on mapping properties. Our main results are the following: Given two Banach spaces $X \neq \{0\}$ and Y, each Lipschitz continuous mapping $F : X \to Y$ gives rise to a mapping $u \mapsto F \circ u$ from $W^{1,p}(\Omega, X)$ to $W^{1,p}(\Omega, Y)$ if and only if Y has the Radon–Nikodým Property. But if in addition F is one-sided Gateaux differentiable, no condition on the space is needed. We also study when weak properties in the sense of duality imply strong properties. Our results are applied to prove embedding theorems, a multi-dimensional version of the Aubin–Lions Lemma and characterizations of the space $W_0^{1,p}(\Omega, X)$.

1. Introduction. Sobolev spaces with values in a Banach space are quite natural objects and occur frequently when treating partial differential equations (see e.g. [Ama95], [Ama01], [DHP03], [CM09], [ADKF14]) and also in probability (see e.g. the recent monograph by Hytönen, van Neerven, Veraar and Weis [HvNVW16]). In the case of a single variable, they are extremely important in the study of time-dependent partial differential equations (see e.g. [CH98] or [Sho97]), but much less seems to have been written about the multivariable counterparts. The purpose of this article is to study the space $W^{1,p}(\Omega, X)$ where X is a Banach space. In some cases scalar proofs just go through and we will only give a reference. But frequently new proofs and ideas are needed. This is in particular the case for *mapping properties*, the main subject of the article.

One question we treat is when each Lipschitz map $F : X \to Y$ leads to a composition mapping $u \mapsto F \circ u : W^{1,p}(\Omega, X) \to W^{1,p}(\Omega, Y)$. It turns out that in general this is equivalent to Y having the Radon–Nikodým Property. However, if we merely consider those Lipschitz continuous F that are

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one-sided Gateaux differentiable, then the result is always true and a chain rule can even be proved. An important case is $F(x) = \|x\|_X$. In this case the chain rule becomes particularly important, since it can be used to lift results from the scalar-valued to the vector-valued case. Special attention is also given to the mapping $F(x) = |x|$ where X is a Banach lattice, e.g. $X = L^r(\Omega)$ or $X = C(K)$, which played a role in [ADKF14]. This mapping and in particular differentiability properties of the projection onto a convex set in Hilbert space have also been studied by Haraux [Har77]. In the finite-dimensional case, such questions were answered thoroughly by Marcus and Mizel [MM72], [MM73], [MM79] (even if F is not globally Lipschitz continuous).

Weak properties in the sense of duality also form an important subject concerning mapping properties. It is the inverse question we ask: Let $u : \Omega \to X$ be a function such that $x' \circ u$ has some regularity property for all $x' \in X'$. Does it follow that u has the corresponding regularity property? In other words we ask whether *weak implies strong*. For example, it is well known that a weakly holomorphic map is holomorphic [Gro53], [ABHN11, Proposition A.3]. The same is true for harmonic maps [Are16]. The analogous result is not true for Sobolev spaces, but we show that, given $1 \le p \le \infty$, a Banach lattice X is weakly sequentially complete if and only if each function $u : \Omega \to X$ such that $x' \circ u \in C^1(\overline{\Omega}, \mathbb{R})$ for all $x' \in X'$ is already in $W^{1,p}(\Omega, X)$. Another positive result concerns Dirichlet boundary conditions: If $u \in W^{1,p}(\Omega, X)$, $1 \le p < \infty$, is such that $x' \circ u \in W_0^{1,p}(\Omega, \mathbb{R})$ for all x' in a separating subset of X' then $u \in W_0^{1,p}(\Omega, X)$. Moreover we show that $u \in W_0^{1,p}(\Omega, X)$ if and only if $\|u\|_X \in W_0^{1,p}(\Omega, \mathbb{R})$.

The paper is organized as follows. We start by investigating when the quotient criterion characterizes the spaces $W^{1,p}(\Omega, X)$. In Section 3 composition with Lipschitz maps is studied, and in Section 4 we add the hypothesis of one-sided Gateaux differentiability. Finally, we apply our results to investigate embedding theorems and weak Dirichlet boundary data in Sections 5–7.

We will denote all norms by $\|\cdot\|_X$, where X is the Banach space to which the norm belongs, and we denote operator norms by $\|\cdot\|_{\mathcal{L}}$. Also $B_X(x, r)$ and $S_X(x, r)$ are the open ball and the sphere of radius r centered at $x \in X$, respectively.

2. The Difference Quotient Criterion and the Radon–Nikodým Property. Let $\Omega \subset \mathbb{R}^d$ be open, X a Banach space and $1 \le p \le \infty$. As in the real-valued case, the first Sobolev space $W^{1,p}(\Omega, X)$ is the space of all functions $u \in L^p(\Omega, X)$ for which the distributional partial derivatives are elements of $L^p(\Omega, X)$, that is, there exist functions $D_j u \in L^p(\Omega, X)$

$(j = 1, \ldots, d)$ such that

$$\int_\Omega u \partial_j \varphi = -\int_\Omega D_j u \, \varphi$$

for all $\varphi \in C_c^\infty(\Omega, \mathbb{R})$, where we write $\partial_j \varphi := \frac{\partial}{\partial x_j}\varphi$ for the classical partial derivative. Analogously to the real-valued case one sees that $D_j u$ is unique and that $D_j u = \partial_j u$ if $u \in C^1(\Omega, X) \cap L^p(\Omega, X)$ with $\partial_j u \in L^p(\Omega, X)$. The space $W^{1,p}(\Omega, X)$ equipped with the norm $\|u\|_{W^{1,p}(\Omega,X)} := \|u\|_{L^p(\Omega,X)} + \sum_j \|D_j u\|_{L^p(\Omega,X)}$ is a Banach space.

We want to establish a criterion for a function $u \in L^p(\Omega, X)$ to be in $W^{1,p}(\Omega, X)$, which is well known if $X = \mathbb{R}$. We start with a look at the following property of functions in $W^{1,p}(\Omega, X)$:

Lemma 2.1. *If $u \in W^{1,p}(\Omega, X)$ with $1 \le p \le \infty$, then there exists a constant C such that for all $\omega \subset\subset \Omega$ and all $h \in \mathbb{R}$ with $|h| < \operatorname{dist}(\omega, \partial\Omega)$ we have*

$$\|u(\cdot + he_j) - u\|_{L^p(\omega,X)} \le C|h| \quad (j = 1, \ldots, d). \tag{2.1}$$

Moreover we can choose $C = \max_{j=1,\ldots,d} \|D_j u\|_{L^p(\Omega,X)}$.

Proof. This can be proven analogously to the real-valued case [Bre11, Proposition 9.3]. Note that the proof is based on regularization arguments which work analogously in the vector-valued case. ∎

It is well known that a function $u \in L^p(\Omega, \mathbb{R})$ $(1 < p \le \infty)$ which satisfies criterion (2.1) is in $W^{1,p}(\Omega, \mathbb{R})$ [Bre11, Proposition 9.3], and that this is false in general if $p = 1$ [Bre11, Chapter 9, Remark 6]. We will refer to (2.1) as the *Difference Quotient Criterion*. We are interested in extending this criterion to Banach spaces. Such theorems have been proven in special cases, e.g. for $p = 2$ and X a Hilbert space [CM09, Lemma A.3], and also if X is reflexive and $\Omega = I$ is an interval [GP06, Proposition 2.2.26]. We will show that the criterion describes the spaces $W^{1,p}(\Omega, X)$ if and only if X has the Radon–Nikodým Property.

For our purposes, it is convenient to define that X has the *Radon–Nikodým Property* if each Lipschitz continuous function f from an interval I to X is differentiable almost everywhere. It turns out that then even every absolutely continuous function is differentiable almost everywhere [ABHN11, Proposition 1.2.4]. By [BL00, Theorem 5.21] this in turn is equivalent to the validity of an X-valued version of the Radon–Nikodým Theorem, which is the usual definition of the Radon–Nikodým Property.

Each reflexive space has the Radon–Nikodým Property, and so does every separable dual. On the other hand, the spaces L^1 and $C(K)$, K compact, have the Radon–Nikodým Property if and only if they are finite-dimensional.

THEOREM 2.2. *Let $1 < p \leq \infty$ and let $u \in L^p(\Omega, X)$ where X is a Banach space that has the Radon–Nikodým Property. Assume that u satisfies the Difference Quotient Criterion* (2.1). *Then $u \in W^{1,p}(\Omega, X)$ and $\|D_j u\|_{L^p(\Omega,X)} \leq C$ for all $j = 1, \dots, d$.*

We will use the fact that $L^p(\Omega, X)$ inherits the Radon–Nikodým Property from X.

THEOREM 2.3 (Sundaresan, Turett, Uhl; see [TU76]). *If (S, Σ, μ) is a finite measure space and $1 < p < \infty$, then $L^p(S, X)$ has the Radon–Nikodým Property if and only if X does.*

Proof of Theorem 2.2. We first consider the case $1 < p < \infty$. Fix a direction $j \in \{1, \dots, d\}$ and let $\omega \subset\subset \Omega$ be bounded. We claim that the distributional derivative of $u_{|\omega}$ exists in $L^p(\omega, X)$ and its norm is bounded by C. For that let $\omega \subset\subset \omega' \subset\subset \Omega$ and $\tau > 0$ be small enough such that the function

$$G : (-\tau, \tau) \to L^p(\omega', X), \qquad t \mapsto u(\cdot + te_j),$$

is well defined. By assumption, G is Lipschitz continuous, and hence differentiable almost everywhere by Theorem 2.3. Fix $0 < t_0 < \operatorname{dist}(\omega, \partial\omega')$ such that

$$G'(t_0) = \lim_{h \to 0} \frac{u(\cdot + (t_0 + h)e_j) - u(\cdot + t_0 e_j)}{h}$$

exists in $L^p(\omega, X)$. Choose a sequence $h_n \to 0$ such that the above convergence holds almost everywhere in ω. Then the function

$$\begin{aligned} g_\omega(\xi) :=& \lim_{n \to \infty} \frac{u(\xi + h_n e_j) - u(\xi)}{h_n} \\ =& \lim_{n \to \infty} \frac{u(\xi - t_0 e_j + (t_0 + h_n)e_j) - u(\xi - t_0 e_j + t_0 e_j)}{h_n} \end{aligned}$$

is an element of $L^p(\omega, X)$ whose norm is bounded by C. Given $\varphi \in C_c^\infty(\omega, \mathbb{R})$ the Dominated Convergence Theorem implies that

$$\begin{aligned} \int_\omega \varphi g_\omega &= \lim_{n \to \infty} \int_\omega \varphi(\xi) \frac{u(\xi + h_n e_j) - u(\xi)}{h_n} \, d\xi \\ &= \lim_{n \to \infty} \int_\omega \frac{\varphi(\xi - h_n e_j) - \varphi(\xi)}{h_n} u(\xi) \, d\xi = -\int_\omega \partial_j \varphi u. \end{aligned}$$

This proves the claim.

Now let $\omega_n \subset\subset \omega_{n+1} \subset\subset \Omega$ be such that $\bigcup_{n \in \mathbb{N}} \omega_n = \Omega$ and let g_{ω_n} be the corresponding functions found in the first step of the proof. These functions may be pieced together to a function $g \in L^p(\Omega, X)$ whose norm is bounded by C. For any $\varphi \in C_c^\infty(\Omega, \mathbb{R})$ there exists an n such that $\varphi \in C_c^\infty(\omega_n, \mathbb{R})$. Thus the first step shows that $g = D_j u$, finishing the case $1 < p < \infty$.

For $p = \infty$ let $\omega \subset\subset \Omega$. Then $u \in L^q(\omega, X)$ for all $q < \infty$. Let $\omega_0 \subset\subset \omega$ and $|h| \leq \operatorname{dist}(\omega_0, \partial\omega)$. Then

$$\|u(\cdot + he_j) - u\|_{L^q(\omega_0, X)} \leq C|h|\lambda(\omega)^{1/q},$$

hence $u \in W^{1,q}(\omega, X)$ with $\|D_j u\|_{L^q(\omega,X)} \leq C\lambda(\omega)^{1/q}$ by the first part of the proof. Letting $q \to \infty$ yields $\|D_j u\|_{L^\infty(\omega,X)} \leq C$, and hence $u \in W^{1,\infty}(\omega, X)$. The proof can now be finished as for $1 < p < \infty$. ■

REMARK 2.4. It was brought to our attention that recently Hytönen, van Neerven, Veraar and Weis [HvNVW16, Proposition 2.5.7] gave a different proof of Theorem 2.2 based on the classical definition of the Radon–Nikodým Property.

We now want to show that the Radon–Nikodým Property of X is not only sufficient for the Difference Quotient Criterion to work, but also necessary. For this, we need the following result on Sobolev functions in one dimension.

PROPOSITION 2.5 (see [CH98, Theorem 1.4.35]). *Let I be an interval and let $1 \leq p \leq \infty$. For $u \in L^p(I, X)$ the following are equivalent:*

(i) $u \in W^{1,p}(I, X)$.
(ii) *u is absolutely continuous, differentiable almost everywhere and $\frac{d}{dt}u \in L^p(I, X)$.*
(iii) *There exist $v \in L^p(I, X)$ and $t_0 \in I$ such that $u(t) = u(t_0) + \int_{t_0}^t v(s)\, ds$ almost everywhere.*

In this case $u' = \frac{d}{dt}u = v$.

THEOREM 2.6. *Let $1 < p \leq \infty$. A Banach space X has the Radon–Nikodým Property if and only if the Difference Quotient Criterion characterizes the space $W^{1,p}(\Omega, X)$.*

Proof. It remains to show that the Difference Quotient Criterion implies the Radon–Nikodým Property. Let $f : I \to X$ be Lipschitz continuous, where without loss of generality I is bounded. We may assume that $\Omega = I$, otherwise extend f to $\mathbb{R}$ so that it remains Lipschitz continuous, embed I^d into Ω and consider the function $\xi \mapsto f(\xi_1)$. It follows that $f \in L^p(I, X)$ satisfies the Difference Quotient Criterion and is thus an element of $W^{1,p}(I, X)$ by assumption. Proposition 2.5 shows that X has the Radon–Nikodým Property. ■

The Difference Quotient Criterion Theorem 2.6 yields our first mapping theorem.

COROLLARY 2.7. *Let $1 < p \leq \infty$, $\Omega \subset \mathbb{R}^d$ be open, and X and Y be Banach spaces, with Y enjoying the Radon–Nikodým Property. Let $F : X \to Y$ be a Lipschitz continuous mapping such that $F(0) = 0$ if Ω has infinite* *measure. Then* $u \mapsto F \circ u$ *defines a mapping*

$$W^{1,p}(\Omega, X) \to W^{1,p}(\Omega, Y).$$

3. Composition with Lipschitz continuous mappings in spaces that have the Radon–Nikodým Property. In this section we will give an alternative proof of the last corollary that also includes the case $p = 1$. It will also contain another characterization of the Radon–Nikodým Property via Sobolev spaces. We first consider the one-dimensional case, in which Proposition 2.5 yields the result right away.

THEOREM 3.1. *Let* $1 \le p \le \infty$ *and let* I *be an interval. Let* X, Y *be Banach spaces such that* Y *has the Radon–Nikodým Property and assume that* $F : X \to Y$ *is Lipschitz continuous with Lipschitz constant* L. *If* $F(0) = 0$ *or* I *is bounded then* $F \circ u \in W^{1,p}(I, Y)$ *for all* $u \in W^{1,p}(I, X)$. *Moreover,* $\|F \circ u(\cdot)\|_X \le L\|u(\cdot)\|_X$ *almost everywhere.*

Proof. Let $u \in W^{1,p}(I, X)$. Proposition 2.5 implies that $F \circ u$ is absolutely continuous and hence differentiable almost everywhere since Y has the Radon–Nikodým Property. By Proposition 2.5 it remains to show that the derivative of $F \circ u$ is an element of $L^p(I, Y)$. This follows easily from the estimate

$$\begin{aligned}\|(F \circ u)'(t)\|_X &= \lim_{h\to 0} \frac{\|F(u(t+h)) - F(u(t))\|_X}{|h|} \\ &\le \limsup_{h\to 0} L\frac{\|u(t+h) - u(t)\|_X}{|h|} = L\|u'(t)\|_X\end{aligned}$$

for almost all $t \in I$. ∎

To show Theorem 3.1 for general domains we will need a higher-dimensional version of Proposition 2.5. It was Beppo Levi [Lev06] who considered functions which are absolutely continuous on lines. This motivates the following terminology: A function $u : \mathbb{R}^d \to X$ has the *BL-property* if for each $j \in \{1, \dots, d\}$ and almost all $(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_d) \in \mathbb{R}^{d-1}$ the function

$$u_j : \mathbb{R} \to X, \qquad t \mapsto u((x_1, \dots, x_{j-1}, t, x_{j+1}, \dots, x_d)),$$

is absolutely continuous and differentiable almost everywhere. Deny and Lions [DL54] showed the connection between Sobolev spaces and functions with the BL-property.

THEOREM 3.2. *Let* $\Omega \subset \mathbb{R}^d$ *be open and* $1 \le p \le \infty$.

(a) *Let* $u \in W^{1,p}(\Omega, X)$. *Then for each* $\omega \subset\subset \Omega$ *there exists* $u^* : \mathbb{R}^d \to X$ *with the BL-property such that* $u^* = u$ *almost everywhere on* ω. *Moreover* $\partial_j u^* = D_j u$ *almost everywhere on* ω.
(b) *Conversely, let* $u \in L^p(\Omega, X)$ *and* $c \ge 0$. *Assume that for each* $\omega \subset\subset \Omega$ *there exists* $u^* : \mathbb{R}^d \to X$ *with the BL-property such that* $u^* = u$ *almost*

everywhere on ω *and* $\|\partial_j u^*\|_{L^p(\omega,X)} \le c$ *for each* $j \in \{1, \dots, d\}$. *Then* $u \in W^{1,p}(\Omega, X)$.

Proof. This can be proven analogously to the real-valued case (see [MZ97, Theorem 1.41]). ■

Combining Theorem 3.2 with the one-dimensional case we obtain a proof of Corollary 2.7 which also includes the case $p = 1$:

THEOREM 3.3. *Suppose that* X, Y *are Banach spaces such that* Y *has the Radon–Nikodým Property and let* $1 \le p \le \infty$. *Let* $F : X \to Y$ *be Lipschitz continuous and assume that* $F(0) = 0$ *if* Ω *has infinite measure. Then* $F \circ u \in W^{1,p}(\Omega, Y)$ *for any* $u \in W^{1,p}(\Omega, X)$.

We give a special example, which will be of interest throughout the rest of this article.

COROLLARY 3.4. *Let* $1 \le p \le \infty$. *If* $u \in W^{1,p}(\Omega, X)$, *then* $\|u\|_X := \|u(\cdot)\|_X \in W^{1,p}(\Omega, \mathbb{R})$.

REMARK 3.5. Pełczyński and Wojciechowski [PW93, Theorem 1.1] have given another proof of Corollary 3.4. This important example will play a major role in many results we will prove about the spaces $W^{1,p}(\Omega, X)$. We will extend it in later sections by computing the distributional derivative of $\|u\|_X$ and showing that $\|\cdot\|_X : W^{1,p}(\Omega, X) \to W^{1,p}(\Omega, \mathbb{R})$ is continuous.

As in the last section, we obtain the converse of Theorem 3.3 and hence a characterization of the Radon–Nikodým Property.

THEOREM 3.6. *Let* Ω *have finite measure,* $1 \le p \le \infty$ *and let* X, Y *be Banach spaces,* $X \ne \{0\}$. *If every Lipschitz continuous* $F : X \to Y$ *gives rise to a mapping* $u \mapsto F \circ u$ *from* $W^{1,p}(\Omega, X)$ *to* $W^{1,p}(\Omega, Y)$, *then* Y *has the Radon–Nikodým Property.*

Proof. Let $f : I \to Y$ be Lipschitz continuous, where without loss of generality I is bounded. Further choose $x_0 \in X$ and $x_0' \in X'$ such that $\langle x_0, x_0' \rangle = 1$. Define a Lipschitz continuous map $F : X \to Y$ via

$$F(x) := f(\langle x, x_0' \rangle).$$

We may assume that $I^d \subset\subset \Omega$. Define $\tilde{u} : \Omega \to X$ via

$$\tilde{u}(\xi) := \xi_1 x_0.$$

Choose $\varphi \in C_c^\infty(\Omega, \mathbb{R})$ such that $\varphi_{|I^d} \equiv 1$ and let $u := \varphi\tilde{u} \in W^{1,p}(\Omega, X)$. By assumption we have $F \circ u \in W^{1,p}(\Omega, Y)$, hence by Theorem 3.2 its ξ_1-derivative exists for almost all $\xi \in I^d$. But for any such ξ this derivative is equal to $\frac{d}{d\xi_1} f(\xi_1)$, hence f is differentiable almost everywhere on I. It follows that Y has the Radon–Nikodým Property. ■

4. Composition with one-sided Gateaux differentiable Lipschitz continuous mappings. In this section we want to drop the Radon–Nikodým Property. If we do so, then there exist Lipschitz continuous mappings $F : X \to Y$ and functions $u \in W^{1,p}(\Omega, X)$ such that the composition $F \circ u$ has no distributional derivative in $L^p(\Omega, Y)$. For this reason we add the assumptions that F is one-sided Gateaux differentiable and show that this is sufficient for $F \circ u \in W^{1,p}(\Omega, Y)$. We will also prove a chain rule in this setting and explicitly compute the distributional derivatives for the cases where F is a norm or a lattice operation.

Let X, Y be Banach spaces. We say that a function $F : X \to Y$ is *one-sided Gateaux differentiable* at x if the right-hand limit

$$D_v^+ F(x) := \lim_{t\to 0+} \frac{1}{t}(F(x + tv) - F(x))$$

exists for every direction $v \in X$. In this case, the left-hand limit

$$D_v^- F(x) := \lim_{t\to 0+} \frac{1}{-t}(F(x - tv) - F(x))$$

exists as well and is given by $D_v^- F(x) = -D_{-v}^+ F(x)$. Let $\Omega \subset \mathbb{R}^d$ be open. For $u : \Omega \to X$ and $\xi \in \Omega$ we denote by

$$\partial_j^\pm u := \lim_{t\to 0\pm} \frac{u(\xi + te_j) - u(\xi)}{t}$$

the one-sided partial derivatives if they exist.

Lemma 4.1 (Chain rule). *Let X, Y be Banach spaces, and let $u : \Omega \to X$, $j \in 1, \dots, d$ and $\xi \in \Omega$ be such that the partial derivative $\partial_j u(\xi)$ exists. If $F : X \to Y$ is one-sided Gateaux differentiable, then*

$$\partial_j^\pm F \circ u(\xi) = D^\pm_{\partial_j u(\xi)} F(u(\xi)).$$

Proof. Let $t > 0$. Then

$$\begin{aligned}
&\frac{1}{t}\big(F(u(\xi + te_j)) - F(u(\xi))\big)\\
&= \frac{1}{t}\big(F(u(\xi + te_j)) - F(u(\xi) + t\partial_j u(\xi))\big) + \frac{1}{t}\big(F(u(\xi) + t\partial_j u(\xi)) - F(u(\xi))\big).
\end{aligned}$$

The first expression on the right can be estimated by

$$\frac{L}{t}\|u(\xi + te_j) - u(\xi) - t\partial_j u(\xi)\|_X \to 0 \quad (t \to 0),$$

and the second expression converges to $D^+_{\partial_j u(\xi)} F(u(\xi))$ as claimed. The left-hand limit can be computed analogously. ∎

Of course if F is one-sided Gateaux differentiable, in general the right and left Gateaux derivatives $D_v^+ F(x)$ and $D_v^- F(x)$ are different. However, if we compose F with a Sobolev function u, something surprising happens:

THEOREM 4.2. *Let $1 \le p \le \infty$ and $u \in W^{1,p}(\Omega, X)$. Suppose that $F : X \to Y$ is Lipschitz continuous and one-sided Gateaux differentiable, and assume furthermore that Ω is bounded or $F(0) = 0$. Then $F \circ u \in W^{1,p}(\Omega, Y)$ and we have the chain rule*

$$D_j(F \circ u) = D^+_{D_j u} F(u) = D^-_{D_j u} F(u).$$

We will need the following consequence of Theorem 3.2.

LEMMA 4.3. *Let $1 \le p \le \infty$, $u \in W^{1,p}(\Omega, X)$ and $j \in \{1, \dots, d\}$ be such that $\partial_j^+ u(\xi)$ exists almost everywhere. Then $D_j u = \partial_j^+ u$ almost everywhere. The same holds for the left-sided derivative.*

Proof. Without loss of generality $j = d$. For $\xi \in \mathbb{R}^d$ we write $\xi = (\hat{\xi}, \xi_d)$ with $\hat{\xi} \in \mathbb{R}^{d-1}$ and $\xi_d \in \mathbb{R}$. Let $\omega_n \subset\subset \Omega$ be such that $\bigcup_{n\in\mathbb{N}} \omega_n = \Omega$. It suffices to show that $D_j u = \partial_j^+ u$ almost everywhere on each ω_n. Fix $n \in \mathbb{N}$ and let u^* be a representative of u on ω_n as in Theorem 3.2. Choose a null set $N \subset \omega_n$ such that $u^* = u$, both $\partial_d^+ u$ and $\partial_d u^*$ exist and $\partial_d u^* = D_d u$ on $\omega_n \setminus N$. By Fubini's Theorem the set

$$\omega_n' := \{\xi \in \omega_n \setminus N : (\hat{\xi}, \xi_d + t_k) \notin N \text{ for some sequence } t_k \downarrow 0\}$$

has full measure in ω_n. For $\xi \in \omega_n'$ we choose a suitable sequence and obtain

$$\begin{aligned}\partial_d^+ u(\xi) &= \lim_{k\to\infty} \frac{u(\xi + t_k e_d) - u(\xi)}{t_k} = \lim_{k\to\infty} \frac{u^*(\xi + t_k e_d) - u^*(\xi)}{t_k} \\ &= \partial_d u^*(\xi) = D_d u(\xi). \ \blacksquare\end{aligned}$$

Proof of Theorem 4.2. First note that $F \circ u \in L^p(\Omega, X)$. Fix a $j \in \{1, \dots, d\}$. The function

$$\Omega \to Y, \quad \xi \mapsto D^{\pm}_{D_j(u(\xi))} F(u(\xi)),$$

is measurable as the limit of a sequence of measurable functions. Since F is Lipschitz continuous, it follows that $D^{\pm}_{D_j(u(\cdot))} F(u(\cdot)) \in L^p(\Omega, Y)$. Fix $\varphi \in C_c^\infty(\Omega, \mathbb{R})$ and let $\omega \subset\subset \Omega$ contain its support. We have to show that

$$\int_\Omega F \circ u \partial_j \varphi = -\int_\Omega D^{\pm}_{D_j u(\xi)} F(u(\xi))\varphi(\xi)\, d\xi.$$

Choose a representative u^* of u on ω as in Theorem 3.2. For every $y' \in Y'$ the function $y' \circ F$ is Lipschitz continuous, hence $\langle F \circ u, y' \rangle \in W^{1,p}(\Omega, \mathbb{R})$ by Theorem 3.3. Further, u^* is partially differentiable almost everywhere on ω and $y' \circ F$ is Gateaux differentiable, hence $\partial_j^{\pm} \langle F \circ u^*(\xi), y' \rangle = \langle D^{\pm}_{\partial_j u^*(\xi)} F(u(\xi)), y' \rangle$ almost everywhere on ω by Lemma 4.1. Applying Lemma 4.3 we obtain

$$D_j \langle F \circ u, y' \rangle = \langle D^{\pm}_{D_j u(\cdot)} F(u(\cdot)), y' \rangle$$

almost everywhere on ω. Thus

$$\Big\langle \int_\Omega F\circ u\partial_j\varphi, y'\Big\rangle = \Big\langle -\int_\Omega D^{\pm}_{D_j u(\xi)}F(u(\xi))\varphi(\xi)\,d\xi, y'\Big\rangle.$$

Since $y'\in X'$ was arbitrary, we obtain the result. ∎

We give a first example: The function $F := \|\cdot\|_X : X\to\mathbb{R}$ is one-sided Gateaux differentiable. The derivatives at $x\in X$ in direction $h\in X$ are

$$D_h^+F(x) = \sup\{\langle h, x'\rangle : x'\in J(x)\},\qquad D_h^-F(x) = \inf\{\langle h, x'\rangle : x'\in J(x)\},$$

where $J(x) = \{x'\in S_{X'}(0,1) : \langle x, x'\rangle = \|x\|_X\}$ is the duality map; see [AGG[+]86, A-II, Remark 2.4]. If $u\in W^{1,p}(\Omega, X)$, $1\le p\le\infty$, we already know that $\|u\|_X\in W^{1,p}(\Omega,\mathbb{R})$ by Corollary 3.4. Here we obtain a formula for the derivative, which will be crucial later.

THEOREM 4.4. *Let $1\le p\le\infty$ and let $u\in W^{1,p}(\Omega, X)$. Then $\|u\|_X\in W^{1,p}(\Omega,\mathbb{R})$ and*

$$D_j\|u(\xi)\|_X = \langle D_j u(\xi), J(u(\xi))\rangle$$

almost everywhere, where $\langle D_j u(\xi), J(u(\xi))\rangle := \langle D_j u(\xi), x'\rangle$ with $x'\in J(u(\xi))$, which does not depend on the choice of $x'\in J(u(\xi))$ for all ξ outside a negligible set.

EXAMPLE 4.5.

(a) Let X be a Hilbert space. Then

$$D_j\|u(\xi)\|_X = \begin{cases}\langle D_j u(\xi), u(\xi)/\|u(\xi)\|_X\rangle & \text{if } u(\xi)\neq 0,\\ 0 & \text{if } u(\xi)=0.\end{cases}$$

In particular, if $X=\mathbb{R}$, then we find the well-known formula $D_j|u| = \operatorname{sign}(u)D_j u$.

(b) Let $X=\ell^1$ and $u(\xi) = (u_n(\xi))_{n\in\mathbb{N}}\in W^{1,p}(\Omega, X)$. Evaluating the nth coordinate we obtain $u_n\in W^{1,p}(\Omega,\mathbb{R})$ with $D_j u_n(\xi) = (D_j u(\xi))_n$ almost everywhere. It follows that

$$D_j\|u(\xi)\|_{\ell^1} = \sum_{n=1}^\infty \operatorname{sign}(u_n(\xi))D_j u_n(\xi).$$

(c) Let $X = C(K)$ for a compact topological space K. For each $\xi\in\Omega$ let $k_\xi\in K$ be such that $u(\xi)$ has a global extremum at k_ξ. Then

$$D_j\|u(\xi)\|_{C(K)} = \operatorname{sign}(u(\xi)(k_\xi))D_j u(\xi)(k_\xi).$$

The following estimate has many interesting consequences (e.g. Theorems 5.1 and 6.13).

COROLLARY 4.6. *Let $1\le p\le\infty$. For all $u\in W^{1,p}(\Omega,X)$ we have $\|u\|_X\in W^{1,p}(\Omega,\mathbb{R})$, and $|D_j\|u(\xi)\|_X|\le\|D_j u(\xi)\|_X$ almost everywhere. In particular, $\big\|\|u\|_X\big\|_{W^{1,p}(\Omega,\mathbb{R})}\le\|u\|_{W^{1,p}(\Omega,X)}$.*

REMARK 4.7. In the case $X = \mathbb{R}$ we have equality in the above inequalities (see Example 4.5(a)). However, if $\dim X \geq 2$, then there exists no $C > 0$ such that $|D_j \|u(\xi)\|_X| \geq C\|D_j u(\xi)\|_X$ for all $u \in W^{1,p}(\Omega, X)$. This can easily be seen by embedding the $\mathbb{R}^2$-valued function $[0, 2\pi] \ni t \mapsto (\sin t, \cos t)$ into $W^{1,p}(\Omega, X)$.

COROLLARY 4.8. *Let $1 \leq p < \infty$. The mapping $\| \cdot \|_X : W^{1,p}(\Omega, X) \to W^{1,p}(\Omega, \mathbb{R})$ is continuous.*

Proof. Let $u_n \to u$ in $W^{1,p}(\Omega, X)$. Have in mind that in a topological space a sequence (x_n) converges to x if and only if every subsequence of (x_n) has a subsequence that converges to x. Hence we may assume that u_n and $D_j u_n$ are dominated by an L^p-function and that $u_n \to u$ and $D_j u_n \to D_j u$ pointwise almost everywhere since every convergent sequence in $L^p(\Omega, X)$ has a subsequence with these properties. It is obvious that $\|u_n\|_X \to \|u\|_X$ in $L^p(\Omega, \mathbb{R})$, hence it only remains to show convergence of the derivatives. Since $|D_j \|u_n\|_X| \leq \|D_j u_n\|_X$ by Corollary 4.6, the functions $D_j \|u_n\|_X$ are uniformly dominated by an L^p-function. To apply the Dominated Convergence Theorem we need to show that they converge almost everywhere to $D_j \|u\|_X$. Let $\xi \in \Omega \setminus N$ where $N \subset \Omega$ is a negligible set such that u_n and $D_j u_n$ converge pointwise and, using Theorem 4.4, simultaneously $D_j \|u_n(\cdot)\|_X = \langle D_j u_n(\cdot), J(u_n(\cdot))\rangle$ $(n \in \mathbb{N})$ and $D_j \|u(\cdot)\|_X = \langle D_j u(\cdot), J(u(\cdot))\rangle$ outside N. Choose $x'_n \in J(u_n(\xi))$. Since we are working with a countable number of measurable functions, we may assume that X is separable. Passing to a subsequence we may assume that (x'_n) converges in the $\sigma(X', X)$-topology to an element $x' \in X'$. One easily sees that $x' \in J(u(\xi))$. It follows that

$$D_j \|u_n(\xi)\|_X = \langle D_j u_n(\xi), x'_n\rangle \to \langle D_j u(\xi), x'\rangle = D_j \|u(\xi)\|_X$$

for all $\xi \in \Omega \setminus N$, finishing the proof. ∎

As a second example we will consider lattice operations. Let E be a real Banach lattice. Typical examples are $L^p(\Omega, \mathbb{R})$ for $1 \leq p \leq \infty$, and $C(K)$ for compact K. We want to examine whether for $u \in W^{1,p}(\Omega, E)$ the functions $u^+ := u(\cdot)^+$, $u^- := u(\cdot)^-$, $|u| := |u(\cdot)|$, etc. are still in $W^{1,p}(\Omega, E)$. Since such lattice operations are Lipschitz continuous, the results of the previous sections imply that lattice operations leave the space $W^{1,p}(\Omega, E)$ invariant if E has the Radon–Nikodým Property, and in the case $E = \mathbb{R}$ this property is a fundamental tool for classical Sobolev spaces. However, $W^{1,p}(\Omega, E)$ may not be invariant under lattice operations in general as the following example shows.

EXAMPLE 4.9. Let $u : (0, 1) \to E = C([0, 1], \mathbb{R})$ be given by

$$u(t)(r) = r - t.$$

We have

$$u(t) = \mathrm{id}_{(0,1)} + \int_0^t u'(s)\,ds,$$

where $u'(t) = -1_{(0,1)} \in L^p((0,1),E)$. Proposition 2.5 implies that $u \in W^{1,p}((0,1),E)$. The function u^+ is given by

$$u^+(t)(r) = \begin{cases} 0 & \text{if } r < t, \\ r-t & \text{if } r \geq t. \end{cases}$$

Assume that $u^+ \in W^{1,p}((0,1),E)$. Then $\frac{d}{dt}u^+(t) \in E$ exists for almost all $t \in (0,1)$ by Proposition 2.5. However, for all $t \in (0,1)$ we actually have

$$\frac{d}{dt}u^+(t)(r) = \begin{cases} 0 & \text{if } r < t, \\ -1 & \text{if } r \geq t, \end{cases}$$

contradicting $\frac{d}{dt}u^+(t) \in E$. Thus $u^+ \notin W^{1,p}((0,1),E)$.

We now want to find a class of Banach lattices E for which the spaces $W^{1,p}(\Omega,E)$ are invariant under lattice operations even though they might not have the Radon–Nikodým Property. For the reader's convenience we will summarize a few necessary facts on Banach lattices and refer to [AGG$^+$86] for a short introduction and to [Sch74] and [MN91] for further information.

Let E be a real Banach lattice. A subset $A \subset E$ is called *downwards directed* if for any $x, y \in A$ there exists $z \in A$ such that $z \leq x, y$. Further, A is called *lower order bounded* if there exists $y \in E$ such that $y \leq x$ for all $x \in A$. We say E has *order continuous norm* if each lower order bounded downwards directed set A converges, that is, there exists an $x_0 \in E$ such that $\inf_{x \in A} \|x - x_0\|_E = 0$. We write $x_0 =: \lim A$.

Now let X be a Banach space. A function $F : X \to E$ is called *convex* if

$$F(\lambda x + (1-\lambda)y) \leq \lambda F(x) + (1-\lambda)F(y)$$

for all $x, y \in X$ and $0 \leq \lambda \leq 1$. This is equivalent to saying that $x' \circ F$ is convex for every $x' \in E'_+$. If F is convex, the difference quotients

$$t \mapsto \frac{F(x+ty) - F(x)}{t}$$

define an increasing function from $\mathbb{R} \setminus \{0\}$ to E. Thus if E has order continuous norm, then F is one-sided Gateaux differentiable and $D_y^- F(x) \leq D_y^+ F(x)$ for all $x, y \in E$. From Theorem 4.2 we deduce the following.

Proposition 4.10. *Let X be a Banach space and $u \in W^{1,p}(\Omega,X)$, where $1 \leq p \leq \infty$. Let $F : X \to E$ be Lipschitz continuous and convex, where E is a Banach lattice with order continuous norm. If Ω is bounded or $F(0) = 0$, then $F \circ u \in W^{1,p}(\Omega,E)$ with*

$$D_j F \circ u = D^+_{D_j u} F(u) = D^-_{D_j u} F(u).$$

Next we consider the function $\vartheta : E \to E$ given by $\vartheta(v) = |v|$, which is convex. We need the following notation. If $v \in E_+$, then

$$E_v := \{w \in E : \exists n \in \mathbb{N} : |w| \le nv\}$$

denotes the *ideal generated by* v. We set

$$|v|^d := \{w \in E : |v| \wedge |w| = 0\}.$$

Assume that E has order continuous norm. Then $E = \overline{E_{v^+}} \oplus \overline{E_{v^-}} \oplus |v|^d$ (see [Sch74, II.8.3] or [AGG$^+$86, C-I.1 and C-I.5]). We denote by P_{v^+}, P_{v^-} and $P_{|v|^d}$ the projections along this decomposition. Define $\operatorname{sign} v : E \to E$ by

$$(\operatorname{sign} v)(w) := P_{v^+}w - P_{v^-}w.$$

EXAMPLE 4.11. Let $E = L^p(\Omega)$ with $1 \le p < \infty$ and let $v \in E$. If we set $\Omega_+ := \{\xi \in \Omega : v(\xi) > 0\}$, $\Omega_- := \{\xi \in \Omega : v(\xi) < 0\}$ and $\Omega_d := \{\xi \in \Omega : v(\xi) = 0\}$ then $E_{v^+} = \{w \in E : w = 0 \text{ on } \Omega_+^c\}$, $E_{v^-} = \{w \in E : w = 0 \text{ on } \Omega_-^c\}$ and $|v|^d = \{w \in E : w = 0 \text{ on } \Omega_d^c\}$. Hence $P_{v^+}w = 1_{\Omega_+}w$, $P_{v^-}w = 1_{\Omega_-}w$ and $P_{|v|^d}w = 1_{\Omega_d}w$. We obtain

$$(\operatorname{sign} v)(w) = \frac{v}{|v|}w \cdot 1_{v\neq 0}.$$

PROPOSITION 4.12. *Assume that E has order continuous norm. Then ϑ is one-sided Gateaux differentiable and*

$$D_w^+\vartheta(v) = (\operatorname{sign} v)w + P_{|v|^d}|w|, \qquad D_w^-\vartheta(v) = (\operatorname{sign} v)w - P_{|v|^d}|w|.$$

Proof. This follows from [AGG$^+$86, C-II, Proposition 5.6]. ∎

With this and Proposition 4.10 we obtain

THEOREM 4.13. *Let E be a Banach lattice with order continuous norm and let $1 \le p \le \infty$. If $u \in W^{1,p}(\Omega, E)$, then $|u| \in W^{1,p}(\Omega, E)$ and*

$$D_j|u| = (\operatorname{sign} u)D_j u.$$

COROLLARY 4.14. *In the setting of Theorem 4.13,*

$$D_j u^+ = P_{u^+}D_j u.$$

Proof. This follows from Theorem 4.13 since $u^+ = \frac{1}{2}(|u| + u)$. ∎

EXAMPLE 4.15.

(a) If $X = \mathbb{R}$, we obtain the well-known formula

$$D_j u^+ = 1_{\{u>0\}}D_j u.$$

(b) Let (S, Σ, μ) be a measure space and let $u \in W^{1,p}(\Omega, L^r(S, \mathbb{R}))$ where $1 \le p \le \infty$ and $1 \le r < \infty$. The norm on $L^r(S, \mathbb{R})$ is order continuous. A pointwise comparison using Corollary 4.14 shows that

$$D_j u^+(\xi) = D_j u(\xi) \cdot 1_{\{s\in S: u(\xi)(s)>0\}}.$$

This was essentially proven directly in [ADKF14, Proposition 4.1].

Finally, we remark that Theorem 4.13 remains true if E is a complex Banach lattice provided $\operatorname{sign} u$ is defined properly [AGG+86, C-II, Proposition 5.6].

5. Embedding theorems. As an application of Corollary 4.6 we show that Sobolev embedding theorems carry over from the real-valued to the vector-valued case. Vector-valued versions of Sobolev–Gagliardo–Nirenberg inequalities are already known [SS05]. Using our work, such theorems follow immediately from the real-valued case, leading to an elegant and short proof.

THEOREM 5.1. *Let $\Omega \subset \mathbb{R}^d$ be open such that we have a continuous embedding $W^{1,p}(\Omega,\mathbb{R}) \hookrightarrow L^r(\Omega,\mathbb{R})$ for some $1 \le p, r \le \infty$. Then we also have a continuous embedding $W^{1,p}(\Omega,X) \hookrightarrow L^r(\Omega,X)$ for any Banach space X, and the norm of the embedding remains the same.*

Proof. Let $u \in W^{1,p}(\Omega,X)$. Then $\|u\|_X \in W^{1,p}(\Omega,\mathbb{R})$ by Corollary 3.4. By assumption it follows that $\|u\|_X \in L^r(\Omega,\mathbb{R})$, and hence by definition of the Bochner space we obtain $u \in L^r(\Omega,X)$. Further, if C is the norm of the embedding in the real-valued case, we use Corollary 4.6 to compute

$$\|u\|_{L^r(\Omega,X)} = \big\|\|u\|_X\big\|_{L^r(\Omega,\mathbb{R})} \le C\big\|\|u\|_X\big\|_{W^{1,p}(\Omega,\mathbb{R})} \le C\|u\|_{W^{1,p}(\Omega,X)}. \ \blacksquare$$

For completeness we also show that Morrey's embedding theorem follows from the real-valued case.

THEOREM 5.2 (Morrey's Embedding Theorem). *Let Ω be open, $1 \le p \le \infty$ and suppose that there exist constants C and $\alpha > 0$ such that*

$$|u(\xi) - u(\eta)| \le C\|u\|_{W^{1,p}(\Omega,\mathbb{R})}|\xi - \eta|^\alpha$$

for all $u \in W^{1,p}(\Omega,\mathbb{R})$ and almost all $\xi, \eta \in \Omega$. Then for all $u \in W^{1,p}(\Omega,X)$ we have

$$\|u(\xi) - u(\eta)\|_X \le C\|u\|_{W^{1,p}(\Omega,X)}|\xi - \eta|^\alpha$$

for almost all $\xi, \eta \in \Omega$. In particular u has a Hölder continuous representative.

Proof. Let $u \in W^{1,p}(\Omega,X)$. Since u is measurable, we may assume that X is separable. Then there exist normed functionals x'_k that are norming for X. Define $u_k := \langle u, x'_k\rangle$. By assumption we have

$$|u_k(\xi) - u_k(\eta)| \le C\|u_k\|_{W^{1,p}(\Omega,\mathbb{R})}|\xi - \eta|^\alpha$$

for all $k \in \mathbb{N}$ and all ξ, η outside a common set of measure zero. Since $\|u_k\|_{W^{1,p}(\Omega,\mathbb{R})} \le \|u\|_{W^{1,p}(\Omega,X)}$, taking the supremum over k gives the claim. $\blacksquare$

Next we prove compactness of the Sobolev embedding. Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary. If $\Omega \subset \mathbb{R}$ is an interval, the following result is known as the Aubin–Lions Lemma [Aub63], [Sho97, Chapter III, Proposition 1.3]. The Aubin–Lions result is very useful in the theory of partial differential equations (see e.g. [AC10]). Many extensions on intervals have been given (see e.g. [Sim87]). Amann [Ama00, Theorem 5.2] gives a multidimensional version if the boundary of Ω is smooth. Here we prove a special case of Amann's result by direct arguments which only require the boundary to be Lipschitz.

THEOREM 5.3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary. Suppose X, Y are Banach spaces such that Y is compactly embedded in X and let $1 \le p < \infty$. Then the embedding*

$$W^{1,p}(\Omega, X) \cap L^p(\Omega, Y) \hookrightarrow L^p(\Omega, X)$$

is compact.

For the proof we will need some auxillary results.

LEMMA 5.4. *Let $\mathcal{F} \subset L^p(\mathbb{R}^d, X)$ be bounded and $\rho \in C_c^\infty(\mathbb{R}^d, \mathbb{R})$. Then there exists a $c > 0$ such that*

$$\|\rho * f(\xi) - \rho * f(\eta)\|_X \le c|\xi - \eta| \quad \text{for all } \xi, \eta \in \mathbb{R}^d \text{ and } f \in \mathcal{F}.$$

Proof. This can be proven analogously to the scalar-valued case (see e.g. [Bre11, Corollary 4.28]). ∎

LEMMA 5.5. *Let $\mathcal{F} \subset W^{1,p}(\mathbb{R}^d, X)$ be bounded and (ρ_n) be a mollifier. Then*

$$\sup_{f \in \mathcal{F}} \|\rho_n * f - f\|_{L^p(\mathbb{R}^d, X)} \to 0 \quad \text{as } n \to \infty.$$

Proof. From Lemma 2.1 we deduce that there exists a $C > 0$ such that for all $h \in \mathbb{R}^d$ we have

$$\|u(\cdot + h) - u\|_{L^p(\mathbb{R}^d, X)} \le C|h|$$

for every $u \in \mathcal{F}$. The result now follows as in the scalar-valued case (see e.g. [Bre11, Step 1 of the proof of Theorem 4.26]). ∎

The next theorem is particularly important, as its scalar-valued counterpart is frequently used in the theory of Sobolev spaces. In the last section we will use it again. For the notion of *uniform Lipschitz boundary* we refer to [Leo09, Definition 12.10]. For bounded sets, uniform Lipschitz boundary is the same as Lipschitz boundary.

THEOREM 5.6 (Extension Theorem). *Let Ω be an open set with uniform Lipschitz boundary. Then for any $1 \le p < \infty$ there exists a bounded linear operator*

$$\mathcal{E} : W^{1,p}(\Omega, X) \to W^{1,p}(\mathbb{R}^d, X)$$

such that $\mathcal{E}(u)_{|\Omega} = u$ for all $u \in W^{1,p}(\Omega, X)$.

Proof. This can be shown analogously to the scalar-valued case (see e.g. [Leo09, Theorem 12.15], and also [Ama95, Theorem 2.4.5]). ∎

Proof of Theorem 5.3. We have to show that

$$B := \{f \in W^{1,p}(\Omega, X) \cap L^p(\Omega, Y) : \|f\|_{W^{1,p}(\Omega,X)} \leq 1, \|f\|_{L^p(\Omega,Y)} \leq 1\}$$

is precompact in $L^p(\Omega, X)$. Let $\mathcal{E}$ be the extension operator in Theorem 5.6. We will argue in two steps: First we show that for any $\omega \subset\subset \Omega$ the set $B_{|\omega}$ is precompact in $L^p(\omega, X)$. For this, let $\varepsilon > 0$ and use Lemma 5.5 to choose $n_0 > \operatorname{dist}(\omega, \partial\Omega)^{-1}$ such that

$$\|\mathcal{E}f - \rho_{n_0} * \mathcal{E}f\|_{L^p(\mathbb{R}^d,X)} \leq \varepsilon$$

for all $f \in B$. For all $\xi \in \overline{\omega}$ we have

$$\|(\rho_{n_0} * \mathcal{E}f)(\xi)\|_Y \leq \|\rho_{n_0}\|_{L^q(\Omega,\mathbb{R})} \|f\|_{L^p(\Omega,Y)} \leq \|\rho_{n_0}\|_{L^q(\Omega,\mathbb{R})} =: C$$

where $1/p + 1/q = 1$. Since $Y \hookrightarrow X$ is compact, the set

$$K := \{y \in Y : \|y\|_Y \leq C\}$$

is precompact in X. Further, by Lemma 5.4 the set

$$H := \{\rho_{n_0} * \mathcal{E}f : f \in B\}$$

is equicontinuous, thus it is precompact in $C(\overline{\omega}, K)$ by the Arzelà–Ascoli Theorem. Consequently, we find $g_j \in L^p(\omega, X)$ with $H \subset \bigcup_{j=1}^m B_{L^p(\omega,X)}(g_j, \varepsilon)$. Let $f \in B$ and choose $j \in 1, \ldots, m$ such that $\|\rho_{n_0} * \mathcal{E}f - g_j\|_{L^p(\omega,X)} < \varepsilon$. Hence

$$\|f - g_j\|_{L^p(\omega,X)} \leq \|\mathcal{E}f - \rho_{n_0} * \mathcal{E}f\|_{L^p(\omega,X)} + \|\rho_{n_0} * \mathcal{E}f - g_j\|_{L^p(\omega,X)} < 2\varepsilon,$$

implying that $B_{|\omega} \subset \bigcup_{j=1}^m B_{L^p(\omega,X)}(g_j, 2\varepsilon)$ is precompact.

Using this we now show that B is precompact in $L^p(\Omega, X)$. Let $\varepsilon > 0$ and use Lemma 5.5 to choose $n_0 \in \mathbb{N}$ such that

$$\|\mathcal{E}f - \rho_{n_0} * \mathcal{E}f\|_{L^p(\mathbb{R}^d,X)} \leq \varepsilon$$

for all $f \in B$. For all $\xi \in \Omega$ we have

$$\|(\rho_{n_0} * \mathcal{E}f)(\xi)\|_X \leq \|\rho_{n_0}\|_{L^q(\mathbb{R}^d,\mathbb{R})} \|\mathcal{E}f\|_{L^p(\mathbb{R}^d,X)} \leq \|\rho_{n_0}\|_{L^q(\mathbb{R}^d,\mathbb{R})} \|\mathcal{E}\|_{\mathcal{L}} =: C$$

independently of $f \in B$. Let $\omega \subset\subset \Omega$ be such that $C|\Omega \setminus \omega|^{1/p} < \varepsilon$. Then $\|\rho_{n_0} * \mathcal{E}f\|_{L^p(\Omega\setminus\omega,X)} < \varepsilon$ for all $f \in B$ by the above estimate. By the first step there exist $g_j \in L^p(\omega, X)$ such that

$$B_{|\omega} \subset \bigcup_{j=1}^m B_{L^p(\omega,X)}(g_j, \varepsilon).$$

Define

$$G_j(\xi) := \begin{cases} g_j(\xi) & \text{if } \xi \in \omega, \\ 0 & \text{otherwise.} \end{cases}$$

Let $f \in B$. Then there exists a $j \in \{1, \dots, m\}$ such that $\|f_{|\omega} - g_j\|_{L^p(\omega,X)} \le \varepsilon$. Thus

$$\begin{aligned}\|f - G_j\|_{L^p(\Omega,X)} &\le \varepsilon + \|f\|_{L^p(\Omega\setminus\omega,X)} \\ &\le \varepsilon + \|\mathcal{E}f - \rho_{n_0} * \mathcal{E}f\|_{L^p(\mathbb{R}^d,X)} + \|\rho_{n_0} * \mathcal{E}f\|_{L^p(\Omega\setminus\omega,X)} \le 3\varepsilon,\end{aligned}$$

proving that $B \subset \bigcup_{j=1}^m B_{L^p(\omega,X)}(G_j, 3\varepsilon)$. This finishes the proof of Theorem 5.3. ∎

6. Weak implies strong and Dirichlet boundary data. Let $\Omega \subset \mathbb{R}^d$ be open. For a vector-valued function $u : \Omega \to X$ it is natural to ask whether a weak regularity property (in the sense of duality) implies the corresponding strong regularity property. For example, if u is locally bounded and $\langle u, x'\rangle$ is harmonic for all x' in a separating subset of X' then u itself is harmonic [Are16, Theorem 5.4]. The analogous property does not hold for Sobolev spaces.

Example 6.1.

(a) The function

$$u : (0,1) \to L^r((0,1), \mathbb{R}) \quad (1 \le r < \infty), \qquad u(t) := 1_{(0,t)},$$

is nowhere differentiable, hence it is not in $W^{1,1}((0,1), L^r((0,1),\mathbb{R}))$. But for each $x' \in L^{r'}((0,1),\mathbb{R})$ we have $\langle u(t), x'\rangle = \int_0^t x'(s)\,ds$, which is in $W^{1,p}((0,1),\mathbb{R})$ for each $p \le r'$ by Proposition 2.5.

(b) Let $A \subset (0,1)$ be Lebesgue nonmeasurable and consider the Hilbert space $\ell^2(A)$ with orthonormal base $e_t := (\delta_{t,s})_{s\in A}$. Consider the function

$$u : (0,1) \to \ell^2(A), \qquad u(t) = \begin{cases} 0, & t \notin A, \\ e_t, & t \in A. \end{cases}$$

Then $(u, x') = 0$ almost everywhere, which defines a function in $W^{1,\infty}((0,1),\mathbb{R})$ for all $x' \in \ell^2(A)$. But u is not even measurable.

However, if we assume more regularity, we obtain a positive result. A space X is called *weakly sequentially complete* if each weak Cauchy sequence in X has a weak limit. Each reflexive space is weakly sequentially complete, but so is even $L^1(\Omega)$.

Proposition 6.2. *Suppose that X is weakly sequentially complete, and let $\Omega \subset \mathbb{R}^d$ be open and bounded. Let $u : \Omega \to X$ be such that $\langle u, x'\rangle \in C^1(\overline{\Omega})$ for all $x' \in X'$. Then $u \in W^{1,\infty}(\Omega, X)$.*

Proof. Since $\langle u, x'\rangle \in C^1(\overline{\Omega})$, it follows that the difference quotient $\frac{1}{h}(u(\xi + he_j) - u(\xi))$ is a weak Cauchy sequence for every $\xi \in \Omega$. The weak sequential completeness of X shows that there exists $u_j(\xi) \in X$ with $\partial_j\langle u(\xi), x'\rangle = \langle u_j(\xi), x'\rangle$ for all $x' \in X'$. The function $\xi \mapsto u_j(\xi)$ is weakly continuous. A weakly continuous function is measurable, which can be shown analogously to [CH98, Corollary 1.4.8]. The functions u and u_j are weakly bounded and hence norm bounded. It follows that $u, u_j \in L^p(\Omega, X)$. Since the integration by parts formula holds weakly and X' separates X, it follows that $u \in W^{1,\infty}(\Omega, X)$ and that $u_j = D_j u$. ∎

Note that it is not enough to show that $\langle u, x' \rangle$ has a representative in $C^1(\overline{\Omega})$ (see Example 6.1(b)). The result is false if we drop the hypothesis that X is weakly sequentially complete, as the following example shows.

EXAMPLE 6.3. The function

$$u : (0,1) \to c_0, \qquad u(t) := \left(\frac{\sin(nt)}{n} \right),$$

is nowhere differentiable, as the only candidate for the derivative is $u'(t) = (\cos(nt))_{n\in\mathbb{N}}$, which is not in c_0 for any $t \in (0,1)$. But for every $x' \in \ell^1 = c_0'$ the function $\langle u, x' \rangle$ is in $C^1([0,1], \mathbb{R}) \subset W^{1,\infty}((0,1), \mathbb{R})$.

A Banach lattice E is weakly sequentially complete if and only if it does not have c_0 as a closed subspace [LT79, Theorem 1.c.4]. Hence, combining the above proposition and example we obtain

COROLLARY 6.4. *Let E be a Banach lattice and $\Omega \subset \mathbb{R}^d$ be open and bounded, and let $1 \le p \le \infty$. Then the following are equivalent:*

(i) *Let $u : \Omega \to X$ be such that $\langle u, x' \rangle \in C^1(\overline{\Omega}, \mathbb{R})$ for all $x' \in X'$. Then $u \in W^{1,p}(\Omega, X)$.*
(ii) *E is weakly sequentially complete.*
(iii) *E does not contain a closed subspace isomorphic to c_0.*

We now turn to Dirichlet boundary conditions, for which we will establish, among others, a result of the type "weak implies strong". From now on let $1 \le p < \infty$. As usual, we define $W_0^{1,p}(\Omega, X)$ as the closure of $C_c^\infty(\Omega, X)$ in the $W^{1,p}(\Omega, X)$-norm. If $u \in W_0^{1,p}(\Omega, X)$ and if $\varphi_n \in C_c^\infty(\Omega, X)$ converges to u, then Corollary 4.8 implies that $\|\varphi_n\|_X$ converges to $\|u\|_X$ in $W^{1,p}(\Omega, \mathbb{R})$. Since $\|\varphi_n\|_X$ is compactly supported, it follows that $\|u\|_X \in W_0^{1,p}(\Omega, \mathbb{R})$ [Bre11, Lemma 9.5]. We will show that the converse is true as well. For that, we will need the following lemmata.

LEMMA 6.5. *Let $u \in W^{1,p}(\Omega, X)$ and $\psi \in W^{1,p}(\Omega, \mathbb{R})$ be such that ψu, $(D_j\psi)u$ and $\psi D_j u$ are in $L^p(\Omega, \mathbb{R})$. Then $\psi u \in W^{1,p}(\Omega, X)$ with*

$$D_j(\psi u) = (D_j \psi) u + \psi D_j u.$$

Proof. If $X = \mathbb{R}$, this follows from [GT01, (7.18)]. Applying this to $\langle u, x' \rangle$ for arbitrary $x' \in X'$ yields the result, since X' separates X. ∎

LEMMA 6.6. *Let $u \in W^{1,p}(\Omega, X)$ and let $\hat{\varphi} \in C_c^\infty(\Omega, \mathbb{R})_+$. Define $\varphi := \hat{\varphi} \wedge \|u\|_X$. Then the function*

$$v := \begin{cases} \frac{u}{\|u\|_X}\varphi & \text{if } u \neq 0, \\ 0 & \text{otherwise} \end{cases}$$

is in $W^{1,p}(\Omega, X)$ and

$$D_j v = \begin{cases} \left(\frac{(D_j u)\|u\|_X - u D_j \|u\|_X}{\|u\|_X^2}\varphi + \frac{u}{\|u\|_X} D_j \varphi\right) & \text{if } u \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Note that the functions

$$\frac{u}{\|u\|_X}\varphi \cdot 1_{u\neq 0}, \quad \frac{D_j u}{\|u\|_X}\varphi \cdot 1_{u\neq 0}, \quad \frac{u}{\|u\|_X} D_j\varphi \cdot 1_{u\neq 0} \quad \text{and} \quad \frac{u D_j \|u\|_X}{\|u\|_X^2}\varphi \cdot 1_{u\neq 0}$$

are all in $L^p(\Omega, X)$. Let $\varepsilon > 0$ and define

$$f_\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+, \qquad t \mapsto \frac{1}{t+\varepsilon}.$$

Then $f_\varepsilon \in C^1(\mathbb{R}_+)$ and $|f'_\varepsilon| \leq 1/\varepsilon^2$. The usual chain rule (see e.g. [Bre11, Proposition 9.5]) yields $f_\varepsilon \circ \|u\|_X \in W^{1,p}(\Omega, \mathbb{R})$ and

$$D_j(f_\varepsilon \circ \|u\|_X) = \frac{-1}{(\|u\|_X + \varepsilon)^2} D_j \|u\|_X.$$

The preceding lemma implies that $\varphi u \in W^{1,p}(\Omega, X)$ and that the usual product rule holds. Using the lemma once more we see that $(f_\varepsilon \circ \|u\|_X)\varphi u \in W^{1,p}(\Omega, X)$ and the usual product rule holds. This means that

$$\int_\Omega (f_\varepsilon \circ \|u\|_X)\varphi u \partial_j \psi = -\int_\Omega \varphi D_j f_\varepsilon \circ \|u\|_X u\psi - \int_\Omega D_j(\varphi u) f_\varepsilon \circ \|u\|_X \psi$$

for all $\psi \in C_c^\infty(\Omega, \mathbb{R})$. Letting $\varepsilon \to 0$, by the Dominated Convergence Theorem we obtain

$$\int_\Omega \varphi \frac{1}{\|u\|_X} u \cdot 1_{u\neq 0} \partial_j \psi = \int_\Omega \varphi \frac{1}{\|u\|_X^2} D_j \|u\|_X u\psi - \int_\Omega D_j(\varphi u) \frac{1}{\|u\|_X}\psi.$$

This proves the claim. ∎

THEOREM 6.7. *Let $1 \leq p < \infty$ and let $u \in W^{1,p}(\Omega, X)$. Then $u \in W_0^{1,p}(\Omega, X)$ if and only if $\|u\|_X \in W_0^{1,p}(\Omega, \mathbb{R})$.*

Proof. It remains to show the "if" part. Let $\hat{\varphi}_n \in C_c^\infty(\Omega, \mathbb{R})_+$ be convergent to $\|u\|_X$ in $W^{1,p}(\Omega, X)_+$ and define $\varphi_n := \hat{\varphi}_n \wedge \|u\|_X$. The function $u_n := \frac{u}{\|u\|_X}\varphi_n \cdot 1_{u\neq 0}$ is in $W^{1,p}(\Omega, X)$ by Lemma 6.6 and is compactly supported. Using standard convolution arguments one shows that $u_n \in W_0^{1,p}(\Omega, X)$. The calculus rules in Lemma 6.6 and the Dominated Convergence Theorem imply that $u_n \to u$ in $W^{1,p}(\Omega, X)$, and thus $u \in W_0^{1,p}(\Omega, X)$. ∎

Using this theorem we can once again elegantly show results for $W_0^{1,p}(\Omega, X)$ using known results in the case $X = \mathbb{R}$. As an example, we can apply Corollary 4.6 to obtain

Corollary 6.8 (Poincaré inequality). *Let $1 \le p < \infty$ and Ω be bounded in direction e_j. Then there exists a constant $C > 0$ such that*

$$\|D_j u\|_{L^p(\Omega,X)} \ge C\|u\|_{L^p(\Omega,X)}$$

for all $u \in W_0^{1,p}(\Omega, X)$. The best such constant coincides with the best constant in the real case.

We will now characterize $W_0^{1,p}(\Omega, X)$ weakly. For that we need the following lemma, which immediately follows from Theorem 6.7.

Lemma 6.9. *Let X be a closed subspace of a Banach space Y and let $u \in W^{1,p}(\Omega, X)$ for some $1 \le p < \infty$. Then $u \in W_0^{1,p}(\Omega, X)$ if and only if $u \in W_0^{1,p}(\Omega, Y)$.*

Theorem 6.10. *Let $u \in W^{1,p}(\Omega, X)$. The following are equivalent:*

(i) $u \in W_0^{1,p}(\Omega, X)$.
(ii) $\langle u, x'\rangle \in W_0^{1,p}(\Omega, \mathbb{R})$ *for every $x' \in X'$.*
(iii) $\langle u, x'\rangle \in W_0^{1,p}(\Omega, \mathbb{R})$ *for every x' in a separating subset of X'.*

Proof. Since u is measurable, it is almost separably valued. The Hahn–Banach Theorem shows that the assumptions are not affected if we replace X by a separable subspace containing the image of u. Hence we may assume that X is separable.

(i)⇒(ii) is trivial.

(ii)⇒(i). The Banach–Mazur Theorem implies that $X \hookrightarrow C[0,1]$ isometrically. In view of the preceding lemma we may assume that $X = C[0,1]$. Now X has a Schauder basis $(b_k)_{k\in\mathbb{N}}$ with coordinate functionals (b_k'). Let $u_n := P_n u = \sum_{k=1}^n \langle u, b_k'\rangle b_k$. Then $u_n \in W_0^{1,p}(\Omega, X)$ by assumption. Further $u_n \to u$ pointwise, and if C is the basis constant of (b_k) then u_n is dominated by $C\|u\|_X$. Hence $u_n \to u$ in $L^p(\Omega, X)$. Analogously one shows that $D_j u_n \to D_j u$ in $L^p(\Omega, X)$, and hence $u \in W_0^{1,p}(\Omega, X)$.

(ii)⇒(iii) is trivial.

(iii)⇒(ii). Consider the space

$$V := \{x' \in X' : \langle u, x'\rangle \in W_0^{1,p}(\Omega, \mathbb{R})\},$$

which by assumption is $\sigma(X', X)$-dense in X'. We will show that it is also closed in the $\sigma(X', X)$-topology. By the Krein–Shmul'yan Theorem [Meg98, Theorem 2.7.11], it suffices to show that $V \cap B_{X'}(0,1)$ is $\sigma(X', X)$-closed. The $\sigma(X', X)$-topology, restricted to this bounded set, is metrizable. Let $x_n' \in V \cap B_{X'}(0,1)$ be such that $x_n' \rightharpoonup^* x'$. Then $\langle u, x_n'\rangle \to \langle u, x'\rangle$ pointwise and the functions $\langle u, x_n'\rangle$ are dominated by $\|u\|_X$. Hence $\langle u, x_n'\rangle \to \langle u, x'\rangle$

in $L^p(\Omega,\mathbb{R})$ by the Dominated Convergence Theorem. The same argument shows that $\langle u, x_n'\rangle \to \langle u, x'\rangle$ in $W^{1,p}(\Omega,\mathbb{R})$. Since $\langle u, x_n'\rangle \in W_0^{1,p}(\Omega,\mathbb{R})$, we have $\langle u, x'\rangle \in W_0^{1,p}(\Omega,\mathbb{R})$. Thus V is $\sigma(X', X)$-closed, which implies (ii). ∎

Example 6.11. Let $X = \ell^r$ for $1 \le r \le \infty$ and let $u = (u_n)_{n\in\mathbb{N}} \in W^{1,p}(\Omega, X)$. Then $u \in W_0^{1,p}(\Omega, X)$ if and only if $u_n \in W_0^{1,p}(\Omega,\mathbb{R})$ for all $n \in \mathbb{N}$.

Finally we describe $W_0^{1,p}(\Omega, X)$ via traces. By using standard convolution arguments Theorem 5.6 yields

Corollary 6.12. *Let $\Omega \subset \mathbb{R}^d$ be an open set with uniform Lipschitz boundary. Then for any $1 \le p < \infty$ the space $W^{1,p}(\Omega, X) \cap C(\overline{\Omega}, X)$ is dense in $W^{1,p}(\Omega, X)$.*

Using this, we can prove the Trace Theorem. On $\partial\Omega$ we consider the $(d-1)$-dimensional Hausdorff measure.

Theorem 6.13 (Trace Theorem). *Let $\Omega \subset \mathbb{R}^d$ be an open set with uniform Lipschitz boundary, $d \ge 2$ and $1 \le p < \infty$. Then there exists a continuous linear operator*

$$\mathrm{Tr}_X : W^{1,p}(\Omega, X) \to L^p(\partial\Omega, X)$$

such that $\mathrm{Tr}_X\, u = u_{|\partial\Omega}$ for all $u \in W^{1,p}(\Omega, X) \cap C(\overline{\Omega}, X)$. Moreover, given $u \in W^{1,p}(\Omega, X)$, we have $u \in W_0^{1,p}(\Omega, X)$ if and only if $\mathrm{Tr}_X\, u = 0$.

Proof. The case $X = \mathbb{R}$ is well known [Leo09, Theorems 15.10 & 15.23]. For $u \in W^{1,p}(\Omega, X) \cap C(\overline{\Omega}, X)$ we define $\mathrm{Tr}_X u := u_{|\partial\Omega}$. This operator and the norm on X commute in the sense that $\|\mathrm{Tr}_X u\|_X = \mathrm{Tr}_{\mathbb{R}} \|u\|_X$. Hence by Corollary 4.6,

$$\|\,\mathrm{Tr}_X\, u\|_{L^p(\partial\Omega,X)} \le \|\mathrm{Tr}_{\mathbb{R}}\,\|_{\mathcal{L}} \|u\|_{W^{1,p}(\Omega,X)}$$

for any $u \in W^{1,p}(\Omega, X) \cap C(\overline{\Omega}, X)$. By Corollary 6.12 we may extend Tr_X to $W^{1,p}(\Omega, X)$. The continuity of the norm on $W^{1,p}(\Omega, X)$ implies that the operator still commutes with the norm as before, hence the “moreover” claim follows from Theorem 6.7. ∎

7. Compact resolvents via Aubin–Lions. As an application of our multidimensional Aubin–Lions Theorem we consider unbounded operators on $L^p(\Omega, H)$. Here $\Omega \subset \mathbb{R}^d$ is open and bounded, and H is a separable Hilbert space.

Let B be a *sectorial* operator on H, that is, satisfying $(-\infty, 0) \subset \rho(B)$ and $\sup_{\lambda<0} \|\lambda(\lambda - B)^{-1}\| < \infty$. It follows from [ABHN11, Proposition 3.3.8] that B is densely defined and $\lim_{\lambda\to-\infty} \lambda(\lambda - B)^{-1}x = x$ for all $x \in H$. For

$1 \le p < \infty$ define $\tilde{B}$ on $L^p(\Omega, H)$ by

$$D(\tilde{B}) = L^p(\Omega, D(B)), \qquad \tilde{B}u = B \circ u.$$

Then $\tilde{B}$ is also sectorial. Now let A be a sectorial operator on $L^p(\Omega, \mathbb{R})$. We want to extend A to a sectorial operator $\tilde{A}$ on $L^p(\Omega, H)$.

LEMMA 7.1 (see [HvNVW16, Theorem 2.1.9]). *Let $T \in \mathcal{L}(L^p(\Omega, \mathbb{R}))$. Then there is a unique bounded operator $\tilde{T}$ on $L^p(\Omega, H)$ such that $\tilde{T}(f \otimes x) = Tf \otimes x$ for all $f \in L^p(\Omega, \mathbb{R})$ and $x \in H$. Moreover $\|\tilde{T}\|_{\mathcal{L}} = \|T\|_{\mathcal{L}}$.*

As a consequence of Lemma 7.1, given $\lambda < 0$ there exists a unique bounded operator $\tilde{R}(\lambda)$ on $L^p(\Omega, H)$ such that $\tilde{R}(\lambda)(f \otimes x) = R(\lambda, A)f \otimes x$ for all $f \in L^p(\Omega, \mathbb{R})$ and $x \in H$. It follows that $(\tilde{R}(\lambda))_{\lambda<0}$ is a pseudoresolvent on $(-\infty, 0)$ and $\lim_{\lambda\to-\infty} \lambda\tilde{R}(\lambda)u = u$ for all $u \in L^p(\Omega, H)$. Since $\ker \tilde{R}(\lambda)$ is independent of λ, it follows that $\tilde{R}(\lambda)$ is injective. Consequently, there exists a unique operator $\tilde{A}$ on $L^p(\Omega, H)$ such that $(-\infty, 0) \subset \rho(\tilde{A})$ and $\tilde{R}(\lambda) = R(\lambda, \tilde{A})$ for all $\lambda < 0$. Thus $\tilde{A}$ is a sectorial operator on $L^p(\Omega, H)$. For tensors $u = f \otimes x$ we have

$$R(\lambda, \tilde{A})R(\lambda, \tilde{B})u = R(\lambda, A)f \otimes R(\lambda, B)x = R(\lambda, \tilde{B})R(\lambda, \tilde{A})u,$$

hence the two resolvents commute. If $\varphi_{\mathrm{sec}}(\tilde{A}) + \varphi_{\mathrm{sec}}(\tilde{B}) < \pi$, a result of Da Prato–Grisvard [Are04, Section 4.2] says that $C = \tilde{A} + \tilde{B}$ is closable and $\overline{C}$ is a sectorial operator.

Assuming that B has compact resolvent, it is not obvious that $\overline{C}$ also has compact resolvent. We will show this if A and B satisfy the conditions of the Dore–Venni Theorem and $D(A) \subset W^{1,p}(\Omega, \mathbb{R})$.

LEMMA 7.2. *Assume that $D(A) \subset W^{1,p}(\Omega, \mathbb{R})$. Then $D(\tilde{A}) \subset W^{1,p}(\Omega, H)$.*

Proof. Let $\lambda < 0$ and $j \in \{1, \dots, d\}$. Then $D_j \circ R(\lambda, A) \in \mathcal{L}(L^p(\Omega, \mathbb{R}))$. By Lemma 7.1 there exists $\tilde{Q}_j \in \mathcal{L}(L^p(\Omega, H))$ such that

$$\tilde{Q}_j(f \otimes x) = D_j(R(\lambda, A)f) \otimes x$$

for all $f \in L^p(\Omega, \mathbb{R})$ and $x \in H$. For every $u \in L^p(\Omega, H)$ there exist linear combinations u_n of tensors such that $u_n \to u$ in $L^p(\Omega, H)$. Then $R(\lambda, \tilde{A})u_n \in W^{1,p}(\Omega, H)$ and $R(\lambda, \tilde{A})u_n \to R(\lambda, \tilde{A})u$ in $L^p(\Omega, H)$. Moreover

$$\|D_j R(\lambda, \tilde{A})u_m - D_j R(\lambda, \tilde{A})u_n\|_{L^p(\Omega,H)} \le \|\tilde{Q}_j\|_{\mathcal{L}} \|u_m - u_n\|_{L^p(\Omega,H)}.$$

Thus $R(\lambda, \tilde{A})u_n$ is a Cauchy sequence in $W^{1,p}(\Omega, H)$. This implies that $R(\lambda, \tilde{A})u \in W^{1,p}(\Omega, H)$. ∎

For the notion of *bounded imaginary powers* we refer to [Are04, Section 4.4] and the references given there.

THEOREM 7.3. *Let A be a sectorial injective operator on $L^p(\Omega, \mathbb{R})$ such that $D(A) \subset W^{1,p}(\Omega, \mathbb{R})$, where $1 < p < \infty$. Let B be a sectorial injective operator on H with compact resolvent. Suppose that both operators have* *bounded imaginary powers and* $\varphi_{\mathrm{bip}}(A) + \varphi_{\mathrm{bip}}(B) < \pi$. *Then* $\tilde{A} + \tilde{B}$ *with domain* $D(\tilde{A}) \cap D(\tilde{B})$ *is closed* (*and hence sectorial*) *and has compact resolvent.*

Proof. It is easy to see that $\tilde{A}$ and $\tilde{B}$ both have bounded imaginary powers and $\varphi_{\mathrm{bip}}(\tilde{A}) \le \varphi_{\mathrm{bip}}(A)$, $\varphi_{\mathrm{bip}}(\tilde{B}) \le \varphi_{\mathrm{bip}}(B)$. It follows from the Dore–Venni Theorem [Are04, Theorem 4.4.8] that $\tilde{A} + \tilde{B}$ is a sectorial operator. Thus

$$D(\tilde{A} + \tilde{B}) = D(\tilde{A}) \cap D(\tilde{B}) \subset W^{1,p}(\Omega, H) \cap L^p(\Omega, D(B))$$

with continuous embedding by the Closed Graph Theorem. By Theorem 5.3 the embedding $W^{1,p}(\Omega, H) \cap L^p(\Omega, D(B)) \hookrightarrow L^p(\Omega, H)$ is compact. This implies that $\tilde{A} + \tilde{B}$ has compact resolvent. ∎

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Wolfgang Arendt, Marcel Kreuter
Institute of Applied Analysis
Ulm University
89069 Ulm, Germany
E-mail: wolfgang.arendt@uni-ulm.de
marcel.kreuter@uni-ulm.de