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Applied Analysis Tutorials

Sheet: 3

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1. Let $(X, d), (Y, e)$ be metric spaces.
   (a) Show that $d : X \times X \to \mathbb{R}$ is continuous.
   (b) Let $A \subset X$. Show that $d(., A) : X \to \mathbb{R}$ is continuous.
   (c) Let $(x^n)_n$ be a sequence in $X \times Y$. Prove that $x^n = (x_1^n, x_2^n)$ converges to $x = (x_1, x_2)$ in $X \times Y$ if and only if $(x_1^n)_n$ converges to x_1 in X and $(y_2^n)_n$ converges to y_2 in Y.

2. (a) Let $(E, ||\cdot||_E)$ be a normed $\mathbb{K}$-space. Show that (E, d_E) is a metric space, where d_E is the corresponding distance on E.
   (b) Let E be a $\mathbb{K}$-vector space and let d be a distance on E such that d is translation invariant and homogenous, i.e. $d(x+z, y+z) = d(x, y)$ for all $x, y, z \in E$ and $d(\lambda x, \lambda y) = |\lambda| d(x, y)$ for all $x, y \in E$ and $\lambda \in \mathbb{K}$. Show that there is a norm $||\cdot||_E$ on E such that $d(x, y) = ||x - y||_E$ (and check the norm properties).

3. (a) Let $E := \mathbb{R}^2$, $||x||_1 := |x_1| + |x_2|, ||x||_2 := \sqrt{x_1^2 + x_2^2}$, and $||x||_\infty := \max\{|x_1|, |x_2|\}$. Show that these norms are equivalent.
   (b) Let E be a $\mathbb{K}$-vector space, $||\cdot||_1$ and $||\cdot||_2$ be two equivalent norms on E and let $x_n, x \in E$. Show that $x_n \to x$ in $(E, ||\cdot||_1)$ if and only if $x_n \to x$ in $(E, ||\cdot||_2)$.

4. Prove Theorem 1.1.43 (Banach's fixed point theorem) using Corollary 1.1.44 (Banach's fixed point theorem for contractions).