Then f_n is a simple function and positive whenever f is positive (the latter follows from the fact that in this case $f^{-1}[B_{n,k}] = \emptyset$ for all k, n).

Moreover, $f_n(x) \to f(x)$. Indeed, if $x \in \Omega$, then there exists an $n_0 \in \mathbb{N}$ such that $|f(x)| \leq 2^{n_0}$. Then $|f_n(x) - f(x)| \leq 2^{-n}$ for all $n \geq n_0$.

If $f \geq 0$ then $f_n \leq f_{n+1}$. Indeed, if $f_n(x) = k2^{-n}$ then $f(x) \in [k2^{-n}, (k+1)2^{-n})$. But then either

$$f(x) \in [(2k)2^{-(n+1)}, (2k+1)2^{-(n+1)}),$$

in which case $f_{n+1}(x) = (2k)2^{-(n+1)} = f_n(x)$, or

$$f(x) \in [(2k+1)2^{-(n+1)}, (2k+2)2^{-(n+1)}),$$

in which case $f_{n+1}(x) = (2k+1)2^{-(n+1)} > f_n(x)$.

In the case where $\mathbb{K} = \mathbb{C}$, we find sequences of simple functions (g_n) and (h_n) that converge to $\operatorname{Re} f$ and $\operatorname{Im} f$, respectively. Then we set $f_n := g_n + ih_n$ for all $n \in \mathbb{N}$. □

We sometimes prefer to work with a slightly more general notion of measurable functions. It has technical advantages to allow functions to take the values ∞ or $-\infty$.

For example, this is natural for stopping times of random processes.

Remark 3.54 (Extended real line)**.** We put $\overline{\mathbb{R}} := \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ which we endow with the σ-algebra $\mathscr{B}(\overline{\mathbb{R}})$, defined as $\sigma(\mathscr{B}(\mathbb{R}) \cup \{\{-\infty\}, \{\infty\}\})$. It follows that a function $f\colon (\Omega, \Sigma) \to \overline{\mathbb{R}}$ is measurable if and only if $f^{-1}[\{\infty\}], f^{-1}[\{-\infty\}] \in \Sigma$ and $f^{-1}[A] \in \Sigma$ for all $A \in \mathscr{B}(\mathbb{R})$.

Similarly, $\mathscr{B}([0,\infty])$ is defined as $\sigma(\mathscr{B}([0,\infty)) \cup \{\{\infty\}\})$. We also remark that Proposition 3.53 generalizes to this situation.

There exist metrics on $\overline{\mathbb{R}}$ resp. $[0,\infty]$ such that $\mathscr{B}(\overline{\mathbb{R}})$ resp. $\mathscr{B}([0,\infty])$ is the Borel σ-algebra for this metric.

Exercise 3.55. Let (Ω, Σ) be a measurable space and $f_n\colon \Omega \to \overline{\mathbb{R}}$ be measurable for all $n \in \mathbb{N}$. Then the functions $\liminf_{n\to\infty} f_n$, $\limsup_{n\to\infty} f_n$, $\inf_{n\in\mathbb{N}} f_n$ and $\sup_{n\in\mathbb{N}} f_n$ are measurable.

Hint: Observe that $\{x \in \Omega : \sup\{f_n(x) : n \in \mathbb{N}\} > a\} = \bigcup_{n\in\mathbb{N}}\{x \in \Omega : f_n(x) > a\}$.

3.7 The Lebesgue integral

Given a measure space (Ω, Σ, μ), we now introduce the Lebesgue integral $\int f \,\mathrm{d}\mu$ for suitable complex-valued, measurable functions that we call 'integrable'. We proceed in several steps and first define the integral for (real-valued) measurable functions f taking values in $[0,\infty]$.

Definition 3.56. Let (Ω, Σ, μ) be a measure space. If $f\colon \Omega \to [0,\infty]$ is a simple function, with standard representation $f = \sum_{k=1}^{n} a_k \mathbb{1}_{A_k}$, one defines the **Lebesgue integral** of f by

$$\int f \,\mathrm{d}\mu := \sum_{k=1}^{n} a_k \mu(A_k)$$

where, by convention, $0 \cdot \infty := 0$ and $\infty + \infty = \infty$.

The convention $0 \cdot \infty = 0$ ensures that the Lebesgue integral of the zero function on $\mathbb{R}$ is zero. Note that $\lambda(\mathbb{R}) = \infty$.

The following lemma shows that the Lebesgue integral for nonnegative simple functions is positively linear and monotone.

Lemma 3.57. *If (Ω, Σ, μ) is a measure space, $f, g\colon \Omega \to [0, \infty]$ are simple functions and $\lambda > 0$, then $\int \lambda f + g \, \mathrm{d}\mu = \lambda \int f \, \mathrm{d}\mu + \int g \, \mathrm{d}\mu$. Moreover, if $f \leq g$, then $\int f \, \mathrm{d}\mu \leq \int g \, \mathrm{d}\mu$.*

Proof. If $\sum_{k=1}^n a_k \mathbb{1}_{A_k}$ is the standard representation of f, then $\sum_{k=1}^n \lambda a_k \mathbb{1}_{A_k}$ is the standard representation of λf. Now $\lambda \int f \, \mathrm{d}\mu = \int \lambda f \, \mathrm{d}\mu$ follows from the definition of the integral. Now let $\lambda = 1$ and $\sum_{l=1}^m b_l \mathbb{1}_{B_l}$ be the standard representation of g. If $\sum_{j=1}^w c_j \mathbb{1}_{C_j}$ is the standard representation of $f + g$, then $C_j = \bigcup_{a_k + b_l = c_j} A_k \cap B_l$. Thus

$$\int f \, \mathrm{d}\mu + \int g \, \mathrm{d}\mu = \sum_{k=1}^n a_k \mu(A_k) + \sum_{l=1}^m b_l \mu(B_l) = \sum_{k=1}^n \sum_{l=1}^m (a_k + b_l) \mu(A_k \cap B_l)$$
$$= \sum_{j=1}^w c_j \mu(C_j) = \int f + g \, \mathrm{d}\mu,$$

where we have used the finite additivity of μ and the fact that the sets $A_k \cap B_l$ are pairwise disjoint.

Now assume that $f \leq g$. Consider the function $g - f$, with the convention that $\infty - \infty = 0$ and $\infty - c = \infty$ for all $c \in [0, \infty)$. Then $g - f$ is a nonnegative, simple function. Hence,

$$\int g \, \mathrm{d}\mu = \int (g - f) + f \, \mathrm{d}\mu = \int g - f \, \mathrm{d}\mu + \int f \, \mathrm{d}\mu \geq \int f \, \mathrm{d}\mu,$$

since the integral of a nonnegative, simple function is clearly nonnegative. $\square$

Lemma 3.58. *Let (Ω, Σ, μ) be a measure space and $f\colon \Omega \to [0, \infty]$ be simple. Then $\nu\colon \Sigma \to [0, \infty]$ defined by $\nu(A) = \int \mathbb{1}_A f \, \mathrm{d}\mu$ is a measure on (Ω, Σ).*

Proof. As $\mathbb{1}_\emptyset f$ is the zero function, the map ν clearly satisfies property (M1). It remains to very (M2). So let (A_k) be a sequence of disjoint sets in Σ. Define $B_n := \bigcup_{k=1}^n A_k$. Then

$$\nu(B_n) = \int \mathbb{1}_{B_n} f \, \mathrm{d}\mu = \int \sum_{k=1}^n \mathbb{1}_{A_k} f \, \mathrm{d}\mu = \sum_{k=1}^n \int \mathbb{1}_{A_k} f \, \mathrm{d}\mu = \sum_{k=1}^n \nu(A_k).$$

Clearly, $B_n \subset B_{n+1}$ for all $n \in \mathbb{N}$ and $A := \bigcup_{n \in \mathbb{N}} B_n = \bigcup_{k \in \mathbb{N}} A_k$. Since f is a simple function, take $f = \sum_{j=1}^m a_j \mathbb{1}_{C_j}$ to be its standard representation.

Observe that

$$
\begin{aligned}
\nu(A) &= \int \mathbb{1}_A f \,\mathrm{d}\mu = \int \sum_{j=1}^{m} a_j \mathbb{1}_{C_j \cap A} \,\mathrm{d}\mu \\
&= \sum_{j=1}^{m} a_j \mu(C_j \cap A) \\
&= \lim_{n\to\infty} \sum_{j=1}^{m} a_j \mu(C_j \cap B_n) \\
&= \lim_{n\to\infty} \int \sum_{j=1}^{m} a_j \mathbb{1}_{C_j \cap B_n} \,\mathrm{d}\mu \\
&= \lim_{n\to\infty} \int \mathbb{1}_{B_n} f \,\mathrm{d}\mu = \lim_{n\to\infty} \nu(B_n) = \sum_{k=1}^{\infty} \nu(A_k)
\end{aligned}
$$

In the previous calculation we used the definition of the integral for simple functions, that it is positively linear, the continuity of the measure μ, and in the last step the identity $\nu(B_n) = \sum_{k=1}^{n} \nu(A_k)$ that was established above. This proves (M2). Hence ν is a measure. $\square$

We can now define the Lebesgue integral by approximation for an arbitrary measurable function $f\colon \Omega \to [0,\infty]$.

Definition 3.59. Let (Ω, Σ, μ) be a measure space and $f\colon \Omega \to [0,\infty]$ measurable. One defines

$$\int f \,\mathrm{d}\mu = \sup\Big\{ \int g \,\mathrm{d}\mu \;:\; 0 \le g \le f,\, g \text{ simple} \Big\}.$$

We say that f is **(Lebesgue) integrable**, if $\int f \,\mathrm{d}\mu < \infty$.

Example 3.60. If δ_a is the Dirac measure on (Ω, Σ) for an $a \in \Omega$, then for all measurable $f\colon \Omega \to [0,\infty]$ we have

$$\int f \,\mathrm{d}\delta_a = f(a)$$

and f is integrable if and only if $f(a) < \infty$.

Proof. First, let f be a simple function. Then $\delta_a(f^{-1}[\{t\}]) = 1$ if $a \in f^{-1}[\{t\}]$, i.e. $f(a) = t$ and $\delta_a[f^{-1}[\{t\}]] = 0$ else. Thus in this case $\int f \,\mathrm{d}\delta_a = f(a)$. Now let $f\colon \Omega \to [0,\infty]$ be a measurable function. If g is a simple function with $g \le f$, then

$$\int_\Omega g \,\mathrm{d}\delta_a = g(a) \le f(a).$$

Taking the supremum over such g, it follows that $\int f \,\mathrm{d}\mu \le f(a)$.

For the reverse inequality, let $g(x) := f(a)\mathbb{1}_{f^{-1}[\{f(a)\}]}(x)$. Since f is measurable and $\{f(a)\} \in \mathscr{B}(\mathbb{R})$, it follows that $f^{-1}[\{f(a)\}] \in \Sigma$. So g is a simple function. Moreover, $g \leq f$. Hence, $\int f \,\mathrm{d}\delta_a \geq \int g \,\mathrm{d}\delta_a = f(a)$. $\square$

Exercise 3.61. Consider the measure space $(\mathbb{N}, \mathscr{P}(\mathbb{N}), \zeta)$. A measurable function $f\colon \mathbb{N} \to [0,\infty]$ is a sequence $(a_n)_{n\in\mathbb{N}} = (f(n))_{n\in\mathbb{N}}$ in $[0,\infty]$. Show that f is integrable if and only if $(a_n) \in \ell^1$ and in this case,

$$\int_{\mathbb{N}} f \,\mathrm{d}\zeta = \sum_{n=1}^{\infty} a_n.$$

Theorem 3.62 (Monotone convergence theorem)**.** *Let (Ω, Σ, μ) be a measure space and $f_n\colon \Omega \to [0,\infty]$ be measurable functions for all $n \in \mathbb{N}$ that are monotonically increasing in n. Then $f(x) := \sup_{n\in\mathbb{N}} f_n(x)$ is measurable and $\int f \,\mathrm{d}\mu = \sup_{n\in\mathbb{N}} \int f_n \,\mathrm{d}\mu$.*

Instead of sup *we could write* lim *in the statement of Theorem 3.62. Of course this requires to consider improper limits, thereby for example setting $f(x)$ to ∞ where $(f_n(x))$ is unbounded.*

Proof. As a consequence of Exercise 3.55, f is measurable.

One deduces from $f_n \leq f$ that

$$\sup_{n\in\mathbb{N}} \int f_n \,\mathrm{d}\mu \leq \int f \,\mathrm{d}\mu.$$

It remains to show the converse inequality. To this end, let $0 \leq \varphi \leq f$ be simple and $\beta > 1$. It suffices to show that

$$\int \varphi \,\mathrm{d}\mu \leq \beta \sup_{n\in\mathbb{N}} \int f_n \,\mathrm{d}\mu.$$

The sufficiency follows from first taking the supremum over all simple $0 \leq \varphi \leq f$ and then taking the infimum over all $\beta > 1$. Define $B_n := \{x \in \Omega : \beta f_n(x) \geq \varphi(x)\}$. Then $B_n \in \Sigma$, $B_n \subset B_{n+1}$ and $\bigcup_{n\in\mathbb{N}} B_n = \Omega$. Moreover, we have $\beta f_n \geq \varphi$ on B_n. Define $\nu(A) := \int \mathbb{1}_A \varphi \,\mathrm{d}\mu$ for all $A \in \Sigma$. Then ν is a measure by Lemma 3.58. Hence

$$\begin{aligned}
\int \varphi \,\mathrm{d}\mu = \nu(\Omega) &= \lim_{n\to\infty} \nu(B_n) \\
&= \lim_{n\to\infty} \int \varphi \mathbb{1}_{B_n} \,\mathrm{d}\mu \\
&\leq \sup_{n\in\mathbb{N}} \int \beta f_n \,\mathrm{d}\mu \\
&= \beta \sup_{n\in\mathbb{N}} \int f_n \,\mathrm{d}\mu.
\end{aligned}$$

This establishes the converse inequality. Hence $\sup_{n\in\mathbb{N}} \int f_n \,\mathrm{d}\mu = \int f \,\mathrm{d}\mu$. $\square$

Remark 3.63. Let $f\colon \Omega \to [0,\infty]$ be measurable. Let (f_n) be a sequence of nonnegative measurable functions such that $f_n \uparrow f$ pointwise. Note that such a sequence exists by Proposition 3.53. It now follows from Theorem 3.62 that

$$\int f \,\mathrm{d}\mu = \sup\left\{\int \varphi \,\mathrm{d}\mu : 0 \leq \varphi \leq f \text{ simple}\right\} = \sup_{n\in\mathbb{N}} \int f_n \,\mathrm{d}\mu.$$

So while the integral for nonnegative measurable functions is defined as the supremum over a very large class of simple functions, it is actually already obtained by the supremum over an approximating sequence as in Proposition 3.53.

Corollary 3.64. *Let (Ω, Σ, μ) be a measure space, $f, g\colon \Omega \to [0,\infty]$ measurable and $\lambda > 0$. Then $\int \lambda f + g \,\mathrm{d}\mu = \lambda \int f \,\mathrm{d}\mu + \int g \,\mathrm{d}\mu$. Moreover, if $f \leq g$, then $\int f \,\mathrm{d}\mu \leq \int g \,\mathrm{d}\mu$.*

Proof. Let (f_n) and (g_n) be sequences of simple functions such that $f_n \uparrow f$ and $g_n \uparrow g$. Then $\lambda f_n + g_n \uparrow \lambda f + g$. Using monotone convergence and Lemma 3.57, we obtain

$$\begin{aligned}\int \lambda f + g \,\mathrm{d}\mu &= \lim_{n\to\infty} \int \lambda f_n + g_n \,\mathrm{d}\mu \\ &= \lim_{n\to\infty} \lambda \int f_n \,\mathrm{d}\mu + \int g_n \,\mathrm{d}\mu = \lambda \int f \,\mathrm{d}\mu + \int g \,\mathrm{d}\mu\end{aligned}$$

The second assertion is clear from the definition and was already observed in the proof of Theorem 3.62. □

Definition 3.65. Let (Ω, Σ, μ) be a measure space. A **null set** is a set $N \subset \Omega$ such that there exists a set $M \in \Sigma$ with $N \subset M$ and $\mu(M) = 0$. Now let $P = P(x)$ be a property which, depending on x, may be true or false. We say that P holds **almost everywhere** or **for almost every $x \in \Omega$**, if $\{x \in \Omega : P(x) \text{ is false}\}$ is a null set. If μ is a probability measure, we also say that P holds **almost surely** if it holds almost everywhere.

Note that the notion of a null set *crucially depends on the measure, of course. So sometimes it is necessary to point out this dependence by speaking about a **μ-null set**.*

Remark 3.66. Note that we do not assume that a null set N is measurable. A measure space in which all null sets are measurable is called **complete**. It is not hard to see that the measure space $(\Omega, \mathscr{M}, \mu^*)$ in the proof of Carathéodory's theorem, Theorem 3.42, is always complete.

We note that there is a straightforward procedure to complete a measure space by minimally enlarging the σ-algebra and thus the domain of the measure to make all null sets measurable.

Example 3.67. Consider $(\mathbb{R}, \mathscr{B}(\mathbb{R}), \delta_0)$. Then almost every (or more precisely, δ_0-a.e.) $x \in \mathbb{R}$ is equal to 0.

Consider $(\mathbb{R}, \mathscr{B}(\mathbb{R}), \lambda)$. Then almost every (or more precisely, λ-a.e.) $x \in \mathbb{R}$ is not equal to 0. In fact, almost every $x \in \mathbb{R}$ is irrational.

Corollary 3.68. *Let (Ω, Σ, μ) be a measure space and $f\colon \Omega \to [0,\infty]$ be measurable. Then $\int f \,\mathrm{d}\mu = 0$ if and only if $f(x) = 0$ for almost every $x \in \Omega$.*

Proof. If f is a simple function, then the assertion is obvious. In the general case, let a measurable $f\colon \Omega \to [0,\infty]$ with $\int f \,\mathrm{d}\mu = 0$ be given. Let (f_n) be an increasing sequence of simple functions converging to f. Such a sequence exists by Proposition 3.53. It follows that $\int f_n \,\mathrm{d}\mu = 0$ for all $n \in \mathbb{N}$. Hence, by the case above, $\{x \in \Omega : f_n(x) \neq 0\}$ is a null set, whence there exists $M_n \in \Sigma$ with $\mu(M_n) = 0$ such that $f_n(x) = 0$ for all $x \notin M_n$. Put $M := \bigcup_{n\in\mathbb{N}} M_n$. Then $M \in \Sigma$ and $\mu(M) \leq \sum_{n=1}^{\infty} \mu(M_n) = 0$. Moreover, $x \notin M$ implies that $f(x) = 0$. Hence $f = 0$ almost everywhere.

If, conversely, $f = 0$ almost everywhere, then there exists a measurable set M with $f(x) = 0$ for all $x \notin M$. If g is a simple function with $0 \leq g \leq f$ then $g^{-1}[\{x\}] \subset M$ for all $x > 0$. By the definition of the integral for simple functions, $\int g \,\mathrm{d}\mu = 0$ and hence, since $g \leq f$ was arbitrary, $\int f \,\mathrm{d}\mu = 0$. □

Exercise 3.69. Let (Ω, Σ, μ) be a measure space and $f\colon \Omega \to [0,\infty]$ be measurable. Show that $\nu\colon \Sigma \to [0,\infty]$, defined by

$$\nu(A) := \int_A f \,\mathrm{d}\mu := \int \mathbb{1}_A f \,\mathrm{d}\mu,$$

defines a measure on (Ω, Σ).

Moreover, show that if $\mu(A) = 0$, then $\nu(A) = 0$. In other words, ν is **absolutely continuous** with respect to μ, which is usually denoted by writing $\nu \ll \mu$.

Remark 3.70. In the setting of Example 3.69 the function f is called the **density** of ν with respect to μ.

In probability theory a density for a distribution is commonly taken with respect to the Lebesgue measure. According to the *Radon–Nikodym theorem* a probability measure $\mathbb{P}$ on $\mathscr{B}(\mathbb{R})$ has a density with respect to the Lebesgue measure if and only if it is absolutely continuous with respect to the Lebesgue measure. Equivalently, this is the case if and only if its distribution function $F(x) := \mathbb{P}((-\infty, x])$ is an absolutely continuous function. Not every continuous real function is absolutely continuous, but every Lipschitz function is.

Theorem 3.71 (Fatou's Lemma). *Let (Ω, Σ, μ) be a measure space, $f_n\colon \Omega \to [0,\infty]$ be measurable and set $f(x) := \liminf_{n\to\infty} f_n(x)$. Then f is measurable and*

$$\int f \,\mathrm{d}\mu \leq \liminf_{n\to\infty} \int f_n \,\mathrm{d}\mu.$$

Proof. Note that $f(x) = \sup_{k\geq 1} \inf_{n\geq k} f_n(x)$. Let $g_k(x) := \inf_{n\geq k} f_n(x)$. Then g_k is measurable by Exercise 3.55. Clearly $g_k \uparrow f$. So it follows from monotone convergence that $\sup_{k\in\mathbb{N}} \int g_k \,\mathrm{d}\mu = \int_\Omega f \,\mathrm{d}\mu$. On the other hand,

$$\int f \,\mathrm{d}\mu = \sup_{k\in\mathbb{N}} \int g_k \,\mathrm{d}\mu \leq \liminf_{n\to\infty} \int f_n \,\mathrm{d}\mu.$$

Indeed, for every $n \geq k$ we have $g_k \leq f_n$ and thus $\int g_k \, d\mu \leq \int f_n \, d\mu$. Since this is true for all $n \geq k$, we have $\int g_k \, d\mu \leq \inf_{n \geq k} \int f_n \, d\mu$. By taking the supremum over $k \in \mathbb{N}$ on both sides, the above inequality follows. $\square$

Definition 3.72. Let (Ω, Σ, μ) be a measure space, $f \colon \Omega \to \mathbb{K}$ be measurable. Then f is called **integrable** if $\int |f| \, d\mu < \infty$. We write $f \in \mathscr{L}^1(\Omega, \Sigma, \mu)$.

If $\mathbb{K} = \mathbb{R}$, we set

$$\int f \, d\mu := \int f^+ \, d\mu - \int f^- \, d\mu.$$

Note that if $|f|$ is integrable then f^+ and f^- are both integrable nonnegative functions. If $\mathbb{K} = \mathbb{C}$, we set

$$\int f \, d\mu = \int \operatorname{Re} f \, d\mu + i \int \operatorname{Im} f \, d\mu.$$

Note that if $|f|$ is integrable, then $\operatorname{Re} f$ and $\operatorname{Im} f$ are both integrable real-valued functions.

If f is an integrable (real or complex) measurable function and $A \in \Sigma$, we define

$$\int_A f \, d\mu := \int f \mathbb{1}_A \, d\mu$$

Lemma 3.73. *Let (Ω, Σ, μ) be a measure space.*

(a) *For all integrable f, we have $|\int f \, d\mu| \leq \int |f| \, d\mu$.*

(b) *If f is integrable and $\lambda \in \mathbb{K}$, then λf is integrable and $\int \lambda f \, d\mu = \lambda \int f \, d\mu$.*

(c) *If f and g are integrable, then $f + g$ is integrable and $\int f + g \, d\mu = \int f \, d\mu + \int g \, d\mu$.*

Remark 3.74. Note that (b) and (c) can be expressed by saying that the integrable functions form a vector space and the map $f \mapsto \int f \, d\mu$ is a linear map from the integrable functions to $\mathbb{K}$.

Proof. Let us first consider the case where $\mathbb{K} = \mathbb{R}$.

(a) We have

$$\left| \int f \, d\mu \right| = \left| \int f^+ \, d\mu - \int f^- \, d\mu \right| \leq \int f^+ \, d\mu + \int f^- \, d\mu = \int |f| \, d\mu,$$

where we have used Corollary 3.64 in the last step.

(b) First note that by Corollary 3.64 one has $\int |\lambda f| \, d\mu = |\lambda| \int |f| \, d\mu < \infty$ if f is integrable. This proves that λf is integrable whenever f is.

Now, if $\lambda > 0$, then $(\lambda f)^+ = \lambda f^+$ and $(\lambda f)^- = \lambda f^-$. Thus, using Corollary 3.64,

$$\begin{aligned}\int \lambda f \,\mathrm{d}\mu &= \int \lambda f^+ \,\mathrm{d}\mu - \int \lambda f^- \,\mathrm{d}\mu \\ &= \lambda \int f^+ \,\mathrm{d}\mu - \lambda \int f^- \,\mathrm{d}\mu = \lambda \int f \,\mathrm{d}\mu.\end{aligned}$$

If, on the other hand, $\lambda < 0$, then $(\lambda f)^+ = -\lambda f^-$ and $(\lambda f)^- = -\lambda f^+$. Thus, in this case,

$$\begin{aligned}\int \lambda f \,\mathrm{d}\mu &= \int -\lambda f^- \,\mathrm{d}\mu - \int -\lambda f^+ \,\mathrm{d}\mu \\ &= \lambda \int f^+ \,\mathrm{d}\mu - \lambda \int f^- \,\mathrm{d}\mu = \lambda \int f \,\mathrm{d}\mu.\end{aligned}$$

(c) Since $|f+g| \le |f| + |g|$, it follows that

$$\int |f+g| \,\mathrm{d}\mu \le \int |f| + |g| \,\mathrm{d}\mu = \int |f| \,\mathrm{d}\mu + \int |g| \,\mathrm{d}\mu < \infty$$

if f and g are integrable.

Moreover, by definition and Corollary 3.64,

$$\begin{aligned}\int f \,\mathrm{d}\mu + \int g \,\mathrm{d}\mu &= \int f^+ \,\mathrm{d}\mu - \int f^- \,\mathrm{d}\mu + \int g^+ \,\mathrm{d}\mu - \int g^- \,\mathrm{d}\mu \\ &= \int f^+ + g^+ \,\mathrm{d}\mu - \int (f^- + g^-) \,\mathrm{d}\mu \\ &\overset{(*)}{=} \int (f+g)^+ \,\mathrm{d}\mu - \int (f+g)^- \,\mathrm{d}\mu \\ &= \int f + g \,\mathrm{d}\mu.\end{aligned}$$

Here, $(*)$ follows from integrating the identity $f^+ + g^+ + (f+g)^- = (f+g)^+ + f^- + g^-$ and using Corollary 3.64.

In the case where $\mathbb{K} = \mathbb{C}$, for (b) and (c) one uses that $\int f \,\mathrm{d}\mu = \int \operatorname{Re} f \,\mathrm{d}\mu + i \int \operatorname{Im} f \,\mathrm{d}\mu$. We omit the easy computations. For (a), we use that for every complex number z one has $|z| = \sup_{t\in\mathbb{R}} \operatorname{Re}(e^{it}z)$. So for every $t \in \mathbb{R}$ one has

$$\operatorname{Re}\Big(e^{it} \int f \,\mathrm{d}\mu\Big) = \operatorname{Re} \int e^{it} f \,\mathrm{d}\mu = \int \operatorname{Re}(e^{it} f) \,\mathrm{d}\mu \le \int |f| \,\mathrm{d}\mu.$$

Taking the supremum over $t \in \mathbb{R}$, statement (a) follows. □

Theorem 3.75 (Dominated convergence theorem)**.** *Let (Ω, Σ, μ) be a measure space and (f_n) be a sequence of integrable functions with the following two properties.*

(a) $\tilde{f}(x) := \lim_{n\to\infty} f_n(x)$ *exists for almost every $x \in \Omega$, say outside the set $N \in \Sigma$ with $\mu(N) = 0$.*

(b) *There exists an integrable function g with $|f_n(x)| \leq g(x)$ for almost every $x \in \Omega$ and all $n \in \mathbb{N}$.*

Then $f\colon \Omega \to \mathbb{K}$, defined by $f(x) = \tilde{f}(x)$ if $x \notin N$ and $f(x) = 0$ if $x \in N$, is integrable and

$$\lim_{n\to\infty} \int |f_n - f| \, d\mu = 0.$$

In particular,

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu.$$

The technicalities with f and $\tilde{f}$ are required as $\tilde{f}$ might not be measurable. Alternatively, to avoid speaking about $\tilde{f}$ and N, one could assume that a measurable $f\colon \Omega \to \mathbb{K}$ is given that is pointwise almost everywhere the limit of the f_n.

Proof. Changing f_n and f on a set of measure zero, we may assume that (a) and (b) hold everywhere. By Proposition 3.30, f is measurable. Since $|f| \leq g$, it follows that f is integrable.

Now observe that $|f_n - f| \leq 2g$ and hence $2g - |f_n - f| \geq 0$. By Fatou's Lemma 3.71,

$$\begin{aligned} \int 2g \, d\mu &= \int \liminf_{n\to\infty} (2g - |f_n - f|) \, d\mu \leq \liminf_{n\to\infty} \int 2g - |f_n - f| \, d\mu \\ &= \int 2g \, d\mu - \limsup_{n\to\infty} \int |f_n - f| \, d\mu. \end{aligned}$$

So $\limsup_{n\to\infty} \int |f_n - f| \, d\mu = 0$, and therefore $\lim_{n\to\infty} \int |f_n - f| \, d\mu = 0$. By Lemma 3.73,

$$\left| \int f_n \, d\mu - \int f \, d\mu \right| \leq \int |f_n - f| \, d\mu \to 0.$$

This proves the claim. □

Example 3.76. Let us give an example that condition (b) in Theorem 3.75 is necessary. Consider $(\mathbb{R}, \mathscr{B}(\mathbb{R}), \lambda)$. If we set $f_n := n\mathbb{1}_{(0,\frac{1}{n})}$, then (f_n) is a sequence of simple functions converging to 0 everywhere. However, $\int_{\mathbb{R}} f_n \, d\lambda \equiv 1 \not\to 0 = \int_{\mathbb{R}} 0 \, d\lambda$.

Exercise 3.77. Consider the situation of Exercise 3.69, i.e. (Ω, Σ, μ) is a measure space, $f\colon \Omega \to [0, \infty]$ is measurable and $\nu(A) := \int_A f \, d\mu$.

Show that g is integrable with respect to ν if and only if gf is integrable with respect to μ and in this case,

$$\int g \, d\nu = \int gf \, d\mu.$$

In the lecture we discussed here how the expected value of random variables is usually computed using densities for the push-forward measure.

We close this section by considering the integration under a push-forward measure, which is of great importance for applications.

Recall that $\mu_\Phi(A) = \mu(\Phi^{-1}[A])$ for all $A \in \mathscr{F}$, see Lemma 3.31.

Theorem 3.78. *Let (Ω, Σ, μ) be a measure space, $(M, \mathscr{F})$ be a measurable space and $\Phi\colon (\Omega, \Sigma) \to (M, \mathscr{F})$ be measurable. We denote the push-forward of μ under Φ by μ_Φ. Then for a measurable $f\colon (M, \mathscr{F}) \to (\mathbb{K}, \mathscr{B}(\mathbb{K}))$ we have*

$f \circ \Phi \in \mathscr{L}^1(\Omega, \Sigma, \mu)$ if and only if $f \in \mathscr{L}^1(M, \mathscr{F}, \mu_\Phi)$. In this case,

$$\int_\Omega f \circ \Phi \, \mathrm{d}\mu = \int_M f \, \mathrm{d}\mu_\Phi.$$

Proof. First, let $f = \sum_{j=1}^n a_k \mathbb{1}_{A_k}$ be a nonnegative, simple function. Then

$$f \circ \Phi = \sum_{j=1}^n a_k \mathbb{1}_{\Phi^{-1}[A_k]}$$

and thus, by the definition of the push-forward measure,

$$\int_\Omega f \circ \Phi \, \mathrm{d}\mu = \sum_{j=1}^n a_k \mu(\Phi^{-1}[A]) = \sum_{j=1}^n a_k \mu_\Phi(A) = \int_M f \, \mathrm{d}\mu_\Phi.$$

It follows that for nonnegative, simple functions the assertion holds true.

Now let $f \colon M \to [0, \infty]$ be measurable and (f_n) be a sequence of simple functions with $f_n \uparrow f$ pointwise. Then, by monotone convergence and the above,

$$\int_\Omega f \circ \Phi \, \mathrm{d}\mu = \sup_{n \in \mathbb{N}} \int_\Omega f_n \circ \Phi \, \mathrm{d}\mu = \sup_{n \in \mathbb{N}} \int_M f_n \, \mathrm{d}\mu_\Phi = \int_M f \, \mathrm{d}\mu_\Phi.$$

This shows that the assertion holds for arbitrary measurable positive f.

Since $|f \circ \Phi| = |f| \circ \Phi$, it follows that $f \in \mathscr{L}^1(M, \mathscr{F}, \mu_\Phi)$ if any only if $f \circ \Phi \in \mathscr{L}^1(\Omega, \Sigma, \mu)$. The general formula follows by splitting real valued functions f into the positive functions f^+ and f^- and complex valued functions f into $\operatorname{Re} f$ and $\operatorname{Im} f$. $\square$

3.8 On the connection between the Lebesgue and Riemann integral

We next compare the Lebesgue integral with the Riemann integral. As is well-known, every continuous function $f \colon [a, b] \to \mathbb{R}$ is Riemann integrable. We now show that such functions are also Lebesgue integrable and the Lebesgue integral agrees with the Riemann integral.

Theorem 3.79. *If $f \colon [a, b] \to \mathbb{K}$ is continuous, then f is a Lebesgue integrable function on $([a, b], \mathscr{B}([a, b]), \lambda)$. Moreover,*

$$\int_{[a,b]} f \, \mathrm{d}\lambda = \mathsf{R}\text{-}\int_a^b f(t) \, \mathrm{d}t.$$

Proof. Let a sequence of partitions $\pi_n := (t_0^{(n)}, \ldots, t_{k_n}^{(n)})$ with $|\pi_n| \to 0$ be given and let $\xi_n = (\xi_1^{(n)}, \ldots, \xi_{k_n}^{(n)})$ be a sequence of associated sample points. Put

$$f_n := \sum_{j=1}^{k_n} f(\xi_j^{(n)}) \mathbb{1}_{[t_{j-1}^{(n)}, t_j^{(n)})}.$$

Then f_n is a simple function and $\int_{[a,b]} f_n \, \mathrm{d}\lambda = S(f, \pi_n, \xi_n)$.

Moreover, (a) $|f_n| \leq \|f\|_\infty$ and the latter is integrable on our measure space and (b) $f_n(t) \to f(t)$ for all $t \in [a,b]$. Indeed, for fixed $t \in [a,b]$, we have $|f_n(t) - f(t)| = |f(\xi_{j_n}^{(n)}) - f(t)|$, where $\xi_{j_n}^{(n)}$ is the sample point in the interval $[t_{j_n-1}^{(n)}, t_{j_n}^{(n)}]$ and j_n is chosen such that t lies in this interval. But then $|\xi_{j_n}^{(n)} - t| \leq |t_{j_n}^{(n)} - t_{j_n-1}^{(n)}| \leq |\pi_n| \to 0$ and hence, by the continuity of f, it follows that $|f(\xi_{j_n}^{(n)}) - f(t)| \to 0$.

Hence the dominated convergence theorem, Theorem 3.75, applies and shows that f is integrable and

$$\int_{[a,b]} f \, \mathrm{d}\lambda = \lim_{n\to\infty} \int_{[a,b]} f_n \, \mathrm{d}\lambda = \lim_{n\to\infty} S(f, \pi_n, \xi_n).$$

Since, on the other hand, the Riemann sums converge to $\text{R-}\int_a^b f(t) \, \mathrm{d}t$, the assertion follows. $\square$

Remark 3.80. Actually, the continuity assumption in Theorem 3.79 is not needed, if one is willing to enlarge the σ-algebra. It can be proved that if $f\colon [a,b] \to \mathbb{K}$ is Riemann integrable then it is almost everywhere equal to a measurable function that is Lebesgue integrable and the Riemann and the Lebesgue integral coincide.

There is also an extension of Theorem 3.79 to improper Riemann integrals. We recall that if $-\infty < a < b \leq \infty$ and $f\colon [a,b) \to \mathbb{R}$ is continuous, then f is called improperly Riemann integrable on $[a,b)$ if the limit $\lim_{r\uparrow b} \text{R-}\int_a^r f(t) \, \mathrm{d}t$ exists. The limit is then called the **improper Riemann integral of f over $[a,b)$** and denoted by $\text{R-}\int_a^{b-} f(t) \, \mathrm{d}t$.

Theorem 3.81. *Let $-\infty < a < b \leq \infty$ and $f\colon [a,b) \to \mathbb{R}$ be continuous, such that the improper Riemann integral $\text{R-}\int_a^{b-} |f(t)| \, \mathrm{d}t$ exists, then f is integrable on $([a,b), \mathscr{B}([a,b)), \lambda)$, the improper Riemann integral $\text{R-}\int_a^{b-} f(t) dt$ exists and*

$$\int_{[a,b)} f \, \mathrm{d}\lambda = \text{R-}\int_a^{b-} f(t) \, \mathrm{d}t.$$

Proof. Pick a sequence $(b_n) \subset (a,b)$ with $b_n \uparrow b$. By monotone convergence and Theorem 3.79,

$$\begin{aligned}\int_{[a,b)} |f| \, \mathrm{d}\lambda &= \lim_{n\to\infty} \int_{[a,b_n]} |f| \, \mathrm{d}\lambda \\ &= \lim_{n\to\infty} \text{R-}\int_a^{b_n} |f(t)| \, \mathrm{d}t = \text{R-}\int_a^{b-} |f(t)| \, \mathrm{d}t < \infty.\end{aligned}$$

It follows that $|f|$ is integrable on $[a,b)$. Moreover, since $\mathbb{1}_{[a,b_n)} f$ converges to f pointwise and $|\mathbb{1}_{[a,b_n)} f| \le |f|$, the dominated convergence theorem yields

$$\int_{[a,b)} f \,\mathrm{d}\lambda = \lim_{n\to\infty} \int_{[a,b_n)} f \,\mathrm{d}\lambda = \lim_{n\to\infty} \text{R-}\int_a^{b_n} f(t)\,\mathrm{d}t = \text{R-}\int_a^{b-} f(t)\,\mathrm{d}t,$$

where we have used Theorem 3.79 in the second step. □

Remark 3.82. Similar results as in Theorem 3.81 also hold for improper Riemann integrals that are improper on the left-hand side or on both sides.

Example 3.83. In Theorem 3.81, the assumption that $\text{R-}\int_{[a,b)} |f(t)|\,\mathrm{d}t$ exists is crucial and cannot be omitted. An example is given by $f\colon [1,\infty) \to \mathbb{R}$, defined by $f(t) = \frac{\sin t}{t}$ In this case, by integration by parts, we obtain

$$\text{R-}\int_1^x \frac{\sin t}{t}\,\mathrm{d}t = \frac{-\cos t}{t}\Big|_1^x - \text{R-}\int_1^x \frac{\cos t}{t^2}\,\mathrm{d}t \to \cos 1 - \text{R-}\int_1^\infty \frac{\cos t}{t^2}\,\mathrm{d}t$$

as $x\to\infty$. The latter improper Riemann integral exists since $|t^{-2}\cos t| \le t^{-2}$ and the latter is integrable. It follows that the improper Riemann integral $\text{R-}\int_1^\infty \frac{\cos t}{t^2}\,\mathrm{d}t$ exists.

On the other hand, on each interval $[k\pi,(k+1)\pi)$, we have $|f(t)| \ge |\sin(t)|((k+1)\pi)^{-1}$. It thus follows that

$$\begin{aligned}\int_{[1,\infty)} f\,\mathrm{d}\lambda &\ge \sum_{k=1}^n \frac{1}{(k+1)\pi} \int_{[k\pi,(k+1)\pi)} |\sin(t)|\,\mathrm{d}\lambda(t) \\ &= \frac{1}{\pi}\Big(\sum_{k=1}^n \frac{1}{k+1}\Big)\cdot \text{R-}\int_0^\pi |\sin(t)|\,\mathrm{d}t.\end{aligned}$$

Since the harmonic series diverges, it follows that f is not integrable on $[1,\infty)$.

Remark 3.84. In what follows, we will also use the 'differential' $\mathrm{d}t$ in Lebesgue integrals instead of the (formally correct) $\mathrm{d}\lambda$. We will thus write

$$\int_{[a,b]} f(t)\,\mathrm{d}t \quad \text{or} \quad \int_a^b f(t)\,\mathrm{d}t$$

to denote the Lebesgue integral of f on the interval $[a,b]$. This is particularly helpful when the function f depends on more than one variable. To have this feature also at hand for general measures, we will frequently write $\int_\Omega f(x)\,\mathrm{d}\mu(x)$ instead of $\int_\Omega f\,\mathrm{d}\mu$ to emphasize that we are integrating with respect to the variable x.

3.9 Integrals depending on a parameter

The main topic of this section is to interchange operations like integration, differentiation and taking limits. This is a topic at the very heart of analysis.

Instead of $[0,1]$ we could use a general metric space here.

Suppose that (Ω, Σ, μ) is a measure space. If we are given a map $f\colon [0,1] \times \Omega \to \mathbb{C}$ such that $f(t, \cdot)$ is integrable for all $t \in [0,1]$, we may define $F(t) := \int_\Omega f(t,x)\,\mathrm{d}\mu(x)$. It is then natural to ask how F depends on the parameter $t \in [0,1]$. In this short section, we use the dominated convergence theorem to prove some results in this direction.

Proposition 3.85. *Let (Ω, Σ, μ) be a measure space. Furthermore, let $f\colon [0,1] \times \Omega \to \mathbb{K}$ be such that the following three properties hold.*

(a) $x \mapsto f(t,x) \in \mathscr{L}^1(\Omega, \Sigma, \mu)$ *for all* $t \in [0,1]$.

(b) $t \mapsto f(t,x)$ *is continuous for almost all* $x \in \Omega$.

(c) *There exists a* $g \in \mathscr{L}^1(\Omega, \Sigma, \mu)$ *such that* $|f(t,x)| \leq g(x)$ *for all* $(t,x) \in [0,1] \times \Omega$.

Then $F\colon [0,1] \to \mathbb{C}$ defined by $F(t) = \int_\Omega f(t,x)\,\mathrm{d}\mu(x)$ is continuous.

Proof. Let $t_n \to t$ in $[0,1]$. Then $f(t_n,x) \to f(t,x)$ for almost all $x \in \Omega$ by (b). Since $|f(t_n,x)| \leq g(x)$ for all $x \in \Omega$ by assumption and $g \in \mathscr{L}^1(\Omega)$, it follows from the dominated convergence theorem, Theorem 3.75, that

$$F(t_n) = \int_\Omega f(t_n,x)\,\mathrm{d}\mu(x) \to \int_\Omega f(t,x)\,\mathrm{d}\mu(x) = F(t).$$

This proves the continuity of F. □

Proposition 3.86. *Let I be an interval in $\mathbb{R}$ and (Ω, Σ, μ) be a measure space. Furthermore, let $f\colon I \times \Omega \to \mathbb{K}$ be such that the following three properties hold.*

(a) $x \mapsto f(t,x) \in \mathscr{L}^1(\Omega, \Sigma, \mu)$ *for all* $t \in I$.

(b) $t \mapsto f(t,x)$ *is differentiable for all* $x \in \Omega$.

(c) *There exists a* $g \in \mathscr{L}^1(\Omega, \Sigma, \mu)$ *such that* $|\frac{\partial}{\partial t} f(t,x)| \leq g(x)$ *for all* $(t,x) \in I \times \Omega$.

Then $F\colon I \to \mathbb{K}$ defined by $F(t) = \int_\Omega f(t,x)\,\mathrm{d}\mu(x)$ is differentiable. Moreover, $\frac{\partial}{\partial t} f(t,x)$ is integrable for all $t \in I$ and

$$F'(t) = \frac{d}{dt} \int_\Omega f(t,x)\,\mathrm{d}\mu(x) = \int_\Omega \frac{\partial}{\partial t} f(t,x)\,\mathrm{d}\mu(x).$$

Proof. Fix $t \in I$ and let (t_n) be a sequence in I that converges to t. Define $h_n, h\colon \Omega \to \mathbb{C}$ by $h_n(x) := (t_n - t)^{-1}(f(t_n,x) - f(t,x))$ and $h(x) = \frac{\partial}{\partial t} f(t,x)$. Then h_n is integrable for every $n \in \mathbb{N}$ as a linear combination of integrable functions. Moreover, $h_n(x) \to h(x)$ for all $x \in \Omega$ by assumption. By the mean-value theorem, $h_n(x) = \frac{\partial}{\partial t} f(\xi_n, x)$ for some ξ_n between

t and t_n. In particular, $|h_n| \leq g$. Thus the dominated convergence theorem shows that h is integrable and

$$\frac{F(t_n) - F(t)}{t_n - t} = \int_\Omega h_n(x)\,\mathrm{d}\mu(x) \to \int_\Omega h(x)\,\mathrm{d}\mu(x) = \int_\Omega \frac{\partial}{\partial t} f(t,x)\,\mathrm{d}\mu(x).$$

This finishes the proof. □

3.10 Product measures

In this section we construct a σ-algebra and a corresponding measure on the product of two suitable measure spaces. Our motivation is to extend the theory in order to deal with iterated integrals. The product measure will allow us to write an iterated integral as a single integral with resprect to the product measure.

Definition 3.87. Let (Ω_k, Σ_k) be a measurable space for $k = 1, \dots, n$. The **product (measurable space)** of the spaces (Ω_k, Σ_k) is the measurable space $(\prod_{k=1}^n \Omega_k, \bigotimes_{k=1}^n \Sigma_k)$, where $\prod_{k=1}^n \Omega_i$ is the Cartesian product of the sets Ω_k, i.e., the set of all tuples $(x_1, \dots, x_n)$ where $x_k \in \Omega_k$ and $\bigotimes_{k=1}^n \Sigma_k$ is generated by the cuboids $A_1 \times \cdots \times A_n$ where $A_k \in \Sigma_k$.

Exercise 3.88. Let (Ω, Σ) and, for $k = 1, \dots, n$, also (Ω_k, Σ_k) be measure spaces. Let $f_k \colon \Omega \to \Omega_k$ be a function and define $f \colon \Omega \to \prod_{k=1}^n \Omega_k$ by $f(x) = (f_1(x), \dots, f_n(x))$. Show that f is $\Sigma / \bigotimes_{k=1}^n \Sigma_k$-measurable if and only if f_k is Σ/Σ_k-measurable for all $k = 1, \dots, n$.

In the following, let $(\Omega_i, \Sigma_i, \mu_i)$ be σ-finite measure spaces for $i = 1, 2$. We define a measure $\mu_1 \otimes \mu_2$ on the σ-algebra $\Sigma_1 \otimes \Sigma_2$ which is the product of the measures μ_1 and μ_2 in the sense that

$$\mu_1 \otimes \mu_2(A \times B) = \mu_1(A)\mu_2(B)$$

for all $A \in \Sigma_1$ and $B \in \Sigma_2$. Note that as μ_1 and μ_2 are σ-finite, by Corollary 3.38 there exists at most one such measure.

Observe that the 2-dimensional Lebesgue measure λ_2 is the product measure of the one-dimensional Lebesgue measure with itself.

For a set $Q \subset \Omega_1 \times \Omega_2$ and $x \in \Omega_1$, $y \in \Omega_2$, we define the **cuts** $[Q]_x$ and $[Q]_y$ by

$$[Q]_x := \{y \in \Omega_2 : (x,y) \in Q\} \quad \text{and} \quad [Q]_y := \{x \in \Omega_1 : (x,y) \in Q\}.$$

Lemma 3.89. *For $x \in \Omega_1$, $y \in \Omega_2$ and $Q \in \Sigma_1 \otimes \Sigma_2$ we have $[Q]_x \in \Sigma_2$ and $[Q]_y \in \Sigma_1$.*

Proof. We put $\mathscr{G} := \{Q \in \Sigma_1 \otimes \Sigma_2 : [Q]_x \in \Sigma_2\}$. We claim that $\mathscr{G}$ is a σ-algebra on $\Omega_1 \times \Omega_2$. Clearly (S1) holds, since $[\Omega_1 \times \Omega_2]_x = \Omega_2$. (S2) and