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Applied Analysis Tutorials

Sheet: 1 (Getting started)

Submission: Tuesday, 20.10.2009 during the next tutorial class.

1. Let (X, d) be a metric space, $(x_n)_{n\in\mathbb{N}}$ be a sequence in X and $x_0 \in X$. Define a new metric $d_1(x, y) := \min\{1, d(x, y)\}$ for $x, y \in X$. Show that (X, d_1) is a metric space and that $x_n \to x_0$ in (X, d) if and only if $x_n \to x_0$ in (X, d_1).

2. Consider a set $S := \{1/n, n \in \mathbb{N}\}$ in a metric space $\mathbb{R}$.
   (a) Is S closed? Is it open? Justify your answers.
   (b) Determine the interior, the closure and the boundary of the set S.
   (c) Is the set $\{n : n \in \mathbb{N}\}$ closed in $\mathbb{R}$?
   (d) Find a metric space (X, d) and a sequence of closed sets in X such that their union is **not** closed.

3. Let (X, d) be a metric space, $x_0 \in X$ and $r > 0$.
   (a) Show that the so called open ball
   $$B(x_0, r) := \{y \in X : d(x_0, y) < r\}$$
   is indeed open.
   (b) Let $(x_n)_n$ and $(y_n)_n$ be two convergent sequences in the metric space X, with $x = \lim_{n\to\infty} x_n$ and $y = \lim_{n\to\infty} y_n$. Show that $d(x_n, y_n) \to d(x, y)$.
   (c) Let $A_\alpha \subset M$ be closed for any $\alpha \in I$, where I is an arbitrary index set. Show that $\cap_{\alpha\in I} A_\alpha$ is closed.

4. (a) Find a metric d for the set S in Exercise 2 so that S equipped with this metric becomes a **complete** metric space!
   (b) Is the new space (S, d) compact?
   (c) Show that the space $\mathbb{Q}$ of rational numbers with the standard metric is not complete.
   (d) If $(x_n)_n$ is a Cauchy sequence in a metric space (X, d) and if for the sequence $(y_n)_n$ in X we have $\lim d(x_n, y_n) = 0$, show that $(y_n)_n$ is also a Cauchy sequence in X.