ulm university universität uulm

Dr. Markus Biegert
Dipl.Ing. S.O. Akindeinde
WS 2009/2010

Applied Analysis Tutorials

Sheet:6

Submission: Tuesday, 24.11.2009 during the next tutorial class.

1. Denote by $\overline{\mathbb{R}}$ the set $\mathbb{R} \cup \{-\infty, \infty\}$. Consider the mapping $g : [-1, 1] \to \overline{\mathbb{R}} = [-\infty, \infty]$ defined by
$$g(x) := \frac{x}{(1-x)(1+x)}.$$
Check that $d(y, z) := |g^{-1}(y) - g^{-1}(z)|$ for all $y, z \in [-\infty, \infty]$ is a well-defined metric on $[-\infty, \infty]$. Characterize open sets in this metric (these are unions of open intervals plus half-closed intervals with + or $-\infty$ at the closed end - prove it!). What are the convergent sequences in the metric? Show that the Borel σ-algebra on $\overline{\mathbb{R}}$ is generated by the sets of the form $[a, \infty]$, where $a \in \bar{\mathbb{R}}$. (10 points)

2. Let Ω be a set and let $(A_\alpha)_{\alpha \in I}$ be a family of σ-algebras on $\Omega, I \neq \emptyset$. Show that $\mathcal{A} := \cap_{\alpha \in I} A_\alpha$ is a σ-algebra on Ω. (5 points)

3. Show that the Borel sets of $\mathbb{R}^n$ are precisely the members of the σ-algebra generated by the compact sets. (5 points)

4. (a) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be linear. Show that $T : \mathbb{R}^n \to \mathbb{R}^n$ is contiunous. (3 points)
   (b) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be linear and invertible. Show that $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is linear. Deduce that T^{-1} is continuous. (2 points)