HOW TWISTED CAN A JORDAN CURVE BE?

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> Everyone knows what a curve is, until he has studied enough mathematics to become confused through the countless number of possible exceptions.
>
> — Felix Klein (1849–1925)

We start by fixing notation. A **closed curve** is a continuous map $\gamma\colon [0,1] \to \mathbb{R}^2$ with $\gamma(0) = \gamma(1)$, and it is a closed **Jordan** curve if in addition $\gamma|_{[0,1)}$ is injective. Similarly one defines a (Jordan) arc, a polygonal (Jordan) arc and a closed polygonal (Jordan) curve. It is well-known, of course, that a closed Jordan curve γ partitions the plane into three connected components: the bounded open **interior** $\operatorname{int}\gamma$, the unbounded open **exterior** $\operatorname{ext}\gamma$ and the compact (image of the) curve itself, which is the common boundary of the first two components. For a captivating historical overview of the struggle to understand curves and toplological dimension we refer to [Cri99], where most of the classical topological results that appear in this note are featured.

For a closed *polygonal* Jordan curve γ there is a simple criterion to check whether a point $x \in \mathbb{R}^2 \setminus \gamma$ is in its interior. This criterion is algorithmic in nature and useful in computational geometry, see for example [O'R98, Section 7.4]. Consider a ray emerging from x towards infinity and count the number of intersections with γ. If the ray is chosen such that it intersects the segments of γ only ouside of their end points (which can be easily arranged for a polygonal Jordan curve) or if intersections with the end points are counted appropriately, the point x is in the interior of γ if and only if the number of intersections is odd. More generally, instead of a ray one may use any suitable polygonal Jordan arc that connects x with a point known to be in the exterior of γ. In fact, in the polygonal setting for all $x \in \operatorname{int}\gamma$, $y \in \operatorname{ext}\gamma$ and $z \in \gamma$ there exists a polygonal Jordan arc connecting x and y that intersects γ exactly in z.

General closed Jordan curves, however, can certainly be mind-boggingly pathological. For example, there are closed Jordan curves in $\mathbb{R}^2$ that are nowhere differentiable and that have positive two-dimensional measure; for several different examples see [Sag94, Chapter 8]. But just how badly can the above criterion for closed polygonal Jordan curves fail in the case of general closed Jordan curves? We say that an arc $\rho\colon [0,1] \to \mathbb{R}^2$ **crosses** the closed Jordan curve γ if $\rho(0) \in \operatorname{int}\gamma$ and $\rho(1) \in \operatorname{ext}\gamma$. The following was proved in [PW12, Theorem 15].

Theorem 1. *There is a closed Jordan curve γ in $\mathbb{R}^2$ such that for every rectifiable arc ρ that crosses γ the intersection $\rho \cap \gamma$ is infinite.*

In fact, in [PW12] it is established that the property in Theorem 1 is generic in the sense of the Baire category theorem, after endowing the set of closed Jordan curves with a suitable complete metric. The construction of a specific closed Jordan curve as described in the theorem is also outlined in [Maj10] and [Pet12].

Academic curiosity and unswerving belief in the pathological prowess of Jordan curves motivates me to ask the following question. Here $\mathcal{H}^s$ denotes the s-dimensional Hausdorff measure in $\mathbb{R}^2$ for $s \in [0,2]$.

Problem A. *Is there a closed Jordan curve γ in $\mathbb{R}^2$ such that for every rectifiable arc ρ that crosses γ one has $\mathcal{H}^1(\rho \cap \gamma) > 0$.*

Clearly Problem A can readily be modified in a number of ways, some of which lead to an easier or harder problem. As a first step one might look for a closed Jordan curve that exhibits the above property as long as ρ is, in addition, a (compact) line segment. It is readily observed that such a curve needs to have positive two-dimensional measure. The standard curves with positive two-dimensional measure by Osgood or Sierpiński–Knopp, however, do not satisfy the property for line segments.

I stumbled across the following result in the recent preprint [Bis16], which almost answers the problem for crossing line segments.

Theorem 2. *There is a closed Jordan curve γ in $\mathbb{R}^2$ such that every line segement that crosses γ has an uncountable intersection with γ. Moreover, it can be arranged that all of these intersections have Hausdorff dimension* 1.

The argument in [Bis16] is constructive and ensures that the intersection with every crossing line segment contains a **Cantor type set**, i.e. a compact, perfect, totally disconnected set. Starting with a circle, in each step the curve is replaced by one in close proximity that wiggles around the previous curve along tiny pieced-together circular arcs. At the end of [Bis16] it is suggested that it might be possible to modify the construction to yield a closed Jordan curve that intersects every crossing line segment in a set of positive 1-dimensional Hausdorff measure, but it is made clear that this is not immediate. Furthermore, Problem A is stated as an open problem in [Bis16].

In the remainder I shall mention several beautiful related results and concepts that I encountered in my attempts to better understand Problem A, even before I had encountered the results in Theorems 1 and 2.

Suppose throughout that γ is a closed Jordan curve in $\mathbb{R}^2$. We shall make use of the following notation. A point $z \in \gamma$ is called **accessible** from $x \in \mathbb{R}^2 \setminus \gamma$ if there exists a Jordan arc from x to z that intersects γ only in z. Similarly, $z \in \gamma$ is called **finitely accessible** (or rectifiably accessible) from $x \in \mathbb{R}^2 \setminus \gamma$ if the connecting Jordan arc can be chosen to be rectifiable. Obviously, if $z \in \gamma$ is (finitely) accessible from $x \in \operatorname{int}\gamma$, then z is (finitely) accessible from every point in $\operatorname{int}\gamma$, and an analogous statement holds for $x \in \operatorname{ext}\gamma$.

Interestingly, every point $z \in \gamma$ is accessible from both the interior and the exterior. This is an immediate consequence of the purely topological Schoenflies theorem (see [Pom92, Corollary 2.9]): there is a homeomorphism $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(\gamma)$ is the unit circle, $f(\operatorname{int}\gamma)$ is the interior of the unit cirle and $f(\operatorname{ext}\gamma)$ its exterior. In fact, a compact set $F \subset \mathbb{R}^2$ is a closed Jordan curve if and only if $\mathbb{R}^2 \setminus F$ has exactly two connected components from both of which every point of F is accessible; see [Kur68, Section §61.II, Theorem 12]. The closed topologist's sine curve [Mun00, p. 381] (i.e. the union of the graph of $\sin(1/x)$ on the interval $(0, 1)$, the set $\{0\} \times (-1, 1]$ and a disjoint Jordan arc joining the points $(0, -1)$ and $(1, \sin 1)$) illustates that the accessibility is essential. In this context it is worthwhile to point out that there is a compact connected set $F \subset \mathbb{R}^2$ such that $\mathbb{R}^2 \setminus F$ has 3 (or more generally $N \in \mathbb{N}$ with $N \geq 3$) connected components each of which has boundary F; see [Bro10] or the Lakes of Wada construction in [Yon17, pp. 60–62].

One readily observes that there are always some points on γ that are finitely accessible from the exterior (e.g. all the points on γ with minimal first coordinate), and similarly there are points on γ that are finitely accessible from the interior. Theorem 1 shows that it is possible (in fact, generic) that no point on γ will be finitely accessible from both the interior and the exterior.

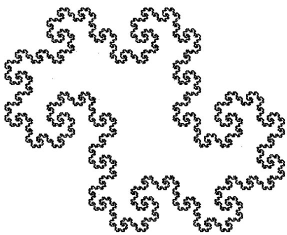
In Veblen's proof of the Jordan curve theorem, it is established along the way that the set of points that are finitely accessible from any given $x \in \mathbb{R}^2 \setminus \gamma$ is dense in γ, see [Veb05, Theorem 10]. In fact, the result is shown to hold with polygonal Jordan arcs. Unsurprisingly, much more can be said. Let us denote the harmonic measure with respect to $\operatorname{int}\gamma$ by ω and with respect to $\operatorname{ext}\gamma$ by ω^*. We shall briefly discuss the definition of the harmonic measure, but for any details we refer to [Pom92, in particular Sections 4.4 and 6.6]. Fix a point $z \in \operatorname{int}\gamma$ and let f be the conformal map from the open unit disk D onto $\operatorname{int}\gamma$ such that $f(0) = z$. By the Caratheodory–Osgood theorem f extends to a homeomorphism between $\overline{D}$ and $\overline{\operatorname{int}\gamma}$. Then for a subset $A \subset \gamma$

one sets $\omega(A) := \frac{1}{2\pi}\mathcal{H}^1(f^{-1}(A))$. The choice of z is not essential as it does not affect the ω nullsets. The measure ω^* is defined similarly with respect to $\operatorname{ext}\gamma$.

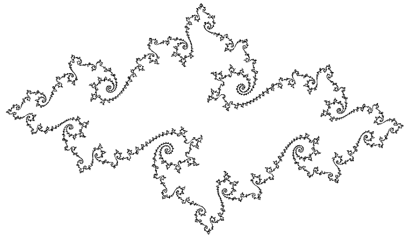
By Lavrentieff's theorem, see [SW42, Theorem 6 and Corollary 8], ω almost every point on γ is finitely accessible from $\operatorname{int}\gamma$, and ω^* almost every point on γ is finitely accessible from $\operatorname{ext}\gamma$. However, the measures ω and ω^* can be mutually singular, which happens if (and only if) the set of points of γ where γ has a tangent is an $\mathcal{H}^1$-nullset [Pom92, Theorem 6.30]. Consequently there exists a nowhere differentiable γ that is star-like with respect to 0 such that ω and ω^* are mutually singular, see [Pom92, Proposition 6.29]. So the mutual singularity of ω and ω^* is clearly much weaker than the property in Theorem 1.

By Marstrand's result in [Mar54, Theorem III] (see [FFJ15] for a survey of related results and [MO16] for a recent strengthening) any Jordan curve γ such that $0 < \mathcal{H}^s(\gamma) < \infty$ for an $s \in (1, 2]$ has the property that at $\mathcal{H}^s$-a.e. point $a \in \gamma$ almost every straight line through a intersects γ in an (uncountable) set of Hausdorff dimension $s - 1$. So fractal Jordan curves are natural candidates when looking for the property in Theorem 2. It is easy to see that the Koch snowflake curve, see for example [Edg08, p. 20], is a closed Jordan curve with Hausdorff dimension $\frac{\log 4}{\log 3} \approx 1.26$ which thus satisfies Marstrand's property, but which certainly does not exhibit the property in Theorem 2 since one can escape from the interor by crossing straight through the tips. Also the boundary of the Knuth–Davis twindragon fractal [Edg08, p. 34], depicted in Figure 1 (a), is a closed Jordan curve by [BW01, Example after Theorem 2.1] with Hausdorff dimension of about 1.52 that does not exhibit the property in Theorem 2, even though by the looks this seems plausible. In fact, by [AS06, Theorems 2.9 and 2.12] the twindragon admits line segments that cross the boundary only once. It would be interesting to identify standard fractal Jordan curves that provably do have the property in Theorem 2. For example, are there such closed Jordan curves that are the Julia set of a rational or polynomial complex function, see [Fal03, Chapter 14] or [Mat95, Section 4.17]? In Figure 1 (b) an example of a quadratic Julia set is depicted that is known to be a closed Jordan curve.

We finally discuss a few select results that establish existence of certain Jordan arcs and closed Jordan curves. There is a characterisation of those subsets of $\mathbb{R}^2$ that can be covered by a Jordan arc by Moore and Kline [MK19]. As every Jordan arc is contained in a closed Jordan curve, see [RSFM15, Corollary 17.23] or [Thu11], it is exactly the same subsets that can be covered by a closed Jordan curve. A special case of the characterisation by Moore and Kline is the Denjoy–Riesz theorem which states that this is possible for any totally disconnected compact set in $\mathbb{R}^2$. For example, by considering a



1 (a) Boundary of the Knuth–Davis twindragon space-filling fractal curve.



1 (b) Julia set for the complex quadratic polynomial $f(z) = z^2 - 0.742 + 0.1i$.

Cantor type set in $\mathbb{R}^2$ with positive 2-dimensional Lebesgue measure, we obtain Osgood's result [Osg03] as an immediate consequence. Extensive information on topological imbeddings of Jordan arcs, closed Jordan curves and totally disconnected compact sets in the plane is provided in [Kel68, Chapters I and II].

In [Kuz96, Theorem 3] extraordinarily pathological Cantor type sets are constructed. For example, in [Kuz96, Theorem 3] it is shown, among other things, that for all $r_1 < r_2$, $C > 0$ and $\varepsilon > 0$ with $\varepsilon < r_2 - r_1$ there exists a Cantor type set $D \subset A := \{x \in \mathbb{R}^2 : r_1 < |x| < r_2\}$ such that every sufficiently smooth arc ρ with curvature bounded by C that connects the inner and outer boundary of the annulus A satisfies $\mathcal{H}^1(\rho \cap D) > r_2 - r_1 - \varepsilon$. In particular, every line segment ρ connecting the inner and outer boundary of A satisfies $\mathcal{H}^1(\rho \cap D) > 0$. By the Denjoy–Riesz theorem there exists a closed Jordan curve γ such that $D \subset \gamma$. Moreover, by [Kuz96, Corollary 2 to Theorem 3] and [RSFM15, Corollary 17.23] one can arrange that $\gamma \subset A$. So $B(0, r_1) \subset \operatorname{int}\gamma$, $\mathbb{R}^2 \setminus B(0, r_2) \subset \operatorname{ext}\gamma$ and γ has very large intersection with all smooth arcs that connect the inner and outer boundary of A and have curvature bounded by C. On its own, however, this abstract construction cannot ensure that the property in Problem A holds since $\operatorname{int}\gamma$ necessarily overlaps with A. Examples of various other Cantor type sets in $\mathbb{R}^2$ that have a large projection in all directions (e.g. the orthogonal projection onto any 1-dimensional subspace is a nontrivial interval) can be found in [Bag59, MT08, Par09, MLVM12, Vas14, RS14]; the set in [Par09] is of Cantor type as a finite union of Cantor type sets due to [Eng78, Theorem 1.3.1].

While the above rambling might not help to tackle Problem A, it at least substanciates how ridiculously twisted a closed Jordan curve can be. I hope also the reader has enjoyed this slightly chaotic journey down the rabbit hole.

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