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Applied Analysis Tutorials

Sheet: 2

Submission: Tuesday, 27.10.2009 during the next tutorial class.

1. Let l^∞ be the space of all real bounded sequences, i.e. $l^\infty := \{(x_n)_{n\in\mathbb{N}}, x_n \in \mathbb{R}, \sup_{n\in\mathbb{N}} |x_n| < \infty\}$. Check that it is a vector space with the usual definition of addition and scalar multiplication. Define a metric (distance) by
$$d(x,y) := \sup_{n\in\mathbb{N}} |x_n - y_n|.$$
Check that this indeed defines a metric. Decide whether (l^∞, d) is separable, complete, compact.

2. Take an arbitrary non-empty set M and define the discrete metric on M by
$$d(x,y) := \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{otherwise.} \end{cases}$$
Determine the convergent sequences in (M,d). Is (M,d) complete? Is it separable? Is it compact? (what do these properties depend on?)

3. Let (M,d) be a metric space.
   (a) Choose $x \in M$. Show that a set $U \subset M$ is a neighborhood of x if and only if $x \in \text{Int}(U)$. (In particular, every open set containing x is a neighborhood of x).
   (b) Prove that the intersection $U \cap V$ of any two neighborhoods U and V of a point $x \in M$ is also a neighborhood of x.
   (c) Show that every convergent sequence is Cauchy. Is the converse always true? If not, give a counter example.

4. Let X, Y, Z be metric spaces.
   (a) Show that $f : X \to Y$ is continuous at $x \in X$ if and only if $\forall \epsilon > 0 \ \exists \ \delta > 0$ such that $d_X(x,y) \leq \delta$ implies $d_Y(f(x), f(y)) \leq \epsilon$.
   (b) Assume that $f : X \to Y$ and $g : Y \to Z$ are continuous. Prove that the composition mapping $g \circ f : X \to Z$ is continuous.