The Functional Calculus for Sectorial Operators and Similarity Methods

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…cum grano salis

"Was endlich die *Deutlichkeit* betrifft, so hat der Leser ein Recht, zuerst die *diskursive* (logische) *Deutlichkeit*, durch Begriffe, dann aber auch eine *intuitive* (ästhetische) *Deutlichkeit*, durch *Anschauungen*, d.i. Beispiele oder andere Erläuterungen in concreto zu fordern. Für die erste habe ich hinreichend gesorgt. Das betraf das Wesen meines Vorhabens, war aber auch die zufällige Ursache, daß ich der zweiten, obzwar nicht so strengen, aber doch billigen Forderung nicht habe Genüge leisten können. Ich bin fast beständig im Fortgange meiner Arbeit unschlüssig gewesen, wie ich es hiermit halten solle. Beispiele und Erläuterungen schienen mir immer nötig und flossen daher auch im ersten Entwurfe an ihren Stellen gehörig ein. Ich sah aber die Größe meiner Aufgabe und die Menge der Gegenstände, womit ich es zu tun haben würde, gar bald ein und, da ich gewahr ward, daß diese ganz allein, im trockenen, bloß scholastischen Vortrage, das Werk schon genug ausdehnen würden, so fand ich es unratsam, es durch Beispiele und Erläuterungen, die nur in populärer Absicht notwendig sind, noch mehr anzuschwellen, zumal diese Arbeit keineswegs dem populären Gebrauche angemessen werden könnte und die eigentlichen Kenner der Wissenschaft diese Erleichterung nicht so nötig haben, (…) ."

I. Kant, *Kritik der reinen Vernunft*, Vorrede zur ersten Auflage[1]

[1]Zitiert nach: Philosophische Bibliothek, Bd. 37a, Meiner, Hamburg 1990, S.11. (Hervorhebungen im Original)

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Introduction and Summary

Schließt man sich der landläufigen Meinung über die Wissenschaft der Mathematik an, so ist diese durch die Sicherheit und die (manchmal sogar "ewige") Wahrheit ihrer Ergebnisse sowie die Präzision ihrer Begriffe vor anderen Wissenschaften ausgezeichnet. Die Mathematik erscheint als eine sich immer weiter ausdehnende Ansammlung von Theoremen und Definitionen, die durch das Band der (deduktiven) Logik miteinander und untereinander verknüpft sind. Dieses Bild der Mathematik hat in Gestalt der "Metamathematik" genannten Disziplin ihre Konkretisierung erfahren, was wiederum zu einer Verfestigung des Bildes auch unter den Mathematikern beitrug. Vertreter dieser Perspektive pflegen "prinzipiell" zu argumentieren: prinzipiell kann jeder mathematische Begriff und jede mathematische Aussage formalisiert werden, prinzipiell kann der Mathematiker durch einen Computer ersetzt werden, etc.. Wir wollen diese Auffassung für den Moment die "reduktionistische" nennen. Natürlich bestreitet auch der Reduktionist nicht, dass die Mathematik eine Entwicklung aufweist, dass sie also *prozessualen* Charakter hat. Ebensowenig wird in Abrede gestellt, dass diese Entwicklung in einem sozialen Raum angesiedelt ist, der wiederum in allerlei Kontexte (politisch, ökonomisch, geistesgeschichtlich, geographisch usw.) eingebettet ist. Nur erklärt der Reduktionist diese anderen Aspekte für unerheblich und nebensächlich; wer über diese Dinge nachdenkt, macht eben nicht mehr Mathematik.

Dies ist deshalb kein blosser Streit um Worte, weil die Aussage, gewisse Dinge seien keine Mathematik mehr, einen normativen Zug besitzt, der — zur Geltung gebracht — das Erscheinungsbild der mathematischen Veröffentlichungen prägt. Als Beispiel sei nur genannt, dass in den mathematischen Periodika kaum je eine kontroverse Debatte zu finden ist. Der Reduktionist würde auch die Notwendigkeit einer solchen bestreiten. Was gibt es an einem korrekt bewiesenen Theorem noch zu diskutieren?[2]

Nun will ich nicht unbedingt einer Mathematik das Wort reden, die die Erforschung ihrer eigenen sozialen und politischen Bedingungen und Kontingenzen als originären Bestandteil ihrer selbst begreift. Ich vertrete lediglich die Auffassung, dass die Mathematik nicht in ihren formalen Aspekten aufgeht. Die Ineinssetzung von Mathematik und ihren formalisierbaren Anteilen ist, m.E., ein Fehler, der in ähnlicher Weise in anderen Wissenschaften begangen wurde (und wird?), und den man im Anschlußan BATESON als Verwechslung von Landschaft und Landkarte bezeichnen kann.[3]

Um im Bild zu bleiben: Das logische Geflecht aus formalen Definitionen, Theoremen und Beweisen ist eine Landkarte, die wir (die Mathematiker) erstel-

[2]Ich übergehe die Schwierigkeit, die sich ergäbe, würde man beim Wort "korrekt" nachhaken.

[3]Vgl. Gregory Bateson, *Geist und Natur*, 5. Aufl., Frankfurt a.M. 1997, S. 40 ff.

len, um uns in der Landschaft (ein jeweiliger Aussschnitt der mathematischen "Welt") zurechtzufinden. Dabei herrscht die gleiche Spannung zwischen "Realität" und ihrer sprachlichen "Fassung" wie in jeder anderen Wissenschaft und sogar in der Alltagswelt. Natürlich bleibt die fundamentale Frage bestehen, ob es Realität jenseits ihrer Beschreibung (also Landschaft jenseits der Karte) überhaupt gibt. Ich nehme hier den pragmatischen Standpunkt ein, dass diese Frage nicht entschieden werden muss, so lange man sinnvoll über die Angemessenheit der Landkarte streiten kann. (Die Frage wird gewissermassen mit den Füssen beantwortet.) Dies wird ermöglicht durch die sogenannten "Intuitionen" oder "Präideen", die — formal nicht fassbar — im Hintergrund stehen und als Kriterien, Leitbilder usw. dienen.

Damit bin ich im Zentrum meiner Vorrede angelangt. Ich meine nicht, dass man die Soziologie der Mathematik als Mathematik ansehen sollte. Aber ich plädiere dafür, dass die Frage, *ob die Landkarte gut ist*, zur Mathematik gehört. Diese Frage hat natürlich viele Konkretisierungen, etwa:

- An welchen Stellen kann man die gegenwärtige Karte verfeinern?
- Wie muss man die Beschriftung der Karte organisieren, um sich auf ihr zurechtzufinden?
- Wo sind "blinde Flecken"? Was "fehlt"?
- Was ist zuviel auf der Karte? Was behindert die Lesbarkeit?

Bei der Verschiedenheit der Fragen und natürlich auch der persönlichen Vorlieben und Schwerpunkte der einzelnen Mathematiker, ist es zu erwarten, dass *unterschiedliche* Entwürfe von Karten miteinander in Konkurrenz treten. Und dass damit eine Diskussion über diese Entwürfe, ihre Vor- und Nachteile entsteht, die sich auch in den mathematischen Veröffentlichungen niederschlägt.

One can read the first part of the following thesis as an attempt to answer the question whether a certain map is appropriate to the landscape which it is supposed to describe. The landscape considered is marked in short by the words "functional calculus for sectorial operators", and we are going to rewrite the map in order to obtain more structure, survey, elegance than we think is achieved by the existing accounts. As a matter of fact, giving an answer to this question is to be understood as a proposal. Since we believe in the evolution of ideas, we hope that our attempt is a stepping stone to a better and better understanding of functional calculus.
Let us briefly sketch the fundamental *intuition* behind the term "functional calculus". Consider the Banach space $X := \mathbf{C}[0,1]$ of complex valued continuous functions on the unit interval. Each function $f \in X$ determines a bounded linear operator

$$M_f = (g \longmapsto fg) : \mathbf{C}[0,1] \longrightarrow \mathbf{C}[0,1]$$

on X called the *multiplication* operator associated with f. Its spectrum is simply the range $f[0,1]$ of f. Given any other continuous function $\psi : \sigma(M_f) \longrightarrow$

$\mathbb{C}$ one can consider the multiplication operator $M_{\psi\circ f}$ associated with $\psi \circ f$. This yields an algebra homomorphism

$$\Phi = (\psi \longmapsto M_{\psi\circ f}) : \mathbf{C}(\sigma(M_f)) \longrightarrow \mathcal{L}(X).$$

Since one has $\Phi(z) = M_f$ and $\Phi((\lambda - z)^{-1}) = R(\lambda, M_f)$ for $\lambda \in \varrho(M_f)$, and the operator $\Phi(\psi)$ is simply multiplication by $\psi(f(z))$, one says that $\Phi(\psi)$ is obtained by "inserting" the operator M_f into the function ψ and writes $\psi(M_f) := \Phi(\psi)$. Generalizing this example to the Banach space $X = \mathbf{C_0}(\mathbb{R})$ one realizes that boundedness of the operators is not an essential requirement. The intuition of functional calculus now consists in the idea that to *every* closed operator A on a Banach space X there corresponds an algebra of complex-valued functions on its spectrum in which the operator A can somehow be "inserted" in a reasonable way. Here "reasonable" means *at least* that $\psi(A)$ should have the expected meaning *if* one expects something, e.g., if $\lambda \in \varrho(A)$ then one expects $(\lambda - z)^{-1}(A) = R(\lambda, A)$ or if A generates a semigroup T then $e^{tz}(A) = T(t)$. (This is just a minimal requirement. There may be other reasonable criteria around.) The obtained mapping $\psi \longmapsto \psi(A)$ is (informally) called a *functional calculus* for A. Unfortunately, up to now there is no overall formalization of this idea. The best thing achieved so far is a case by case construction. For example, if one knows that the operator A is "essentially" a multiplication operator, then it is straightforward to construct a functional calculus. By one version of the spectral theorem, this is the case if A is a normal operator on a Hilbert space. (We have put an account of the spectral theorem into Appendix C.) If one is not in such a lucky position, then, depending on certain properties of A, one can give constructions of $\psi(A)$ like

$$\psi(A) = \frac{1}{2\pi i} \int_\Gamma \psi(z) R(z, A)\, dz$$

(but there are others, compare §7, Chapter 1.). The idea behind doing so is that one takes an integral representation of the function ψ in terms of other functions φ (in our example: a Cauchy integral with the "other" functions being elementary rationals) where $\varphi(A)$ should have a straightforward meaning. Then one inserts A into the "known" part and makes sure that the definition is still meaningful.

A classical example of this procedure occurs when A is a bounded operator and ψ is a holomorphic function defined on an open superset of $\sigma(A)$. Then our Cauchy integral is perfectly reasonable (with an appropriate Γ). The functional calculus we arrive at is called the *Dunford-Riesz* calculus, see [Con90, Chapter VII,§4]. This can be generalized to situations when A is no longer bounded or if the set $\mathbb{C} \setminus \mathrm{dom}(\psi)$ touches the spectrum of A, say, in some isolated points (which we call the critical points for the moment). One can retain the Cauchy integral definition by requiring stronger integrability properties for ψ provided one knows something about the growth behaviour of the resolvent of A in the vicinity of the critical points. (Compare §3 and the comments to it in §7 of Chapter 1. There you find also more historical remarks and other types of functional calculi.)

One example for such a growth condition is given by the concept of "sectoriality" of an operator. A sectorial operator A has its spectrum contained in some sector S_ω with the numbers $\|\lambda R(\lambda, A)\|$ being uniformly bounded outside every larger sector. These operators play a prominent role in the theory of elliptic and parabolic partial differential equations. Already in the 1960s the so-called "fractional powers" A^α (for $\alpha \in \mathbb{C}$) of a sectorial operator A were defined (see [KS59], [Bal60], [Yos60], [Kat60]) and have been subject to extensive research ever since. They are of great importance in the theory of non-autonomous evolution equations (see [Kn71] or [Paz83]). The purely imginary powers link the fractional powers to interpolation theory ([Tri95], [MCSA01]) and they play an important role, by the celebrated Dore-Venni Theorem [DV87], for the so-called "Maximal Regularity Problem". However, up to now there has been no development of the the theory of fractional powers into a full account of a functional calculus, not even in the recent monograph [MCSA01]. Such an account became possible when the "natural functional calculus for sectorial operators" (our terminology) entered the stage. McIntosh had developed this functional calculus in his work on the Cauchy singular integral operator on a Lipschitz curve (see his seminal paper [McI86] and the lecture notes [ADM96]). It was the primary goal to obtain *unbounded* operators through the functional calculus, a goal which was not so common at that time. McIntosh remarks in [McI86] that the theory of fractional powers can be recovered by his functional calculus. However, the main focus of his research was on the *boundedness of the H^∞-calculus* which, with the help of Yagi's ideas from [Yag84], could be shown to be equivalent to certain so called *quadratic estimates* in the Hilbert space case. This focus remained in the subsequent attempts to generalize the results from Hilbert space to $\mathbf{L}^p$-spaces and general Banach spaces, as in [Bd92], [CDMY96], [Fra97] and [FM98].

In the first part of our thesis we will give a *systematic account* of the natural functional calculus for sectorial operators. We will — in contrast to the literature on the topic — omit any density assumptions on the operator. Moreover, we do not even require injectivity in the first place but show how the functional calculus can be extended to larger and larger algebras of functions according to spectral properties (injectivity, invertibility, boundedness) of the operator. We unify certain approximation procedures known in the literature by introducing the concept of *sectorial approximation*. Furthermore, we emphasize the importance of the so-called *composition rule*

$$(f \circ g)(A) = f(g(A))$$

which in our opinion is the basic ingredient which makes the functional calculus a working tool.

In Chapter 2 we will follow McIntosh's remark and show that the theory of fractional powers can be developed in a straightforward and elegant way (§1 and §2), once one has set up the basic properties of the functional calculus. The same holds for the fundamentals of holomorphic semigroups (§3) and operator logarithms (§4).

As has already been pointed out, the first part of the thesis is a "rewriting of the map" on the landscape of functional calculus for sectorial operators. The subsequent second part is dedicated to single results. It can be considered as a refinement of the map and a discovery of new paths and connections which had previously been hidden.

One of these hidden connections is certainly the operator logarithm. It seems that, apart from an old theorem of NOLLAU in [Nol69] and a few recent articles ([Boy94], [Oka00b], [Oka00a]), there have been no further results on that topic. In Chapter 3 we complement Nollau's theorem and show that the secoriality angle of A can be recovered from spectral properties of $\log A$, see Theorem 3.9. This immediately yields a proof (and a generalization) of a celebrated theorem of PRÜSS and SOHR from [PS90] (without setting up the Mellin transform calculus), see Corollary 3.12 and Theorem 3.14. Next, we give an example of an injective sectorial operator A on a UMD space X such that the purely imaginary powers A^{is} are all bounded (i.e., $A \in \mathrm{BIP}(X)$) but the type of the group $(A^{is})_{s\in\mathbb{R}}$ is strictly *larger than* π. Whether this is possible or not has (at least implicitly) been an open question for more than ten years. In the final section of Chapter 3 we give — by functional calculus methods — a characterization of group generators on Hilbert spaces (Theorem 3.26). This theorem generalizes a recent result of LIU (from [Liu00]) and provides a much more accessible proof of a theorem of BOYADZHIEV and DELAUBENFELS from [Bd94]. Combining this theorem with the previous result on logarithms we recover the important result of MCINTOSH (see [McI86]) that for an injective sectorial operator on a Hilbert space, boundedness of the H^∞-calculus and boundedness of the imaginary powers are equivalent (Corollary 3.31).

The connections of the functional calculus and *similarity* questions on Hilbert spaces are twofold. One is given by von Neumann's inequality (see [Pau86] or [Pis01]) which says that a contraction T on a Hilbert space H satisfies

$$\|p(T)\| \leq \sup\{|p(z)| \mid |z| \leq 1\}.$$

Via the Cayley transform one can conclude from this that if A is an injective m-accretive operator, its H^∞-calculus must be bounded. Thus, information on the numerical range of A (depending on the particular scalar product) gives information on the functional calculus which does not depend on the particular scalar product.

On the other hand the quadratic estimates which we have mentioned above in connection with MCINTOSH's and YAGI's work, can be reinterpreted as a construction of equivalent scalar products, see Proposition 4.17, Corollary 4.21, and Theorem 4.23. For example, assume that $-A$ is injective and generates a bounded holomophic semigroup on the Hilbert space H. It follows from MCINTOSH's results that the natural H^∞-calculus for A is bounded if and only if the singular integral

$$\left(\int_0^\infty \left\|A^{\frac{1}{2}}e^{-tA}x\right\|^2 \, dt\right)^{\frac{1}{2}} \quad (x \in H)$$

defines an equivalent Hilbert norm on H. As is easily seen, the semigroup $(e^{-tA})_{t\geq 0}$ is contractive with respect to this new norm. In Chapter 4 we give an account of this result and use it to deduce a far stronger similarity result (Theorem 4.26, thereby generalizing a theorem of LEMERDY (in [LM98b]) and FRANKS (in [Fra97]). Moreover, our proof does not require the deep result of PAULSEN on which the proofs in [LM98b] and [Fra97] rest. A consequence of the similarity theorem so obtained is a characterization of variational operators modulo similarity (Corollary 4.27). Here, an operator is called variational if it can be obtained via an elliptic form. We show in addition, that a variational operator always has the square root property with respect to some equivalent scalar product. The original square root problem has a long history and could be solved only recently in [AHL$^+$01]. (See Remark 4.16 for more information on the square root problem and §7 of Chapter 4 for historical remarks on other similarity problems.)

Whereas in Chapter 4 we used results from the functional calculus to obtain a similarity theorem, in Chapter 5 we reverse this process. Recall that every bounded linear operator A on a Hilbert space H has a canonical decomposition

$$A = \frac{A - A^*}{2} + \frac{A + A^*}{2}$$

as sum of a skew-adjoint and a selfadjoint operator. This decomposition reflects the canonical decomposition of the elements of the numerical range of A in real and imaginary parts. Furthermore, the commutator

$$[\frac{A - A^*}{2}, \frac{A + A^*}{2}] = \frac{1}{2}(AA^* - A^*A)$$

of both summands is selfadjoint. For unbounded operators such a decomposition fails, in general, for many reasons.
We show that, if A generates a strongly continuous *group* on H, one can always find an equivalent scalar product such that the above decomposition remains valid (Theorem 5.9). This is done by a simple application of "Liapunov's direct method" (see [DKn74]) to the group generated by A. Thus, a group generator on a Hilbert space can always be viewed as a bounded perturbation of a skew-adjoint operator (after changing the scalar product). This allows us to give a second (and very elegant) proof of the Boyadzhiev-deLaubenfels Theorem (already mentioned above) on the boundedness of the H^∞-calculus for group generators on Hilbert space. Finally we combine our decomposition result with a theorem of FATTORINI to prove the following: If $-A$ generates a cosine function on the Hilbert space H one can find an equivalent scalar product on H such that the numerical range of A is contained in a horizontal parabola (Corollary 5.18).

Right at the beginning of each chapter we have put a short overview on the things to come. The last section of each chapter contains miscellaneous remarks and comments as well as historical informations. We have put a reference for a statement directly into the text only when we do not give a proof. All other references are in the corresponding "Comments" sections. Let me point out that (hopefully) all notations and background results are contained in the chapters comprising the "Appendix". We have refrained from giving numbers to definitions. Instead, we have put an extensive index and a nomenclature of notation at the end. Whenever a new notion is defined, we will indicate this by using boldface letters as in

> "An operator A is called **sectorial** if ...".

Some of the presented results have been published (or are about to be) in [ABH01], [Haa01], [Haa02], and [Haa03].

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Am Ende möchte ich den folgenden Personen meinen tiefen Dank aussprechen. Ohne sie wäre nicht nur diese Arbeit nicht, wie sie jetzt ist. Ich kann nicht in Worte fassen, was ich ihnen verdanke: Dietlinde Haase, Annette Kaune, Stefan Gabler, Frank Fischer, Hans Kiesl, Frieder Günther, Karin Bey und Rainer Nagel.

Organon*

*The ancient Greek word "Organon" has been used of old to denote a certain collection of ARISTOTLE's writings which deal with logic, grammar, and the philosophy of language. In these books, ARISTOTLE develops systematically the results and problems in that parts of science which until the sixteenth century have been considered the most fundamental. FRANCIS BACON's *Novum Organon* consciously alluding to its ancient predecessor plays a similar role for the development of modern science since the Renaissance.

In using this title for the two chapters to come we do not want to insinuate that we regard the treated topics as like fundamental. However, there is a resemblance with the Aristotelian corpus which may justify the use of the word. First, the theory presented is in fact *basic* at least for the chapters which follow. Second, the emphasis in our presentation is on *systematic development of theory*. This means that we have tried to give an account of the natural functional calculus for sectorial operators as systematical and as complete as possible without being too artificial.

First Chapter
The Functional Calculus for Sectorial Operators

In §1 the basic theory of sectorial operators is developed including the concept of sectorial approximation. In §2 we introduce some notation for certain spaces of holomorphic functions on sectors. A functional calculus for sectorial operators is constructed in two steps in §3. Fundamental properties including the composition rule are proved. In §4 we give natural extensions of the functional calculus to larger function spaces in case the operator satisfies certain spectral conditions in 0 and/or ∞. In this way a panorama of functional calculi is developed. In §5 we provide some fundamental boundedness and approximation results including the important approximation technique of MCINTOSH. The §6 is devoted to the boundedness of the natural H^∞-calculus. It is examined how boundedness of the calculus on a small function algebra relates to the boundedness on a large one. We deal with the uniqueness question and present the example of an operator whose natural H^∞-calculus is not bounded. Finally we provide an important lemma by KALTON and WEIS.

§1 Sectorial Operators

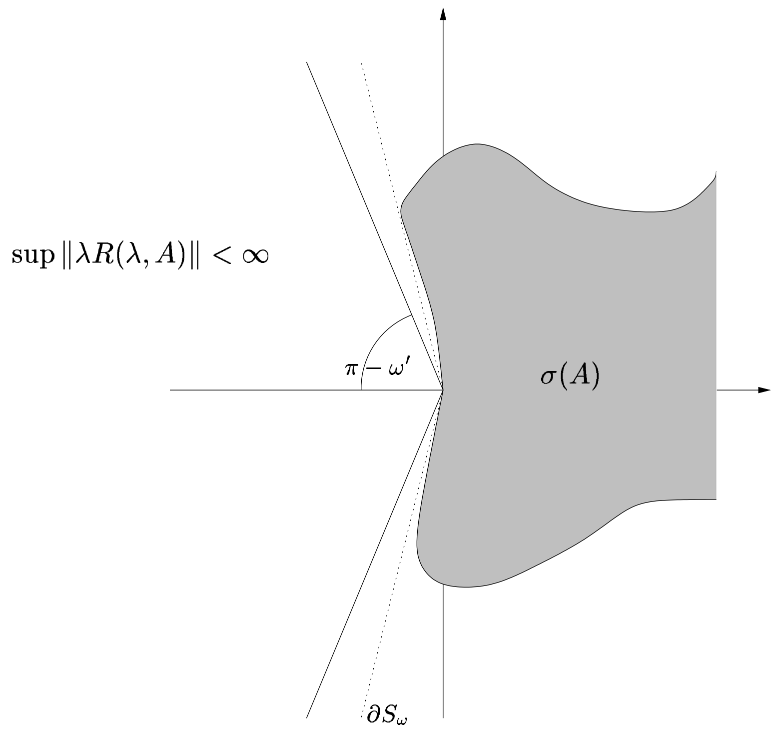
In the following, X always denotes a (nontrivial) Banach space and A a (single-valued linear) operator on X. (Note the "Agreement" on page 137 in Appendix A.)
For $0 < \omega \leq \pi$ let

$$S_\omega := \{z \in \mathbb{C} \mid z \neq 0 \,\text{and}\, |\arg z| < \omega\}$$

denote the open sector symmetric about the positive real axis with opening angle ω. To cover also the case $\omega = 0$ we define $S_0 := (0, \infty)$. An operator A is called **sectorial** of angle $\omega < \pi$ (in short: $A \in \operatorname{Sect}(\omega)$) if

1) $\sigma(A) \subset \overline{S_\omega}$ and
2) $M(A, \omega') := \sup\{\|\lambda R(\lambda, A)\| \mid \lambda \notin \overline{S_{\omega'}}\} < \infty$ for all $\omega < \omega' < \pi$.

The following picture illustrates this notion:



A family of operators $(A_\iota)_\iota$ is called **uniformly sectorial** of angle ω, if $A_\iota \in \mathrm{Sect}(\omega)$ for each ι, and $\sup_\iota M(A_\iota, \omega') < \infty$ for all $\omega < \omega' < \pi$. Finally, we call

$$\omega_A := \min\{0 \le \omega < \pi \mid A \in \mathrm{Sect}(\omega)\}$$

the **spectral angle** (or **sectoriality angle**) of A.

Proposition 1.1. *Let A be a closed operator on a Banach space X.*

a) *If $(-\infty, 0) \subset \varrho(A)$ and $M(A) := M(A, \pi) := \sup_{t>0} \left\|t(t+A)^{-1}\right\| < \infty$, then $M(A) \ge 1$ and $A \in \mathrm{Sect}(\pi - \arcsin(\frac{1}{M(A)}))$.*

Let $A \in \mathrm{Sect}(\omega)$ for some $0 \le \omega < \pi$.

b) *If A is injective, then $A^{-1} \in \mathrm{Sect}(\omega)$, and the* **fundamental identity**

$$\lambda \left(\lambda + A^{-1}\right)^{-1} = I - \frac{1}{\lambda}\left(\frac{1}{\lambda} + A\right)^{-1} \tag{1.1}$$

holds for all $0 \ne \lambda \in \mathbb{C}$. In particular, $M(A^{-1}, \omega') \le 1 + M(A, \omega')$ for all $\omega < \omega' \le \pi$.

c) *Let $n \in \mathbb{N}$ and $x \in X$. Then one has*

$$\begin{array}{lcl} x \in \overline{\mathcal{D}(A)} & \iff & \lim_{t\to\infty} t^n(t+A)^{-n}x = x, \ \textit{and} \\ x \in \overline{\mathcal{R}(A)} & \iff & \lim_{t\to 0} A^n(t+A)^{-n}x = x. \end{array}$$

d) *We have $\mathcal{N}(A) \cap \overline{\mathcal{R}(A)} = 0$. If $\overline{\mathcal{R}(A)} = X$, then A is injective.*

e) *The identity $\mathcal{N}(A^n) = \mathcal{N}(A)$ holds for all $n \in \mathbb{N}$.*

f) *The family of operators* $\{(A+\delta)(A+\varepsilon+\delta)^{-1} \mid \varepsilon > 0,\ \delta \geq 0\}$ *is uniformly sectorial of angle* ω. *The family* $(\varepsilon A)_{\varepsilon \geq 0}$ *is uniformly sectorial. In fact* $M(\varepsilon A, \omega') = M(A, \omega')$ *for all* $\omega < \omega' \leq \pi$ *and all* $\varepsilon > 0$.

g) *Let* $\varepsilon > 0$, $n, m \in \mathbb{N}$ *and* $x \in X$. *Then we have*

$$\left(A(A+\varepsilon)^{-1}\right)^n x \in \mathcal{D}(A^m) \quad \Longleftrightarrow \quad x \in \mathcal{D}(A^m).$$

h) *If the Banach space* X *is reflexive, one has* $\overline{\mathcal{D}(A)} = X$ *and*

$$X = \mathcal{N}(A) \oplus \overline{\mathcal{R}(A)}.$$

i) *If* $0 \neq \lambda \in \mathbb{C}$ *with* $|\arg \lambda| + \omega < \pi$, *then* $\lambda A \in \operatorname{Sect}(\omega + |\arg \lambda|)$ *and* $M(\lambda A, \omega') \leq M(A, \omega' - |\arg \lambda|)$ *for each* $|\arg \lambda| + \omega < \omega' < \pi$.

Proof. Ad *a)*. Set $M := M(A)$. For each $x \in D(A)$ and $\lambda > 0$ we have

$$\|x\| \leq \frac{1}{\lambda} M \|Ax\| + M \|x\|.$$

This implies $\|x\| \leq M \|x\|$ for all $x \in \mathcal{D}(A)$. If $M < 1$, then $\mathcal{D}(A) = 0$, but this is impossible, since $X \neq 0$ and $\varrho(A) \neq \emptyset$.
Let $\lambda_0 < 0$ and $M' > M$. For each $\mu \in \mathbb{C}$ with $|\mu - \lambda_0| \leq \frac{|\lambda_0|}{M'}$ we have $\mu \in \varrho(A)$ and $R(\mu, A) = \sum_n (\mu - \lambda_0)^n R(\lambda_0, A)^{n+1}$ (Neumann series, see Proposition A.7). Hence it follows that

$$\begin{aligned} |\mu| \, \|R(\mu, A)\| &\leq \frac{|\mu|}{|\lambda_0|} \sum_n \frac{|\mu - \lambda_0|^n}{|\lambda_0|^n} M^{n+1} \leq M \frac{|\mu|}{|\lambda_0|} \sum_n \left(\frac{M}{M'}\right)^n \\ &\leq \left(1 + \frac{|\mu - \lambda_0|}{|\lambda_0|}\right) \frac{M}{1 - \frac{M}{M'}} \leq \left(1 + \frac{1}{M'}\right) \frac{MM'}{M' - M} = \frac{(M'+1)M}{M' - M}. \end{aligned}$$

Now, if we choose $\omega' > \pi - \arcsin \frac{1}{M}$ and define $M' := \frac{1}{\sin(\pi - \omega')}$, then it is clear that $M' > M$. Choose $\mu \in \mathbb{C}$ such that $\pi \geq |\arg \mu| \geq \omega'$ and define $\lambda_0 := \operatorname{Re} \mu$. Then $\lambda_0 < 0$ and $|\mu - \lambda_0| \leq \frac{|\lambda_0|}{M'}$, whence $\|\mu R(\mu, A)\| \leq (M'+1)M/(M'-M)$.
Ad *b)*. The identity (1.1) is true for all operators and all $\lambda \neq 0$ (cf. Lemma A.5 in the Appendix). The statement follows.
Ad *c)*. Obviously, the reverse directions of the two biimplications hold. For $x \in \mathcal{D}(A)$ one uses the identity $x = t(t+A)^{-1}x + \frac{1}{t}[t(t+A)^{-1}]Ax$. After repeatedly inserting this identity into itself one arrives at

$$x = [t(t+A)^{-1}]^n x + \frac{1}{t} \sum_{k=1}^{n} [t(t+A)^{-1}]^k Ax.$$

This shows $\lim_{t \to \infty} [t(t+A)^{-1}]^n x = x$ for $x \in \mathcal{D}(A)$. By the uniform boundedness of the operators $([t(t+A)^{-1}]^n)_{t>0}$ this is eventually true for all $x \in \overline{\mathcal{D}(A)}$. The second implication is handled similarly.
Ad *d)*. This is an immediate consequence of *c)*.
Ad *e)*. Evidently, $\mathcal{N}(A) \subset \mathcal{N}(A^n)$. But if $x \in \mathcal{N}(A^n)$ and $n \geq 2$, then in particular $x \in \mathcal{D}(A^{n-1})$ and one has $0 = (t+A)^{-1}A^n x = A(t+A)^{-1}A^{n-1}x$ for all $t > 0$. Because of $A^{n-1}x \in \mathcal{R}(A)$ one can apply *c)* to obtain $A^{n-1}x = 0$. By repeating this argument one finally arrives at $Ax = 0$.
Ad *f)*. Define (for the moment) $A_\delta := A + \delta$, and let $\omega' > \omega$. From $\lambda \in S_{\pi - \omega'}$ it follows that $\lambda + \delta \in S_{\pi - \omega'}$, and because of

$$\lambda(\lambda + A_\delta)^{-1} = \frac{\lambda}{\delta + \lambda} (\delta + \lambda)(A + \delta + \lambda)^{-1}$$

the family $(A_\delta)_{\delta \geq 0}$ is uniformly sectorial with angle ω and $M(A_\delta) \leq M(A)$.
Choose $\varepsilon > 0$. Because $\lambda \in S_{\pi - \omega'}$ we have $\frac{\lambda}{1+\lambda}\varepsilon + \delta \in S_{\pi - \omega'}$. For these λ one obtains

$$\lambda + A_\delta(A_\delta + \varepsilon)^{-1} = (\lambda(A_\delta + \varepsilon) + A_\delta)(A_\delta + \varepsilon)^{-1} = (\lambda + 1)\left(A_\delta + \frac{\lambda}{1+\lambda}\varepsilon\right)(A_\delta + \varepsilon)^{-1}.$$

Hence the operator $\lambda + A_\delta(A_\delta + \varepsilon)^{-1}$ is invertible and

$$\begin{aligned}\lambda[\lambda + A_\delta(A_\delta+\varepsilon)^{-1}]^{-1} &= \frac{\lambda}{1+\lambda}(A_\delta+\varepsilon)[A_\delta + \frac{\lambda}{1+\lambda}\varepsilon]^{-1}\\ &= \frac{\lambda}{1+\lambda}(A_\delta + \frac{\lambda}{1+\lambda}\varepsilon + \frac{\varepsilon}{1+\lambda})[A_\delta + \frac{\lambda}{1+\lambda}\varepsilon]^{-1}\\ &= \frac{\lambda}{1+\lambda}I + \frac{1}{1+\lambda}\left[\frac{\lambda\varepsilon}{1+\lambda}\left(A_\delta + \frac{\lambda\varepsilon}{1+\lambda}\right)^{-1}\right].\end{aligned}$$

The uniform sectoriality of the family $(A_\delta)_\delta$ now gives the first statement of *f)*. The second is due to the fact, that sectors are invariant under dilations with positive factors.
Ad *g)*. We consider first the reverse implication. To prove it we select $x \in \mathcal{D}(A^m)$. Then $A(A+\varepsilon)^{-1}x = x - \varepsilon(A+\varepsilon)^{-1}x \in \mathcal{D}(A^m)$, and an iteration of this argument yields $(A(A+\varepsilon)^{-1})^n x \in \mathcal{D}(A^m)$. The proof of the direction $\Rightarrow$ can also be reduced to the case $n = 1$ which is proved by induction on m.
Ad *h)*. Let $x \in X$ and X be reflexive. The sequence $(n(n+A)^{-1}x)_{n\in\mathbb{N}}$ is bounded, hence it has a weakly convergent subsequence $n_k(n_k + A)^{-1}x \rightharpoonup y$. This means that $A(n_k + A)^{-1}x \rightharpoonup x - y$. Now, $(n_k + A)^{-1}x \to 0$ even strongly. Because the graph of A is closed and a linear subspace of $X \oplus X$, it is weakly closed, whence $x - y = 0$. But this means that x lies in the weak closure of $\mathcal{D}(A)$ which is the same as the strong closure, for $\mathcal{D}(A)$ is a subspace of X. Altogether it follows that $x \in \overline{\mathcal{D}(A)}$.
Let $x \in \mathcal{N}(A) \cap \overline{\mathcal{R}(A)}$. Then we have $0 = Ax = \lim_{t\to 0}(t+A)^{-1}Ax = \lim_{t\to 0} A(t+A)^{-1}x = x$ by *c)*. Therefore, the sum is direct. For arbitrary $x \in X$, by the reflexivity of X on can find a sequence $t_n \searrow 0$ and a y such that $t_n(t_n + A)^{-1}x \rightharpoonup y$. But we have $t_n A(t_n + A)^{-1}x \to 0$. The graph of A is weakly closed, hence $y \in \mathcal{N}(A)$. This implies that $A(t_n + A)^{-1}x \to x - y\,(w)$. Therefore, $x - y$ is in the weak closure of $\mathcal{R}(A)$ which is identical to the strong closure $\overline{\mathcal{R}(A)}$, for $\mathcal{R}(A)$ is a subspace of X. It follows that $x \in \mathcal{N}(A) + \overline{\mathcal{R}(A)}$.
Ad *i)*. If $\mu \notin \overline{S_{\omega'}}$, then clearly $\mu\lambda^{-1} \notin \overline{S_{\omega'-|\arg\lambda|}}$. Hence $\mu R(\mu, \lambda A) = (\mu\lambda^{-1})R(\mu\lambda^{-1}, A)$ is uniformly bounded by $M(A, \omega' - |\arg\lambda|)$ for such μ. ☐

Let $A \in \operatorname{Sect}(\omega)$ on the Banach space X. We define $Y := \overline{\mathcal{R}(A)}$ and denote by B the part of A in Y, i.e.,

$$\mathcal{D}(B) := \mathcal{D}(A) \cap \overline{\mathcal{R}(A)} \quad \text{with} \quad By := Ay \quad (y \in \mathcal{D}(B)).$$

It is easy to see that $B \in \operatorname{Sect}(\omega)$ on Y with $M(\omega', B) \le M(\omega', A)$ for all $\omega < \omega' \le \pi$. Moreover, B is injective. Therefore, we call B the **injective part** of A. The identity

$$\mathcal{D}(B^n) = \mathcal{D}(A^n) \cap Y \qquad (n \in \mathbb{N})$$

is easily proved by induction. Proposition 1.1 yields $\mathcal{N}(A) \cap \overline{\mathcal{R}(A)} = \mathcal{N}(A) \cap Y = 0$, and by a short argument one obtains

$$R(\lambda, A)(x \oplus y) = \frac{1}{\lambda}x \oplus R(\lambda, B)y$$

for all $x \in \mathcal{N}(A)$, $y \in Y$.

A uniformly sectorial sequence $(A_n)_{n\in\mathbb{N}}$ of angle ω is called a **sectorial approximation** on S_ω for the operator A, if

$$\lambda \in \varrho(A) \quad \text{and} \quad R(\lambda, A_n) \to R(\lambda, A) \text{ in } \mathcal{L}(X). \tag{1.2}$$

for some $\lambda \notin \overline{S_\omega}$. From Proposition A.18 it follows that in this case (1.2) is true for *all* $\lambda \notin \overline{S_\omega}$. Moreover, A itself is sectorial of angle ω.
If $(A_n)_n$ is a sectorial approximation for A on S_ω, we write $A_n \to A\ (S_\omega)$ and speak of **sectorial convergence**.

Proposition 1.2. *a) If $A_n \to A$ (S_ω) and all A_n as well as A are injective, then $A^{-1} \to A^{-1}$ (S_ω).*

b) If $A_n \to A$ (S_ω) and $A \in \mathcal{L}(X)$, then $A_n \in \mathcal{L}(X)$ for large n and $A_n \to A$ in norm.

c) If $A_n \to A$ (S_ω) and $0 \in \varrho(A)$, then $0 \in \varrho(A_n)$ for large n.

d) If $(A_n)_n \subset \mathcal{L}(X)$ is uniformly sectorial of angle ω and if $A_n \to A$ in norm, then $A_n \to A$ (S_ω).

e) If $A \in \mathrm{Sect}(S_\omega)$, then $(A+\varepsilon)_{\varepsilon>0}$ is a sectorial approximation for A on S_ω.

f) If $A \in \mathrm{Sect}(S_\omega)$, then $(A_\varepsilon)_{0<\varepsilon\le 1}$, where

$$A_\varepsilon := (A+\varepsilon)\,(1+\varepsilon A)^{-1},$$

is a sectorial approximation for A on S_ω.

Proof. Assertion *a)* follows from *b)* in Proposition 1.1. If $A_n \to A\,(S_\omega)$ and $A \in \mathcal{L}(X)$, then $(1+A_n)^{-1} \to (1+A)^{-1}$ in norm. But the set of bounded invertible operators on X is open with the inversion mapping being continuous, hence eventually $(1+A_n)^{-1}$ is invertible, and $(1+A_n) \to (1+A)$ in norm. This gives *b)*. To prove *c)* define $B_n := A_n^{-1}$ and $B := A^{-1}$. Then B_n and B are possibly multivalued operators. However, by the Spectral Mapping Theorem for the resolvent (Proposition A.12) and Lemma A.5 we have $(1+B)^{-1}, (1+B_n)^{-1} \in \mathcal{L}(X)$ and $(1+B_n)^{-1} \to (1+B)^{-1}$ in norm. With the same argument as in the proof of *b)* we conclude that $B_n \in \mathcal{L}(X)$ for large n. For *d)* suppose that $(A_n)_n$ is uniformly sectorial with $A_n \to A$ in norm. Then $(1+A_n) \to (1+A)$ in norm and $\sup_n \left\|(1+A_n)^{-1}\right\| < \infty$. This implies $(1+A)^{-1} \in \mathcal{L}(X)$ and $(1+A_n)^{-1} \to (1+A)^{-1}$ in norm. Finally, we prove *e)* and *f)*. Let A be sectorial. The uniform sectoriality of $(A+\varepsilon)_{\varepsilon>0}$ has been shown in the proof of part *f)* of Prop. 1.1. But it is clear that $(1+\varepsilon+A)^{-1} \to (1+A)^{-1}$ in norm, whence *e)* follows. Assertion *f)* results from Prop. 1.1, part *f)* and the identity $(A+\varepsilon)(1+\varepsilon A)^{-1} = \varepsilon^{-1}(A+\varepsilon)(A+\varepsilon+\frac{1-\varepsilon^2}{\varepsilon})^{-1}$. ☐

Remark 1.3. Although we have assumed throughout the section that A is single-valued, the *definition* of sectoriality makes perfect sense even if A is multivalued. Although we deal mostly with single-valued operators, sometimes it is quite illuminating to have the multivalued case in mind. Therefore, we will speak of a "multivalued sectorial operator" whenever it is convenient. Note that the fundamental identity (1.1) still holds in the multivalued case and implies readily that the inverse of a sectorial operator is sectorial with the same angle. One has $x \in \overline{\mathcal{D}(A)} \;\Leftrightarrow\; \lim_{t\to\infty} t(t+A)^{-1}x = x$ in the multivalued case as well (cf. *c)* of Prop. 1.1). This shows $A0 \cap \overline{\mathcal{D}(A)} = 0$. Most statements of this section remain true in the multivalued case, at least after adapting notation a little bit. As a rule, one has to replace expressions of the form $B(B+\lambda)$ by $I - \lambda(B+\lambda)^{-1}$. For example we obtain that $A_\varepsilon = \frac{1}{\varepsilon} - \left(\frac{1-\varepsilon^2}{\varepsilon}\right)(1+\varepsilon A)^{-1}$ is a sectorial approximation of A by bounded and invertible operators (cf. *f)* of Proposition 1.2).

§ 2 Spaces of Holomorphic Functions

In the next section we will construct a functional calculus for sectorial operators. This will be done by proceeding along the lines of the well-known Dunford calculus for bounded operators which can be found, e.g., in [Con90, Chapter VII, §4] or in [Rud91, Chapter 10]). So the starting point for the construction is a Cauchy integral. Since the spectrum of a sectorial operator in general is unbounded, one has to integrate along infinite lines (here: the boundary of a sector). As a matter of fact, this is only possible for a restricted collection of functions. Dealing with these functions requires some notation which we introduce in this section.

Let us write $\mathcal{O}(\Omega)$ to denote the space of all holomorphic functions on the open set $\Omega \subset \mathbb{C}$. Suppose A denotes a sectorial operator of angle ω on a Banach space X. We want to define operators of the form

$$f(A) := \frac{1}{2\pi i} \int_\Gamma f(z) R(z, A)\, dz$$

where $f \in \mathcal{O}(S_\varphi)$ ($\varphi > \omega$), and the path Γ "surrounds" the sector S_ω in the positive sense. This means in particular that — considered as a curve on the Riemann sphere — Γ passes through the point ∞. To give meaning to the above integral, the function f should have a quick decay at ∞.
Thus we call f **regularly decaying at** ∞, if $f(z) = O(|z|^\alpha)$ for $|z| \to \infty$ for some $\alpha < 0$. (Analogously, f is said to be **regularly decaying at** 0, if $f(z) = O(|z|^\alpha)$ for $|z| \to 0$ and some $\alpha > 0$.) By the sectoriality of A, the function f being regularly decaying at ∞ guarantees integrability at infinity, at least if Γ is eventually straight.
In 0 we have two possibilities for the time being. If f is holomorphic in 0, i.e., if f allows a holomorphic continuation to a neighborhood of 0, one can choose the path Γ in such a way that it avoids the point 0. If this is not possible, we have no choice but to demand that f is regular at 0.

It is therefore natural to consider the so-called **Dunford-Riesz class** on S_φ, defined by

$$\mathcal{DR}(S_\varphi) := \{f \in H^\infty(S_\varphi) \mid f \text{ is regularly decaying at } 0 \text{ and at } \infty\},$$

where

$$H^\infty(S_\varphi) := \{f \in \mathcal{O}(S_\varphi) \mid f \text{ is bounded}\}$$

is the Banach algebra of all bounded, holomorphic functions on S_φ. Obviously, $\mathcal{DR}(S_\varphi)$ is an algebra ideal in the algebra $H^\infty(S_\varphi)$. With each $f(z)$ also the function $f(1/z)$ is a member of $\mathcal{DR}(S_\varphi)$.[1]

Lemma 1.4. *Let $0 < \varphi \le \pi$ and let $f : S_\varphi \longrightarrow \mathbb{C}$ be holomorphic. The following assertions are equivalent:*

(i) *The function f belongs to $\mathcal{DR}(S_\varphi)$.*

(ii) *There is $C \ge 0$ and $s > 0$ such that $|f(z)| \le C \min(|z|^s, |z|^{-s})$ for all $z \in S_\varphi$.*

(iii) *There is $C \ge 0$ and $s > 0$ such that $|f(z)| \le C \frac{|z|^s}{1+|z|^{2s}}$ for all $z \in S_\varphi$.*

[1]We simply write z for the coordinate function on $\mathbb{C}$. Hence the symbols $f(z)$ and f are used interchangeably.

(iv) *There is* $C \geq 0$ *and* $s > 0$ *such that* $|f(z)| \leq C \left(\frac{|z|}{1+|z|^2}\right)^s$ *for all* $z \in S_\varphi$.

Proof. We omit the proof. □

Recalling the introductory remarks of this section, also the set

$$\mathcal{DR}_0(S_\varphi) := \{f \in H^\infty(S_\varphi) \mid \ f \text{ is holomorphic in } 0 \text{ and regularly decaying at } \infty\}$$

is of some importance. The next lemma gives a useful characterization.

Lemma 1.5. *Let* $0 < \varphi \leq \pi$ *and* $f : S_\varphi \longrightarrow \mathbb{C}$ *holomorphic. The following assertions are equivalent.*

(i) *The function* f *belongs to* $\mathcal{DR}(S_\varphi) + \mathcal{DR}_0(S_\varphi)$.

(ii) *The function* f *is bounded and has the following two properties.*

1) $f(z) = O(|z|^\alpha)$ $(z \to \infty)$ *for some* $\alpha < 0$, *and*
2) $f(z) - c = O(|z|^\beta)$ $(z \to 0)$ *for some* $\beta > 0$ *and some* $c \in \mathbb{C}$.

Proof. The implication $(i) \Rightarrow (ii)$ is clear. To prove the converse, pick $\beta > 0$ and $c \in \mathbb{C}$ such that $f(z) - c = O(|z|^\beta)$ for $z \to 0$. Without restriction we can assume that $\beta < 1$. Then

$$f(z) = \frac{c}{1+z} + \frac{f(z)-c}{1+z} + \frac{z}{1+z} f(z) \in \mathcal{DR}_0 + \mathcal{DR} + \mathcal{DR}$$

□

Note that the constant function **1** is *not* included in the algebra (!) $\mathcal{DR}(S_\varphi) + \mathcal{DR}_0(S_\varphi)$. Adding the space of constant functions, we obtain the algebra

$$\mathcal{DR}_{\text{ext}}(S_\varphi) := \mathcal{DR}(S_\varphi) + \mathcal{DR}_0(S_\varphi) + \mathbb{C}\mathbf{1},$$

called the **extended Dunford-Riesz class**.

Lemma 1.6. *Let* $0 < \varphi \leq \pi$ *and* $f : S_\varphi \longrightarrow \mathbb{C}$ *holomorphic. The following assertions are equivalent:*

(i) *The function* f *belongs to* $\mathcal{DR}_{\text{ext}}(S_\varphi)$.

(ii) *There is* $h \in \mathcal{DR}(S_\varphi)$ *and* $g, \tilde{g} \in \mathcal{DR}_0(S_\varphi)$ *such that*

$$f(z) = h(z) + g(z) + \tilde{g}(z^{-1}).$$

(iii) *The function* f *is bounded and has the following properties:*

1) $f(z) - d = O(|z|^\alpha)$ $(z \to \infty)$ *for some* $\alpha < 0$ *and some* $d \in \mathbb{C}$.
2) $f(z) - c = O(|z|^\beta)$ $(z \to 0)$ *for some* $\beta > 0$ *and some* $c \in \mathbb{C}$.

Proof. Assume (i). Then $f(z) = c + F(z) + G(z)$ for $c \in \mathbb{C}$, $F \in \mathcal{DR}$ and $G \in \mathcal{DR}_0$. Hence

$$f(z) = F(z) + (G(z) + H(z)) + H(z^{-1})$$

where $H(z) = c/(1+z)$. This proves (ii). The implication $(ii) \Rightarrow (iii)$ is clear, and $(iii) \Rightarrow (i)$ follows from Lemma 1.5. □

If follows from Lemma 1.6 that $\mathcal{DR}_{\text{ext}}(S_\varphi)$ is *invariant under inversion*, i.e., if $f(z) \in \mathcal{DR}_{\text{ext}}(S_\varphi)$, then also $f(z^{-1}) \in \mathcal{DR}_{\text{ext}}(S_\varphi)$.

If the precise sector S_φ is understood from the context, we will simply write $\mathcal{DR}$ instead of $\mathcal{DR}(S_\varphi)$ in the sequel. The same applies to the other function spaces $\mathcal{DR}_0, \mathcal{DR}_{\text{ext}}, H^\infty \dots$. Given $0 \le \omega < \pi$ we let

$$\mathcal{DR}[S_\omega] := \bigcup_{\omega<\varphi\le\pi} \mathcal{DR}(S_\varphi)$$

and use a similar notation for the other function spaces.

Let us end with a last convention. For $0 < \varphi \le \pi$ we define

$$\begin{aligned}\mathcal{O}_{\text{c}}(S_\varphi) := \{f \in \mathcal{O}(S_\varphi) \mid\ & f \text{ is bounded on } S_\varphi \cap \{r \le |z| \le R\} \\ & \text{for all } 0 < r < R < \infty\}\end{aligned}$$

to be the set of all holomorphic functions on S_φ which are bounded on proper annuli.

§ 3 The Natural Functional Calculus

In this section A always denotes a sectorial operator of angle ω on a Banach space X. We pursue our idea of defining a functional calculus by means of a Cauchy integral.

Functional Calculus by Cauchy-type Integrals.
Let $0 < \varphi < \pi$ und $\delta > 0$. We call $\Gamma_\varphi = \partial S_\varphi$ the boundary of the sector S_φ, oriented in the positive sense, i.e.,

$$\Gamma_\varphi := -\mathbb{R}_+ e^{i\varphi} \oplus \mathbb{R}_+ e^{-i\varphi}.$$

Besides, we denote by $\Gamma_{\varphi,\delta} := \partial(S_\varphi \cup B_\delta(0))$ the positively oriented boundary of $S_\varphi \cup B_\delta(0)$, i.e.,

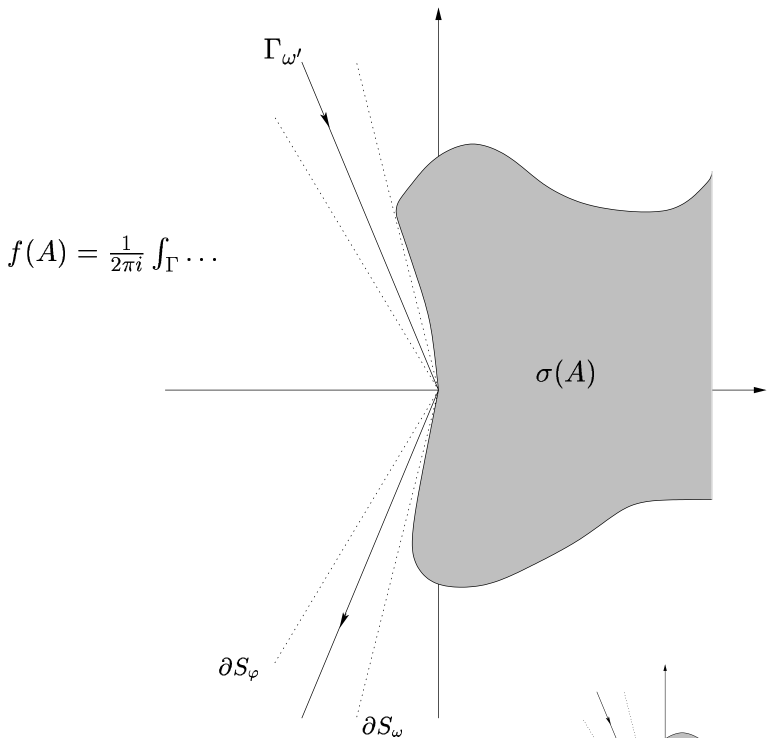
$$\Gamma_{\varphi,\delta} := -(\delta,\infty)e^{i\varphi} \oplus (\delta,\infty)e^{-i\varphi} \oplus \delta e^{i(\varphi,2\pi-\varphi)}.$$

For $\omega < \varphi < \pi$, $f \in \mathcal{DR}(S_\varphi)$, and $g \in \mathcal{DR}_0(S_\varphi)$ we can now define

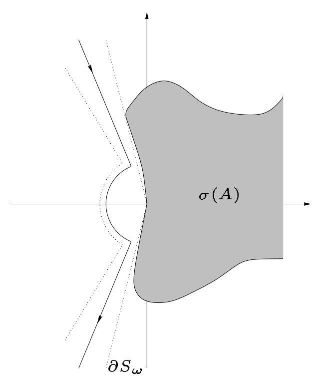
$$f(A) := \frac{1}{2\pi i}\int_{\Gamma_{\omega'}} f(z)\,R(z,A)\,dz \quad \text{and} \tag{1.3}$$

$$g(A) := \frac{1}{2\pi i}\int_{\Gamma_{\omega',\delta}} g(z)\,R(z,A)\,dz, \tag{1.4}$$

where $\omega < \omega' < \varphi$ and $\delta > 0$ are chosen in such a way that g allows a holomorphic continuation to a neighborhood of $\overline{B_\delta(0)}$. The following picture illustrates the definition of $f(A)$.



Similarly, the picture for $g(A)$ looks like this:



Let us answer the obvious questions. The definitions are independent of the auxiliary parameters ω' and δ. This is a standard argument using Cauchy's integral theorem. The two definitions agree for functions $f = g \in \mathcal{DR} \cap \mathcal{DR}_0$ in the intersection of both function spaces, for in this case one can let δ tend to zero in the integral (1.4) obtaining the expression (1.3) in the limit. Hence, by

$$h(A) := f(A) + g(A) \quad \text{for} \quad h = f + g \in \mathcal{DR}(S_\varphi) + \mathcal{DR}_0(S_\varphi)$$

a linear mapping

$$(h \longmapsto h(A)) : \mathcal{DR}(S_\varphi) + \mathcal{DR}_0(S_\varphi) \longrightarrow \mathcal{L}(X)$$

is defined. We summarize the basic properties of this mapping.

Proposition 1.7. *Let $A \in \mathrm{Sect}(\omega)$ and $\varphi > \omega$. Then the following assertions hold.*

a) If $x \in \mathcal{N}(A)$ and $f \in \mathcal{DR}(S_\varphi) + \mathcal{DR}_0(S_\varphi)$, then $f(A)x = f(0)x$.

b) The mapping $(h \mapsto h(A)) : \mathcal{DR}(S_\varphi) + \mathcal{DR}_0(S_\varphi) \longrightarrow \mathcal{L}(X)$ is a homomorphism of algebras.

c) For $\lambda \notin \overline{S_\varphi}$ the identity $R(\lambda, A) = \left(\frac{1}{\lambda - z}\right)(A)$ holds.

d) *We have* $(A-\nu)R(\lambda,A)R(\mu,A) = \left(\frac{(z-\nu)}{(\lambda-z)(\mu-z)}\right)(A)$ *for all* $\lambda,\mu \notin \overline{S_\varphi}$ *and* $\nu \in \mathbb{C}$.

e) *If* B *is a closed operator commuting with the resolvents of* A*, then* B *also commutes with* $f(A)$*. In particular,* $f(A)$ *commutes with* A.

f) *Let* B *denote the injective part of* A*. Then* $Y := \overline{\mathcal{R}(A)}$ *is invariant under the action of* $f(A)$*, and one has* $f(B) = f(A)|_Y$.

Proof. Ad *a)*. For $x \in \mathcal{N}(A)$ and $f \in \mathcal{DR} \cup \mathcal{DR}_0$ we have $\int_\Gamma f(z)R(z,A)x\,dz = \int_\Gamma f(z)/z\,dz\,x$. In case $f \in \mathcal{DR}$ one can approximate Γ by closed finite paths which are entirely contained in S_φ. The according integrals vanish by Cauchy's theorem. Analogously, in the case where $f \in \mathcal{DR}_0$ one writes Γ as a sum $\Gamma = \partial B_\delta(0) \oplus \Gamma_1$, with Γ_1 again is contained in the sector S_φ. Cauchy's integral formula then yields $\int_\Gamma f(z)/z\,dz = 2\pi i f(0)$.
Ad *b)*. The proof is a simple application of Fubini's theorem.
Ad *c)*. Let $f(z) := (\mu - z)^{-1}$. Then $f \in \mathcal{DR}_0(S_\varphi)$, hence

$$f(A) = \frac{1}{2\pi i}\int_{\Gamma_{\omega',\delta}} f(z)R(z,A)\,dz.$$

Now, choose $R > |\lambda|$ such that $|R\sin\omega'| > |\mathrm{Im}\,\lambda|$, and form the path $\Gamma := -\Gamma_{\omega',R}$. Due to Cauchy's integral formula one obtains $\int_{\Gamma_{\omega',\delta}} f(z)R(z,A)\,dz + \int_\Gamma f(z)R(z,A)\,dz = 2\pi i R(\mu,A)$. Using again Cauchy's theorem we see that $\int_{\Gamma-n} f(z)R(z,A)\,dz$ is independent of $n \in \mathbb{N}$. (The choice of R guarantees that always $\mu \notin \Gamma - n$.) Letting $n \to \infty$ yields $\int_{\Gamma-n} f(z)R(z,A)\,dz \to 0$. This proves the claim.
Ad *d)*. The assertion results from *b), c)*, and the identity $\frac{z-\nu}{(\mu-z)(\lambda-z)} = -\frac{1}{\mu-z} + \frac{\lambda-\nu}{(\mu-z)(\lambda-z)}$.
Ad *e)*. This is immediate.
Ad *f)*. The statement follows from the fact that Y is $R(\lambda,A)$-invariant, with $R(\lambda,A)|_Y = R(\lambda,B)$ for all $\lambda \in \varrho(A)$. ◻

In the next step we will extend this functional calculus to a wider class of functions. Of course we could immediately define $f(A)$ for $f \in \mathcal{DR}_{\mathrm{ext}}$ since $\mathbf{1} \notin \mathcal{DR} + \mathcal{DR}_0$. Then, Proposition 1.7 remains true. We do not elaborate on this since we will make a bigger jump in a moment.

The Natural Functional Calculus.
We keep the overall assumption on the operator A. The extension of the basic functional calculus described above should be as intuitive as possible. Since this cannot be done by a Cauchy-type integral any more[2] we have to use a trick. We define

$$\mathcal{A}(S_\varphi) := \{f : S_\varphi \longrightarrow \mathbb{C} \mid \exists n \in \mathbb{N} : \frac{f(z)}{(1+z)^n} \in \mathcal{DR}(S_\varphi) + \mathcal{DR}_0(S_\varphi)\},$$

and $\mathcal{A}[S_\omega] := \bigcup_{\varphi>\omega} \mathcal{A}(S_\varphi)$. If ω is understood, we write simply $\mathcal{A}$ instead of $\mathcal{A}[S_\omega]$.

Lemma 1.8. *Let* $f : S_\varphi \longrightarrow \mathbb{C}$ *be holomorphic. The following assertions are equivalent:*

(i) *The function* f *belongs to* $\mathcal{A}(S_\varphi)$.

[2]There cannot be, e.g., a Cauchy integral definition yielding the identity operator. For if $I = \frac{1}{2\pi i}\int_\Gamma f(z)R(z,A)\,dz$ for some f and some Γ, then $\mathcal{D}(A)$ must be dense which is not true in general.

(ii) The function f belongs to $\mathcal{O}_c(S_\varphi)$ and has the following two properties:

1) $f(z) = O(|z|^\alpha)$ $(z \to \infty)$ for some $\alpha \in \mathbb{R}$, and

2) $f(z) - c = O(|z|^\beta)$ $(z \to 0)$ for some $\beta > 0$ and some $c \in \mathbb{C}$.

(iii) There is $c \in \mathbb{C}$, $n \in \mathbb{N}$, and $F \in \mathcal{DR}$ such that $f(z) = c + (1+z)^n F(z)$.

If f is bounded, one can choose $n = 1$ in (iii).

In particular, $\mathcal{A}(S_\varphi)$ is an algebra of functions containing every rational function with poles outside of $\overline{S_\varphi}$.

Proof. *(i)⇒(ii)*: If $f(z)(1+z)^{-n} = F(z) + G(z)$, where $F \in \mathcal{DR}, G \in \mathcal{DR}_0$, then clearly condition *1)* is satisfied. The function $(1+z)^n G(z) - G(0)$ is holomorphic in 0, whence regularly decaying at 0. Hence, $f = (1+z)^n F(z) + ((1+z)^n G(z) - G(0)) + G(0)$ satisfies *2)* with $c = G(0)$.
(ii)⇒(iii): Choose α, β, c as in *(ii)*, and let $n > \alpha$. Then clearly $(f(z) - c)(1+z)^{-n} \in \mathcal{DR}$. If f is bounded, one can take $\alpha = 0$, hence $n = 1$ will do the job. The implication *(iii)⇒(i)* is trivial. □

For $f \in \mathcal{A}(S_\varphi)$ we now define

$$f(A) := (1+A)^n \left(\frac{f(z)}{(1+z)^n} \right)(A)$$

where n is such that $f(z)(1+z)^{-n} \in \mathcal{DR}(S_\varphi) + \mathcal{DR}_0(S_\varphi)$. By Proposition 1.7 this definition is independent of the special n. We call the mapping

$$(f \longmapsto f(A)) : \mathcal{A}[S_\omega] \longrightarrow \{\text{closed operators on } X\}$$

the **natural functional calculus** for A on S_ω. (For a discussion of this terminology, see §4.)

Proposition 1.9. *Let $f \in \mathcal{A}(S_\varphi)$. Then the following assertions hold.*

a) The operator $f(A)$ is closed.

b) If A is bounded, then also $f(A)$ is bounded. If in addition A is invertible, then the mapping $g \longmapsto g(A)$ is equal to the usual Dunford calculus.

c) If $T \in \mathcal{L}(X)$ commutes with A, it also commutes with $f(A)$. If $f(A) \in \mathcal{L}(X)$, then $f(A)$ commutes with A.

d) If $\overline{\mathcal{D}(A)} = X$ and $f(z)(1+z)^{-n} \in \mathcal{DR} + \mathcal{DR}_0$, then $D(A^n)$ is a core for $f(A)$.

e) Let also $g \in \mathcal{A}$. Then

$$f(A) + g(A) \subset (f+g)(A) \quad \text{and} \quad f(A)\,g(A) \subset (fg)(A).$$

Furthermore, $\mathcal{D}((fg)(A)) \cap \mathcal{D}(g(A)) = \mathcal{D}(f(A)g(A))$.

f) We have $(\mathbf{1})(A) = I$, $\left(\frac{f(z)}{\lambda - z}\right)(A) = f(A)R(\lambda, A)$ for all $\lambda \notin \overline{S_\varphi}$, and $((z-\mu)f(z))(A) = (A-\mu)f(A)$ for all $\mu \in \mathbb{C}$.

g) If A is injective and $f(z^{-1}) \in \mathcal{A}$, then the **inversion rule** *$f(A) = f(z^{-1})(A^{-1})$ holds.*

h) If $f, f^{-1} \in \mathcal{A}$, then $(f^{-1})(A) = f(A)^{-1}$; in particular, $f(A)$ is injective.

i) Let B denote the injective part of A on $Y := \overline{\mathcal{R}(A)}$. Then $\mathcal{D}(f(B)) = \mathcal{D}(f(A)) \cap Y$ with $f(B) = f(A)|_Y$. Hence the identity $f(A)(x \oplus y) = f(0)x \oplus f(B)y$ holds for all $x \in \mathcal{N}(A), y \in \overline{\mathcal{R}(A)}$.

Proof. We choose (once and for all) a natural number n such that $F(z) := f(z)(1+z)^{-n} \in \mathcal{DR} + DR_0$. Assertion *a)* follows from Lemma A.3, the operator $(1+A)^n$ being closed. Assertion *b)* is clear. To prove assertion *c)*, we compute

$$Tf(A) = T(1+A)^n F(A) \subset (1+A)^n TF(A) = (1+A)^n F(A)T = f(A)T$$

This is correct by *e)* of Prop. 1.7.
We prove *d)*. The operator $F(A)$ commutes with $(1+A)^{-n}$, hence $\mathcal{D}(A^n)$ is $F(A)$-invariant. This gives $\mathcal{D}(A^n) \subset \mathcal{D}(f(A))$. For arbitrary $x \in \mathcal{D}(f(A))$ we have $T_t(x) := (t(t+A)^{-1})^n x \to x$ with $t \to \infty$. As a matter of fact, $T_t(x) \in \mathcal{D}(A^n)$ and, by *c)*, also $f(A)T_t x = T_t f(A)x \to f(A)x$. To prove *e)*, we choose m such that $G(z) := g(z)(1+z)^{-m} \in \mathcal{DR} + \mathcal{DR}_0$. Without restriction we can assume $m = n$. Then we compute

$$\begin{aligned} f(A) + g(A) &= (1+A)^n F(A) + (1+A)^n G(A) \\ &\subset (1+A)^n (F(A) + G(A)) = (1+A)^n (F+G)(A) = (f+g)(A) \\ \text{and} \quad f(A)g(A) &= (1+A)^n f(A)(1+A)^n G(A) \\ &\subset (1+A)^n (1+A)^n F(A)G(A) = (1+A)^{2n}(FG)(A) = (fg)(A). \end{aligned}$$

Let $x \in \mathcal{D}(g(A)) \cap \mathcal{D}((fg)(A))$, i.e. $G(A)x \in \mathcal{D}(A^n)$ and $F(A)G(A)x = (FG)(A)x \in \mathcal{D}(A^{2n})$. From $F(A)(1+A)^n \subset (1+A)^n F(A)$ it follows that $F(A)g(A)x = F(A)(1+A)^n G(A)x = (1+A)^n (FG)(A)x \in \mathcal{D}(A^n)$. Hence, $x \in \mathcal{D}(f(A))$ and *e)* is proved.
Ad *f)*. We have $(\mathbf{1})(A) = (1+A)\left(\frac{1}{1+z}\right)(A) = (1+A)(1+A)^{-1} = I$, by *c)* of Prop.1.7. Again by *c)* of Prop.1.7 and *e)*, we know that $f(A)R(\lambda, A) \subset (f(z)(\lambda - z)^{-1})(A)$ with $\mathcal{D}((f(z)(\lambda - z)^{-1})(A) = \mathcal{D}(f(A)R(\lambda, A)) \cap \mathcal{D}(R(\lambda, A)) = \mathcal{D}(f(A)R(\lambda, A))$. Finally,

$$\begin{aligned} ((z-\mu)f(z))(A) &= (1+A)^{n+2}\left(\frac{f(z)(z-\mu)}{(1+z)^{n+2}}\right)(A) = (1+A)^{n+2}[(A-\mu)(1+A)^{-2}]F(A) \\ &= (A-\mu)(1+A)^{n+2}(1+A)^{-2}F(A) = (A-\mu)(1+A)^n F(A) \\ &= (A-\mu)f(A). \end{aligned}$$

Here we have used *d)* of Prop.1.7 and Lemma A.19.
To prove *g)* we first assume that $f \in \mathcal{DR}(S_\varphi)$. Then $f(z^{-1}) \in \mathcal{DR}$ again, and the identity $f(A) = f(z^{-1})(A^{-1})$ follows from a simple change of variable (namely $w := z^{-1}$) in the defining integral (1.3) with the help of the fundamental identity (1.1). In the general situation the hypothesis implies that f is bounded and has a limit d in ∞. We write $f(z^{-1}) = d + (1+z)G(z)$ with $G \in \mathcal{DR}$. A short computation yields $G(z^{-1}) = \frac{z}{(1+z)}(f(z) - d)$. Hence,

$$\begin{aligned} f(z^{-1})(A^{-1}) &= d + (1+A^{-1})G(A^{-1}) = d + (1+A^{-1})G(z^{-1})(A) \\ &= d + (1+A^{-1})\left(\left(\frac{z}{1+z}\right)(f(z) - d)\right)(A) \\ &= d + (1+A^{-1})A(1+A)^{-1}(f(A) - d) \\ &= d + (f(A) - d) = f(A), \end{aligned}$$

where we heavily used *e)* and *f)* and the fact that the claim is already proved for $G \in \mathcal{DR}$.
Ad *h)*. By *e)* and *f)* we know that $f(A)(f^{-1})(A) \subset I$ with $\mathcal{D}(I) \cap \mathcal{D}(f^{-1}(A)) = \mathcal{D}(f(A)(f^{-1})(A))$. This gives $f^{-1}(A) \subset f(A)^{-1}$ The reverse inclusion follows from symmetry.
The proof of *i)* is easy and will be omitted. □

Recall that for *any* operator A with nonempty resolvent set there is a definition of $r(A)$ where r is a rational function on $\mathbb{C}$ with all its poles outside of $\sigma(A)$ (see Section A.5 in the Appendix). Now, from *f)* of Proposition 1.9 we can see that this general definition agrees with the "new" definition of $r(A)$ through the $\mathcal{A}(S_\varphi)$-functional calculus, when A is sectorial of angle $\omega < \varphi$. We state this as a corollary.

Corollary 1.10. *Let $A \in \operatorname{Sect}(\omega)$ and r a rational function on $\mathbb{C}$ with all its poles outside of $\overline{S_\omega}$. Then the definitions of "$r(A)$" through the functional calculus and the general one given in Appendix A lead to the same operator.*

Further Properties.

In general, a law of the form "$f(A) + g(A) = (f + g)(A)$" will be false, but one can hope to have the equation for special choices of f and g. We define

$$H(A) := \{f \mid f \in \mathcal{A}[S_\omega],\ f(A) \in \mathcal{L}(X)\}.$$

The following corollary shows the usefulness of this definition.

Corollary 1.11. *Let $A \in \operatorname{Sect}(\omega)$. Then the following assertions hold.*

a) The identities

$$f(A) + g(A) = (f + g)(A) \quad \textit{and} \quad f(A)g(A) = (fg)(A)$$

hold for all $f \in \mathcal{A}$ and $g \in H(A)$. In particular, if $f(z) = c + (1 + z)^n F(z)$ for some $c \in \mathbb{C}, F \in \mathcal{DR}$, then $f(A) = c + (1 + A)^n F(A)$.

b) If $h \in \mathcal{A}$, $f \in H(A)$, and $f(A)$ is injective, we have

$$f(A)^{-1} h(A) f(A) = h(A)$$

either if $f^{-1} \in \mathcal{A}$ or if $\varrho(h(A)) \neq \emptyset$.

Proof. Ad *a)*. By *e)* of Prop.1.9, $f(A) + g(A) \subset (f + g)(A)$ and $(f + g)(A) + (-g)(A) \subset f(A)$. Hence we have $\mathcal{D}(f(A) + g(A)) = \mathcal{D}(f(A)) = \mathcal{D}((f + g)(A))$. The second identity follows again from *e)* of Prop.1.9. Note that $\mathcal{D}(g(A)) = X$ by assumption. The additional statement is now immediate.
We prove *b)*. By applying *c)* of Proposition 1.9 twice, we see that $f(A)$ commutes with $h(A)$. This yields $f(A)^{-1}h(A)f(A) \supset f(A)^{-1}f(A)h(A) = h(A)$. If $f^{-1} \in \mathcal{A}$ we can apply *e)* and *h)* of Proposition 1.9 to conclude $f(A)^{-1}h(A)f(A) = f^{-1}(A)h(A)f(A) \subset (f^{-1}hf)(A) = h(A)$. If $\varrho(h(A)) \neq \emptyset$ we can apply Proposition A.11. □

As a consequence we immediately obtain the following result.

Proposition 1.12. *Let A be a sectorial operator of angle ω. Then $H(A)$ is a subalgebra of $\mathcal{A}[S_\omega]$ containing all rational functions which are bounded on $\overline{S_\omega}$, The mapping*

$$(f \longmapsto f(A)) : H(A) \longrightarrow \mathcal{L}(X)$$

is a homomorphism of algebras.

Next, we will examine the behaviour of the natural functional calculus with respect to the composition of functions.

Corollary 1.13. *Let $g \in \mathcal{A}(S_\varphi)$ and $\mu \notin \overline{g(S_\varphi)}$. Then $(\mu - g(z))^{-1} \in \mathcal{A}$ and $\mu - g(A)$ is injective with*

$$\left(\frac{1}{\mu - g(z)}\right)(A) = (\mu - g(A))^{-1}.$$

In particular, $\mu \in \varrho(g(A))$ if and only if $(\mu - g(z))^{-1} \in H(A)$.

Proof. By Proposition 1.9, part *h)* we only have to show $(\mu - g(z))^{-1} \in \mathcal{A}$. But if n is chosen such that $(1+z)^{-n}(g(z) - g(0)) \in \mathcal{DR}$, we have

$$(1+z)^{-n}\left(\frac{1}{\mu - g(z)} - \frac{1}{\mu - g(0)}\right) = \frac{1}{\mu - g(0)} \frac{g(z) - g(0)}{(1+z)^n} \frac{1}{\mu - g(z)} \in \mathcal{DR},$$

since $(\mu - g(z))^{-1} \in H^\infty(S_\varphi)$. $\square$

Remark 1.14. Corollaries 1.11 and 1.13 were proved using only the *formal properties* of the natural functional calculus (and not its definition). So we will have the same results whenever a functional calculus with the same formal properties is at hand. This observation will prove useful in the sequel (see Corollary 1.18 below).

The Composition Rule.

We will now study the so-called **composition rule** by which we mean an identity of the form

$$(f \circ g)(A) = f(g(A)). \tag{1.5}$$

It is this identity which we consider the most fundamental, making the functional calculus "work". As a matter of fact, the rule as it stands does not make sense, unless we require some additional hypotheses. In short, these can be described in the sentence "*everything* should make sense". Basically, we require A to be sectorial and $g(A)$ and $(f \circ g)(A)$ be defined by the natural functional calculus for sectorial operators. Moreover, also $g(A)$ should be sectorial and $f(g(A))$ be defined by the natural functional calculus. We obtain the following result.

Proposition 1.15. *Let $0 \le \omega < \varphi < \pi$, $0 \le \omega' < \varphi' < \pi$, and $g \in \mathcal{A}(S_\varphi)$ such that $g(S_\varphi) \subset \overline{S_{\omega'}}$. Assume $A \in \mathrm{Sect}(\omega)$ and $g(A) \in \mathrm{Sect}(\omega')$. Then each $f \in \mathcal{A}(S_{\varphi'})$ with $f \circ g \in \mathcal{A}(S_\varphi)$ satisfies the composition rule*

$$(f \circ g)(A) = f(g(A)).$$

Proof. Without restriction one can assume $f(0) = 0$. Define $c := g(0)$. We choose n and m large enough such that

$$\frac{f(z)}{(1+z)^n} \in \mathcal{DR}(S_{\varphi'}), \ \frac{f(g(z)) - f(c)}{(1+z)^m} \in \mathcal{DR}(S_{\varphi'}), \quad \text{and} \quad \frac{g(z) - c}{(1+z)^m} \in \mathcal{DR}(S_\varphi).$$

Let $\Gamma' := \Gamma_{\omega_1'}$ where $\omega' < \omega_1' < \varphi'$ and $\Gamma := \Gamma_{\omega_1}$ where $\omega < \omega_1 < \varphi$. Then

$$\begin{aligned}(1+A)^{-m} f(g(A))(1+g(A))^{-n} &= (1+A)^{-m}\left(\frac{f(z)}{(1+z)^n}\right)(g(A)) \\ &= (1+A)^{-m} \frac{1}{2\pi i} \int_{\Gamma'} \frac{f(\lambda)}{(1+\lambda)^n} R(\lambda, g(A))\, d\lambda\end{aligned}$$

For $0 \ne \lambda \in \Gamma'$ the identity

$$R(\lambda, g(A)) = (\lambda - g(z))^{-1}(A) = \frac{1}{\lambda - c} + \left(\frac{g_1(z)}{(\lambda - c)(\lambda - g(z))}\right)(A)$$

holds by Corollaries 1.13 and 1.11, where we have written $g_1(z) := g(z) - c$. We conclude that

$$\begin{aligned}
&(1+A)^{-m}\frac{1}{2\pi i}\int_{\Gamma'}\frac{f(\lambda)}{(1+\lambda)^n}R(\lambda, g(A))\,d\lambda \\
&= (1+A)^{-m}\left[\frac{1}{2\pi i}\int_{\Gamma'}\frac{f(\lambda)}{(1+\lambda)^n}\frac{1}{\lambda-c}\,d\lambda + \frac{1}{2\pi i}\int_{\Gamma'}\frac{f(\lambda)}{(1+\lambda)^n}\left(\frac{g_1(z)}{(\lambda-c)(\lambda-g(z))}\right)(A)\,d\lambda\right] \\
&= (1+A)^{-m}\frac{f(c)}{(1+c)^n} + \frac{1}{2\pi i}\int_{\Gamma'}\frac{f(\lambda)}{(1+\lambda)^n}\left(\frac{g_1(z)}{(\lambda-c)(\lambda-g(z))(1+z)^m}\right)(A)\,d\lambda.
\end{aligned}$$

(We have used Cauchy's theorem to simplify the first summand.) Treating the second summand separately yields

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\Gamma'}\frac{f(\lambda)}{(1+\lambda)^n}\left(\frac{g_1(z)}{(\lambda-c)(\lambda-g(z))(1+z)^m}\right)(A)\,d\lambda \\
&= \frac{1}{2\pi i}\int_{\Gamma'}\frac{f(\lambda)}{(1+\lambda)^n}\frac{1}{2\pi i}\int_{\Gamma}\frac{g_1(z)}{(\lambda-c)(\lambda-g(z))(1+z)^m}R(z,A)\,dz\,d\lambda \\
&\overset{(1)}{=} \frac{1}{2\pi i}\int_{\Gamma}\left(\frac{1}{2\pi i}\int_{\Gamma'}\frac{f(\lambda)}{(1+\lambda)^n}\left[\frac{1}{\lambda-g(z)}-\frac{1}{\lambda-c}\right]d\lambda\right)\frac{1}{(1+z)^m}R(z,A)\,dz \\
&\overset{(2)}{=} \frac{1}{2\pi i}\int_{\Gamma}\left(\frac{f(g(z))}{(1+g(z))^n}-\frac{f(c)}{(1+c)^n}\right)\frac{1}{(1+z)^m}R(z,A)\,dz \\
&\overset{(3)}{=} \left(\frac{f(g(z))}{(1+g(z))^n(1+z)^m}-\frac{f(c)}{(1+c)^n(1+z)^m}\right)(A) \\
&= \left(\frac{f(g(z))}{(1+g(z))^n(1+z)^m}\right)(A) - (1+A)^{-m}\frac{f(c)}{(1+c)^n},
\end{aligned}$$

where we used Fubini's theorem in (1) and Cauchy's theorem in (2). (We postpone the justification for applying Fubini's theorem until the end of the proof.) Equality (3) holds because the function in brackets is in the Dunford-Riesz class:

$$\begin{aligned}
&\left(\frac{f(g(z))}{(1+g(z))^n}-\frac{f(c)}{(1+c)^n}\right)\frac{1}{(1+z)^m} \\
&= \frac{f(g(z))-f(c)}{(1+g(z))^n(1+z)^m} + f(c)\frac{(1+c)^n-(1+g(z))^n}{(1+g(z))^n(1+z)^m(1+c)^n} \\
&= \frac{f(g(z))-f(c)}{(1+g(z))^n(1+z)^m} + f(c)\frac{g_1(z)}{(1+z)^m}\left(\sum_{k=0}^{n-1}(1+c)^{k-n}(1+g(z))^{-(k+1)}\right) \in \mathcal{DR}\,.
\end{aligned}$$

(Recall that the function $(1+g(z))^{-1}$ is bounded.) Putting together what we know by now yields

$$\begin{aligned}
&(1+A)^{-m}f(g(A))(1+g(A))^{-n} \\
&= \left(\frac{f(g(z))}{(1+g(z))^n(1+z)^m}\right)(A) = \left(\frac{f(g(z))}{(1+z)^m}\right)(A)(1+g(A))^{-n}
\end{aligned}$$

(by another application of Corollaries 1.11 and 1.13). Multiplying both sides from the left with the operator $(1+g(A))^n(1+A)^m$ gives

$$f(g(A)) = (1+g(A))^n(f\circ g)(A)(1+g(A))^{-n}$$

Now we can apply part b) of Corollary 1.11 to conclude that $f(g(A)) = (f\circ g)(A)$.

Thus we are left to show that the application of Fubini's theorem was justified. In order to do this one has, after estimating the resolvent, to consider the function

$$F(\lambda, z) := \frac{f(\lambda)g_1(\lambda)}{(1+\lambda)^n(\lambda-c)(\lambda-g(z))(1+z)^m z}$$

and prove its product integrability. The representation

$$F(\lambda, z) = \left(\frac{f(\lambda)}{(1+\lambda)^n\lambda}\right)\left(\frac{\lambda}{(\lambda-c)(\lambda-g(z))}\right)\left(\frac{g_1(z)}{z(1+z)^m}\right)$$

shows that $c \neq 0$ is harmless, since $\lambda/(\lambda - g(z))$ is uniformly bounded because of the condition $g(z) \in \overline{S_{\omega'}}$ and $\lambda \in \Gamma'$. If $c = 0$, we have

$$\begin{aligned} F(\lambda, z) &= \left(\frac{f(\lambda)}{(1+\lambda)^n \lambda}\right)\left(\frac{1}{\lambda - g(z)}\right)\left(\frac{g(z)}{z(1+z)^m}\right) \\ &= \left(\frac{f(\lambda)}{(1+\lambda)^n \lambda^{1+\alpha}}\right)\left(\frac{\lambda^\alpha g(z)^{1-\alpha}}{\lambda - g(z)}\right)\left(\frac{g(z)^\alpha}{z(1+z)^m}\right). \end{aligned}$$

Here, $0 < \alpha < 1$ is chosen in such a way that the first factor remains integrable. Then, the middle term is still uniformly bounded, hence F is integrable. □

§4 Extensions According to Spectral Conditions

To define the basic functional calculus for functions in $\mathcal{DR}_{\text{ext}}$ the single-valuedness of the operator A is not really needed. In fact the Cauchy integrals (1.3) and (1.4) make sense even if A is multivalued, provided the necessary resolvent estimates hold. Moreover, one always has the inversion rule

$$f(z^{-1})(A) = f(A^{-1})$$

for $f \in \mathcal{DR}_{\text{ext}}$.

[Note that from Lemma 1.6 it follows that $f(z^{-1}) \in \mathcal{DR}_{\text{ext}}$ whenever $f \in \mathcal{DR}_{\text{ext}}$. Assume first that $f \in \mathcal{DR}_0$. Define $F(z) := f(z^{-1})$ and $c := f(0)$. Then

$$F(z) = c - \frac{c}{1+z} + \frac{F(z)}{1+z} + \frac{z}{1+z}(F(z) - c)$$

where the last two summand are contained in $\mathcal{DR}$. Using the inversion rule for functions in $\mathcal{DR}$ we obtain

$$\begin{aligned} &F(A) = c - c(1+A)^{-1} + \frac{f(z)}{1+z^{-1}}(A^{-1} + \frac{z^{-1}}{1+z^{-1}}(f(z) - c)(A^{-1}) \\ &= c[1 + (1+A)^{-1}] + (f(z) - \frac{c}{1+z})(A^{-1}) \\ &= c(1+A^{-1})^{-1} + f(A^{-1}) - c(1+A^{-1})^{-1} = f(A^{-1}). \end{aligned}$$

In general, f is of the form $f = c + F + G$ where $F \in \mathcal{DR}$ and $G \in \mathcal{DR}_0$. Since the inversion rule holds for either F and G, the claim follows.]

The situation changes when one wants to go over to the class $\mathcal{A}$. Here, single-valuedness was essential in a way[3]. However, single-valuedness is a spectral condition at ∞ (namely: $\infty \notin P\tilde{\sigma}(A)$). Hence, we see that "improving" the operator by a spectral condition at ∞ allowed us to extend the functional calculus to the class $\mathcal{A}$. Not surprisingly, the symmetry of the original situation vanished, i.e., the characterizing conditions for functions in $\mathcal{A}$ at 0 and at ∞ are not analogous.

We will now remedy this drawback by imposing the "dual" condition to single-valuedness. That is we require the operator to be *injective*. This will allow us to extend the functional calculus to a class $\mathcal{B}$ thereby restoring the symmetry the $\mathcal{A}$-calculus lacked. We will do this in some detail because the situation of injective sectorial operators is of special importance. Later on we will shortly

[3] This is due to the fact that the definition $f(A) := (1+A)^n(f(z)(1+z)^{-n})(A)$ should not depend on n and yield a single-valued operator. Discarding the second requirement one can in fact construct a functional calculus for multivalued operators, see [MSP00].

browse through the other possibilities of extending the natural functional calculus by imposing further spectral conditions.

The Case when A is Injective.
We now consider an injective sectorial operator $A \in \mathrm{Sect}(\omega)$ on the Banach space X. If X is reflexive, then automatically $\overline{\mathcal{D}(A)} = \overline{\mathcal{R}(A)} = X$. This implies already $\overline{\mathcal{D}(A^n) \cap \mathcal{R}(A^n)} = X$ for each $n \in \mathbb{N}$.
[Let $x \in X$. Then $A^n(t+A)^{-n}(1+tA)^{-n}x \in \mathcal{D}(A^n) \cap \mathcal{R}(A^n)$. Now, by Proposition 1.1, we have $\lim_{t\to 0} A^n(t+A)^{-n}(1+tA)^{-n}x = x$, if $x \in \overline{\mathcal{D}(A)} \cap \overline{\mathcal{R}(A)}$.]
We define $\tau(z) := z(1+z)^{-2} \in \mathcal{DR}$ and

$$\mathcal{B}(S_\varphi) := \{f : S_\varphi \longrightarrow \mathbb{C} \mid \text{there is } n \in \mathbb{N}\ \tau(z)^n f(z) \in \mathcal{DR}(S_\varphi)\},$$

where $0 < \varphi \leq \pi$ and $\mathcal{B} := \mathcal{B}[S_\omega] := \bigcup_{\varphi > \omega} \mathcal{B}(S_\varphi)$. Obviously, $\mathcal{B}(S_\varphi)$ is an algebra of functions which contains $\mathcal{A}(S_\varphi)$. A holomorphic function f on S_φ belongs to $\mathcal{B}$ if and only if f has at most polynomial growth at 0 and at ∞ (and is bounded in between). In particular,

$$H^\infty(S_\varphi) = \{f \mid f : S_\varphi \longrightarrow \mathbb{C}\ f \text{ is holomorphic and bounded}\}$$

is a subalgebra of $\mathcal{B}(S_\varphi)$. Note that with f also $f(z^{-1})$ belongs to $\mathcal{B}$.
To define $f(A)$ for $f \in \mathcal{B}$ we will use a similar trick as in the case of the $\mathcal{A}$-calculus. We let $\Lambda := \Lambda_A := \tau(A)^{-1} = (A(1+A)^{-2})^{-1}$ and note that

$$\Lambda_A = (1+A)^2 A^{-1} = (1+A)A^{-1}(1+A) = (2 + A + A^{-1}) = \Lambda_{A^{-1}}$$

with $\mathcal{D}(\Lambda) = \mathcal{D}(A) \cap \mathcal{R}(A)$. Now we define

$$f(A) := \Lambda^n (\tau^n f)(A)$$

for $f \in \mathcal{B}(S_\varphi)$, where n is chosen such that $\tau^n f = z^n f(z)/(1+z)^{2n} \in \mathcal{DR}$. By Proposition 1.9 this definition does not depend on n. Apart from that, if $f \in \mathcal{A}$ then also $f \in \mathcal{B}$ and both the "new" definition of $f(A)$ and the "old" one yield the same operator.
[Let $f = (1+z)^n F(z)$ with $F \in \mathcal{DR} + \mathcal{DR}_0$. Then $\tau^n f \in \mathcal{DR}$ and one can compute

$$\begin{aligned} f(A)_{new} &= \Lambda^n(\tau^n f)(A) = \Lambda^n \left(z^n(1+z)^{-n}F(z)\right)(A) \\ &= \Lambda^n (A(1+A)^{-1})^n F(A) = (1+A)^n F(A) = f(A)_{old}.] \end{aligned}$$

Also note that Λ_A^n is a closed operator since its inverse $(\Lambda_A^n)^{-1} = \tau(A)^n$ is bounded. Let us summarize the properties of the $\mathcal{B}$-calculus.

Proposition 1.16. *Let $f \in \mathcal{B}(S_\varphi)$. Then the following assertions hold.*

a) The operator $f(A)$ is closed.

b) If A is bounded and invertible then $f(A) \in \mathcal{L}(X)$, and the mapping $(g \longmapsto g(A))$ is equal to the usual Dunford calculus.

c) If $T \in \mathcal{L}(X)$ commutes with A, it also commutes with $f(A)$. If $f(A) \in \mathcal{L}(X)$, then $f(A)$ commutes with A.

d) If $\overline{\mathcal{D}(A)} = X = \overline{\mathcal{R}(A)}$ and $\tau^n f \in \mathcal{DR}$, then $D(A^n) \cap \mathcal{R}(A^n)$ is a core for $f(A)$.

e) *Let also* $g \in \mathcal{B}$. *Then*

$$f(A) + g(A) \subset (f+g)(A) \quad \textit{and} \quad f(A)\,g(A) \subset (fg)(A).$$

Furthermore, $\mathcal{D}((fg)(A)) \cap \mathcal{D}(g(A)) = \mathcal{D}(f(A)g(A))$.

f) *We have also* $f(z^{-1}) \in \mathcal{B}$, *and* A *satisfies the inversion rule* $f(A) = f(z^{-1})(A^{-1})$.

g) *If* $f, f^{-1} \in \mathcal{B}$, *then* $f(A)$ *is injective with* $(f^{-1})(A) = f(A)^{-1}$.

Proof. *a)* and *b)* are clear. The first assertion of *c)* follows from the fact that by $TA \subset AT$ also $TA^{-1} \subset A^{-1}T$, hence $T\Lambda \subset \Lambda T$ holds. If $f(A) \in \mathcal{L}(X)$, then the statement just proved implies that $f(A)$ commutes with the resolvents of A, whence $f(A)A \subset Af(A)$. The proofs of *d)* and *e)* are analogous to the ones of the corrseponding assertions in Proposition 1.9. To prove *f)* we first note that $\tau(z^{-1}) = \tau(z)$. If $F := \tau^n f \in \mathcal{DR}$, then $F(z^{-1}) = \tau(z)^n f(z^{-1})$. Since we already know that $F(A) = F(z^{-1})(A^{-1})$, it follows that

$$f(z^{-1})(A^{-1}) = \Lambda^n_{A^{-1}} F(z^{-1})(A^{-1}) = \Lambda^n_A F(A) = f(A)$$

The proof of *g)* again is completely analogous to the proof of the corresponding statement in Proposition 1.9. □

Remark 1.17. A law of the form $(zf(z))(A) = Af(A)$ can not hold for all $f \in \mathcal{B}$ (cf. part *f)* of Proposition 1.9). For, by *f)* of Proposition 1.16, we certainly have $(z^{-1})(A) = A^{-1}$. Therefore, $(zz^{-1})(A) = (\mathbf{1})(A) = I \neq AA^{-1} = A(z^{-1})(A)$ in general.

As mentioned in Remark 1.14, we immediately obtain the following corollary.

Corollary 1.18. *Let* A *be an injective sectorial operator with angle* ω *on the Banach space* X. *We define*

$$H(A) := \{f \mid f \in \mathcal{B}[S_\omega],\ f(A) \in \mathcal{L}(X)\}.$$

The following assertions hold.

a) *The identities*

$$(f+g)(A) = f(A) + g(A) \quad \textit{and} \quad (fg)(A) = f(A)g(A).$$

hold for all $f \in \mathcal{B}$ *and* $g \in H(A)$.

b) *The set* $H(A)$ *is a subalgebra of* $\mathcal{B}$ *and the mapping*

$$(f \longmapsto f(A)) : H(A) \longrightarrow \mathcal{L}(X)$$

is a homomorphism of algebras.

c) *If* $h \in \mathcal{B}$, $f \in H(A)$ *and* $f(A)$ *is injective, then* $f(A)^{-1}h(A)f(A) = h(A)$ *in case that either* $\varrho(h(A)) \neq \emptyset$ *or* $f^{-1} \in \mathcal{B}$.

d) *If* $g \in \mathcal{B}(S_\varphi)$ *and* $\mu \notin \overline{g(S_\varphi)}$ *then* $\mu - g(A)$ *is injective with* $(\mu - g(A))^{-1} = (\mu - g(z))^{-1}(A)$. *In particular,* $\mu \in \varrho(g(A))$ *if and only if* $(\mu - g(z))^{-1} \in H(A)$.

One could complain about using the symbol "$H(A)$" again, since it is already defined in §3. This is a slight abuse of notation which will not cause any trouble. If A is not injective we use the meaning given to "$H(A)$" in § 3. If A is injective we use the one given here.

Recall that for *any* injective operator A there is a definition of $p(A)$ where $p \in \mathbb{C}[z, z^{-1}]$ is a polynomial in the variables z and z^{-1} (see Section A.6). The following proposition shows that we meet this general definition with our $\mathcal{B}$-calculus.

Proposition 1.19. *Let $A \in \operatorname{Sect}(\omega)$ be injective and $p = \sum_{k\in\mathbb{Z}} a_k z^k \in \mathbb{C}[z, z^{-1}]$ a polynomial. Then $p \in \mathcal{B}$ and $p(A) = \sum_{k\in\mathbb{Z}} a_k A^k$.*

Proof. We can assume that $p \notin \mathbb{C}[z]$. Hence we can write $p(z) = \sum_{k=-n}^{m} a_k z^k$ where $a_{-n} \neq 0$ and $n \geq 1$. It follows almost immediately from the definition that $p(A) = \Lambda^n(\tau^n p)(A)$. Now,

$$
\begin{aligned}
\Lambda^n(\tau^n p)(A) &= \Lambda^n \left((\sum_{k=0}^{m+n} a_{k-n} z^k)(1+z)^{-2n} \right)(A) = \Lambda^n \left(\sum_{k=0}^{m+n} a_{k-n} z^k \right)(A)(1+A)^{-2n} \\
&\overset{!}{=} (1+A)^{2n} A^{-n} (\sum_{k=0}^{m+n} a_{k-n} A^k)(1+A)^{-2n} \\
&\subset (1+A)^{2n} (\sum_{k=0}^{m+n} a_{k-n} A^k) A^{-n} (1+A)^{-2n} \\
&= (\sum_{k=0}^{m+n} a_{k-n} A^k)(1+A)^{2n}(1+A)^{-2n} A^{-n} = (\sum_{k=0}^{m+n} a_{k-n} A^k) A^{-n} \\
&\overset{!}{=} \sum_{k=-n}^{m} a_k A^k = \sum_{k=-n}^{m} (a_k z^k)(A) \subset (\sum_{k=-n}^{m} a_k z^k)(A) = p(A)
\end{aligned}
$$

Here we have used Lemma A.24, Lemma A.23, Lemma A.19, and Corollary 1.11. □

In the following we will be concerned with the composition rule $(f \circ g)(A) = f(g(A))$. For the sake of completeness we incorporate the statement of Proposition 1.15 into the formulation of the next proposition.

Proposition 1.20. *Let $0 \leq \omega < \varphi \leq \pi$, $0 \leq \omega' < \varphi' \leq \pi$, and $g : S_\varphi \longrightarrow \mathbb{C}$ such that $g(S_\varphi) \subset \overline{S_{\omega'}}$ Assume that $A \in \operatorname{Sect}(\omega)$. Then the composition rule*

$$f(g(A)) = (f \circ g)(A)$$

holds in each of the following cases:

1) $g \in \mathcal{A}(S_\varphi)$, $f \in \mathcal{A}(S_{\varphi'})$, $f \circ g \in \mathcal{A}(S_\varphi)$.
2) $g \in \mathcal{A}(S_\varphi)$, *$g(A)$ injective,* $f \in \mathcal{B}(S_{\varphi'})$, $f \circ g \in \mathcal{A}(S_\varphi)$.
3) *A injective,* $g \in \mathcal{B}(S_\varphi)$, $f \in \mathcal{A}(S_{\varphi'})$, $f \circ g \in \mathcal{B}(S_\varphi)$.
4) *A injective,* $g \in \mathcal{B}(S_\varphi)$, *$g(A)$ injective,* $f \in \mathcal{B}(S_{\varphi'})$, $f \circ g \in \mathcal{B}(S_\varphi)$.

In this it is always assumed that $g(A) \in \operatorname{Sect}(\omega')$.

Proof. Assertion *1)* was proved in Proposition 1.15. One can reduce *2)* to *1)*. In fact, choose n such that $\tau^n f \in \mathcal{DR}$. By definition and *1)* we obtain

$$\begin{aligned} f(g(A)) &= \Lambda^n_{g(A)} \left(\frac{f(z) z^n}{(1+z)^{2n}} \right) (g(A)) = \Lambda^n_{g(A)} \left(\frac{(f \circ g) g^n}{(1+g)^{2n}} \right) (A) \\ &= \Lambda^n_{g(A)} (1+A)^m \left(\frac{(f \circ g) g^n}{(1+z)^m (1+g)^{2n}} \right) (A) \\ &= \Lambda^n_{g(A)} (1+A)^m \Lambda^{-n}_{g(A)} \left(\frac{(f \circ g}{(1+z)^m} \right) (A) \\ &\overset{(1)}{=} (1+A)^m \left(\frac{f \circ g}{(1+z)^m} \right) (A) = (f \circ g)(A) \end{aligned}$$

where in (1) we have used Corollary 1.18 and the fact that $\Lambda^{-n}_{g(A)} = (g/(1+g)^2)^n(A)$ with $(g/(1+g)^2)^n \in H(A)$.
In the same manner one can reduce *4)* to *3)*.
We prove *3)*. The proof is similar to the proof of *1)*. One can assume $f(0) = 0$ and choose n, m large enough in order to have $f/(1+z)^n \in \mathcal{DR}$ and $(f \circ g)\tau^m \in \mathcal{DR}$. The paths Γ and Γ' surround the sectors S_ω and $S_{\omega'}$, respectively. Then we have

$$\begin{aligned} f(g(A)) &= (1+g(A))^n \left(\frac{f}{(1+z)^n} \right) (g(A)) \\ &= (1+g(A))^n \Lambda^m_A \Lambda^{-m}_A \frac{1}{2\pi i} \int_{\Gamma'} \frac{f(\lambda)}{(1+\lambda)^n} R(\lambda, g(A))\, d\lambda \\ &= (1+g(A))^n \Lambda^m_A \frac{1}{2\pi i} \int_{\Gamma'} \frac{f(\lambda)}{(1+\lambda)^n} \Lambda^{-m}_A R(\lambda, g(A))\, d\lambda \\ &= (1+g(A))^n \Lambda^m_A \frac{1}{2\pi i} \int_{\Gamma'} \frac{f(\lambda)}{(1+\lambda)^n} \left(\frac{z^m}{(1+z)^{2m}(\lambda - g(z))} \right) (A)\, d\lambda \\ &= (1+g(A))^n \Lambda^m_A \frac{1}{(2\pi i)^2} \int_{\Gamma'} \int_{\Gamma} \frac{f(\lambda)}{(1+\lambda)^n} \frac{z^m}{(1+z)^{2m}(\lambda - g(z))} R(z, A)\, dz\, d\lambda \\ &\overset{(1)}{=} (1+g(A))^n \Lambda^m_A \frac{1}{2\pi i} \int_{\Gamma} \frac{f(g(z))\ z^m}{(1+g(z))^n (1+z)^{2m}} R(z, A)\, dz \\ &= (1+g(A))^n \Lambda^m_A \left(\frac{f(g(z))\ z^m}{(1+g(z))^n (1+z)^{2m}} \right) (A) \\ &\overset{(2)}{=} (1+g(A))^n \Lambda^m_A (1+g(A))^{-n} \left(\frac{f(g(z))\ z^m}{(1+z)^{2m}} \right) (A) \\ &\overset{(3)}{=} \Lambda^m_A \left(\frac{f(g(z))\ z^m}{(1+z)^{2m}} \right) (A) = (f \circ g)(A). \end{aligned}$$

Equality (1) is an application of Cauchy's integral theorem. Before, one has to interchange the order of integration, and this can be justified in a way similar to the analogous situation in the proof of Proposition 1.15. Finally, Equalities (2) and (3) come from Corollary 1.18. □

The Case when $\mathbf{0} \in \boldsymbol{\rho}(\mathbf{A})$.

If one assumes not only injectivity but invertibility of the operator A, one can extend the functional calculus to a class of functions for which no growth condition at 0 has to be required. Define

$$\mathcal{C}(S_\varphi) := \{ f \in \mathcal{O}_c(S_\varphi) \mid f = O(|z|^\alpha)\ (z \to \infty) \text{ for some } \alpha > 0 \}$$

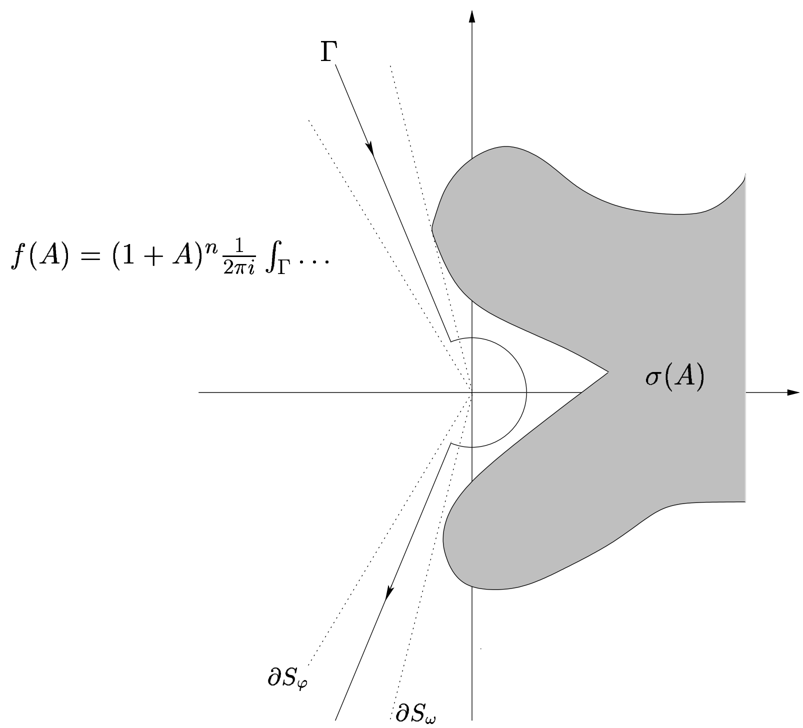
and, as usual, $\mathcal{C} := \mathcal{C}[S_\omega] := \bigcup_{\varphi > \omega} \mathcal{C}(S_\varphi)$. For $A \in \mathrm{Sect}(\omega)$ satisfying $0 \in \varrho(A)$ and $f \in \mathcal{C}(S_\varphi)$ $(\varphi > \omega)$ we define

$$f(A) := (1+A)^n \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{(1+z)^n} R(z, A)\, dz$$

where n is such that $f(z) = O(|z|^\alpha)$ $(z \to \infty)$ and $\alpha < n$. The path Γ surrounds the set

$$S_{\omega'}(\varepsilon', \infty) := \{z \in \mathbb{C} \mid |\arg z| < \omega',\ |z| > \varepsilon'\}$$

(where $\omega < \omega' < \varphi$) in the positive sense. Here, $\varepsilon > \varepsilon' > 0$ is small enough in order to have $\sigma(A) \subset \overline{S_\omega(\varepsilon, \infty)}$. Thus, the situation looks like this:



As usual, Cauchy's theorem shows that the definition is independent of the choices of ω' and ε'. The identities

$$\frac{1}{1+z} R(z, A) = (1 + A)^{-1} R(z, A) - \frac{1}{1+z} (1 + A)^{-1}, \tag{1.6}$$

$$A^{-1} z R(z, A) = -A^{-1} + R(z, A) \tag{1.7}$$

together with Cauchy's theorem show the definition also to be independent of the chosen n and to be coherent with the previously defined functional calculus for injective sectorial operators.
As a matter of fact, one can prove in a similar way a theorem which is analogous to Propositions 1.9 and 1.16. One defines

$$H(A) := \{f \mid f \in \mathcal{C}[S_\omega],\ f(A) \in \mathcal{L}(X)\}$$

and obtains a result analogous to Corollary 1.18. The composition rule $f(g(A)) = (f \circ g)(A)$ is now to prove in the following five cases.

1) A arbitrary, $g \in \mathcal{A}$, $0 \in \varrho(g(A))$, $f \in \mathcal{C}$, $f \circ g \in \mathcal{A}$.

2) A injective, $g \in \mathcal{B}$, $0 \in \varrho(g(A))$, $f \in \mathcal{C}$, $f \circ g \in \mathcal{B}$.
3) $0 \in \varrho(A)$, $g \in \mathcal{C}$, $f \in \mathcal{A}$, $f \circ g \in \mathcal{C}$.
4) $0 \in \varrho(A)$, $g \in \mathcal{C}$, $g(A)$ injective, $f \in \mathcal{B}$, $f \circ g \in \mathcal{C}$.
5) $0 \in \varrho(A)$, $g \in \mathcal{C}$, $0 \in \varrho(g(A))$, $f \in \mathcal{C}$, $f \circ g \in \mathcal{C}$.

Of course, the proofs are analogous to the ones given for Proposition 1.15 and Proposition 1.20. Note, that the rule $f(A) = f(z^{-1})(A^{-1})$ is an instance of the composition rule.

The Remaining Cases.
We now want to complete the picture according to the following table:

A sectorial &	A multivalued	A single-valued	$A \in \mathcal{L}(X)$
A^{-1} multivalued	$\mathcal{DR}_{\mathrm{ext}}$	$\mathcal{A}$	
A injective		$\mathcal{B}$	
$0 \in \varrho(A)$		$\mathcal{C}$	

Define $\tilde{\mathcal{A}} := \{f(z^{-1}) \mid f \in \mathcal{A}\}$ and $\tilde{\mathcal{C}} := \{f(z^{-1}) \mid f \in \mathcal{C}\}$. Guided by the inversion rule $f(A) = f(z^{-1})(A^{-1})$ we can define $f(A)$ in the cases

1) A injective (but possibly multivalued), $f \in \tilde{\mathcal{A}}$, and
2) A injective and bounded, $f \in \tilde{\mathcal{C}}$.

This gives consistent extensions and we arrive at the following situation.

A sectorial &	A multivalued	A single-valued	$A \in \mathcal{L}(X)$
A^{-1} multivalued	$\mathcal{DR}_{\mathrm{ext}}$	$\mathcal{A}$	
A injective	$\tilde{\mathcal{A}}$	$\mathcal{B}$	$\tilde{\mathcal{C}}$
$0 \in \varrho(A)$		$\mathcal{C}$	

In the case $A \in \mathcal{L}(X)$ we can clearly drop any growth condition at ∞ for the functions under consideration. However, if A is not injective, we still have to impose a growth condition at 0. So we define

$$\mathcal{D}(S_\varphi) := \{f \in \mathcal{O}_{\mathrm{c}}(S_\varphi) \mid \; f - c \text{ is regularly decaying at } 0 \text{ for some } c \in \mathbb{C}\}$$

For $f \in \mathcal{D}(S_\varphi)$ we can define $f(A)$ by

$$f(A) := c + \frac{1}{2\pi i} \int_\Gamma (f(z) - c) R(z, A)\, dz$$

where c is such that $f - c$ is regularly decaying at 0, and Γ surrounds the bounded sector

$$S_{\omega'}(0, R) := \{z \in \mathbb{C} \mid \; |\arg z| < \omega', \; |z| < R\}$$

(with $\omega < \omega' < \varphi$, $r(A) < R$) in the positive sense.
Clearly, this definition extends the $\mathcal{A}$-calculus. Letting $\tilde{\mathcal{D}} := \{f(z^{-1}) \mid f \in \mathcal{D}\}$ we define $f(A) := f(z^{-1})(A^{-1})$ for $f \in \tilde{\mathcal{D}}$ and $0 \in \varrho(A)$. This yields the almost complete mosaic:

A sectorial &	A multivalued	A single-valued	$A \in \mathcal{L}(X)$
A^{-1} multivalued	$\mathcal{DR}_{\mathrm{ext}}$	$\mathcal{A}$	$\mathcal{D}$
A injective	$\tilde{\mathcal{A}}$	$\mathcal{B}$	$\tilde{\mathcal{C}}$
$0 \in \varrho(A)$	$\tilde{\mathcal{D}}$	$\mathcal{C}$	

Of course, the inclusions $\tilde{\mathcal{D}} \subset \mathcal{C}$ and $\mathcal{D} \subset \tilde{\mathcal{C}}$ give rise to a pair of dual compatibility questions.

[Only one of the two dual situations has to be checked. If we deal with $A \in \mathcal{L}(X)$ injective and $f \in \mathcal{D}$, we have to prove the identity

$$c + \frac{1}{2\pi i}\int_{\Gamma_1}(f(z)-c)R(z,A)\,dz = (1+A^{-1})\frac{1}{2\pi i}\int_{\Gamma_2}\frac{f(w^{-1})}{1+w}R(w,A^{-1})\,dw,$$

where Γ_1, Γ_2 are suitable. One can assume $\Gamma_1 = \{z^{-1} \mid z \in \Gamma_2\}$. Hence the claim follows by a change of variables ($w = z^{-1}$) with the help of the fundamental identity (1.1).]

The case when $A \in \mathcal{L}(X), 0 \in \varrho(A)$ is the case of the usual Dunford calculus. We let

$$\mathcal{E}(S_\varphi) := \mathcal{O}_{\mathrm{c}}(S_\varphi)$$

and define $f(A)$ by the Cauchy integral

$$f(A) := \frac{1}{2\pi i}\int_\Gamma f(z)R(z,A)\,dz$$

where Γ surrounds the sector $S_{\omega'}(\varepsilon, R)$ in the positive sense. Here $\omega < \omega' < \varphi$, ε is small and R is large enough. Again we have to check compatibility and prove the inversion rule $f(A^{-1}) = f(z^{-1})(A)$ for $f \in \mathcal{E}$. We arrive at the complete picture

A sectorial &	A multivalued	A single-valued	$A \in \mathcal{L}(X)$
A^{-1} multivalued	$\mathcal{DR}_{\mathrm{ext}}$	$\mathcal{A}$	$\mathcal{D}$
A injective	$\tilde{\mathcal{A}}$	$\mathcal{B}$	$\tilde{\mathcal{C}}$
$0 \in \varrho(A)$	$\tilde{\mathcal{D}}$	$\mathcal{C}$	$\mathcal{E}$

of compatible functional calculi, satisfying the inversion rule. All combinations of these functional calculi give rise to an instance of the composition rule $(f \circ g)(A) = f(g(A))$. This is a total of 81 cases to check. With the help of the inversion rule this number is more or less halved. If one is only interested in single-valued operators, 36 cases remain. (Here the dual situations are sparse and the inversion rule does not help that much (saving: 6 cases). The $\mathcal{D}$-calculus and the $\mathcal{E}$-calculus only yield bounded operators, so there is another saving of 6 cases. There remain 24 cases to check.) The proofs are of course analogous to the proofs already given in Propositions 1.15 and 1.20, but we do not know if there is a single argument which covers all cases.

Our systematic introduction of functional calculi for sectorial operators has now come to an end. Depending on how "good" the operator A is in terms of certain spectral conditions we have given meaning to the symbol "$f(A)$" where "f" denotes a holomorphic function belonging to a certain function space. In each case we call the resulting mapping ($f \longmapsto f(A)$) simply the **natural functional calculus** for the sectorial operator A. For a given operator A and a given function f to say that "$f(A)$" *is defined by the natural functional calculus for sectorial operators* (in short: **NFCSO**) simply means that f satisfies the necessary regularity conditions in 0 and ∞ corresponding to the spectral properties of A according to the *complete picture* shown above.

§ 5 Boundedness and Approximation

We now will prove some results on boundedness and approximation which will be used in later chapters.

Sectorial Approximation.
We begin with the investigation of how the functional calculus behaves with respect to *sectorial approximation* (see Proposition 1.2 and the definitions before it).

Proposition 1.21. *Let $A \in \operatorname{Sect}(\omega)$ and $(A_n)_n$ a sectorial approximation of A on S_ω. We assume that the approximants are at least as "good" as A, in the sense of § 4. Then the following assertions hold.*

a) If $f \in \mathcal{DR}_{\text{ext}}$ then $f(A_n) \to f(A)$ in norm.

b) If $A \in \mathcal{L}(X)$ and $f \in \mathcal{D}$ then $f(A_n) \to f(A)$ in norm. Dually, if $0 \in \varrho(A)$ and $f \in \tilde{\mathcal{D}}$ then $f(A_n) \to f(A)$ in norm. The same conclusion holds in case $A, A^{-1} \in \mathcal{L}(X)$ and $f \in \mathcal{E}$.

c) If $f(A)$ is defined by the NFCSO, $f(A_n) \in \mathcal{L}(X)$ and $f(A_n) \to T \in \mathcal{L}(X)$ strongly, then $f(A) = T$.

Proof. The assertions *a)* and *b)* follow from the Dominated Convergence Theorem.
To prove *c)* we have to consider several cases. By "duality" we can restrict ourselves to the cases $f \in \mathcal{A}, \mathcal{B}, \mathcal{C}$. Because the proofs are all similar we only treat the case $f \in \mathcal{B}$. This means that we can find $m \in \mathbb{N}$ such that $\tau^m f \in \mathcal{DR}$. Hence $(\tau^m f)(A_n) \to (\tau^m f)(A)$ in norm. Since $f(A_n) \in \mathcal{L}(X)$, we have $(\Lambda_{A_n})^{-m} f(A_n) = (\tau^m f)(A_n)$. Sectorial convergence implies $\Lambda_{A_n}^{-m} \to \Lambda_A^{-m}$ in norm. This gives $\Lambda_A^{-m} T = (\tau^m f)(A)$, whence $T = f(A)$ follows. □

Boundedness.
Let $0 < \varphi < \pi$ and $f \in \mathcal{DR}_{\text{ext}}(S_\varphi)$. For each $0 < \omega' < \varphi$ we define

$$C(f, \omega') := \inf \ |c| + \frac{1}{2\pi} \int_{\Gamma_{\omega'}} |g(z)| \ \frac{|dz|}{|z|} + \frac{1}{2\pi} \int_{\Gamma_{\omega', \delta}} |h(z)| \ \frac{|dz|}{|z|}$$

where the infimum is taken over all representations $f = c + g + h$ with $c \in \mathbb{C}$, $g \in \mathcal{DR}$, $h \in \mathcal{DR}_0$ and $\delta > 0$ such that h is holomorphic on a neighbourhood of $\overline{B_\delta(0)}$. Each sectorial operator $A \in \operatorname{Sect}(\omega)$ satisfies the **fundamental estimate**

$$\|f(A)\| \le C(f, \omega') M(A, \omega'), \tag{1.8}$$

where $\omega < \omega' < \varphi$, as an elementary computation shows. This has an important consequence.

Proposition 1.22. *Let $A \in \operatorname{Sect}(\omega)$ and $0 \le \psi$ with $\psi + \omega < \pi$. Let $f \in \mathcal{DR}_{\text{ext}}(S_\varphi)$ where $\psi + \omega < \varphi < \pi$. Then we have*

$$\|f(\lambda A)\| \le M(A, \omega' - \psi)C(f, \omega')$$

for all $\psi + \omega < \omega' < \varphi$ and all $0 \ne \lambda \in \mathbb{C}$ such that $|\arg \lambda| \le \psi$. In particular we have $\sup_{t>0} \|f(tA)\| < \infty$.

Proof. Note that $\lambda A \in \operatorname{Sect}(\omega + |\arg \lambda|) \subset \operatorname{Sect}(\omega + \psi)$ by *i)* of Proposition 1.1. Now, the fundamental estimate (1.8) yields $\|f(\lambda A)\| \le M(\lambda A, \omega')C(f, \omega') \le M(A, \omega' - |\arg \lambda|)C(f, \omega') \le M(A, \omega' - \psi)C(f, \omega')$. □

Here is a similar result.

Lemma 1.23. *Let $(f_\alpha)_\alpha \subset \mathcal{DR}(S_\varphi)$, and assume there are $C, s, c_\alpha > 0$ such that*

$$|f_\alpha(z)| \le C \frac{|c_\alpha z|^s}{1 + |c_\alpha z|^{2s}} \tag{1.9}$$

for each α and each $z \in S_\varphi$. If $A \in \operatorname{Sect}(\omega)$ and $\omega < \omega' < \varphi$, then

$$\sup_\alpha \|f_\alpha(A)\| \le CM(A, \omega')c(s, \omega')$$

where $c(s, \omega') := \frac{1}{2\pi} \int_{\Gamma_{\omega'}} \frac{|z|^s}{1+|z|^{2s}} \frac{|dz|}{|z|}$

Proof. The claim follows from a simple change of variables in the integral, which makes the scalar c_α vanish. □

We will apply Lemma 1.23 in Proposition 1.29 below.

Approximation of Functions.

In the approximation results we want to prove, *pointwise* convergence of holomorphic functions is the usual hypothesis. Thereby we will be in need of the following fact, known as **Vitali's theorem**: Assume $\Omega \subset \mathbb{C}$ is open and connected, and $(f_\alpha)_\alpha$ is a locally bounded net of holomorphic functions on Ω. If the set $\{z \in \Omega \mid f_\alpha(z) \text{ converges}\}$ has a limit point in Ω, then f_α converges to a holomorphic function uniformly on compact subsets of Ω.

[For *sequences*, an elegant proof can be found in [ABHN01], Theorem A.5. However, one can easily modify the proof given there to obtain the result for nets.]

Lemma 1.24. *Let $A \in \operatorname{Sect}(\omega)$ and $\omega < \varphi \le \pi$. Let $(f_\alpha)_\alpha \subset \mathcal{DR}(S_\varphi)$ be a net of functions converging pointwise to a function f. Assume that there are $C, s > 0$ such that*

$$|f_\alpha(z)| \le C \frac{|z|^s}{1 + |z|^{2s}} \tag{1.10}$$

for each α and each $z \in S_\varphi$. Then $f \in \mathcal{DR}(S_\varphi)$ and $\|f_\alpha(A) - f(A)\| \to 0$.

Proof. Because of (1.10) the family $(f_\alpha)_\alpha$ is locally bounded. Vitali's theorem implies that f is holomorphic, hence $f \in \mathcal{DR}(S_\varphi)$. Moreover, $f_\alpha \to f$ uniformly on compact sets. Now, the claim follows from a version of the Dominated Convergence Theorem. □

The next result is an application of the foregoing.

Lemma 1.25. *Let $[a, b] \subset \mathbb{R}$, $F : [a, b] \times S_\varphi \longrightarrow \mathbb{C}$ continuous such that $F(t, .) : S_\varphi \longrightarrow \mathbb{C}$ is holomorphic for each $t \in [a, b]$. Assume there are $C, s > 0$ such that*

$$|F(t, z)| \leq C \frac{|z|^s}{1 + |z|^{2s}}$$

for all $t \in [a, b]$, $z \in S_\varphi$. We define $f(z) := \int_a^b F(t, z)\, dt$. Then the following assertions hold.

a) $f \in \mathcal{DR}(S_\varphi)$;

b) $(t \longmapsto F(t, A)) : [a, b] \longrightarrow \mathcal{L}(X)$ *is continuous.*

c) $f(A) = \int_a^b F(t, A)\, dt$.

Proof. Assertion *a)* is clear and *b)* is an immediate consequence of Lemma 1.24. Furthermore we have

$$\begin{aligned}(2\pi i) \int_a^b F(t, A) &= \int_a^b \int_\Gamma F(t, z)\, R(z, A)\, dz\, dt \\ &\overset{\text{Fub.}}{=} \int_\Gamma \left(\int_a^b F(t, z)\, dt \right) R(z, A)\, dz = \int_\Gamma f(z) R(z, A)\, dz \\ &= (2\pi i) f(A).\end{aligned}$$

This proves *c)*. □

The next result is known as the **Convergence Lemma**. Note that one has $\mathcal{D}(A) \subset \mathcal{D}(f(A))$ for all $f \in H^\infty(S_\varphi) \cap \mathcal{A}$, and $\mathcal{D}(A) \cap \mathcal{R}(A) \subset \mathcal{D}(f(A))$ for all $f \in H^\infty(S_\varphi)$ if A is injective.

[If $f \in H^\infty \cap \mathcal{A}$ we have $f(z)(1+z)^{-1} \in f(0)(1+z)^{-1} + \mathcal{DR}$. This gives $f(A)(1+A)^{-1} \in \mathcal{L}(X)$, whence $\mathcal{D}(A) \subset \mathcal{D}(f(A))$ follows. In the second case consider $f(z)z(1+z)^{-2}$.]

Proposition 1.26. *Let $A \in \mathrm{Sect}(\omega)$, $\omega < \varphi \leq \pi$, and $(f_\alpha)_\alpha \subset H^\infty(S_\varphi)$. Assume $\sup_\alpha \|f_\alpha\|_\infty < \infty$ and that the limit $f(z) := \lim_\alpha f_\alpha(z)$ exists pointwise on S_φ. Suppose that $f_\alpha(A)$ and $f(A)$ are defined by the NFCSO. Then*

$$f_\alpha(A)x \to f(A)x$$

for all $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$. Moreover the following assertions hold:

a) If A is injective, $f_\alpha(A) \in \mathcal{L}(X)$, and $f_\alpha(A) \to T \in \mathcal{L}(X)$ strongly, then $f(A) = T$.

b) If A is densely defined with dense range and $\sup_n \|f_\alpha(A)\| < \infty$, then $f(A) \in \mathcal{L}(X)$ and $f_\alpha(A) \to f(A)$ strongly.

Proof. Vitali's theorem implies that $f \in H^\infty(S_\varphi)$. Define $g(z) := f(z)z/(1+z)^2$ and $g_\alpha(z) := f_\alpha(z)z/(1+z)^2$. The net $(g_\alpha)_\alpha \subset \mathcal{DR}(S_\varphi)$ obviously satisfies the hypotheses of Lemma 1.24. Hence $f_\alpha(A)A(1+A)^{-2} = g_\alpha(A) \to g(A) = f(A)A(1+A)^{-2}$ in norm. So $f_\alpha(A)x \to f(A)x$ for all $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$. Assume that A is injective, $f_\alpha(A) \in \mathcal{L}(X)$, and $f_\alpha(A) \to T \in \mathcal{L}(X)$ strongly. Let $x \in X$. Then $g_\alpha(A)x =: y_\alpha \in \mathcal{D}(\Lambda_A)$ for each α and $y_\alpha \to y := g(A)x$. Now, $\Lambda_A y_\alpha = f_\alpha(A)x \to Tx$, by assumption. Since the operator Λ_A is closed, we have $y \in \mathcal{D}(\Lambda_A)$ and $\Lambda_A y = Tx$. But this means $x \in \mathcal{D}(f(A))$ with $f(A)x = Tx$. The proof of assertion *b)* is now trivial. □

Remark 1.27. One can wonder whether *a)* remains to be true if we drop the hypothesis that A is injective. (In this case we must of course assume $f_\alpha, f \in \mathcal{A}$.) Then we can still conclude that $f(A) \in \mathcal{L}(X)$. However, $T = f(A)$ is not true in general, but only $T = f(A)$ on $\overline{\mathcal{R}(A)}$ and $\mathcal{R}(T - f(A)) \subset \mathcal{N}(A)$.

[Take $x \in \mathcal{R}(A)$. Then $(1+A)^{-1}x \in \mathcal{D}(A) \cap \mathcal{R}(A)$. Hence, $y_\alpha := (f_\alpha(z)(1+z)^{-1})(A)x = f_\alpha(A)(1+A)^{-1}x \to f(A)(1+A)^{-1}x = (f(z)(1+z)^{-1})(A)x =: y$, by Proposition 1.26. Since $f_\alpha(A) \in \mathcal{L}(X)$, $y_\alpha \in \mathcal{D}(A)$ and $(1+A)y_\alpha = f_\alpha(A)x \to Tx$. Because $(1+A)$ is closed, we have $y \in \mathcal{D}(A)$ and $(1+A)y = Tx$. This means that $x \in \mathcal{D}(f(A))$ and $f(A)x = Tx$. Since always $\mathcal{D}(A) \subset \mathcal{D}(f(A))$ for $f \in \mathcal{A} \cap H^\infty$ and $X = \mathcal{D}(A) + \mathcal{R}(A)$ we obtain $X \subset \mathcal{D}(f(A))$, whence $f(A) \in \mathcal{L}(X)$. From above we see that $T = f(A)$ on $\mathcal{R}(A)$. Let $x \in \mathcal{D}(A)$. Then $f(A)x, Tx \in \mathcal{D}(A)$ as well and $A(f(A)x - Tx) = f(A)Ax - TAx = 0$.]

The above conclusion is in fact the best one can expect. Let, e.g., A be a sectorial operator on a reflexive space X such that $\mathcal{N}(A) \neq 0$ and let $P : X \longrightarrow \overline{\mathcal{R}(A)}$ the projection along $\mathcal{N}(A)$ (see *h)* of Proposition 1.1). With $f := \mathbf{1}$ and $f_n(z) := z(\frac{1}{n} + z)^{-1}$ we have $f_n(A) \to P$ strongly, but $f(A) = I \neq P$.

McIntosh's Approximation Technique.
Let $\psi \in \mathcal{DR}(S_\varphi)$ and define $\psi_t(z) := \psi(tz)$ for $z \in S_\varphi$ and $t > 0$. Furthermore, we define

$$\psi_{a,b}(z) := \int_a^b \psi(tz)\,\frac{dt}{t} \quad (0 < a < b < \infty) \quad \text{and} \quad \alpha := \int_0^\infty \psi(t)\,\frac{dt}{t}.$$

Finally, we define for $t > 0$ the function

$$\psi_t := (z \longmapsto \psi(tz)) \in \mathcal{DR}(S_\varphi)$$

Let us choose once and for all constants $C, s > 0$ such that

$$|\psi(z)| \leq C \frac{|z|^s}{1 + |z|^{2s}} \quad (z \in S_\varphi).$$

Lemma 1.28. *Let ψ, α, C, s as above. Then*

$$|\psi(tz)| \leq C \frac{|z|^s}{1 + |z|^{2s}} \max\{t^s, t^{-s}\}$$

for all $z \in S_\varphi$ and all $t > 0$. Furthermore, the following assertions hold.

a) $\psi_{a,b} \in H^\infty(S_\varphi)$ *for all* $0 < a < b < \infty$.

b) $\sup_{a,b} \|\psi_{a,b}\|_\varphi < \infty$.

c) $\psi_{a,b} \to \alpha\mathbf{1}$ *for* $(a,b) \to (0,\infty)$ *uniformly on compact subsets of* S_φ.

Proof. To prove the claimed inequality it suffices to show that

$$\frac{|tz|^s}{1 + |tz|^{2s}} \;\leq\; \frac{|z|^s}{1 + |z|^{2s}} \max\{t^s, t^{-s}\}$$

for all $z \in S_\varphi, t > 0$. This amounts to the fact that $\sup_{x>0} \frac{1+x}{t^{-s} + t^s x} \leq \max\{t^s, t^{-s}\}$ for all $t > 0$ which can be verified by elementary calculus. From the inequality just proved, assertion *a)* follows readily. Part *b)* follows from

$$|\psi_{a,b}(z)| \leq \int_a^b C \frac{|tz|^s}{1 + |tz|^{2s}} \frac{dt}{t} \leq C \int_0^\infty \frac{t^s}{1 + t^{2s}} \frac{dt}{t} = \frac{C}{s} \int_0^\infty \frac{dt}{1 + t^2} = \frac{C\pi}{2s}.$$

Now, *c)* is just an application of Vitali's theorem. ∎

Proposition 1.29. *Let ψ, α, C, s as above. Let $f \in H^\infty(S_\varphi)$ and $A \in \mathrm{Sect}(\omega)$ with $\omega < \varphi$. Then the following assertions are true:*

a) *The mapping $(t \longmapsto (f\psi_t)(A)) : (0,\infty) \longrightarrow \mathcal{L}(X)$ is continuous.*

b) $\psi_{a,b}(A) = \int_a^b \psi(tA)\,\frac{dt}{t}$ *for all $[a,b] \subset (0,\infty)$.*

c) $\lim_{a,b} \psi_{a,b}(A)x = \int_0^\infty \psi(tA)x\frac{dt}{t} = \alpha x$ *for all $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$.*

d) $\sup_{t>0} \|(f\psi_t)(A)\| \le \|f\|_\infty CM(A,\omega')c(s,\omega')$, *where $\omega < \omega' < \varphi$ is arbitrary and $c(s,\omega')$ is as in Lemma 1.23.*

e) $\sup_{t>0} \int_0^\infty \|\psi(tA)\theta(rA)\|\ \frac{dr}{r} < \infty$, *where $\theta \in \mathcal{DR}[S_\omega]$ is arbitrary.*

Proof. Choose $[a,b] \subset (0,\infty)$. Define $F(t,z) := f(z)\psi(tz)$ on $[a,b] \times S_\varphi$. Lemma 1.28 shows that F satisfies the hypothesis of Lemma 1.25. This proves *a)* and *b)*. Part *c)* is an easy consequence of Lemma 1.28 and the Convergence Lemma 1.26. To prove part *d)*, simply apply Lemma 1.23. This is possible since we have

$$|(f\psi_t)(z)| \le \|f\|_\varphi\, C\frac{|tz|^s}{1+|tz|^{2s}}$$

for all $t > 0,\ z \in S_\varphi$. Assertion *e)* is a little more involved. We can assume $\theta \in \mathcal{DR}(S_\varphi)$ without loss of generality. Choose $\omega < \omega' < \varphi$ and define $\Gamma := \Gamma_{\omega'}$.

$$\begin{aligned}\int_0^\infty \|\psi(tA)\theta(rA)\|\ \frac{dr}{r} &\lesssim \int_0^\infty \int_\Gamma |\psi(tz)\theta(rz)|\,\frac{|dz|}{|z|}\,\frac{dr}{r} = \int_0^\infty \int_\Gamma \left|\psi(z)\theta(rt^{-1}z)\right|\,\frac{|dz|}{|z|}\,\frac{dr}{r}\\ &\le \int_\Gamma \int_0^\infty \left|\theta(re^{i\arg z})\right|\,\frac{dr}{r}\,|\psi(z)|\,\frac{|dz|}{|z|} \le \int_\Gamma \frac{|\psi(z)|}{|z|}\,|dz|\,\max_{\varepsilon=\pm 1}\left(\int_0^\infty \left|\theta(re^{\varepsilon i\omega'})\right|\,\frac{dr}{r}\right).\end{aligned}$$

The constant hidden in the symbol "$\lesssim$" is of course $M(A,\omega')/2\pi$. ∎

For most cases, the approximation on $\mathcal{D}(A) \cap \mathcal{R}(A)$ given by *c)* of Proposition 1.29 is sufficient. But sometimes it is good to know a little more. Unfortunately, it is a lot of work to achieve this little gain of knowledge.

Lemma 1.30. *Let $\psi_{a,b}$ defined as above. Then* $\sup_{a,b} \|\psi_{a,b}(A)\| < \infty$.

Proof. We write

$$\begin{aligned}\psi_{a,b}(A) &= \left(\mathbf{1} - \int_0^a \psi(tz)\frac{dt}{t} - \int_b^\infty \psi(tz)\frac{dt}{t}\right)(A)\\ &= \left(\mathbf{1} - \left[\int_0^a \psi(tz)\frac{dt}{t} - \frac{az}{1+az}\right] - \left[\int_b^\infty \psi(tz)\frac{dt}{t} - \frac{1}{1+bz}\right]\right)(A)\\ &\quad - aA(1+aA)^{-1} - (1+bA)^{-1}\end{aligned}$$

The last two summands are bounded independent of a and b, since A is sectorial.

Claim 1. There is a constant D such that

$$\left|\int_0^a \psi(tz)\frac{dt}{t} - \frac{az}{1+az}\right| \le D\frac{|az|^s}{1+|az|^{2s}}$$

for all $a > 0, z \in S_\varphi$. (Here s is as chosen above.)

Proof of Claim. We have $|\psi(z)| \le C\frac{|z|^s}{1+|z|^{2s}}$ for all $z \in S_\varphi$. Define

$$C_1 := \sup_{w \in S_\varphi}\left|\frac{1}{1+w}\right| = \sup_{w\in S_\varphi}\left|\frac{w}{1+w}\right|, \quad \text{and} \quad g_a(z) := \int_0^a \psi(tz)\,\frac{dt}{t}.$$

Then

$$h_a := g_a - \frac{az}{1+az} = \frac{g_a}{1+az} + \frac{az}{1+az}(g_a - \mathbf{1})$$

We can estimate

$$|g_a(z)| \le C \int_0^a \frac{|tz|^s}{1+|tz|^{2s}} \frac{dt}{t} = \frac{C}{s} \int_0^{|az|^s} \frac{1}{1+t^2}\, dt = \frac{C}{s} \arctan |az|^s \le \frac{C}{s} \min(|az|^s, \frac{\pi}{2})$$

If $|az| \le 1$, this yields

$$|h_a(z)| \le \frac{C}{s} |az|^s + C_1 |az| \le 2(\frac{C}{s} + C_1) \frac{|az|^s}{1+|az|^{2s}}$$

If $|az| \ge 1$, we obtain

$$\left| \frac{g_a(z)}{1+az} \right| \le \frac{CC_1\pi}{2s} \frac{1}{|az|} \le \frac{CC_1\pi}{s} \frac{|az|^s}{1+|az|^{2s}}.$$

(The last step is due to the inequality $1 + x^{2s} \le 2x^{s+1}$ which is true for $s \le 1$ and all $x \ge 1$.) It remains to estimate $|g_a(z) - 1|$ for $|az| \ge 1$. Now,

$$\begin{aligned} |g_a(z) - 1| &= \left| \int_0^a \psi(tz) \frac{dt}{t} - \int_0^\infty \psi(t) \frac{dt}{t} \right| \\ &\le \left| \int_0^{|az|} \psi(te^{i \arg z}) \frac{dt}{t} - \int_0^{|az|} \psi(t) \frac{dt}{t} \right| + \left| \int_{|az|}^\infty \psi(t) \frac{dt}{t} \right| \\ &=: \qquad T_1 \qquad + \qquad T_2. \end{aligned}$$

The first summand T_1 can be estimated by applying Cauchy's theorem. In fact, letting $\gamma_1 := [0, |az|]e^{i \arg z}$, $\gamma_2 := |az|\, e^{i[0, \arg z]}$, and $\gamma_3 := [0, |az|]$, we obtain

$$\begin{aligned} T_1 &= \left| \int_{\gamma_1 - \gamma_3} \psi(z) \frac{dz}{z} \right| = \left| \int_{\gamma_2} \psi(z) \frac{dz}{z} \right| = \left| \int_0^{\arg z} \psi(|az|\, e^{ir}) \frac{|az|\, ie^{ir}}{|az|\, e^{ir}}\, dr \right| \\ &= \left| \int_0^{\arg z} \psi(|az|\, e^{ir})\, dr \right| \le \varphi\, C \frac{|az|^s}{1+|az|^{2s}}. \end{aligned}$$

For the second summand we have

$$T_2 = \left| \int_{|az|}^\infty \psi(t) \frac{dt}{t} \right| \le \frac{C}{s} \int_{|az|^s}^\infty \frac{1}{1+t^2}\, dt = \frac{C}{s}(\frac{\pi}{2} - \arctan |az|^s) \le \frac{C\pi}{2s} \frac{|az|^s}{1+|az|^{2s}}$$

if $|az| \ge 1$. (This is due to the inequality $\frac{\pi}{2} - \arctan x \le \frac{\pi}{2} \frac{x}{1+x^2}$ which holds for $x \ge 1$ and can be verified by elementary arguments.) This completes the proof of the Claim 1.

Claim 2. There is a constant D' such that

$$\left| \int_b^\infty \psi(tz) \frac{dt}{t} - \frac{1}{1+bz} \right| \le D' \frac{|bz|^s}{(1+|bz|)^{2s}}$$

for all $b > 0$, $z \in S_\varphi$.

Proof of Claim. We reduce the assertion to Claim 1. Let $\tilde{\psi}(z) := \psi(z^{-1})$. Then we have

$$\begin{aligned} \left| \int_b^\infty \psi(tz) \frac{dt}{t} - \frac{1}{1+bz} \right| &\overset{s=t^{-1}}{=} \left| \int_0^{b^{-1}} \tilde{\psi}(sz^{-1}) \frac{ds}{s} - \frac{(bz)^{-1}}{1+(bz)^{-1}} \right| \\ &\le D' \frac{|bz|^{-s}}{1+|bz|^{-2s}} = D' \frac{|bz|^s}{1+|bz|^{2s}}, \end{aligned}$$

where we have applied Claim 1 to the function $\tilde{\psi}$ with $a := b^{-1}$ and z^{-1} instead of z.

Claim 1 and Claim 2 now show that Lemma 1.23 is applicable. □

Combining Lemma 1.30 with the Convergence Lemma 1.26 and *c)* of Proposition 1.29 we obtain the following result.

Proposition 1.31. (McIntosh Approximation)
Let $A \in \mathrm{Sect}(\omega)$, and $\psi \in \mathcal{DR}[S_\omega]$ such that $\int_0^\infty \psi(t)\,dt/t = 1$. Then

$$\int_a^b \psi(tA)x\frac{dt}{t} = \left(\int_a^b \psi(tz)\frac{dt}{t}\right)(A)x \overset{a\to 0, b\to\infty}{\longrightarrow} \int_0^\infty \psi(tA)x\,\frac{dt}{t} = x$$

for all $x \in \overline{\mathcal{D}(A) \cap \mathcal{R}(A)}$.

§ 6 The Boundedness of the H^∞-Calculus

Let $A \in \mathrm{Sect}(\omega)$ be a sectorial operator on the Banach space X and let $\varphi > \omega$. Suppose we are given a subalgebra $\mathcal{F} \subset H^\infty(S_\varphi)$ such that $f(A)$ is defined by the NFCSO for each $f \in \mathcal{F}$. (This is a restriction only if A is not injective.) We say that **the natural $\mathcal{F}$-calculus for A is bounded** if $f(A) \in \mathcal{L}(X)$ for all $f \in \mathcal{F}$ and

$$\|f(A)\| \leq C\,\|f\|_\varphi \qquad (f \in \mathcal{F}) \tag{1.11}$$

for some constant $C \geq 0$. Here, $\|f\|_\varphi$ is shorthand for

$$\|f\|_\varphi := \|f\|_{\infty, S_\varphi} = \sup\{|f(z)| \mid z \in S_\varphi\} \tag{1.12}$$

We call $\inf\{C \geq 0 \mid \text{(1.11) holds}\}$ the **bound** of the natural $\mathcal{F}$-calculus.
If $\mathcal{F}$ is a closed subalgebra of $H^\infty(S_\varphi)$ and A is injective, the Closed Graph Theorem together with part *a)* of the Convergence Lemma (Proposition 1.26) yields the existence of a constant C satisfying (1.11) if only $f(A) \in \mathcal{L}(X)$ for all $f \in \mathcal{F}$.

In the sequel we will examine how boundedness of the natural $\mathcal{F}$-calculus for some $\mathcal{F}$ implies the same for a different (larger) $\mathcal{F}$, and how the appertaining bounds are related. In doing so we will need to use the notation and the results of Appendix D.

Theorems on Boundedness of Functional Calculi.
We will first deal with sectorial operators having dense domain and dense range. In this situation the Convergence Lemma (Proposition 1.26) is most powerful.

Proposition 1.32. *Let $A \in \mathrm{Sect}(\omega)$ with dense domain and dense range. Let $\omega < \varphi < \pi$ and $C \geq 0$. The following assertions are equivalent.*

(i) The natural $\mathcal{DR}(S_\varphi)$-calculus for A is bounded with bound C.

(ii) The natural $H^\infty(S_\varphi)$-calculus for A is bounded with bound C.

(iii) The natural $H^\infty(S_\varphi) \cap \mathbf{C_0}(\overline{S_\varphi})$-calculus for A is bounded with bound C.

(iv) The natural $\mathcal{R}_0^\infty(S_\varphi)$-calculus for A is bounded with bound C.

(v) The natural $\mathcal{R}^\infty(S_\varphi)$-calculus for A is bounded with bound C.

Proof. Let us first note that the implications $(ii) \Rightarrow (i)$ and $(ii) \Rightarrow (v) \Rightarrow (iv)$ are trivial. By Proposition D.3 the space $\mathcal{R}_0^\infty(S_\varphi)$ is uniformly dense in $H^\infty(S_\varphi) \cap \mathbf{C_0}(\overline{S_\varphi})$. Applying part *a)* of the Convergence Lemma (Proposition 1.26) yields the implication $(iv) \Rightarrow (iii)$. (Note that here

only injectivity of A is needed.) To show the implication $(iii) \Rightarrow (i)$ let $f \in \mathcal{DR}(S_\varphi)$. Define $f_n(z) := f(z + \frac{1}{n})$. Obviously, $f_n \in H^\infty(S_\varphi) \cap \mathbf{C}_0(\overline{S_\varphi})$, $\|f_n\|_\varphi \leq \|f\|_\varphi$ and $f_n \to f$ pointwise on S_φ. Now (i) follows from another application of the Convergence Lemma.
Finally, we show $(i) \Rightarrow (ii)$. Let $f \in H^\infty(S_\varphi)$. Define $\psi_s(z) := z^s/(1+z)^{2s}$ for $s > 0$. Then $\psi_s, f\psi_s \in \mathcal{DR}(S_\varphi)$. Moreover,

$$\|f\psi_s\|_\varphi \leq \|f\|_\varphi K^s$$

where $K := \|\psi_1\|_\varphi$. Using (i) gives $\|(f\psi_s)(A)\| \leq C\|f\|_\infty K^s$. Now we apply the Convergence Lemma (Prop. 1.26) and infer that $f(A) \in \mathcal{L}(X)$ with $(f\psi_s)(A) \to f(A)$ strongly if $s \searrow 0$. Since $\lim_{s\to 0} K^s = 1$ we obtain $\|f(A)\| \leq C\|f\|_\infty$. □

The questions become more difficult, if we drop the density assumptions. Recall that for a sectorial operator A we can always form the uniformly sectorial family

$$A_\varepsilon := (\varepsilon + A)(1 + \varepsilon A)^{-1}$$

which is a sectorial approximation of A and consists of bounded invertible operators (see Proposition 1.2).

Lemma 1.33. *Let $A \in \operatorname{Sect}(\omega)$, $\omega \leq \varphi < \pi$, and $C \geq 0$. The following assertions are equivalent.*

(i) The natural $\mathcal{R}^\infty(S_\varphi)$-calculus for A is bounded with bound C.

(ii) The natural $\mathcal{R}^\infty(S_\varphi)$-calculus for A_ε is bounded with bound C, for each $\varepsilon > 0$.

Proof. Let $r_\varepsilon(z) = (\varepsilon + z)(1 + \varepsilon z)^{-1} \in \mathcal{R}^\infty(S_\varphi)$. Then $A_\varepsilon = r_\varepsilon(A)$. If $r \in \mathcal{R}^\infty(S_\varphi)$ then $r \circ r_\varepsilon \in \mathcal{R}^\infty(S_\varphi)$ with $\|r \circ r_\varepsilon\|_\varphi \leq \|r\|_\varphi$, since r_ε maps S_φ into S_φ. The composition rule yields $r(A_\varepsilon) = (r \circ r_\varepsilon)(A)$, whence we have proved the implication $(i) \Rightarrow (ii)$.
The reverse implication follows from the fact that $r(A_\varepsilon) \to r(A)$ in norm, by *a)* of Proposition 1.21. (Note that $\mathcal{R}^\infty(S_\varphi) \subset \mathcal{DR}_{\text{ext}}(S_\varphi)$.) □

One should note that $\varphi = \omega$ is allowed in Lemma 1.33.

Proposition 1.34. *Let $A \in \operatorname{Sect}(\omega)$, $\omega < \varphi < \pi$ and $C \geq 0$. We define K to be the closure of S_φ in $\mathbb{C}_\infty$. The following assertions are equivalent.*

(i) The natural $\mathcal{A}(S_\varphi) \cap \mathbf{C}(K)$-calculus for A is bounded with bound C.

(ii) The natural $\mathcal{R}^\infty(S_\varphi)$-calculus for A is bounded with bound C.

If A is injective, one can replace $\mathcal{A}(S_\varphi)$ by $H^\infty(S_\varphi)$ in a).

Proof. The implication $(i) \Rightarrow (ii)$ is obviously true. To prove the reverse implication, take $f \in \mathcal{A}(S_\varphi) \cap \mathbf{C}(K)$. By Proposition D.3 there is a sequence $(r_n)_n \subset \mathcal{R}^\infty(S_\varphi)$ such that $\|r_n - f\|_\varphi \to 0$. By Lemma 1.33, the hypothesis (ii) implies that the natural $\mathcal{R}^\infty(S_\varphi)$-calculus for A_ε is bounded with bound C, for each $\varepsilon > 0$. Since each A_ε is bounded and invertible, we can apply Proposition 1.32 to conclude that the natural $H^\infty(S_\varphi)$-calculus for A_ε is bounded with bound C for each $\varepsilon > 0$. This implies that $r_n(A_\varepsilon) \to f(A_\varepsilon)$ *uniformly* in $\varepsilon > 0$. From Proposition 1.21 we see that $r_n(A_\varepsilon) \to r_n(A)$ for $\varepsilon \searrow 0$ for each n. A standard argment from functional analysis now implies that there is $T \in \mathcal{L}(X)$ with $f(A_\varepsilon) \to T$. Another application of Proposition 1.21 yields that $T = f(A)$.
The same proof works in the case where A is injective and $f \in H^\infty(S_\varphi) \cap \mathbf{C}(K)$. □

Proposition 1.35. *Let $A \in \operatorname{Sect}(\omega)$, $\omega \leq \varphi < \pi$ and $C \geq 0$. Assume that the natural $\mathcal{R}_0^\infty(S_\varphi)$-calculus for A is bounded with bound C. If A is densely defined and $\varphi \leq \frac{\pi}{2}$ then the natural $\mathcal{R}^\infty(S_\varphi)$-calculus for A is bounded with bound C.*

Proof. Let $r_n := \frac{n}{n+z}$. Then $\|r_n\|_\varphi = 1$ since $\varphi \le \frac{\pi}{2}$. Moreover, $r_n(A)x \to x$ for $n \to \infty$ (by Proposition 1.1), since $\mathcal{D}(A)$ is dense in X. Thus for a function $r \in \mathcal{R}^\infty(S_\varphi)$ we have

$$\|r(A)x\| = \lim_n \|r(A)r_n(A)x\| = \lim_n \|(rr_n)(A)x\| \le C\,\|rr_n\|_\varphi\,\|x\| \le C\,\|r\|_\varphi\,\|x\|$$

for all $x \in X$. □

We do not know if one can omit the assumption $\overline{\mathcal{D}(A)} = X$ or $\varphi \le \frac{\pi}{2}$.

Uniqueness of the Functional Calculus

Let us turn (for the moment) to a more general situation. Assume that A is a closed operator on a Banach space X, $\Omega \subset \mathbb{C}$ is open and $\mathcal{F} \subset H^\infty(\Omega)$ is a subalgebra containing the rationals $r_\lambda(z) = (\lambda - z)^{-1}$ for $\lambda \notin \overline{\Omega}$. We say that a mapping $\Phi : \mathcal{F} \to \mathcal{L}(X)$ is a **bounded $\mathcal{F}$-calculus** for A if the following conditions are satisfied:

1) The mapping Φ is a homomorphism of algebras.
2) We have $\Phi(r_\lambda) = R(\lambda, A)$ for all $\lambda \notin \overline{\Omega}$.
3) If $\mathbf{1} \in \mathcal{F}$ then $\Phi(\mathbf{1}) = I$.
4) There is $C \ge 0$ such that $\|\Phi(f)\| \le C\,\|f\|_\Omega$ for all $f \in \mathcal{F}$.

Here, $\|f\|_\Omega$ denotes the supremum norm of f on Ω. We say that a sequence $(f_n)_n \subset \mathcal{F}$ **converges boundedly and pointwise** on Ω to a function f if $f_n \to f$ pointwise on Ω and $\sup_n \|f_n\|_\Omega < \infty$. We say that Φ is continuous with respect to bounded and pointwise convergence (in short: **b.p.-continuous**), if it has the following property.

5) If $f_n, f \in \mathcal{F}$ such that $f_n \to f$ boundedly and pointwise on Ω, then $\Phi(f_n) \to \Phi(f)$ strongly on X.

Note that if $\mathbf{1} \notin \mathcal{F}$ we can always extend Φ to $\mathcal{F}' = \mathcal{F} \oplus \mathbb{C}\mathbf{1}$ satisfying 3 and retaining all other properties.

Lemma 1.36. *Let A be a closed operator on the Banach space X. Let $0 < \omega < \pi$ and assume that Φ is a bounded $H^\infty(S_\omega)$-calculus for A with bound $C \ge 0$. Then A is sectorial with $\omega_A \le \omega$.*
Take a sector S_φ with $\omega_A < \varphi$ and $\omega \le \varphi$ and denote by K the closure of S_φ in $\mathbb{C}_\infty$. Then the following assertions hold.

a) *$\Phi(f) = f(A)$ for all $f \in \mathcal{A}(S_\varphi) \cap \mathbf{C}(K)$.*

b) *If A is injective, then $\Phi(f) = f(A)$ for $f \in H^\infty(S_\varphi) \cap \mathbf{C}(K)$.*

c) *If A has dense domain and dense range then the natural $H^\infty(S_\varphi)$-calculus for A is bounded with bound C.*

Proof. The sectoriality of A follows by applying Φ to the rationals $\frac{\lambda}{\lambda - z}$. By Proposition D.3, we have $\Phi(f) = f(A)$ for $f \in \mathcal{R}^\infty(S_\omega)$. The assertions *a)* and *b)* now follow from Proposition 1.34. Part *c)* is immediate from Proposition 1.32. □

Proposition 1.37. *Let A be a closed operator on a Banach space X and let $0 < \omega < \pi$. Then there is at most one bounded and bp.-continuous $H^\infty(S_\omega)$-calculus Φ for A. If such a Φ exists, the operator A is sectorial with $\omega_A \le \omega$ and has dense domain and dense range. Moreover, Φ coincides with the natural functional calculus on $H^\infty(S_\varphi)$, where $\varphi > \omega$ with $\varphi \ge \omega$.*

Proof. Uniqueness follows from Proposition D.4. Define $f_n(z) := n(n+z)^{-1}$. Then $f_n \to \mathbf{1}$ pointwise on S_ω and $\sup_n \|f_n\|_\omega < \infty$. Since Φ is b.p.-continuous, $n(n+A)^{-1} = \Phi(f_n) \to \Phi(\mathbf{1}) = I$ strongly whence A is densely defined. The same argument with $f_n(z) = z(\frac{1}{n}+z)^{-1}$ yields that A has dense range. Take $\varphi > \omega_A$ with $\varphi \geq \omega$. From Lemma 1.36 it follows that the natural $H^\infty(S_\varphi)$-calculus for A is bounded. By uniqueness, the natural calculus must coincide with Φ. □

Remark 1.38. Let $A \in \mathrm{Sect}(\omega)$ with dense domain and range and let $\varphi > \omega$. Proposition 1.32 implies that if A has *some* bounded $H^\infty(S_\varphi)$-calculus with bound C, then the *natural* $H^\infty(S_\varphi)$-calculus for A is bounded with bound C.

An Operator without a Bounded H^∞-Calculus

We will sketch the construction of an operator A on a separable Hilbert space H such that the following conditions are satisfied:

1) The operator A is invertible and sectorial of angle 0.
2) The semigroup generated by $-A$ is immediately compact, see Definition 4.23 in Chapter II of [EN00].
3) The natural $H^\infty(S_{\frac{\pi}{2}})$-calculus of A is not bounded.

By the McIntosh-Yagi Theorem 3.31 below, this implies that $A \notin \mathrm{BIP}(H)$ (see §4 for definition) and the natural $H^\infty(S_\varphi)$-calculus for A is not bounded for *any* $0 < \varphi \leq \pi$.

The construction uses the notion of a **conditional basis**. This is a sequence $(e_n)_{n\geq 1}$ in H with the following properties:

1) For every $x \in H$ there is a unique sequence $(x_n)_n \subset \mathbb{C}$ such that $x = \sum_{n=1}^\infty x_n e_n$ in H.
2) There is a sequence $(\theta_n)_n \subset \{-1, 1\}$ and a vector $x \in H$ such that $x = \sum_{n=1}^\infty x_n e_n$ but the series $\sum_n \theta_n x_n e_n$ does not converge.

This means that $(e_n)_n$ is a **Schauder basis** of H which is not unconditional, see Part I, Chapter I of [LT96] for definitions. By Proposition 2.b.11 of [LT96], Part I, each separable Hilbert space has a conditional basis.

If $(e_n)_n$ is a Schauder basis of H, one can define the projections

$$P_n : \sum_{k=1}^\infty x_k e_k \longmapsto \sum_{k=1}^n x_k e_k$$

and obtains $P_n \in \mathcal{L}(H)$ for each n with $M_0 := \sup_n \|P_n\|$ being finite. The number M_0 is called the **basis constant** of the basis $(e_n)_n$. Given a scalar sequence $a = (a_n)_n \subset \mathbb{C}$ we define

$$\|a\| := \limsup_n |a_n| + \sum_{n\geq 1} |a_{n+1} - a_n|$$

which may be infinite. With a we associate an operator A on H by

$$\mathcal{D}(A) := \{x = \sum_k x_k e_k \in H \mid \sum_k a_k x_k e_k \text{ converges}\}$$

and $Ax = \sum_k a_k x_k e_k$ for $x \in \mathcal{D}(A)$. An easy partial summation argument — see Lemma 2.2 in [Ven93] — yields that $A \in \mathcal{L}(H)$ if $\|a\| < \infty$, and in this case we have $\|A\| \le M_0 \, \|a\|$. Furthermore, if in addition we have $\lim_n a_n = 0$, then $\|P_n A - A\| \to 0$ whence A is compact.

Assume now that $(a_n)_n$ is a *strictly increasing* sequence of positive real numbers with $a_1 > 0$ and $\lim_n a_n = \infty$. Given $\lambda \notin [a_1, \infty)$ we form the sequence $r_\lambda := (\frac{1}{\lambda - a_n})_n$ and compute

$$\|r_\lambda\| = \sum_{k\ge 1} \left| \frac{1}{\lambda - a_{k+1}} - \frac{1}{\lambda - a_k} \right| = \sum_{k\ge 1} \left| \int_{a_k}^{a_{k+1}} \frac{dt}{(\lambda - t)^2} \right| \le \int_{a_1}^{\infty} \frac{dt}{|\lambda - t|^2}.$$

Hence, the operator associated to r_λ is bounded (even compact) and it is easily seen that this operator equals $R(\lambda, A)$. In particular, $\sigma(A) \subset [a_1, \infty)$. For $\lambda = |\lambda| \, e^{i\varphi}$, $0 < |\varphi| \le \pi$, we obtain

$$\|\lambda R(\lambda, A)\| \le M_0 \, |\lambda| \, \|r_\lambda\| \le M_0 \int_0^\infty \frac{|\lambda| \, dt}{|\lambda - t|^2} = M_0 \int_0^\infty \frac{dt}{|e^{i\varphi} - t|^2}.$$

This shows that $A \in \mathrm{Sect}(0)$ and A has compact resolvent. Since the semigroup generated by $-A$ is holomorphic (see §3 of Chapter 2), it is immediately compact by Theorem 4.29 in Chapter II of [EN00].

Let $\mathcal{D} := \mathrm{span}\{e_n \mid n \in \mathbb{N}\} \subset \mathcal{D}(A)$. Since the projections P_n commute with A, we easily see that

$$f(A) \left(\sum_{k=1}^{n} x_n e_n \right) = \sum_{k=1}^{n} f(a_n) x_n e_n$$

for each $n \in \mathbb{N}, (x_k)_{k=1}^n \in \mathbb{C}^n$ and each $f \in H^\infty[S_0]$. Given $\varphi > 0$ we conclude that the natural $H^\infty(S_\varphi)$-calculus for A is bounded if and only if $\sum_{k=1}^{\infty} f(a_n) x_n e_n$ converges whenever $\sum_{k=1}^{\infty} x_n e_n$ does.

We now specialize the sequence a to $a_n := 2^n$. Since the basis $(e_n)_n$ is assumed to be conditional, we pick a sequence $(\theta_n)_n \subset \{-1, 1\}$ and a vector $x = \sum_{k=1}^{\infty} x_n e_n \in H$ such that $\sum_{k=1}^{\infty} \theta_n x_n e_n$ does not converge. Employing a result of CARLESON (from [Gar81, Section VII.1]) one can find $f \in H^\infty(S_{\frac{\pi}{2}})$ such that $f(2^n) = \theta_n$ for all n. By the remarks above, we obtain that $f(A) \notin \mathcal{L}(H)$.

The Kalton-Weis Lemma.
The next result, due to KALTON and WEIS, is of great importance and shows that a bounded $\mathcal{DR}$-calculus implies a large amount of "unconditionality".

Lemma 1.39. (Kalton-Weis)
Let $\omega < \varphi < \pi$ and $f \in \mathcal{DR}(S_\varphi)$. Then there is a constant $D > 0$ such that the following holds. If $A \in \mathrm{Sect}(\omega)$ such that the natural $\mathcal{DR}(S_\varphi)$-calculus for A is bounded with bound C, then

$$\left\| \sum_{k\in\mathbb{Z}} a_k \, f(t2^k A) \right\| \le CD \, \|a\|_\infty$$

for all $t > 0$ and all finite sequences $a = (a_k)_{k\in Z} \subset \mathbb{C}$.

Proof. Since $f \in \mathcal{DR}(S_\varphi)$ there is $C' > 0$ and $s > 0$ such that $|f(z)| \le C' \frac{|z|^s}{1+|z|^{2s}}$ for all $z \in S_\varphi$. (See Lemma 1.4.) Let $a = (a_k)_{k\in\mathbb{Z}}$ be a finite sequence of complex numbers such that $\|a\|_\infty \le 1$ and let $t > 0$. We estimate

$$\begin{aligned}\left\|\sum_k a_k f(t2^k A)\right\| &= \left\|\left(\sum_k a_k f(t2^k z)\right)(A)\right\| \le C \sup_{z\in S_\varphi}\left|\sum_k a_k f(t2^k z)\right|\\ &\le C \sup_{z\in S_\varphi}\sum_k \left|f(2^k z)\right| \le CC' \sup_{z\in S_\varphi}\sum_k \frac{\left|2^k z\right|^s}{1+|2^k z|^{2s}}\\ &= CC' \sup_{t>0}\sum_k \frac{(2^k t)^s}{1+(2^k t)^{2s}} = CC' \sup_{1\le t\le 2}\sum_k \frac{(2^k t)^s}{1+(2^k t)^{2s}}\end{aligned}$$

Thus with $D := C' \sup_{1\le t\le 2}\sum_k \frac{(2^k t)^s}{1+(2^k t)^{2s}}$ the lemma is proved. □

§ 7 Comments

§1 Sectorial Operators. Most of the material is adapted from the monograph [MCSA01, Section 1.2] where also the standard examples are presented. The notions "uniform sectoriality" and "sectorial approximation" are new but of course only unify methods used in the literature. E.g., the family of operators $A_\varepsilon = (A+\varepsilon)(1+\varepsilon A)^{-1}$ is used in [Prü93, Section 8.1] and [LM98a, Section 2] and is called "Nollau approximation" in [Oka00b]. Basics on multivalued sectorial operators can be found in [MSP00, Section 2].

§2 Spaces of Holomorphic Functions. The name "Dunford-Riesz class" together with the notation $\mathcal{DR}$ is taken from [Uit98, Section 1.3.3.1] and [Uit00, Section 2]. Other notations for the same class are used in the literature, e.g., $\Psi(S_\varphi)$ in [CDMY96] and $H_0^\infty(S_\varphi)$ in [LM98a].

§3/§4 The Natural Functional Calculus with Extensions. The fundamental intention of a "functional calculus" (or "operational calculus" as it is called in the older literature) is to somehow "insert" a given operator into a scalar function. The "mother of all functional calculi" so to speak, is provided by the spectral theorem for bounded normal operators on a Hilbert space (see Appendix C). The aim to obtain a similar tool for general bounded operators on a Banach space lead to the so called Dunford-Riesz calculus, see [DS58, Section VII.3] for the mathematics and [DS58, Section VII.11] for some historical remarks. This turned out to be only a special case of a general construction in Banach algebras an account of which can be found in [Con90, Chapter VII, §4].
The first approach to a functional calculus for *unbounded* operators was to reduce it to the bounded case by an application of a resolvent/elementary rational function, cf. [DS58, Section VII.9] or [ADM96, Lecture 2]. This functional calculus is sometimes called **Taylor** calculus. Here as in the bounded case, only functions are used which are holomorphic in a neighborhood of the spectrum, where ∞ has to be considered a member of the spectrum if A is unbounded (see Section A.2).
Defining $f(A)$ by the usual Cauchy integral even for functions f such that $\mathbb{C} \setminus \operatorname{dom}(f)$ intersects $\sigma(A)$ nontrivially yields a *singular integral*. To overcome this difficulty one uses the "regularization trick" on which our natural functional calculus rests. This technique appears in [Bad53] for strip type operators (see

Chapter 3, §1). For sectorial operators it was used by McIntosh in [McI86] and has become folklore since.

More or less *systematic* accounts of the natural functional calculus for sectorial operators can be found, e.g., in [ADM96], [Uit98] or [Wei]. Our presentation differs from these approaches in two decisive points. First, they mostly consider only injective operators, with the obvious disadvantage that the usual fractional powers of an arbitrary sectorial operator (see Chapter 2) can not be treated by their methods. Also they can deal with a bounded (but not injective) sectorial operator only in an artificial way (via its inverse).

The second main difference between our and other treatments lies in that we do not make any density assumptions on either the domain or the range of the operator. This is motivated by the fact that there are important examples of operators which are not densely defined, like the Dirichlet Laplacian on $\mathbf{C}(\overline{\Omega})$ where Ω is some open and bounded subset of $\mathbb{R}^n$, compare [ABHN01, Section 6.1]. Moreover, it seems to be advantageous to discard density assumptions in a systematic treatment because then one is not tempted to prove things by approximation and closure arguments (which usually are a tedious matter).

The focus on injective operators (with dense range and dense domain) in the literature is of course not due to unawareness but a matter of convenience. As one can see, e.g., by comparing the proofs of Proposition 1.15 and Proposition 1.20, working only with injective operators often makes life easier. And McIntosh's seminal paper [McI86] outlines how to treat the matter discarding the injectivity assumption.

While the main properties of the natural functional calculus as enumerated, e.g., in Proposition 1.9 or Proposition 1.16 are folklore, the statements of Corollaries 1.11 and 1.18 are new as they stand, and especially the *composition rule*, formulated and proved in Propositions 1.15 and 1.20 has not been previously proved in this generality. We believe that only the composition rule opens the door for a fruitful use of the natural functional calculus beyond the mere definition.

Other Functional Calculi. Although not particularly important for our purposes, we mention that there are (a lot of) other functional calculi around in the literature. The common pattern is this: Suppose you are given an operator A and a function f for which you would like to define $f(A)$. Take some (usually: integral) representation of f in terms of other functions g for which you already "know" $g(A)$. Then plug in A into the known parts and hope that the formulas still make sense. Obviously, our Cauchy integral-based natural calculus is of this type (the known parts are resolvents). Other examples are:

1) The **Hirsch** functional calculus, based on a representation

$$f(z) = a + \int_{\mathbb{R}_+} \frac{z}{1+tz}\,\mu(dt)$$

where μ is a suitable complex measure on $\mathbb{R}_+$. Details can be found in [MCSA01, Chapter 4].

2) The **Phillips** calculus, based on the Laplace transform

$$f(z) = \int_{\mathbb{R}_+} e^{-zt}\,\mu(dt)$$

where μ is a finite complex measure on $\mathbb{R}_+$. (Here, the "known" part is $e^{-zt}(A) = e^{-tA}$, i.e., $-A$ is assumed to generate a bounded C_0-semigroup.) The main reference for this is [HP74, Chapter XV].
3) The **Mellin transform** calculus as developed in [PS90] and [Uit00].
4) A functional calculus based on the **Poisson integral formula**, see [Bd91] and [deL87].

(Clearly this list of functional calculi is far from being complete.) Of course, each of these calculi can be extended by the "regularization trick". For the Phillips calculus, this is done in [Uit98, Section 1.3.4].

Let us mention that there are first attempts to define a functional calculus for *multivalued* sectorial operators, see [Ala91] and [MSP00].

§5 Boundedness and Approximation. Whereas parts *a)* and *b)* of Proposition 1.21 are well-known, part *c)* is new (but of course not a big deal). The definition of the numbers $C(f, \omega')$ as well as Proposition 1.22 were created in order to unify certain arguments in the literature. The payoff will appear in §3 of Chapter 2. Lemma 1.24 is a refinement of [McI86, Section 4, Theorem part a)]. Part *b)* of the Convergence Lemma (Proposition 1.26) is [McI86, Section 5, Theorem] or [CDMY96, Lemma 2.1] or [ADM96, Theorem D] and practically contained in all other papers on H^∞-calculus. Part *a)* is our contribution to the matter, see also Remark 1.27. Lemma 1.25 was invented to pave the ground for McIntosh's approximation technique (Proposition 1.31) which is used in [McI86] and the subsequent works of MCINTOSH and his co-workers. Lemma 1.28 and Proposition 1.29 give, more or less, a systematized account of results of [ADM96]. To prove the uniform boundedness of the operators $\psi_{a,b}(A)$, formulated in Lemma 1.30, is a (not worked-out) exercise in Section (E) of [ADM96]. Since our proof seems to be quite involved, it would be interesting if there is an essentially shorter one.
The full power of Proposition 1.31 will only be used in the proof of Proposition 2.12. Whereas in [ADM96] it seems also an essential ingredient in deriving the McIntosh-Yagi results, we found that Lemma 1.30 is in fact not needed, see Lemma 4.18 up to Corollary 4.21.

§6 The Boundedness of the H^∞-calculus. Proposition 1.32 covers results from the literature. The proof of the implication $(i) \Rightarrow (ii)$ is from [CDMY96], the proof of the implication $(iv) \Rightarrow (iii)$ is inspired by [LM98a] and [LM98b], where Runge's theorem is invoked. Lemma 1.33 up to Proposition 1.35 are new. Discussing *uniqueness* of the functional calculus seems to be a delicate matter (see also the "Concluding Remarks" below). Since usually the only thing which links the functional calculus to the operator is its behaviour on rational functions, a continuity assumption with respect to *bounded and pointwise* convergence of functions seems necessary to obtain uniqueness, at least if the function algebra is large. This is reflected in Proposition 1.37. Lemma

1.36 is the best we could achieve if such a continuity property is missing. But it leaves open a number of questions: what happens for functions f in $H^\infty(S_\varphi) \cap \mathcal{A}(S_\varphi) \setminus \mathbf{C}(K)$ if A is just sectorial (part *a*)? Or if $f \in H^\infty(S_\varphi) \setminus \mathbf{C}(K)$ and A is injective (part *b)*?

The first example of an operator without a bounded H^∞-calculus appears in [MY90]. Our construction is taken from [LM98a, Theorem 4.1] and [LM00] which carry on ideas from [BC91] and [Ven93]. The Kalton-Weis Lemma 1.39 (our terminology!) is Lemma 4.1 of [KW01].

Bounded H^∞-calculus and Maximal Regularity. Historically, the natural functional calculus for sectorial operators was developed by McIntosh in his work on the Cauchy singular integral operator on a Lipschitz curve, see his seminal paper [McI86], especially the sections 9 and 10, as well as the lecture notes [ADM96]. The main focus was on the boundedness of the H^∞-calculus which, with the help of Yagi's ideas from [Yag84], could be shown to be equivalent to certain so called *quadratic estimates* in the Hilbert space case. (We will reprove some of these results in Theorem 3.31 and Theorem 4.23, see also the comments to Chapter 4.) This focus pertained in the subsequent attempts to generalize the results from Hilbert space to $\mathbf{L}^p$-spaces and general Banach spaces, as it was done in [Bd92], [CDMY96], [Fra97] and [FM98].
Meanwhile, the boundedness of the H^∞-calculus was observed to be related to the problem of *maximal* $\mathbf{L}^p$*-regularity* coming from the theory of evolution equations. (See [Dor93] for an exposition of the problem.) In an abstract form it reduces to the problem to determine when the sum of two C_0-semigroup generators on a Banach space is closed. The celebrated Dore-Venni Theorem in [DV87] (see also [Mon99] and [MCSA01, Chapter 8]) gives a handable condition requiring that the operators have bounded imaginary powers (see the end of Section 4). Now, a sectorial operator satisfies this condition if it has a bounded H^∞-calculus. More interaction between the two concepts can be seen from the lecture notes [LM98a]. Recently, the long-desired characterization of operators with the maximal regularity property (on UMD spaces) was obtained by Weis in [Wei01] (see also [AB02]) and in [KW01] this result is reproved by functional calculus methods.

Concluding Remarks. It might be considered unsatisfying that we have never defined formally what a functional calculus is[4]. This is because we feel that the notion of functional calculus is basically *intuitive* and should remain so — at least for the moment — in order to keep its full power. In fact there are formal definitions in the literature, as, e.g., in [Bd94], but these attempts are *ad hoc* and one can usually find a functional calculus (in the intuitive sense) which does not meet the given formal definition. This indicates that the time for a "good" definition has not yet come and the theory developed here may be seen as a contribution to the strive for finding one some time in the future.
The lack of a formal definition is particularly present if one asks the question of uniqueness. However, as seen in Section 6, uniqueness is delicate (even if a formal definition is at hand) and seems not to be handable if situations become

[4]The definition on page 48 is to be understood as a "local" definition.

more general (missing density, continuity ...). Unfortunately, the terminology used in the literature almost automatically raises the uniqueness question, namely when speaking of operators "which have a bounded H^∞-functional calculus", where obviously it is meant that a very special functional calculus (most often the one we call the natural functional calculus) is bounded. The cited phrase does only make sense if one knows what *a* H^∞-functional calculus is (in general, not some special one) and what it means that this functional calculus is bounded.

We have tried to resolve this problem and to contribute to a more concise terminology in future times by introducing the concept of the *natural* functional calculus. A mathematical object concretely constructed and widely used should have a definite name. This has the convenient effect that one is not forced to give a formal definition of the notion of "functional calculus" (see above) and causes the uniqueness question to vanish (or at least to fade). In fact we do not consider the uniqueness problem very important. (We are in good company, with almost no paper elaborating on the matter.) This is because — philosophically speaking — *there is only one functional calculus*. Since all explicitly constructed functional calculi somehow rely on integral representations of scalar functions (see above) and these integral representations are compatible, it is not daring to expect that also the resulting functional calculi are.

However, this is only intuitive reasoning and it remains to prove rigorously the compatibility (with composition rules, of course!) of all known (and future) functional calculi. As long as there is no "Great Unifying Theory" (and some mathematicians believe that there never will be one), this appears to be a tedious task.

Second Chapter

Fractional Powers and Related Topics

In this chapter, the basic theory of fractional powers A^α of a sectorial operator A is presented. This is done (quite elegantly) by making use of the functional calculus developed in Chapter 1. In §1 we introduce fractional powers with positive real part. The section comprises proofs for the scaling property, the power laws, the moment inequality, and the Balakrishnan representation. Furthermore, we examine the behaviour of $(A+\varepsilon)^\alpha$ for variable ε and of $A^\alpha x$ with variable α. In §2 we generalize the results from §1 to fractional powers with arbitrary real part. (Here, the operator A has to be injective.) A bit of a detour, the definition and the fundamental properties of holomorphic semigroups are presented in §3. The usual generator/semigroup correspondence is extended to the multivalued case. Once again, the account is elegant by using the theory of functional calculus. In §4 the logarithm of an injective sectorial operator A is defined. We prove an important spectral result originally due tue NOLLAU and consider the connection of $\log A$ to the family of imaginary powers $(A^{is})_{s\in\mathbb{R}}$ of A.

§ 1 Fractional Powers with Positive Real Part

In this section X always denotes a Banach space and A a sectorial operator of angle ω on X. (Recall the general agreement on terminology, see page 137.)

Observe that for each $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > 0$ the function $f(z) = z^\alpha$ is contained in the class $\mathcal{A}(S_\varphi)$ for any $0 < \varphi \le \pi$. In fact, one has $z^\alpha(1+z)^{-n} \in \mathcal{DR}$ if $n > \operatorname{Re}\alpha$. So the definition

$$A^\alpha := (z^\alpha)(A) = (1+A)^n \left(\frac{z^\alpha}{(1+z)^n}\right)(A) \qquad (0 < \operatorname{Re}\alpha < n)$$

is reasonable. We call A^α the **fractional power** with exponent α of A.

Proposition 2.1. *Let A be a sectorial operator on the Banach space X. Then the following assertions hold.*

a) If A is bounded, then also A^α is, and the mapping

$$(\alpha \longmapsto A^\alpha) : \{\operatorname{Re}\alpha > 0\} \longrightarrow \mathcal{L}(X)$$

is holomorphic.

b) *If* $n > \operatorname{Re}\alpha > 0$*, then* $D(A^n) \subset D(A^\alpha)$*, and the mapping*

$$(\alpha \longmapsto A^\alpha x) : \{0 < \operatorname{Re}\alpha < n\} \longrightarrow X$$

is holomorphic for each $x \in D(A^n)$*.*

c) **(First Power Law)** *The identity*

$$A^{\alpha+\beta} = A^\alpha A^\beta$$

holds for all $\operatorname{Re}\alpha, \operatorname{Re}\beta > 0$*. In particular,* $\mathcal{D}(A^\gamma) \subset \mathcal{D}(A^\alpha)$ *for* $0 < \operatorname{Re}\alpha < \operatorname{Re}\gamma$*.*

d) *One has* $\mathcal{N}(A^\alpha) = \mathcal{N}(A)$ *for all* $\operatorname{Re}\alpha > 0$*.*

e) *If is A injective, then* $(A^{-1})^\alpha = (A^\alpha)^{-1}$*. If* $0 \in \varrho(A)$*, then also* $0 \in \varrho(A^\alpha)$*.*

f) *If* $T \in \mathcal{L}(A)$ *commutes with* A*, then it also commutes with* A^α*.*

g) *Let* $\operatorname{Re}\alpha, \operatorname{Re}\beta > 0$*,* $x \in X$*, and* $\varepsilon > 0$*. Then*

$$\big(A(A+\varepsilon)^{-1}\big)^\alpha x \in \mathcal{D}(A^\beta) \quad \Longleftrightarrow \quad x \in \mathcal{D}(A^\beta).$$

h) *If A is densely defined, then* $\mathcal{D}(A^n)$ *is a core for* A^α*, where* $0 < \operatorname{Re}\alpha < n$*.*

i) *If the Banach space* X *is reflexive, then*

$$A^\alpha(x \oplus y) = 0 \oplus B^\alpha y$$

for all $x \in \mathcal{N}(A), y \in \overline{\mathcal{R}(A)}$*, where* $B := A|_{\overline{\mathcal{R}(A)}}$ *is the injective part of A.*

Proof. Part *a)* easily follows from the definition and the Dominated Convergence Theorem. The same applies to *b)*, if one notes that $A^\alpha x = (z^\alpha/(1+z)^n)(A)(1+A)^n x$ for $x \in \mathcal{D}(A^n)$ and $0 < \operatorname{Re}\alpha < n$.
The power law is a little more involved. From the general statements on the functional calculus (Proposition 1.9) we know that $A^\alpha A^\beta \subset A^{\alpha+\beta}$, with $\mathcal{D}(A^\alpha A^\beta) = \mathcal{D}(A^{\alpha+\beta}) \cap \mathcal{D}(A^\beta)$.
Let $n > \operatorname{Re}\alpha, \operatorname{Re}\beta$ be fixed. We define $\Phi_\alpha := (z^\alpha/(1+z)^n)(A) \in \mathcal{L}(X)$ and Φ_β analogously. Let $x \in \mathcal{D}(A^{\alpha+\beta})$. Then $\Phi_\alpha \Phi_\beta x = \frac{z^{\alpha+\beta}}{(1+z)^{2n}}(A)x \in \mathcal{D}(A^{2n})$. From this it follows that

$$A^n(1+A)^{-2n}\Phi_\beta x = \frac{z^{n+\beta}}{(1+z)^{3n}}(A)x = \frac{z^{n-\alpha}}{(1+z)^n}(A)\Phi_\alpha\Phi_\beta x \in \mathcal{D}(A^{2n}).$$

Applying Proposition 1.1, part *g)*, we obtain $(1+A)^{-n}\Phi_\beta x \in \mathcal{D}(A^{2n})$. But this gives $\Phi_\beta x \in \mathcal{D}(A^n)$, hence $x \in \mathcal{D}(A^\beta)$.
Assertion *d)* follows from $A^n x = A^{n-\alpha}A^\alpha x$ and part *e)* of Proposition 1.1.
We prove *e)*. Note that $f(z) := z^{-\alpha} \in \mathcal{B}$. Hence by *f)* and *g)* of Proposition 1.16 we have $(A^{-1})^\alpha = f(z^{-1})(A^{-1}) = f(A) = (f^{-1}(A))^{-1} = (A^\alpha)^{-1}$. The second assertion follows from the first because of *a)*.
Assertion *f)* is a special case of part *c)* of Proposition 1.9.
We prove *g)*. Recall that $A(A+\varepsilon)^{-1}$ is bounded and sectorial and commutes with A. Hence, by *f)*, $(A(A+\varepsilon)^{-1})^\alpha$ commutes with A, whence it commutes with A^β by another application of *f)*. This gives one implication. Let $(A(A+\varepsilon)^{-1})^\alpha x \in \mathcal{D}(A^\beta)$ and choose $n > \operatorname{Re}\alpha, \operatorname{Re}\beta$. Applying the implication just proved with α replaced by $n - \alpha$, we obtain $(A(A+\varepsilon)^{-1})^n x \in \mathcal{D}(A^\beta)$. From this we conclude that $(A(A+\varepsilon)^{-1})^n \Phi_\beta x \in \mathcal{D}(A^n)$, where $\Phi_\beta := (z^\beta/(1+z)^n)(A)$. By Proposition 1.1, part *g)* this implies $\Phi_\beta x \in \mathcal{D}(A^n)$, whence $x \in \mathcal{D}(A^\beta)$.
Part *h)* follows from *d)* of Proposition 1.9. The statement in *i)* is obvious. □

Proposition 2.2. (Scaling Property)
Let $A \in \operatorname{Sect}(\omega)$ for some $0 < \omega < \pi$ on the Banach space X. Let $0 < \alpha < \frac{\pi}{\omega}$. Then $A^\alpha \in \operatorname{Sect}(\alpha\omega)$. In particular, we have $\omega_{A^\alpha} = \alpha\omega_A$.

Proof. Let $\lambda \notin \overline{S_{\alpha\omega}}$ and define $f_\lambda(z) := (\lambda - z^\alpha)^{-1}$. By Corollary 1.13, $\lambda \in \varrho(A^\alpha)$ if and only if $f_\lambda \in H(A)$. Define

$$\psi_\lambda(z) := \frac{\lambda}{\lambda - z^\alpha} - \frac{|\lambda|^{\frac{1}{\alpha}}}{z + |\lambda|^{\frac{1}{\alpha}}} = \frac{\lambda z + |\lambda|^{\frac{1}{\alpha}} z^\alpha}{(\lambda - z^\alpha)(z + |\lambda|^{\frac{1}{\alpha}})}.$$

Obviously, $\psi_\lambda \in \mathcal{DR}(S_\theta)$ for each $0 < \theta < \pi$. Then

$$f_\lambda(z) = \frac{1}{\lambda - z^\alpha} = \frac{1}{\lambda}\left(\frac{|\lambda|^{\frac{1}{\alpha}}}{z + |\lambda|^{\frac{1}{\alpha}}} + \psi_\lambda(z)\right) \in H(A).$$

So we have shown that $\lambda \in \varrho(A^\alpha)$ and $R(\lambda, A^\alpha) = f_\lambda(A)$. To prove sectoriality it suffices to show $\sup_{\lambda \notin \overline{S_{\alpha\varphi}}} \|\psi_\lambda(A)\| < \infty$ for $\omega < \varphi < \pi/\alpha$. By the sectoriality of A there is a constant c such that

$$\|\psi_\lambda(A)\| \le c \int_\Gamma |\psi_\lambda(z)| \, \frac{|dz|}{|z|}.$$

Employing the scaling property $\psi_{t^\alpha\lambda}(tz) = \psi_\lambda(z)$ for $t > 0$ we see that it is sufficient to prove the uniform boundedness of $\psi_\lambda(A)$ für $\lambda \notin \overline{S_{\alpha\varphi}}$, $|\lambda| = 1$. This is easily done. □

Corollary 2.3. *Let $(A_n)_n \subset \operatorname{Sect}(\omega)$ be uniformly sectorial for some $0 < \omega < \pi$. Let $0 < \alpha < \frac{\pi}{\omega}$. Then $(A_n^\alpha)_n \subset \operatorname{Sect}(\alpha\omega)$ is also uniformly sectorial. Furthermore, from $A_n \to A$ (sect) it follows that $A_n^\alpha \to A^\alpha$ (sect).*

The next result is an immediate consequence of Proposition 2.2 and the general composition rule of Proposition 1.15.

Proposition 2.4. *Let $A \in \operatorname{Sect}(\omega)$ for some ω. Assume $0 < \alpha < \frac{\pi}{\omega}$ and $\omega < \varphi < \frac{\pi}{\alpha}$. If $f \in \mathcal{DR}(S_{\alpha\varphi})$ ($f \in \mathcal{A}(S_{\alpha\varphi})$), the function $f(z^\alpha)$ is in $\mathcal{DR}(S_\varphi)$ (in $\mathcal{A}(S_\varphi)$), and the identity*

$$f(A^\alpha) = (f(z^\alpha))(A).$$

holds.

Corollary 2.5. (Second Power Law) *Let $A \in \operatorname{Sect}(\omega)$ with $0 < \omega < \pi$, and let $0 < \alpha < \frac{\pi}{\omega}$. Then*

$$(A^\alpha)^\beta = A^{\alpha\beta}$$

for all $\operatorname{Re}\beta > 0$.

Corollary 2.6. *Assume $A \in \operatorname{Sect}(\omega)$, $\operatorname{Re}\gamma > 0$, and $x \in \mathcal{D}(A^\gamma)$. Then the mapping*

$$(\alpha \longmapsto A^\alpha x) : \{0 < \operatorname{Re}\alpha < \operatorname{Re}\gamma\} \longrightarrow X$$

is holomorphic.

Proof. We can assume $\gamma > 0$ without restriction. Choose $n > \gamma$. Then $A^\alpha x = B^{\alpha n/\gamma} x$ and $x \in \mathcal{D}(B^n)$ where $B := A^{\gamma/n}$. Now the claim follows from *b)* of Proposition 2.1. □

We now compare the operators A^α and $(A + \varepsilon)^\alpha$ for $\varepsilon > 0$.

Proposition 2.7. *Let* $0 < \operatorname{Re}\alpha < 1$ *and* $\varepsilon > 0$. *Then* $T_\varepsilon := ((z+\varepsilon)^\alpha - z^\alpha)(A) \in \mathcal{L}(X)$. *Moreover, we have* $A^\alpha + T_\varepsilon = (A+\varepsilon)^\alpha$ *and* $\|T_\varepsilon\| \le C\varepsilon^{\operatorname{Re}\alpha}$, *where* $C = C(\alpha, \varphi, M(A,\varphi))$ *for* $\omega < \varphi < \pi$.

Proof. Let $\psi_1(z) := (z+1)^\alpha - z^\alpha - (z+1)^{-1}$. A short computation shows that $\psi_1 \in \mathcal{DR}$. Hence $\psi_\varepsilon(z) := \varepsilon^\alpha \psi_1(z/\varepsilon) = (z+\varepsilon)^\alpha - z^\alpha - \varepsilon^\alpha(\varepsilon^{-1}z+1)^{-1}$ is also contained in $\mathcal{DR}$. This shows that $(z+\varepsilon)^\alpha - z^\alpha \in H(A)$. By part *a)* of Corollary 1.11 and the composition rule we obtain $A^\alpha + T_\varepsilon = (A+\varepsilon)^\alpha$. Furthermore, we have

$$\varepsilon^{-\alpha}T_\varepsilon = \varepsilon^{-\alpha}\psi_\varepsilon(A) - (1+\varepsilon^{-1}A)^{-1} = \psi_1(\varepsilon^{-1}A) - (1+\varepsilon^{-1}A)^{-1}.$$

The claim now follows from $\sup_{\varepsilon>0}\left\|(1+\varepsilon^{-1}A)^{-1}\right\| = M(A)$ and $\left\|\psi_1(\varepsilon^{-1}A)\right\| \le M(A,\varphi)C(\psi_1,\varphi)$ for $\omega < \varphi < \pi$ (apply Proposition 1.22 with $\psi = 0$). □

Remark 2.8. With the help of the Balakrishnan representation (see below) the last proposition can be improved with respect to the constant C. Namely, one can explicitly determine a constant which only depends on $M(A)$, $\operatorname{Re}\alpha$, and $|\sin\alpha\pi|$, see [MCSA01, Proposition 5.1.14].

The last proposition implies in particular that $\mathcal{D}((A+\varepsilon)^\alpha) = \mathcal{D}(A^\alpha)$ for $\varepsilon > 0$ and $0 < \operatorname{Re}\alpha < 1$. However, this is true for *all* $\operatorname{Re}\alpha > 0$ as it is shown by the next result.

Proposition 2.9. *Let* $\operatorname{Re}\alpha > 0$ *and* $\varepsilon > 0$. *Then the following assertions hold.*

a) $\mathcal{D}(A^\alpha) = \mathcal{D}((A+\varepsilon)^\alpha)$.

b) $A^\alpha\left((A+\varepsilon)^{-1}\right)^\alpha = \left(A(A+\varepsilon)^{-1}\right)^\alpha$.

c) $\lim_{\varepsilon\to 0}(A+\varepsilon)^\alpha x = A^\alpha x$ *for each* $x \in \mathcal{D}(A^\alpha)$.

Proof. Apply the composition rule to the functions $f(z) := (z+\varepsilon)^{-1}$ and $g(z) := z^\alpha$ to obtain $(z+\varepsilon)^{-\alpha}(A) = \left((A+\varepsilon)^{-1}\right)^\alpha \in \mathcal{L}(X)$. Hence $(z+\varepsilon)^{-\alpha} \in H(A)$ and

$$(A(A+\varepsilon)^{-1})^\alpha = \left(\frac{z^\alpha}{(z+\varepsilon)^\alpha}\right)(A) = z^\alpha(A)(z+\varepsilon)^{-\alpha}(A) = A^\alpha((A+\varepsilon)^{-1})^\alpha$$

by the composition rule again, whence *b)* is proved. Furthermore, we have

$$\left((A+\varepsilon)^{-1}\right)^\alpha = (z+\varepsilon)^{-\alpha}(A) = ((z+\varepsilon)^\alpha(A))^{-1} = ((A+\varepsilon)^\alpha)^{-1}$$

by the composition rule and *h)* of Proposition 1.9. Together with *b)* this shows that $\mathcal{D}((A+\varepsilon)^\alpha) = \mathcal{R}(((A+\varepsilon)^{-1})^\alpha) \subset \mathcal{D}(A^\alpha)$. Let us prove the other inclusion of *a)*. Choose $x \in \mathcal{D}(A^\alpha)$ and $n > \operatorname{Re}\alpha$. Applying the first power law and *b)* we obtain

$$\begin{aligned}\mathcal{D}(A^n) \ni (A+\varepsilon)^{-n}A^\alpha x = A^\alpha(A+\varepsilon)^{-n}x &= A^\alpha((A+\varepsilon)^{-1})^\alpha((A+\varepsilon)^{-1})^{n-\alpha}x\\ &= (A(A+\varepsilon)^{-1})^\alpha((A+\varepsilon)^{-1})^{n-\alpha}x.\end{aligned}$$

This gives $((A+\varepsilon)^{-1})^{n-\alpha}x \in \mathcal{D}(A^n)$ by *g)* of Proposition 2.1. From this it follows that $x = (A+\varepsilon)^{n-\alpha}((A+\varepsilon)^{-1})^{n-\alpha}x \in \mathcal{D}((A+\varepsilon)^\alpha)$.
We are left to show *c)*. The family of operators $(A+\varepsilon)(A+1)^{-1}$ is a sectorial approximation of $A(A+1)^{-1}$ (apply *c)* of Proposition 1.2 together with *f)* of Proposition 1.1). Proposition 1.21 together with *b)* implies that

$$(A+\varepsilon)^\alpha((A+1)^{-1})^\alpha = ((A+\varepsilon)(A+1)^{-1})^\alpha \to (A(A+1)^{-1})^\alpha = A^\alpha((A+1)^\alpha)^{-1}$$

in norm. In particular we have $\lim_{\varepsilon\searrow 0}(A+\varepsilon)^\alpha x = A^\alpha x$ for all $x \in \mathcal{D}((A+1)^\alpha) = \mathcal{D}(A^\alpha)$. □

Remark 2.10. The last result together with the rule $(A^\alpha)^{-1} = (A^{-1})^\alpha$ (if A is injective) reduces the definition of A^α for (general) sectorial operators A to the one for operators $A \in \mathcal{L}(X)$ with $0 \in \varrho(A)$ where the usual Dunford calculus is at hand. Therefore, the last result can be viewed as an "interface" to the literature where often the fractional powers are defined in a different way. Instead of proving a statement for fractional powers directly with recourse to the definition one can proceed in three steps:

1) The validity of the statement is proved for $A \in \mathcal{L}(X)$ with $0 \in \varrho(A)$.
2) One shows that in case A is injective the statement for A follows from the statement for A^{-1}.
3) One shows that the statement is true for A if it is true for all $A + \varepsilon$ ($0 < \varepsilon$ small).

In fact, many proofs follow this scheme.

Corollary 2.11. *One has $\mathcal{D}(A^\alpha) \subset \overline{\mathcal{D}(A)}$ and $\mathcal{R}(A^\alpha) \subset \overline{\mathcal{R}(A)}$ for each $\operatorname{Re}\alpha > 0$ and each sectorial operator A.*

Proof. Assume first that A is bounded, i.e., $A \in \mathcal{L}(X)$. Then with Γ being an appropriate *finite* path, we have

$$A^\alpha = \frac{1}{2\pi i}\int_\Gamma z^\alpha R(z,A)\,dz = \frac{1}{2\pi i}\int_\Gamma z^{\alpha-1} AR(z,A)\,dz$$

by Cauchy's theorem. This gives $\mathcal{R}(A^\alpha) \subset \overline{\mathcal{R}(A)}$ if $A \in \mathcal{L}(X)$. For arbitrary A we apply this to the operator $A(A+1)^{-1}$ and obtain

$$\mathcal{R}(A^\alpha) = \mathcal{R}[A^\alpha((A+1)^\alpha)^{-1}] = \mathcal{R}[(A(A+1)^{-1})^\alpha] \subset \overline{\mathcal{R}(A(A+1)^{-1})} = \overline{\mathcal{R}(A)}.$$

(Here we used *b)* of the last proposition.) Finally we conclude from this and *a)* of Proposition 2.9 that

$$\mathcal{D}(A^\alpha) = \mathcal{D}[(A+1)^\alpha] = \mathcal{R}[((A+1)^{-1})^\alpha] \subset \overline{\mathcal{R}((A+1)^{-1})} = \overline{\mathcal{D}(A)}.$$

This proves the statement. □

We will now be concerned with the so called **moment inequality**.

Proposition 2.12. *Let $0 \le \omega < \omega' < \pi$ and $d > 0$. Then there is a constant $C = C(\omega', d)$ with the following property. If $A \in \operatorname{Sect}(\omega)$ and $0 < \operatorname{Re}\alpha < \operatorname{Re}\gamma < \operatorname{Re}\beta$ with $d = \operatorname{Re}(\beta - \alpha)$, then*

$$\|A^\gamma x\| \le \frac{C(\omega',d)e^{\omega' \max(|\operatorname{Im}(\beta-\gamma)|,|\operatorname{Im}(\gamma-\alpha)|)}}{\theta(1-\theta)}\,\|A^\alpha x\|^{1-\theta}\,\|A^\beta x\|^\theta \qquad (x \in \mathcal{D}(A^\beta)), \tag{2.1}$$

where $\theta \in (0,1)$ is given by $\operatorname{Re}\gamma = (1-\theta)\operatorname{Re}\alpha + \theta\operatorname{Re}\beta$.

Proof. Choose $\psi_0 \in \mathcal{DR}$ such that $\psi_1(z) := \psi_0(z)z^{\beta-\alpha} \in \mathcal{DR}$ and $\int_0^\infty \psi(t)\,dt/t = 1$, where $\psi(z) = \psi_0(z)z^{\beta-\gamma} = \psi_1(z)z^{\alpha-\gamma} \in \mathcal{DR}$. (See below for an example.) By Proposition 1.31 we have $y = \int_0^\infty \psi(tA)y\,dt/t$ for each $y \in \overline{\mathcal{D}(A) \cap \mathcal{R}(A)}$. Let $x \in \mathcal{D}(A^\beta)$. From Corollary 2.11 we know that $A^\gamma x \in \overline{\mathcal{D}(A)} \cap \overline{\mathcal{R}(A)} = \overline{\mathcal{D}(A) \cap \mathcal{R}(A)}$. Hence we have

$$\begin{aligned}
A^\gamma x &= \int_0^\infty \psi(tA)A^\gamma x\,\frac{dt}{t} = \int_0^\infty t^{-1-\gamma}(\psi(z)z^\gamma)(tA)x\,dt \\
&= \int_0^\varepsilon t^{-1-\gamma}\psi_0(tA)(tA)^\beta x\,dt + \int_\varepsilon^\infty t^{-1-\gamma}\psi_1(tA)(tA)^\alpha x\,dt \\
&= \int_0^\varepsilon t^{-1+(\beta-\gamma)}\psi_0(tA)A^\beta x\,dt + \int_\varepsilon^\infty t^{-1-(\gamma-\alpha)}\psi_1(tA)A^\alpha x\,dt
\end{aligned}$$

for all $\varepsilon > 0$. This gives

$$\begin{aligned}\|A^\gamma x\| &\le \int_0^\varepsilon t^{-1+(\operatorname{Re}\beta-\operatorname{Re}\gamma)}\,dt\ (\sup_{t>0}\|\psi_0(tA)\|)\ \|A^\beta x\| \\ &\quad + \int_\varepsilon^\infty t^{-1-(\operatorname{Re}\gamma-\operatorname{Re}\alpha)}\,dt\ (\sup_{t>0}\|\psi_1(tA)\|)\ \|A^\alpha x\| \\ &= \tilde{C}\left(\frac{\varepsilon^{\operatorname{Re}\beta-\operatorname{Re}\gamma}}{\operatorname{Re}\beta-\operatorname{Re}\gamma}\|A^\beta x\| + \frac{\varepsilon^{-(\operatorname{Re}\gamma-\operatorname{Re}\alpha)}}{\operatorname{Re}\gamma-\operatorname{Re}\alpha}\ \|A^\alpha x\|\right)\end{aligned}$$

for all $\varepsilon > 0$, where $\tilde{C} := \max(\sup_{t>0}\|\psi_0(tA)\|, \sup_{t>0}\|\psi_1(tA)\|)$. Taking the minimum with respect to ε yields $\varepsilon := (\|A^\alpha x\| / \|A^\beta x\|)^{\frac{1}{\operatorname{Re}\beta-\operatorname{Re}\alpha}}$. Inserting this we obtain

$$\|A^\gamma x\| \le \tilde{C}\left(\frac{1}{\theta(1-\theta)d}\right)\|A^\beta x\|^\theta\,\|A^\alpha x\|^{1-\theta}$$

We now give an example for a function ψ_0 as was needed in the above argument. Let

$$\tilde{\psi}_0(z) := \frac{z\,z^{-i\operatorname{Im}(\beta-\gamma)}}{(1+z)^{2+d}},\quad \tilde{\psi}(z) := \frac{z^{1+\operatorname{Re}(\beta-\gamma)}}{(1+z)^{2+d}},\quad \tilde{\psi}_1(z) := \frac{z^{1+d}z^{i\operatorname{Im}(\gamma-\alpha)}}{(1+z)^{2+d}}$$

and

$$c^{-1} := \int_0^\infty \tilde{\psi}(t)\,dt/t = \int_0^\infty \frac{t^{\operatorname{Re}(\beta-\gamma)}}{(1+t)^{2+d}}\,dt \ge \int_1^\infty \frac{1}{(1+t)^{2+d}} = \frac{1}{(1+d)2^{1+d}}.$$

Then we can take $\psi_0 := c\tilde{\psi}_0$, $\psi := c\tilde{\psi}$, and $\psi_1 = c\tilde{\psi}_1$.
Finally, we try to obtain more information about the value of $\tilde{C}$. Note first that $c \le (1+d)2^{1+d}$. Next, we estimate $\left\|\tilde{\psi}_0(tA)\right\|$ by

$$\left\|\tilde{\psi}_0(tA)\right\| \le \frac{M(A,\omega')}{2\pi}\int_{\Gamma_{\omega'}} \frac{\left|z^{-i\operatorname{Im}(\beta-\gamma)}\right|\,|dz|}{|1+z|^{2+d}} = \frac{M(A,\omega')}{2\pi}e^{\omega'|\operatorname{Im}(\beta-\gamma)|}\int_{\Gamma_{\omega'}}\frac{|dz|}{|1+z|^{2+d}}.$$

Similarly, we obtain

$$\left\|\tilde{\psi}_1(tA)\right\| \le \frac{M(A,\omega')}{2\pi}\int_{\Gamma_{\omega'}} \frac{|z|^d\left|z^{-i\operatorname{Im}(\gamma-\alpha)}\right|\,|dz|}{|1+z|^{2+d}} = \frac{M(A,\omega')}{2\pi}e^{\omega'|\operatorname{Im}(\gamma-\alpha)|}\int_{\Gamma_{\omega'}}\frac{|z|^d\,|dz|}{|1+z|^{2+d}}.$$

Since $\sup_{z\in\Gamma_{\omega'}}|z/1+z| = \sup_{z\in\Gamma_{\omega'}}|1/1+z| = \max(1, |\sin\omega'|^{-1})$, we have

$$\tilde{C} \le M(A,\omega')\frac{(1+d)2^{1+d}\max(1,|\sin\omega'|^{-d})}{2\pi}\left(\int_{\Gamma_{\omega'}}\frac{|dz|}{|1+z|^2}\right)e^{\omega'\max(|\operatorname{Im}(\beta-\gamma)|,|\operatorname{Im}(\gamma-\alpha|)}.$$

Hence, by defining $C(\omega',d) := M(A,\omega')\max(1,|\sin\omega'|^{-d})\frac{(1+d)2^d}{d\pi}\int_{\Gamma_{\omega'}}\frac{|dz|}{|1+z|^2}$, the proposition is proved. □

We now turn to an integral representation which historically was one of the first approaches to fractional powers.

Proposition 2.13. (Balakrishnan-Representation)
Let $A \in \operatorname{Sect}(\omega)$ on the Banach space X and let $0 < \operatorname{Re}\alpha < 1$. Then

$$A^\alpha x = \frac{\sin\alpha\pi}{\pi}\int_0^\infty t^{\alpha-1}\,(t+A)^{-1}Ax\,dt \tag{2.2}$$

for all $x \in \mathcal{D}(A)$. More general, we have

$$A^\alpha x = \frac{\Gamma(m)}{\Gamma(\alpha)\Gamma(m-\alpha)}\int_0^\infty t^{\alpha-1}[A(t+A)^{-1}]^m x\,dt \tag{2.3}$$

where $0 < \operatorname{Re}\alpha < n \le m$ and $x \in \mathcal{D}(A^n)$.

Proof. Assume first that $0 < \operatorname{Re}\alpha < 1$. For $x \in \mathcal{D}(A)$ we have

$$A^\alpha x = \frac{1}{2\pi i}\int_{\Gamma_\varphi} z^{\alpha-1} R(z,A)Ax\,dz, \tag{2.4}$$

where $\omega < \varphi < \pi$. In fact, we can compute

$$\begin{aligned} A^\alpha x &= \left(\frac{z^\alpha}{z+\varepsilon}\right)(A)(\varepsilon + A)x = \left(\frac{z^\alpha}{z+\varepsilon}\right)(A)Ax + \varepsilon\left(\frac{z^\alpha}{(z+\varepsilon)(z+1)}\right)(A)(1+A)x \\ &= \frac{1}{2\pi i}\int_{\Gamma_\varphi} z^{\alpha-1}\left(\frac{z}{z+\varepsilon}\right)R(z,A)Ax\,dz + \frac{\varepsilon}{2\pi i}\int_{\Gamma_\varphi}\left(\frac{z^\alpha}{(z+\varepsilon)(z+1)}\right)R(z,A)(1+A)x\,dz, \end{aligned}$$

where $\varepsilon > 0$. Letting $\varepsilon \searrow 0$, the second summand vanishes and we obtain (2.4). Note that the function $z \longmapsto z^{\alpha-1}R(z,A)Az$ is integrable on the boundary Γ of the sector: $R(z,A)Az$ is bounded in 0 and $O(|z|^{-1})$ for $z \to \infty$. The functions $z \longmapsto (z/z+\varepsilon)$ are bounded on Γ_φ uniformly in ε, hence the Dominated Convergence Theorem is applicable.
Starting from (2.4) we "deform" the path Γ_φ onto the negative real axis. This means that the opening angle φ of Γ_φ is enlarged until the angle π is reached. Cauchy's theorem ensures that the integral does not change its value during this deforming procedure. Lebegue's theorem shows that the limit is exactly (2.2).
So we have proved the first part of the Proposition. For the second we assume $m = n$ and $n-1 < \operatorname{Re}\alpha < n$. Then we write

$$\begin{aligned} A^\alpha x = A^{\alpha-(n-1)}A^{n-1}x &= \frac{\sin(\alpha-n+1)\pi}{\pi}\int_0^\infty t^{\alpha-n}(t+A)^{-1}A^n x\,dt \\ &\overset{\text{Int.b.p.}}{=} \frac{\sin((\alpha-n+1)\pi)(n-1)!}{\pi(\alpha-n+1)(\alpha-n+2)\dots(\alpha-1)}\int_0^\infty t^{\alpha-1}(t+A)^{-n}A^n x\,dt \\ &= \frac{\Gamma(n)}{\Gamma(\alpha)\Gamma(n-\alpha)}\int_0^\infty t^{\alpha-1}(t+A)^{-n}A^n x\,dt, \end{aligned}$$

where $x \in \mathcal{D}(A^n)$. Here we used the standard formulae for the Gamma function $z\Gamma(z) = \Gamma(z+1)$ and $\sin(\pi z)/\pi = 1/(\Gamma(z)\Gamma(1-z))$. An holomorphy argument allows us to replace the assumption $n-1 < \operatorname{Re}\alpha$ by $0 < \operatorname{Re}\alpha < n = m$. Thus we have proved that (2.3) holds for $n = m$.
To prove the general statement we use induction on m. The assertion is already known for $n = m$. Define

$$c_m := \frac{\Gamma(m)}{\Gamma(m-\alpha)\Gamma(\alpha)} \quad \text{and} \quad I_m := \int_0^\infty t^{\alpha-1}[A(t+A)^{-1}]^m x\,dt.$$

Then we have

$$\begin{aligned} I_m &\overset{\text{I.b.p.}}{=} \frac{t^\alpha}{\alpha}[A(t+A)^{-1}]^m x|_0^\infty + \frac{m}{\alpha}\int_0^\infty t^\alpha[A(t+A)^{-1}]^m(t+A)^{-1}x\,dt \\ &= \frac{m}{\alpha}\int_0^\infty t^\alpha[A(t+A)^{-1}]^m(t+A)^{-1}x\,dt \\ &= \frac{m}{\alpha}\int_0^\infty t^{\alpha-1}([A(t+A)^{-1}]^m x - [A(t+A)^{-1}]^{m+1})x\,dt \\ &= \frac{m}{\alpha}(I_m - I_{m+1}). \end{aligned}$$

This means that $\frac{m}{m-\alpha}I_{m+1} = I_m$. Since $c_m\frac{m}{m-\alpha} = c_{m+1}$, the induction is complete. □

Corollary 2.14. *Let $0 < \alpha < n$. Then $\sup_{t>0}\left\|(t(t+A)^{-1})^\alpha\right\| \le M(A)^n$ and $\sup_{t>0}\left\|(A(t+A)^{-1})^\alpha\right\| \le (M(A)+1)^n$.*

Proof. Let $0 < \alpha < 1$. Then we have

$$\begin{aligned} \left((t+A)^{-1}\right)^\alpha &= \frac{\sin\pi\alpha}{\pi}\int_0^\infty s^{\alpha-1}(t+A)^{-1}[s+(t+A)^{-1}]^{-1}\,ds \\ &= \frac{\sin\pi\alpha}{\pi}\int_0^\infty s^{\alpha-1}(st+1+A)^{-1}\,ds, \quad \text{whence} \\ \left\|\left((t+A)^{-1}\right)^\alpha\right\| &\le \frac{\sin\pi\alpha}{\pi}\int_0^\infty s^{\alpha-1}(st+1)^{-1}\,dt M(A) = \frac{1}{t^\alpha}M(A). \end{aligned}$$

The general statement follows with the help of an easy induction argument. The proof of the second assertion is similar, see [MCSA01, Remark 5.1.2]. □

Corollary 2.15. *Let $A \in \operatorname{Sect}(\omega)$ and $0 < \operatorname{Re}\alpha < 1$. Then*

$$A^\alpha x = \frac{\sin \pi\alpha}{\pi} \int_0^\infty t^{-\alpha}(t + A^{-1})^{-1} x \, dt$$

for $x \in \mathcal{D}(A)$. (The operator A^{-1} may be multivalued, see Remark 1.3.)

Proof. Starting from the Balakrishnan representation (2.2) we obtain

$$\begin{aligned} A^\alpha x &= \frac{\sin \pi\alpha}{\pi} \int_0^\infty s^\alpha (s+A)^{-1} Ax \frac{ds}{s} \overset{t=1/s}{=} \frac{\sin \pi\alpha}{\pi} \int_0^\infty t^{-\alpha} (\frac{1}{t} + A)^{-1} Ax \frac{dt}{t} \\ &= \frac{\sin \pi\alpha}{\pi} \int_0^\infty t^{-\alpha} (I - \frac{1}{t}(\frac{1}{t} + A)^{-1}) x \frac{dt}{t} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty t^{-\alpha} t (t + A^{-1})^{-1} x \frac{dt}{t} \\ &= \frac{\sin \pi\alpha}{\pi} \int_0^\infty t^{-\alpha} (t + A^{-1})^{-1} x \, dt \end{aligned}$$

by the fundamental identity (1.1). □

Proposition 2.16. *Assume $A \in \operatorname{Sect}(\omega)$, $x \in \mathcal{D}(A)$, and $0 \le \varphi < \frac{\pi}{2}$. Then the following assertions hold.*

a) $x \in \overline{\mathcal{R}(A)} \iff \lim_{\alpha \to 0,\, \alpha \in S_\varphi} A^\alpha x = x$.

b) $Ax \in \overline{\mathcal{D}(A)} \iff \lim_{\alpha \to 1,\, \alpha \in 1 - S_\varphi} A^\alpha x = Ax$.

Proof. We prove *a)*. The implication "⇐" is immediate from Corollary 2.11. To prove the reverse direction we use the Balakrishnan representation 2.2 and write

$$\begin{aligned} \|A^\alpha x - Ax\| &= \left\| \frac{\sin \alpha\pi}{\pi} \int_0^\infty t^{\alpha-1}((t+A)^{-1}Ax - \frac{1}{t+1}x) dt \right\| \\ &\le \left| \frac{\sin \alpha\pi}{\pi} \right| \int_0^L t^{\operatorname{Re}\alpha - 1} \left\| (t+A)^{-1}Ax - x \right\| + t^{\operatorname{Re}\alpha} \|x\| \, dt \\ &\quad + \left| \frac{\sin \alpha\pi}{\pi} \right| \int_L^\infty t^{\operatorname{Re}\alpha - 2} \left\| t(t+A)^{-1}Ax - \frac{t}{1+t}x \right\| dt \\ &\le \left| \frac{\sin \alpha\pi}{\alpha\pi} \right| \frac{|\alpha|}{\operatorname{Re}\alpha} \Big(L^{\operatorname{Re}\alpha} \sup_{t \le L} \left\| (t+A)^{-1}Ax - x \right\| + \frac{\operatorname{Re}\alpha}{\operatorname{Re}\alpha + 1} L^{\operatorname{Re}\alpha + 1} \|x\| \\ &\quad + \frac{\operatorname{Re}\alpha}{1 - \operatorname{Re}\alpha} L^{\operatorname{Re}\alpha - 1} (M \|Ax\| + \|x\|) \Big) . \end{aligned}$$

Note that $\sin \alpha\pi / \alpha\pi$ is continuous in 0 and that $|\alpha| / \operatorname{Re}\alpha$ is bounded by $(\cos\varphi)^{-1}$. Due to $x \in \overline{\mathcal{R}(A)}$ we can choose the number L in such a way that $\sup_{t \le L} \left\| (t+A)^{-1}Ax - x \right\|$ is small. For a fixed L the other summands tend to zero as $\operatorname{Re}\alpha \to 0$.
The proof of *b)* requires similar arguments, see [MCSA01, p.62]. □

Remark 2.17. Let $A \in \operatorname{Sect}(\omega)$ and $0 < \alpha < 1$. Then $A^\alpha \in \operatorname{Sect}(\alpha\omega)$ as we know from Proposition 2.2. By applying the same technique as in the proof of Proposition 2.13 one can obtain a Balakrishnan-type representation for the resolvent of A^α, i.e.,

$$R(\lambda, A^\alpha) = \frac{-\sin \alpha\pi}{\pi} \int_0^\infty \frac{t^\alpha}{(\lambda - t^\alpha e^{i\pi\alpha})(\lambda - t^\alpha e^{-i\pi\alpha})} (t+A)^{-1} \, dt$$

for $|\arg \lambda| > \alpha\pi$. One can deduce $M(A^\alpha) \le M(A)$ from this, see [MCSA01, (5.24) and (5.25)] and [Tan79, (2.23)].

§ 2 Fractional Powers with Arbitrary Real Part

To introduce fractional powers with arbitrary real part, i.e., in order to define

$$A^\alpha := (z^\alpha)(A) \qquad (\alpha \in \mathbb{C}).$$

we have to assume that the sectorial operator A is *injective*, because $z^\alpha \in \mathcal{B}(S_\varphi)$ for all $\alpha \in \mathbb{C}$ and each $0 < \varphi < \pi$.

Proposition 2.18. *Let $A \in \operatorname{Sect}(\omega)$ be injective. Then the following assertions hold.*

a) The operator A^α is injective with $(A^\alpha)^{-1} = A^{-\alpha} = (A^{-1})^\alpha$ for each $\alpha \in \mathbb{C}$.

b) We have $A^\alpha A^\beta \subset A^{\alpha+\beta}$ with $\mathcal{D}(A^\beta) \cap \mathcal{D}(A^{\alpha+\beta}) = \mathcal{D}(A^\alpha A^\beta)$ for all $\alpha, \beta \in \mathbb{C}$.

c) If $\overline{\mathcal{D}(A)} = X = \overline{\mathcal{R}(A)}$, then $A^{\alpha+\beta} = \overline{A^\alpha A^\beta}$ for all $\alpha, \beta \in \mathbb{C}$.

d) If $0 < \operatorname{Re}\alpha < 1$ then

$$A^{-\alpha}x = \frac{\sin \pi\alpha}{\pi} \int_0^\infty t^{-\alpha}(t+A)^{-1}x\, dt$$

for all $x \in \mathcal{R}(A)$.

e) If $\alpha \in \mathbb{R}$ satisfying $|\alpha| < \frac{\pi}{\omega}$, then $A^\alpha \in \operatorname{Sect}(|\alpha|\,\omega)$ and one has

$$(A^\alpha)^\beta = A^{\alpha\beta}$$

for all $\beta \in \mathbb{C}$.

f) Let $\operatorname{Re}\alpha_0, \operatorname{Re}\alpha_1 > 0$. Then $\mathcal{D}(A^{\alpha_1}) \cap \mathcal{R}(A^{\alpha_0}) \subset \mathcal{D}(A^\alpha)$ for each α with $-\operatorname{Re}\alpha_0 < \operatorname{Re}\alpha < \operatorname{Re}\alpha_1$. The mapping

$$(\alpha \longmapsto A^\alpha x) : \{\alpha \mid -\operatorname{Re}\alpha_0 < \operatorname{Re}\alpha < \operatorname{Re}\alpha_1\} \longrightarrow X$$

is holomorphic for each $x \in \mathcal{D}(A^{\alpha_1}) \cap \mathcal{R}(A^{\alpha_0})$.

Proof. Part *a)* follows from *f)* and *g)* of Proposition 1.16 in the same way as *e)* of Proposition 2.1. Part *b)* is immediate from *e)* of Proposition 1.16.
We prove *c)*. For $1 < n \in \mathbb{N}$ we define

$$\tau_n(A) := n(n+A)^{-1} - \frac{1}{n}(\frac{1}{n}+A)^{-1} = (n+A)^{-1}\left(\frac{n-1}{n}\right)A(\frac{1}{n}+A)^{-1}.$$

Then it is easy to see that for $k \in \mathbb{N}$ the following assertions hold.

a) $\sup_n \left\|\tau_n(A)^k\right\| < \infty$.

b) $\lim_{n\to\infty} \tau_n(A)^k x = x$ for $x \in \overline{\mathcal{D}(A)} \cap \overline{\mathcal{R}(A)}$.

c) $\mathcal{R}(\tau_n(A)^k) \subset \mathcal{D}(A^k) \cap \mathcal{R}(A^k)$, and $\overline{\mathcal{D}(A^k) \cap \mathcal{R}(A^k)} = \overline{\mathcal{D}(A)} \cap \overline{\mathcal{R}(A)}$.

We now choose $k \in \mathbb{N}$ such that $\mathcal{D}(A^k) \cap \mathcal{R}(A^k) \subset \mathcal{D}(A^\beta)$. Let $x \in \mathcal{D}(A^{\alpha+\beta})$ and define $x_n := \tau_n(A)^k x$. Then $x_n \in \mathcal{D}(A^{\alpha+\beta}) \cap \mathcal{D}(A^\beta) \subset \mathcal{D}(A^\alpha A^\beta)$ and $x_n \to x$. Furthermore, $A^\alpha A^\beta x_n = \tau_n(A)^k A^{\alpha+\beta}x \to A^{\alpha+\beta}x$. This shows the claim.
Part *d)* follows from Corollary 2.15 by replacing A by A^{-1}; part *e)* follows from *a)* and Proposition 2.2 together with the composition rule (Proposition 1.20).
To prove *f)*, let $\operatorname{Re}\alpha_0, \operatorname{Re}\alpha_1 > 0$, $x \in \mathcal{D}(A^{\alpha_1}) \cap \mathcal{R}(A^{\alpha_0})$, and $-\operatorname{Re}\alpha_0 < \alpha < \operatorname{Re}\alpha_1$. If $\alpha \notin i\mathbb{R}$, then it is clear from Proposition 2.1 that $x \in D(A^\alpha)$. Assume $\alpha = is \in i\mathbb{R}$. Choose $n \in \mathbb{N}$ such that $\operatorname{Re}\alpha_0, \operatorname{Re}\alpha_1 < n$ and define $B := A^{1/n}$. Then $x \in \mathcal{D}(B) \cap \mathcal{R}(B)$, hence $x \in \mathcal{D}(B^{ins}) = \mathcal{D}(A^{is}) = \mathcal{D}(A^\alpha)$. Since $A^\alpha x = A^{\alpha+\alpha_0}A^{-\alpha_0}x$ the second statement follows from Corollary 2.6. ☐

For actual computations the following integral representations are of great importance.

Proposition 2.19. (Komatsu-Representation)
Let $A \in \operatorname{Sect}(\omega)$ be injective. The identities

$$\begin{aligned} A^\alpha x &= \frac{\sin \pi\alpha}{\pi}\Big[\frac{1}{\alpha}x - \frac{1}{1+\alpha}A^{-1}x + \int_0^1 t^{\alpha+1}(t+A)^{-1}A^{-1}x\,dt \\ &\qquad + \int_1^\infty t^{\alpha-1}(t+A)^{-1}Ax\,dt\Big] \qquad (2.5)\\ &= \frac{\sin \pi\alpha}{\pi}\Big[\frac{1}{\alpha}x + \int_0^1 t^{-\alpha}(1+tA)^{-1}Ax\,dt \\ &\qquad - \int_0^1 t^{\alpha}(1+tA^{-1})^{-1}A^{-1}x\,dt\Big] \qquad (2.6)\end{aligned}$$

hold for $|\operatorname{Re}\alpha| < 1$ and $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$.

Note that the second formula is symmetric in A and A^{-1} whence it can again be seen that $A^{-\alpha}x = (A^{-1})^\alpha x$ for $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$.

Proof. We assume first that $0 < \operatorname{Re}\alpha < 1$. Starting from the Balakrishnan-representation (2.2) we obtain

$$\begin{aligned} \frac{\pi}{\sin\alpha\pi}A^\alpha x &= \int_0^\infty t^{\alpha-1}(t+A)^{-1}Ax\,dt = \int_0^1 t^{\alpha-1}(t+A)^{-1}Ax\,dt + \int_1^\infty t^{\alpha-1}(t+A)^{-1}Ax\,dt \\ &= \int_0^1 t^{\alpha-1}(1+tA^{-1})^{-1}x\,dt + \int_1^\infty t^{\alpha-1}(t+A)^{-1}Ax\,dt \\ &= \int_0^1 t^{\alpha-1}(1-(1+tA^{-1})^{-1}tA^{-1})x\,dt + \int_1^\infty t^{\alpha-1}(t+A)^{-1}Ax\,dt \\ &= \frac{1}{\alpha}x - \int_0^1 t^{\alpha}(1+tA^{-1})^{-1}A^{-1}x\,dt + \int_1^\infty t^{\alpha-1}(t+A)^{-1}Ax\,dt \qquad (*) \\ &= \frac{1}{\alpha}x - \int_0^1 t^{\alpha}(1+tA^{-1})^{-1}A^{-1}x\,dt + \int_0^1 t^{-\alpha}(1+tA)^{-1}Ax\,dt \end{aligned}$$

for $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$, where in the last step we have replaced t by t^{-1} in the second integral. This last formula makes sense even for $-1 < \operatorname{Re}\alpha < 1$. Hence, by holomorphy, we obtain (2.6) for $|\operatorname{Re}\alpha| < 1$. The representation (2.5) now follows from $(*)$ with the help of the identity $t^{-1}(t+A)^{-1} = t^{-1}A^{-1} - (t+A)^{-1}A^{-1}$. □

A particularly important subclass of the injective sectorial operators are the invertible ones.

Proposition 2.20. *Let $A \in \operatorname{Sect}(\omega)$ and assume $0 \in \varrho(A)$. Then the mapping*

$$(\alpha \longmapsto A^{-\alpha}) : \{\operatorname{Re}\alpha > 0\} \longrightarrow \mathcal{L}(X)$$

is holomorphic. For every $0 < \varphi < \frac{\pi}{2}$,

$$\sup\{\|A^{-\alpha}\| \mid 0 < \operatorname{Re}\alpha < 1,\ |\arg\alpha| \le \varphi\} < \infty. \qquad (2.7)$$

Moreover, we have

$$x \in \overline{\mathcal{D}(A)} \quad \Longleftrightarrow \quad \lim_{\alpha\to 0, |\arg\alpha|\le\varphi} A^{-\alpha}x = x.$$

Proof. The first assertion follows from $A^{-\alpha} = (A^{-1})^{\alpha}$ and *a)* of Prop.2.1. To prove (2.7) we employ *d)* of Prop. 2.18 and obtain

$$\left\|A^{-\alpha}x\right\| \le K\frac{|\sin\pi\alpha|}{\pi}\int_0^\infty t^{-\operatorname{Re}\alpha}(t+1)^{-1}\,dt\ \|x\| = K\frac{|\sin\pi\alpha|}{\sin(\pi\operatorname{Re}\alpha)}\|x\|,$$

where $K := \sup_{t>0}\left\|(t+1)(t+A)^{-1}\right\| < \infty$. Hence $\left\|A^{-\alpha}\right\| \le K\left|\frac{\sin\pi\alpha}{\pi\alpha}\right|\frac{\pi\operatorname{Re}\alpha}{\sin(\pi\operatorname{Re}\alpha)}\frac{|\alpha|}{\operatorname{Re}\alpha}$. If $A^{-\alpha}x \to x$ then $x \in \overline{\mathcal{D}(A)}$ by Corollary 2.11. The second implication follows from *a)* of Prop. 2.16. □

Proposition 2.21. *Let $-A$ be the generator of an exponentially stable semigroup $T = (T(t))_{t\ge0}$. Then $A \in \operatorname{Sect}(\frac{\pi}{2})$ and $0 \in \varrho(A)$. Furthermore,*

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}T(t)\,dt \tag{2.8}$$

for all $\operatorname{Re}\alpha > 0$.

Proof. If $-A$ generates a *bounded* semigroup, it is sectorial of angle $\pi/2$. This follows from Proposition A.27. By exponential stability there are $M \ge 1, \varepsilon > 0$ such that $\|T(t)\| \le Me^{-\varepsilon t}$ for all $t \ge 0$. The resolvent of $-A$ can be computed via the Laplace transform of the semigroup, i.e.,

$$(u+A)^{-1}x = \int_0^\infty e^{-ut}T(s)x\,dt$$

for each $x \in X$ and each $u > -\varepsilon$. In particular, $0 \in \varrho(A)$. To establish the identity (2.8) for all $\operatorname{Re}\alpha > 0$, it suffices to prove it for $0 < \alpha < 1$ (by holomorphy). Using *d)* of Proposition 2.18 we obtain for $x \in X$

$$\begin{aligned} A^{-\alpha}x &= \frac{\sin\pi\alpha}{\pi}\int_0^\infty u^{-\alpha}(u+A)^{-1}x\,du = \frac{\sin\pi\alpha}{\pi}\int_0^\infty u^{-\alpha}\int_0^\infty e^{-ut}T(t)x\,dt\,du \\ &\overset{\text{Fub.}}{=} \frac{\sin\pi\alpha}{\pi}\int_0^\infty\left(\int_0^\infty u^{-\alpha}e^{-ut}\,du\right)T(t)x\,dt \\ &\overset{s=u/t}{=} \frac{\sin\pi\alpha}{\pi}\int_0^\infty\int_0^\infty\left(\frac{s}{t}\right)^{-\alpha}e^{-s}\,ds\,t^{-1}T(t)x\,dt = \frac{\sin\pi\alpha}{\pi}\Gamma(1-\alpha)\int_0^\infty t^{\alpha-1}T(t)x\,dt. \end{aligned}$$

Now, the well known formula $\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin\alpha\pi}$ (valid for all $\alpha \in \mathbb{C}\setminus\mathbb{Z}$) yields (2.8). □

Note that in the last proposition we did not assume the semigroup to be strongly continuous in 0.

§3 Holomorphic Semigroups

In this section we survey the basic properties of (bounded) holomorphic semigroups. The material is (almost) standard and is only included because it fits nicely into our functional calculus framework.

Let A be a sectorial operator of angle $0 \le \omega < \frac{\pi}{2}$. In contrast to other sections we allow the operator A *to be multivalued*. For $0 \ne \lambda \in \mathbb{C}$ with $|\arg\lambda| < \frac{\pi}{2} - \omega$ the function $e^{-\lambda z}$ clearly is in $\mathcal{DR}_0(S_\varphi)$ for each φ satisfying $\omega < \varphi < \frac{\pi}{2} - |\arg\lambda|$. [For $z \in S_\varphi$ we have $\left|e^{-\lambda z}\right| = e^{-\operatorname{Re}(\lambda z)} = e^{-|\lambda||z|\cos(\arg\lambda+\arg z)} \le e^{-|\lambda|\cos(|\arg\lambda|+\varphi)|z|}$.] This allows us to define

$$e^{-\lambda A} := (e^{-\lambda z})(A) \in \mathcal{L}(X). \tag{2.9}$$

Proposition 2.22. *Let A be a multivalued sectorial operator of angle ω with $\omega < \frac{\pi}{2}$. Then the following assertions hold.*

a) $e^{-\lambda A}e^{-\mu A} = e^{-(\lambda+\mu)A}$ *for all* $\lambda, \mu \in S_{\frac{\pi}{2}-\omega}$.

b) *The mapping*
$$(\lambda \longmapsto e^{-\lambda A}) : S_{\frac{\pi}{2}-\omega} \longrightarrow \mathcal{L}(X)$$
is holomorphic.

c) *Let* $0 < \varphi < \frac{\pi}{2} - \omega$. *Then*
$$\sup\{\|e^{-\lambda A}\| \mid |\arg \lambda| \le \varphi\} < \infty.$$
More precisely, one has $\|e^{-\lambda A}\| \le M(A, \omega')C(e^{-z}, \omega')$ *for all* $|\arg \lambda| \le \varphi$ *and* $\varphi + \omega < \omega' < \frac{\pi}{2}$.

d) *The identity*
$$(\mu + A)^{-1} = \int_0^\infty e^{-\mu t}e^{-tA}\, dt$$
holds true for all $\operatorname{Re}\mu > 0$.

e) *For all* $\lambda \in S_{\frac{\pi}{2}-\omega}$ *we have* $\mathcal{R}(e^{-\lambda A}) \subset \bigcap_{n\in\mathbb{N}} \mathcal{D}(A^n)$.

f) *If* $x \in \overline{\mathcal{D}(A)}$ *then*
$$\lim_{\lambda\to 0, |\arg \lambda|\le\varphi} e^{-\lambda A}x = x$$
for each $0 < \varphi < \frac{\pi}{2} - \omega$.

Proof. Assertion *a)* follows from the multiplicativity of the $\mathcal{DR}_0$-functional calculus (see Prop. 1.7). The statement in *b)* is easily proved by using the definition of $e^{-\lambda A}$ as a Cauchy integral and applying the Dominated Convergence Theorem. We obtain
$$\frac{d^n}{d\lambda^n}(e^{-\lambda A}) = ((-z)^n e^{-\lambda z})(A).$$
Assertion *c)* is a special case of Prop. 1.22 in noting that $e^{-\lambda A} = (e^{-z})(\lambda A)$ (which is an instance of the composition rule). Given $\operatorname{Re}\mu > 0$ we compute
$$\begin{aligned}\int_0^\infty e^{-\mu t}e^{-tA}\, dt &= \int_0^\infty e^{-\mu t}\frac{1}{2\pi i}\int_\Gamma e^{-tz}R(z,A)\, dz = \frac{1}{2\pi i}\int_\Gamma\int_0^\infty e^{-\mu t}e^{-tz}\, dt\, R(z,A)\, dz \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{1}{\mu + z}R(z,A)\, dz = (\mu + A)^{-1}\end{aligned}$$
by *c)* of Proposition 1.7. (Here, Γ is an appropriate path avoiding 0, see Chapter I, §3.)
To prove *e)*, take $n \in \mathbb{N}$ and $\lambda \in S_{\frac{\pi}{2}-\omega}$. Since both functions $(1+z)^{-n}$ and $(1+z)^n e^{-\lambda z}$ are contained in $\mathcal{DR}_0$ we obtain $e^{-\lambda A} = (1+A)^{-n}((1+z)^n e^{-\lambda z})(A)$ by multiplicativity.
We prove *f)*. Choose $0 < \varphi < \pi/2 - \omega$. Then $(e^{-\lambda z}(1+z)^{-1})(A) \to (1+A)^{-1}$ in norm for $S_\varphi \ni \lambda \to 0$ by the Dominated Convergence Theorem. Hence $e^{-\lambda A}x = (e^{-\lambda z}(1+z)^{-1})(A)z \to x$ for $x \in \mathcal{D}(A)$ and $z \in (1+A)x$. By the uniform boundedness proved in *c)* we obtain $e^{-\lambda A}x \to x$ even for all $x \in \overline{\mathcal{D}(A)}$. This completes the proof. □

Proposition 2.23. *Let* $A \in \operatorname{Sect}(\omega)$ *for* $\omega < \frac{\pi}{2}$ *and assume that A is single valued. Then we have* $f(A)e^{-\lambda A} \in \mathcal{L}(X)$ *for all* $\lambda \in S_{\frac{\pi}{2}-\omega}$ *and* $f \in \mathcal{A}[S_\omega]$. *Moreover, let* $0 < \varphi < \frac{\pi}{2} - \omega$ *and* $\operatorname{Re}\alpha > 0$. *Then*
$$\sup\left\{|\lambda|^{\operatorname{Re}\alpha}\|A^\alpha e^{-\lambda A}\| \mid \arg \lambda \le \varphi\right\} < \infty.$$
More precisely, one has $\|A^\alpha e^{-\lambda A}\| \le M(A, \omega' - \varphi)C(z^\alpha e^{-z}, \omega')e^{(\operatorname{Im}\alpha \arg \lambda)}\, |\lambda|^{-\operatorname{Re}\alpha}$ *for all* $|\arg \lambda| \le \varphi$ *and* $\varphi + \omega < \omega' < \frac{\pi}{2}$.

Proof. Take $f \in \mathcal{A}[S_\omega]$. Then $f(z)e^{-\lambda z}$ is regularly decaying at ∞ and $f(z)e^{-\lambda z} - f(0)$ is regularly decaying at 0, hence $f(z)e^{-\lambda z} \in H(A)$ (see Remark 1.14). By a standard functional calculus argument we obtain $f(A)e^{-\lambda A} = (f(z)e^{-\lambda z})(A) \in \mathcal{L}(X)$. The last statement follows from Proposition 1.22 in noting that
$\lambda^\alpha A^\alpha e^{-\lambda A} = (z^\alpha e^{-z})(\lambda A)$ (composition rule). □

Let $0 < \theta \leq \frac{\pi}{2}$. A mapping $T : S_\theta \longrightarrow \mathcal{L}(X)$ is called a **bounded holomorphic (degenerate) semigroup** (of angle θ) if it has the following properties:

1) The **semigroup law** $T(\lambda)T(\mu) = T(\lambda + \mu)$ holds for all $\lambda, \mu \in S_\theta$.
2) The mapping $T : S_\theta \longrightarrow \mathcal{L}(X)$ is holomorphic.
3) The mapping T satisfies $\sup_{\lambda \in S_\varphi} \|T(\lambda)\| < \infty$ for each $0 < \varphi < \theta$.

(Note that, by *a), b)* and *c)* of Proposition 2.22, $(e^{-\lambda A})_{\lambda \in S_\theta}$ is a bounded holomorphic semigroup if $A \in \mathrm{Sect}(\omega)$ and $\theta := \frac{\pi}{2} - \omega > 0$.) By holomorphy, T is uniquely determined by its values on $(0, \infty)$. Moreover, if the semigroup law holds for real values, then it holds for all λ (see the proof of Proposition 3.7.2 in [ABHN01]). Also by holomorphy and the semigroup law, the space $\mathcal{N}_T := \mathcal{N}(T(\lambda))$ is independent of $\lambda \in S_\theta$. Hence either each or none of the operators $T(\lambda)$ is injective.

If we restrict a bounded holomorphic semigroup T to the positive real axis $(0, \infty)$, we obtain a bounded semigroup as defined in Section A.7. This semigroup has a **generator** B which is defined via its resolvent by

$$R(\lambda, B) := \int_0^\infty e^{-\lambda t} T(t)\, dt$$

for $\mathrm{Re}\,\lambda > 0$. Let $A := -B$. Then we have $A0 = \mathcal{N}_T$ by (A.1) on page 150. Hence, A is single-valued if and only if $T(w)$ is injective for one/all $w \in S_\theta$.

The m.v. operator A is sectorial (at least of angle $\frac{\pi}{2}$), since

$$\left\|\lambda(\lambda + A)^{-1}\right\| = \|\lambda R(\lambda, B)\| \leq (\sup_{t>0} \|T(t)\|) \frac{|\lambda|}{\mathrm{Re}\,\lambda}$$

for all $\mathrm{Re}\,\lambda > 0$. But even more is true: The m.v. operator A is sectorial of angle $\frac{\pi}{2} - \theta$, and we have $T(\lambda) = e^{-\lambda A}$ for all $\lambda \in S_\theta$.

[Let $\varphi \in (-\theta, \theta)$ and consider the bounded holomorphic semigroup $T(e^{i\varphi}\cdot)$ on $S_{\theta - |\varphi|}$. We know already that there is a m.v. operator $A_\varphi \in \mathrm{Sect}(\pi/2)$ such that $(\lambda + A_\varphi)^{-1} = \int_0^\infty e^{-\lambda t} T(te^{i\varphi})\, dt$ for all $\mathrm{Re}\,\lambda > 0$. Let $\Gamma = (0, \infty)e^{i\varphi}$. By Cauchy's theorem,

$$(s+A)^{-1} = \int_0^\infty e^{-st} T(t)\, dt = \int_\Gamma e^{-sz} T(z)\, dz = e^{i\varphi} \int_0^\infty e^{-se^{i\varphi}t} T(te^{i\varphi})\, dt = e^{i\varphi}(se^{i\varphi} + A_\varphi)^{-1}$$

for all $s > 0$. Hence $(s + A)^{-1} = (s + e^{-i\varphi} A_\varphi)^{-1}$ for all $s > 0$. This gives $e^{i\varphi} A = A_\varphi$. Since φ ranges between $-\theta$ and θ and each A_φ is sectorial of angle $\pi/2$ we obtain that A is sectorial of angle $\pi/2 - \theta$. Employing *d)* of Proposition 2.22 and the injectivity of the Laplace transform we conclude $T(t) = e^{-tA}$ for all $t > 0$. By holomorphy, this implies $T(\lambda) = e^{-\lambda A}$ for all $\lambda \in S_\theta$.]

Combining the above considerations with Proposition 2.22 we can summarize:

Proposition 2.24. *There is a one-one correspondence between m.v. sectorial operators A of angle $\omega < \frac{\pi}{2}$ and bounded holomorphic semigroups T on $S_{\frac{\pi}{2}-\omega}$ given by the relations*

$$T(z) = e^{-zA} \ \ (z \in S_{\frac{\pi}{2}-\omega}), \qquad (\lambda + A)^{-1} = \int_0^\infty e^{-\lambda t} T(t)\, dt \ \ (\operatorname{Re}\lambda > 0).$$

The operator A is single-valued if and only if $T(z)$ is injective for one/all $z \in S_{\frac{\pi}{2}-\omega}$.

Remark 2.25. An **exponentially bounded holomorphic semigroup** of angle $0 < \theta \leq \frac{\pi}{2}$ is a holomorphic mapping $T : S_\theta \longrightarrow \mathcal{L}(X)$ with the semigroup property and such that $\{T(\lambda) \mid \lambda \in S_\varphi, \ |\lambda| \leq 1\}$ is bounded for each $0 < \varphi < \theta$. Given such a semigroup for each $0 < \varphi < \theta$ one can find $M_\varphi \geq 1, w_\varphi \geq 0$ such that

$$\|T(\lambda)\| \leq M_\varphi e^{w_\varphi \operatorname{Re}\lambda}$$

for $\lambda \in S_\varphi$. In particular, $T|_{(0,\infty)}$ is an exponentially bounded semigroup in the sense of Section A.7. Thus it has a generator $-A$, which is characterized by

$$(\lambda + A)^{-1} = \int_0^\infty e^{-\lambda t} T(t)\, dt$$

which holds for sufficiently large $\operatorname{Re}\lambda$. Given $0 < \varphi < \theta$ and w_φ as above, we have

$$e^{-w_\varphi \lambda}\, T(\lambda) = e^{-\lambda(A + w_\varphi)}$$

for $\lambda \in S_\varphi$. Thus, all statements on exponentially bounded holomorphic semigroups can be reduced to statements on bounded holomorphic semigroups. For example, given $\varphi \in (-\theta, \theta)$, the mapping $(t \longmapsto T(e^{i\varphi}t))$ is an exponentially bounded semigroup with generator $-e^{i\varphi}A$. (See the proof before Proposition 2.24.)

From Corollary A.29 we know that the **space of strong continuity** of $T|_{(0,\infty)}$ is exactly $\overline{\mathcal{D}(A)}$. Employing Proposition 2.24 and part *f)* of Proposition 2.22 we see that even

$$\lim_{S_\varphi \ni \lambda \to 0} T(\lambda)x = x$$

for $x \in \overline{\mathcal{D}(A)}$ and each $0 \leq \varphi < \theta$.

Let $A \in \operatorname{Sect}(\omega)$ on the Banach space X and assume that $0 \in \varrho(A)$. We know from Proposition 2.20 that $(A^{-z})_{\operatorname{Re} z > 0}$ is an exponentially bounded holomorphic semigroup of angle $\frac{\pi}{2}$, with $\overline{\mathcal{D}(A)}$ as its space of strong continuity. In the next section we will identify its generator.

§ 4 The Logarithm and the Imaginary Powers

We return to our terminological agreement that "operator" always is to be read as "single-valued operator" (see the agreement on p. 137). In fact, we will work with an injective, single-valued operator $A \in \operatorname{Sect}(\omega)$, where $\omega < \pi$.

Since the function $\log z$ is contained in the class $\mathcal{B}(S_\varphi)$ for each $0 < \varphi \le \pi$ it is reasonable to call

$$\log A := (\log z)(A)$$

the **logarithm** of the operator A. Because of $\log(z^{-1}) = -\log z$ we have $\log(A^{-1}) = -\log A$. The next result is quite fundamental.

Lemma 2.26. (Nollau)
Let $A \in \mathrm{Sect}(\omega)$ be injective. If $|\mathrm{Im}\,\lambda| > \pi$ then $\lambda \in \varrho(\log A)$ and

$$R(\lambda, \log A) = \int_0^\infty \frac{-1}{(\lambda - \log t)^2 + \pi^2}(t+A)^{-1}\,dt. \tag{2.10}$$

Hence $\|R(\lambda, \log A)\| \le \frac{M(A)\pi}{|\mathrm{Im}\,\lambda| - \pi}$.

We call the formula (2.10) the **Nollau representation** of the resolvent of $\log A$.

Proof. Assume first that $A, A^{-1} \in \mathcal{L}(X)$. We choose $\omega < \varphi < \pi$, $a > 0$ small and $b > 0$ large enough and denote by Γ the positively oriented boundary of the bounded sector $S_\varphi(a,b)$. Then

$$\begin{aligned}
(\lambda - \log z)^{-1}(A) &= \frac{1}{2\pi i}\int_\Gamma \frac{1}{\lambda - \log z} R(z, A)\,dz \\
&= \frac{1}{2\pi i}\int_a^b \frac{e^{-i\varphi}}{\lambda - \log t + i\varphi} R(te^{-i\varphi}, A)\,dt + \frac{1}{2\pi}\int_{-\varphi}^{\varphi} \frac{be^{is}}{\lambda + \log b - is} R(be^{is}, A)\,ds \\
&\quad - \frac{1}{2\pi i}\int_a^b \frac{e^{i\varphi}}{\lambda - \log t - i\varphi} R(te^{i\varphi}, A)\,dt - \frac{1}{2\pi}\int_{-\varphi}^{\varphi} \frac{ae^{is}}{\lambda + \log a - is} R(ae^{is}, A)\,ds \\
&\overset{(1)}{=} \int_a^b \frac{-1}{(\lambda - \log t)^2 + \pi^2}(t+A)^{-1}\,dt + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{be^{is}}{\lambda + \log b - is} R(be^{is}, A)\,ds \\
&\quad - \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{ae^{is}}{\lambda + \log a - is} R(ae^{is}, A)\,ds \\
&\overset{(2)}{=} \int_0^\infty \frac{-1}{(\lambda - \log t)^2 + \pi^2}(t+A)^{-1}\,dt =: J(A),
\end{aligned}$$

where we have let $\varphi \to \pi$ in (1) and $a \to 0, b \to \infty$ in (2). Hence we have proved the claim in the special case of A and A^{-1} being bounded. In the general case, define $A_\varepsilon := (A+\varepsilon)(1+\varepsilon A)^{-1}$. Then $(A_\varepsilon)_{\varepsilon>0}$ is a sectorial approximation of A (see Proposition 1.2). Let $f(z) := (\lambda - \log z)^{-1}$. Obviously $f \in H^\infty(S_\varphi)$ for $\varphi > \omega$. We have already shown that $f(A_\varepsilon) = J(A_\varepsilon) \in \mathcal{L}(X)$. It follows from the Dominated Convergence Theorem that

$$f(A_\varepsilon) \to J(A) = \int_0^\infty \frac{-1}{(\lambda - \log t)^2 + \pi^2}(t+A)^{-1}\,dt.$$

in norm. Applying Proposition 1.21 we obtain $f(A) = J(A)$. From *d)* of Corollary 1.18 we conclude that in fact $f(A) = (\lambda - \log A)^{-1}$. Having shown this we compute

$$\begin{aligned}
\|R(\lambda, \log A)\| &\le \int_0^\infty \frac{M(A)}{|(\lambda - \log t)^2 + \pi^2|}\frac{dt}{t} = \int_{\mathbb{R}} \frac{M(A)}{|(i\,\mathrm{Im}\,\lambda - s)^2 + \pi^2|}\,ds \\
&= \int_{\mathbb{R}} \frac{M(A)}{\sqrt{(s^2 - ((\mathrm{Im}\,\lambda)^2 - \pi^2)^2 + 4s^2(\mathrm{Im}\,\lambda)^2}}\,ds \\
&\le \int_{\mathbb{R}} \frac{M(A)}{(s^2 + ((\mathrm{Im}\,\lambda)^2 - \pi^2)}\,ds = \frac{M(A)\pi}{\sqrt{(\mathrm{Im}\,\lambda)^2 - \pi^2}} \le \frac{M(A)\pi}{|\mathrm{Im}\,\lambda| - \pi}
\end{aligned}$$

where we have used the inequalities

$$\begin{aligned}
(s^2 - ((\mathrm{Im}\,\lambda)^2 - \pi^2))^2 + 4s^2(\mathrm{Im}\,\lambda)^2 &\ge (s^2 - ((\mathrm{Im}\,\lambda)^2 - \pi^2))^2 + 4s^2((\mathrm{Im}\,\lambda)^2 - \pi^2) \\
&= (s^2 + ((\mathrm{Im}\,\lambda)^2 - \pi^2))^2, \qquad \text{and}
\end{aligned}$$

$$\sqrt{(\mathrm{Im}\,\lambda)^2 - \pi^2} \ge |\mathrm{Im}\,\lambda| - \pi$$

which are easy to verify. $\square$

Proposition 2.27. *Let $A \in \mathrm{Sect}(\omega)$ be injective. If $|\mathrm{Im}\,\lambda| > \omega$ then $\lambda \in \varrho(\log A)$ and for each $\varphi > \omega$ there is a constant M_φ such that*

$$\|R(\lambda, \log A)\| \le \frac{M_\varphi}{|\mathrm{Im}\,\lambda| - \varphi}$$

for $|\mathrm{Im}\,\lambda| > \varphi$. In fact $M_\varphi = \pi M(A^{\frac{\pi}{\varphi}})$. Furthermore, $\mathcal{D}(\log A) \subset \overline{\mathcal{D}(A)} \cap \overline{\mathcal{R}(A)}$.

Proof. Let $\alpha := \pi/\varphi$. Then $A^\alpha \in \mathrm{Sect}(\alpha\omega)$ is injective. The composition rule yields $\log(A^\alpha) = \alpha \log(A)$. Applying Nollaus's Lemma 2.26 we obtain $\|R(\mu, \alpha \log A)\| \le M(A^\alpha)\pi(|\mathrm{Im}\,\mu| - \pi)^{-1}$ for $|\mathrm{Im}\,\mu| > \pi$. Letting $\lambda = \mu/\alpha$ we arrive at $\|R(\lambda, \log A)\| \le M(A^\alpha)\pi(|\mathrm{Im}\,\lambda| - \varphi)^{-1}$ for $|\mathrm{Im}\,\lambda| > \pi/\alpha = \varphi$. From the Nollau-representation (2.10) for $R(\cdot, \log A)$ it is immediate that $\mathcal{D}(\log A) \subset \overline{\mathcal{D}(A)}$. But $-\log A = \log A^{-1}$, hence also $\mathcal{D}(\log A) \subset \overline{\mathcal{D}(A^{-1})} = \overline{\mathcal{R}(A)}$. □

As promised, we can now identify the generator of the holomorphic semigroup $(A^{-z})_{\mathrm{Re}\,z>0}$, where A is a sectorial and invertible operator.

Proposition 2.28. *Let $A \in \mathrm{Sect}(\omega)$ with $0 \in \varrho(A)$. Then $-\log A$ is the generator of the holomorphic semigroup $(A^{-z})_{\mathrm{Re}\,z>0}$. In particular, $\overline{\mathcal{D}(\log A)} = \overline{\mathcal{D}(A)}$.*

Proof. By semigroup theory there is $c > 0$ such that $(e^{-ct}A^{-t})_{t>0}$ is a bounded semigroup. Clearly it suffices to show that $(\lambda + \log A)^{-1} = \int_0^\infty e^{-\lambda t} A^{-t}\, dt$ for some λ with $\mathrm{Re}\,\lambda > c$ and $|\mathrm{Im}\,\lambda > \pi|$. So choose such a λ and let $0 < a < b < \infty$. Then

$$\begin{aligned}\int_a^b e^{-\lambda t} A^{-t}\, dt &= \int_a^b e^{-\lambda t} \frac{1}{2\pi i} \int_\Gamma z^{-t} R(z, A)\, dz\, dt = \frac{1}{2\pi i} \int_\Gamma \int_a^b e^{-\lambda t} z^{-t}\, dt R(z, A)\, dz \\ &= \frac{1}{2\pi i} \int_\Gamma \frac{e^{-\lambda a} z^{-a} - e^{-\lambda b} z^{-b}}{\lambda + \log z} R(z, A)\, dz \\ &= e^{-\lambda a} A^{-a} (\lambda + \log A)^{-1} - e^{-\lambda b} (\lambda + \log A)^{-1},\end{aligned}$$

where Γ is an appropriate path avoiding 0 (see Chapter I, §4). The last equality is due to the fact that $(\lambda + \log z)^{-1}(A) = (\lambda + \log A)^{-1} \in \mathcal{L}(X)$ by Nollau's Lemma 2.26. Since $\mathrm{Re}\,\lambda > c$ we have $\left\|e^{-\lambda b} A^{-b}\right\| \to 0$ for $b \to \infty$. From $\mathcal{D}(\log A) \subset \overline{\mathcal{D}(A)}$ and Proposition 2.20 we see that $e^{-\lambda a} A^{-a} (\lambda + \log A)^{-1} \to (\lambda + \log A)^{-1}$ strongly. This finishes the proof. □

Remark 2.29. Suppose that A is a *bounded* sectorial operator. As a matter of fact, the fractional powers $(A^z)_{\mathrm{Re}\,z>0}$ of A form a quasi-bounded holomorphic semigroup as well. What is its generator? If A is injective, the answer is of course $\log A$. But if A is not injective, the generator is not single-valued and can not be obtained by *our* functional calculus. Since also in this case the generator should be $\log A$ in a sense, we feel the necessity of a sophisticated functional calculus for multivalued operators. However, we do not further pursue this matter here.

We end our investigation of the logarithm with the following nice observation. Consider the function $f(z) := \log z - \log(z + \varepsilon) = \log(z(z + \varepsilon)^{-1})$. A short computation reveals that f is holomorphic in ∞ with $f(\infty) = 0$. Hence, if $A \in \mathrm{Sect}(\omega)$ and $0 \in \varrho(A)$, then $f(A) \in \mathcal{L}(X)$, i.e., $f \in H(A)$. From the usual rules of functional calculus we see that $\log(A + \varepsilon)$ is a bounded perturbation of $\log A$. In particular, $\mathcal{D}(\log A) = \mathcal{D}(\log(A + \varepsilon))$.

We now turn to the **imaginary powers** A^{is} of an injective operator $A \in \mathrm{Sect}(\omega)$.

Proposition 2.30. *Let $A \in \mathrm{Sect}(\omega)$ be injective and let $0 \neq s \in \mathbb{R}$. Then the following assertions hold.*

a) If $A^{is} \in \mathcal{L}(X)$ then $\mathcal{D}(A^\alpha) \subset \mathcal{D}(A^{\alpha+is})$ for all $\alpha \in \mathbb{C}$. Conversely, if $\mathcal{D}(A^\alpha) \subset \mathcal{D}(A^{\alpha+is})$ for $\alpha \in \{-1, 1\}$, then $A^{is} \in \mathcal{L}(X)$.

b) If $0 \in \varrho(A)$ and $\varepsilon > 0$, then

$$(A + \varepsilon)^{is} = (1 + \varepsilon A^{-1})^{is} A^{is}.$$

In particular, $\mathcal{D}((A+\varepsilon)^{is}) = \mathcal{D}(A^{is})$. Moreover, $(A+\varepsilon)^{is}x \to A^{is}x$ for $\varepsilon \to 0$ and $x \in \mathcal{D}(A^{is})$.

c) The space $D := \mathcal{D}((A + \varepsilon)^{is})$ is independent of $\varepsilon > 0$. If $x \in D$ and $\lim_{\varepsilon\to 0}(A + \varepsilon)^{is}x =: y$ exists, then $x \in \mathcal{D}(A^{is})$ with $y = A^{is}x$.

d) If $A^{is} \in \mathcal{L}(X)$, then also $(A + \varepsilon)^{is} \in \mathcal{L}(X)$ for all $\varepsilon > 0$. Moreover, $\sup_{0\leq\varepsilon\leq 1} \left\|(A + \varepsilon)^{is}\right\| < \infty$ and $\lim_{\varepsilon\to 0}(A + \varepsilon)^{is}x = A^{is}x$ for all $x \in \overline{\mathcal{R}(A)}$.

e) Let $0 \in \varrho(A)$. If $A^{is} \in \mathcal{L}(X)$, then $\sup_{0<\alpha<1} \left\|A^{-\alpha+is}\right\| < \infty$ and

$$\lim_{\alpha\searrow 0} A^{-\alpha+is}x = A^{is}x$$

for all $x \in \overline{\mathcal{D}(A)}$. Conversely, if $\sup_{0<\alpha<1} \left\|A^{-\alpha+is}\right\| < \infty$ and $\overline{\mathcal{D}(A)} = X$, then $A^{is} \in \mathcal{L}(X)$.

Proof. Ad *a)*. Assume $A^{is} \in \mathcal{L}(X)$ and let $\alpha \in \mathbb{C}$. Then $A^{is}A^\alpha \subset A^{\alpha+is}$, hence $\mathcal{D}(A^\alpha) = \mathcal{D}(A^{is}A^\alpha) \subset \mathcal{D}(A^{\alpha+is})$. Conversely, assume $\mathcal{D}(A) \subset \mathcal{D}(A^{1+is})$ and $\mathcal{R}(A) \subset \mathcal{D}(A^{-1+is})$. Since always $\mathcal{D}(A^{1+is}) \cap \mathcal{D}(A) = \mathcal{D}(A^{is}A)$, we have $\mathcal{D}(A) = \mathcal{D}(A^{is}A)$. Hence $\mathcal{R}(A) \subset \mathcal{D}(A^{is})$. Similarly we obtain $\mathcal{D}(A) = \mathcal{R}(A^{-1}) \subset \mathcal{D}(A^{is})$. But $\mathcal{D}(A) + \mathcal{R}(A) = X$, whence $A^{is} \in \mathcal{L}(X)$.
Ad *b)*. Since $A(A + \varepsilon)^{-1}$ is bounded and invertible, we have $(A(A + \varepsilon)^{-1})^{is} \in \mathcal{L}(X)$. But

$$(1 + \varepsilon A^{-1})^{-is} = (A(A + \varepsilon)^{-1})^{is} = \left(\frac{z}{z + \varepsilon}\right)^{is}(A)$$

by the composition rule. Hence $(z(z + \varepsilon)^{-1})^{is} \in H(A)$. Therefore,

$$(A + \varepsilon)^{is}(1 + \varepsilon A^{-1})^{-is} = \left[(z + \varepsilon)^{is}\left(\frac{z}{z + \varepsilon}\right)^{is}\right](A) = (z^{is})(A) = A^{is}$$

(see Corollary 1.18). From this we conclude

$$(A + \varepsilon)^{is} = (1 + \varepsilon A^{-1})^{is}(A + \varepsilon)^{is}(1 + \varepsilon A^{-1})^{-is} = (1 + \varepsilon A^{-1})^{is}A^{is}$$

by *c)* of Corollary 1.18. The last statement follows from the fact that $(1 + \varepsilon A^{-1})^{is} \to I$ in norm for $\varepsilon \to 0$ (apply *b)* of Proposition 1.21).
Ad *c)*. The first statement is immediate from *b)*. Now assume $x \in D$ and $\lim_{\varepsilon\searrow 0}(A + \varepsilon)^{is}x =: y$ exists. Now

$$(\tau(z)z^{is})(A) = \left(\left(\frac{z}{z + \varepsilon}\right)^{is}\tau(z)\right)(A + \varepsilon)^{is}x \to \tau(A)y \qquad (\varepsilon \searrow 0)$$

by Lemma 1.24. But this is just to say that $A^{is}x = y$.
Ad *d)*. Assume $A^{is} \in \mathcal{L}(X)$. Then

$$\mathcal{D}(A + \varepsilon) = \mathcal{D}(A) \subset \mathcal{D}(A^{1+is}) = \mathcal{D}((A + \varepsilon)^{1+is})$$

by *a)* and Proposition 2.9. Since $(A + \varepsilon)^{-1+is}$ is bounded, we also have $\mathcal{D}((A + \varepsilon)^{-1}) = X \subset \mathcal{D}((A + \varepsilon)^{-1+is})$. Applying *a)* again we obtain $(A + \varepsilon)^{is} \in \mathcal{L}(X)$. From

$$(A + \varepsilon)^{is}x = \left[(A + \varepsilon)^{\frac{1}{2}+is} - A^{\frac{1}{2}+is}\right](A + \varepsilon)^{-\frac{1}{2}}x + A^{is}\left[A(A + \varepsilon)^{-1}\right]^{\frac{1}{2}}x$$

we conclude $\sup_{0\le\varepsilon\le 1}\left\|(A+\varepsilon)^{is}\right\| < \infty$ with the help of Proposition 2.7 and Corollary 2.14. Now, let $x = A^{\frac{1}{2}}y \in \mathcal{R}(A^{\frac{1}{2}})$. Then the first summand is $O(\varepsilon^{\frac{1}{2}})$. The second summand can be written as $(A(A+\varepsilon)^{-1})A^{is}(A+\varepsilon)^{\frac{1}{2}}y$ and this tends to $A^{is}A^{\frac{1}{2}}y = A^{is}x$ for $\varepsilon \searrow 0$, since $A^{is} \in \mathcal{L}(X)$, $(A+\varepsilon)^{\frac{1}{2}}y \to A^{\frac{1}{2}}y$ (by Proposition 2.9), and $A(A+\varepsilon)^{-1} \to I$ strongly on $\overline{\mathcal{R}(A)}$ (by Proposition 1.1).
Ad *e)*. The statements follow almost immediately from Proposition 2.20. □

Example 7.3.3 in [MCSA01] which goes back to KOMATSU yields a *bounded* sectorial operator A on a Banach space X such that $A^{is} \notin \mathcal{L}(X)$ for all $0 \neq s \in \mathbb{R}$. This shows in particular that in general $\mathcal{D}(A^{is}) \neq \mathcal{D}((A+\varepsilon)^{is})$.

The next result shows that — in a way — the operator $i\log A$ can be considered the "generator" of the operator family $(A^{is})_{s\in\mathbb{R}}$. Recall the notation $\Lambda_A^{-1} = A(1+A)^{-2}$ for an injective sectorial operator A and note that $(A^{is}\Lambda_A^{-1})_{s\in\mathbb{R}}$ is a strongly continuous family of bounded operators on X.

Proposition 2.31. *Let $A \in \operatorname{Sect}(\omega)$ be injective. For each $\theta > \omega$ there is a constant $C_\theta \ge 0$ such that*
$$\left\|A^{is}\Lambda_A^{-1}\right\| \le M(A,\theta)C_\theta e^{|s|\theta}$$
for all $s \in \mathbb{R}$. Moreover, one has
$$(\lambda - i\log A)^{-1} = \Lambda_A \int_0^\infty e^{-\lambda s}A^{is}\Lambda_A^{-1}\,ds \tag{2.11}$$
for all $\operatorname{Re}\lambda > \omega$.

Proof. Choose $\theta > \omega$. Then $\left\|A^{is}\Lambda_A^{-1}\right\| = \left\|(z^{is}\tau(z))(A)\right\| \le C(f,\theta)M(A,\theta)$, where $f(z) = z^{is}\tau(z)$ (see (1.8) on page 40). But
$$C(f,\theta) \le \frac{1}{2\pi}\int_{\Gamma_\theta}\frac{\left|z^{is}\right|}{|1+z|^2}\,|dz| = \frac{1}{2\pi}\int_{\Gamma_\theta}\frac{e^{-s\arg z}}{|1+z|^2}\,|dz| \le \frac{e^{|s|\theta}}{2\pi}\int_{\Gamma_\theta}\frac{1}{|1+z|^2}\,|dz|\,.$$
For $0 < a < b$ we compute
$$\begin{aligned}
\int_a^b e^{-\lambda s}A^{is}\Lambda_A^{-1}\,ds &= \int_a^b e^{-\lambda s}\frac{1}{2\pi i}\int_\Gamma \frac{z^{is}z}{(1+z)^2}R(z,A)\,dz \\
&= \frac{1}{2\pi i}\int_\Gamma \frac{z}{(1+z)^2}\left(\int_a^b z^{is}e^{-\lambda s}\,ds\right)R(z,A)\,dz \\
&= \frac{1}{2\pi i}\int_\Gamma \frac{z}{(1+z)^2}\frac{e^{-(\lambda-i\log z)a} - e^{-(\lambda-i\log z)b}}{\lambda - i\log z}R(z,A)\,dz \\
&= \frac{e^{-\lambda a}}{2\pi i}\int_\Gamma \frac{z^{ia}z}{(1+z)^2(\lambda-i\log z)}R(z,A)\,dz - \\
&\qquad \frac{e^{-\lambda b}}{2\pi i}\int_\Gamma \frac{z^{ib}z}{(1+z)^2(\lambda-i\log z)}R(z,A)\,dz \\
&= e^{-\lambda a}A^{ia}\Lambda_A^{-1}R(\lambda, i\log A) - e^{-\lambda b}A^{ib}\Lambda_A^{-1}R(\lambda, i\log A)
\end{aligned}$$
The second summand tends to 0 in norm for $b \to \infty$, since $\operatorname{Re}\lambda > \omega$. The first summand tends to $\Lambda_A^{-1}R(\lambda, i\log A)$ strongly for $a \to 0$ (by *f)* of Proposition 2.18). Hence we obtain
$$\int_0^\infty e^{-\lambda s}A^{is}\Lambda_A^{-1}\,ds = \Lambda_A^{-1}R(\lambda, i\log A)$$
which immediately implies (2.11). □

Corollary 2.32. *Let $A \in \operatorname{Sect}(\omega)$ be injective. The following assertions are equivalent.*

(i) $\overline{\mathcal{D}(A) \cap \mathcal{R}(A)} = X$ *and* $A^{is} \in \mathcal{L}(X)$ *for all* $s \in \mathbb{R}$.

(ii) The operators $(A^{is})_{s\in\mathbb{R}}$ *form a* C_0*-group of bounded operators on* X.

(iii) The operator $i \log A$ *generates a* C_0*-group* $(T(s))_{s\in\mathbb{R}}$ *of bounded operators on* X.

In this case we have $T(s) = A^{is}$ *for all* $s \in \mathbb{R}$.

Proof. $(i) \Rightarrow (ii)$. Define $T(s) := A^{is}$ for $s \in \mathbb{R}$. Obviously, T is a group. For every $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$, $T(\cdot)x$ is continuous. This follows from Proposition 2.18. Since $\overline{\mathcal{D}(A) \cap \mathcal{R}(A)} = X$, we have that for *each* $x \in X$ the trajectory $T(\cdot)x$ is at least strongly measurable. From Theorem 10.2.3 of [HP74] we infer that T is strongly continuous on $(0, \infty)$, but this implies readily that T is strongly continuous on the whole real line.
$(ii) \Rightarrow (iii)$. This follows from Proposition 2.31, since the hypothesis allows us to put the operator Λ_A^{-1} in front of the integral.
$(iii) \Rightarrow (i)$. Assume that $i \log A$ generates the C_0-group T. In particular, $\mathcal{D}(\log A)$ must be dense in X. Now, $\mathcal{D}(\log A) \subset \overline{\mathcal{D}(A)} \cap \overline{\mathcal{R}(A)} = \overline{\mathcal{D}(A) \cap \mathcal{R}(A)}$ by Proposition 2.27. Thus, we are left to show $T(s) = A^{is}$ for all $s \in \mathbb{R}$. Employing Proposition 2.31 we obtain

$$\int_0^\infty e^{-\lambda s} A^{is} \Lambda_A^{-1}\, ds = \Lambda_A^{-1} R(\lambda, i \log A) = \int_0^\infty e^{-\lambda s} \Lambda_A^{-1} T(s)\, ds$$

for $\operatorname{Re} \lambda$ large. The Uniqueness Theorem for the Laplace transform implies $A^{is}\Lambda_A^{-1} = \Lambda_A^{-1} T(s)$ for all $s > 0$. Multipying both sides of this equation with Λ_A yields $A^{is} = T(s)$ for $s > 0$. From this we infer $A^{-is} = (A^{is})^{-1} = T(s)^{-1} = T(-s)$ for all $s > 0$, and the proof is complete. □

If A satisfies the equivalent conditions in Corollary 2.32 we say that A has **bounded imaginary powers** and write $A \in \operatorname{BIP}(X)$. Note that $\overline{\mathcal{D}(A) \cap \mathcal{R}(A)} = X$ is included in this definition; there exist operators A such that $A^{is} \in \mathcal{L}(X)$ for all s and even $\sup_{s\in\mathbb{R}} \|A^{is}\| < \infty$, but $\overline{\mathcal{D}(A) \cap \mathcal{R}(A)} \neq X$ (see Example 7.3.1 in [MCSA01]). If $A \in \operatorname{BIP}(X)$, by semigroup theory, we know that

$$\theta_A := \inf\{\theta \geq 0 \mid \exists C : \|A^{is}\| \leq C\, e^{|s|\theta} \ (s \in \mathbb{R})\} < \infty.$$

We will write $A \in \operatorname{BIP}(X, \theta)$ if $A \in \operatorname{BIP}(X)$ and $\theta \geq \theta_A$. Note that this notation slightly differs from the terminology used in [MCSA01, Definition 8.1.1]. The first part of the celebrated Prüss-Sohr theorem states that always $\theta_A \geq \omega_A$ if $A \in \operatorname{BIP}(X)$. We will reprove (and in fact generalize) this result in Chapter 3 §3, see Corollary 3.12.

Remark 2.33. Suppose that $(A^{is})_{s>0} \subset \mathcal{L}(X)$ is strongly continuous and norm bounded in 0. This means that $T(s) := A^{is}$ is an exponentially bounded semigroup in the terminology of Section A.7. It follows from Proposition 2.31 that the operator $i \log A$ is the generator of T even in this situation. The same conclusion holds (with an appropriate notion of "generator") in case that $(s \longmapsto A^{is})$ is continuous only with respect to a "very weak" topology, like those considered in [Küh01b] and [Küh01a]. In any case, Proposition 2.31 implies that $(A^{is}\Lambda_A^{-1})_{s\in\mathbb{R}}$ is an exponentially bounded Λ_A^{-1}-regularized group with generator $i \log A$, see [deL94] and compare also [Oka00b, Addendum] and [Oka00a].

§ 5 Comments

A Recent Monograph on Fractional Powers. Beginning with the seminal papers by KRASNOSELSKII and SOBOLEVSKI [KS59], BALAKRISHNAN [Bal60], YOSIDA [Yos60], KATO [Kat60], around 1960, fractional powers have been the subject of extensive research. A first attempt to exhaustively present the theory was undertaken by KOMATSU in a series of papers [Kom66], [Kom67], [Kom69b], [Kom69a], [Kom70] since around 1965. Although the fractional powers were widely used in the theory of evolution equations (an early example is KREIN's book [Kn71], see also [Paz83]), the first monograph to appear on the theory of fractional powers was [MCSA01] in 2001. Without doubt, this book is a milestone. However, it has some considerable drawbacks. The authors are quite fond of proving results in such a way that as much as possible remains valid on general sequentially complete locally convex spaces, and on these spaces "non-negativity" of an operator is not the same as "sectoriality" (as it is in Banach spaces, see Proposition 1.1, *a)*). To this aim they base their construction on the Balakrishnan-representation, which is a real integral. As a consequence, they are forced to give a definition of fractional powers "by cases" as is sketched in Remark 2.10. This leads to fuzzy proofs, compare their proof of the scaling property. (On the other hand they provide a proof of the spectral mapping theorem $\sigma(A^\alpha) = \sigma(A)^\alpha$ which we did not.) Apart from elegance, another drawback is that their construction is not embedded in a large functional calculus (although they make constant use of an extension of the Hirsch functional calculus). The powerful composition rule therefore is not at hand.

§1 Fractional Powers with Positive Real Part. The results presented in this section are standard and in fact all included in [MCSA01, Chapter V]. The proof of Proposition 2.2 is from [AMN97, Proposition 3.5]. After proving Proposition 2.7 we learned that the same already had been done in [LM01, Lemma 3.3] The idea for the proof of the moment inequality (Proposition 2.12) is from [Wei]. However, since in this text the operator is assumed to have dense domain and dense range we had to adapt the proof to the general situation. Here we needed McIntosh's approximation technique provided in Proposition 1.31. The proof of the Balakrishnan-representation and its corollaries is from [MCSA01].

§2 Fractional Powers with Arbitrary Real Part. Like in the previous section, the results are well-known. We took Proposition 2.20 and Proposition 2.21 from [Ama95, Section III.4.6].

§3 Holomorphic Semigroups. This section is inspired by [Lun95]. The generalization to the multivalued case seems to be new as it stands. However, there are similar results in [FY99, Chapter III]

§4 The Logarithm and the Imaginary Powers. The first reference for Lemma 2.26 is NOLLAU's paper [Nol69]. Our proof is an adaptation of [MCSA01, Lemma 10.1.5] and [Oka00b, Lemma 5.1] (for the norm estimate). Proposition 2.27 is stated and proved in the Hilbert space case in [MCSA01, Theorem

10.1.6] (without using any Hilbert space properties). Our proof is slightly different. The basic facts on the imaginary powers collected in Proposition 2.30 are extracted from [MCSA01, Chapter 7]. Part *d)* is a perturbation result, see also §4 in Chapter 3. Proposition 2.31 is new, as far as we know. Corollary 2.32 is essentially in [MCSA01, Theorem 10.1.3 and Theorem 10.1.4]. An earlier account can be found in [Oka00b, Addendum] and [Oka00a]. From [Uit98, Proposition 2.2.31] have we learned the argument in the proof of the implication $(i) \Rightarrow (ii)$.

Fractional Powers and Interpolation Spaces. The connection between fractional powers and *real* interpolation spaces in a general setting was (maybe first?) examined in [Kom67]. KOMATSU could show that for a densely defined, sectorial operator A the identity

$$\mathcal{D}(A^\alpha) = (A, \mathcal{D}(A))_{\mathrm{Re}\,\alpha,p}$$

for some $1 \le p < \infty$ and some $0 < \mathrm{Re}\,\alpha < 1$ implies the identity for *all* $0 < \mathrm{Re}\,\alpha < 1$. Moreover, if in addition it is assumed that $0 \in \varrho(A)$, then $A \in \mathrm{BIP}(X)$ (see [Kom67, Proposition 2.9] and compare [MCSA01, Corollary 11.5.3]).[1]
The Hilbert space case was treated by LIONS in his influencing paper [Lio62] where he showed that for m-accretive operators, the spaces $\mathcal{D}(A^\alpha)$ coincide with the real interpolation spaces $(X, \mathcal{D}(A))_{\mathrm{Re}\,\alpha,2}$ constructed by the trace method. In his proof he utilizes KATO's generalization of the Heinz Inequality to m-accretive operators in [Kat61b] and the fact that the result is true for positive selfadjoint operators. (Implicit in this is the identity of complex and real interpolation spaces on Hilbert spaces.)
Around 1970, starting with the paper [See71], it was realized that the hypothesis $A \in \mathrm{BIP}(X)$ is the right one to imply equality of the domains of fractional powers $\mathcal{D}(A^\alpha)$ with the *complex* interpolation spaces. This fact can be found in [Tri95, Section 1.15.3], a recent account is [MCSA01, Theorem 11.6.1]. However, both references leave open the question if the converse statement holds, i.e., if an equality $\mathcal{D}(A^\alpha) = [X, \mathcal{D}(A)]_{1-\alpha}$ for some $0 < \alpha < 1$ already implies $A \in \mathrm{BIP}(X)$. At least in Hilbert spaces, this must be true, as KOMATSU's result shows.
Altough the first edition of TRIEBEL's book appeared in 1978 and KOMATSU's paper is even older, YAGI in his 1984 paper [Yag84] does not mention them. He reproved the stated equivalence for operators on Hilbert spaces. The sufficiency of the assumption $A \in \mathrm{BIP}(H)$ is proved in remarking that KATO's proof of the Heinz Inequality remains valid under this hypothesis. YAGI gave an equivalent characterization in terms of so called **quadratic estimates** for the operator A and its adjoint A^* which were generalized afterwards by MCINTOSH in [McI86].
More results on interpolation theory in connection with quadratic estimates and H^∞-calculus can be found in [AMN97] and [Dor99].

[1]Note that since there are *bounded* operators without bounded imaginary powers, the second consequence does not hold without the additional assumption.

Problemata**

**In using the title "Problemata", we want to indicate that in the chapters to come we deal with single questions, problems and results. (A systematic treatment would have to put the things into a different order.) Like with the word "Organon" before, we borrow the name from an Aristotelian collection of books called *Problemata Physica*.

Third Chapter
The Logarithm and the Characterization of Group Generators

This chapter opens the stage for a closer examination of the logarithm of a sectorial operator. In §1 we introduce strip type operators as an abstract concept which bundles up the spectral properties of operator logarithms. In §2 we develop (in an *ad hoc* manner) a natural functional calculus for strip type operators including composition rules. As a result we obtain that an injective sectorial operator is uniquely determined by its logarithm (Corollary 3.8). In §3 it is proved that the spectral height of $\log A$ always *equals* the spectral angle of A. This is used in §4 to obtain a new proof of a theorem of PRÜSS and SOHR. In §5 we construct an example of an injective sectorial operator $A \in \mathrm{BIP}(X)$ on an UMD space X such that the group type of $(A^{is})_{s\in\mathbb{R}}$ is strictly greater than π. In §6 we give a characterization of C_0-group generators on Hilbert spaces involving the boundedness of the natural H^∞-calculus on a strip. As a corollary we obtain the result of MCINTOSH and YAGI that for a Hilbert space operator A with bounded imaginary powers, the natural H^∞-calculus is bounded on each sector bigger than the spectral sector.

§1 Strip Type Operators

For $\omega > 0$ we denote by

$$H_\omega := \{z \in \mathbb{C} \mid |\mathrm{Im}\, z| < \omega\}$$

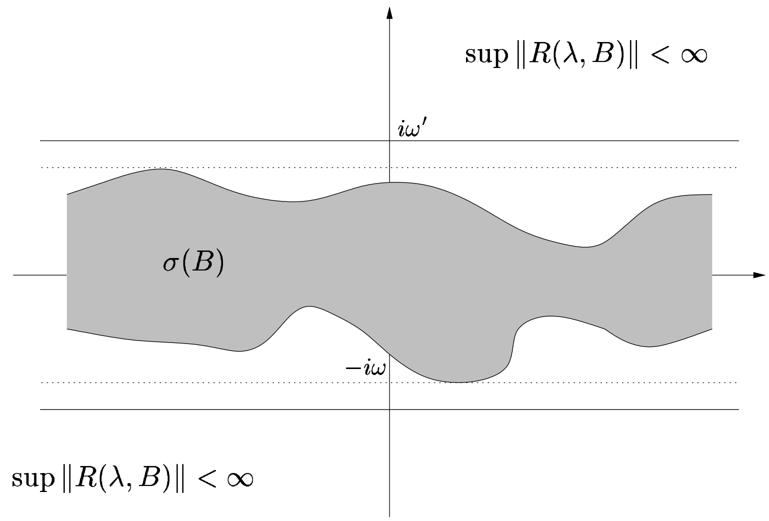
the **horizontal strip** of height ω which is symmetric with respect to the real axis. In the case $\omega = 0$ we define $H_0 := \mathbb{R}$. An operator B on a Banach space X is said to be a **strip type operator** of height ω (in short: $B \in \mathrm{Strip}(\omega)$), if

1) $\sigma(B) \subset \overline{H_\omega}$ and
2) $L(B, \omega') := \sup\{\|R(\lambda, B)\| \mid |\mathrm{Im}\,\lambda| \geq \omega'\} < \infty$ for all $\omega < \omega'$.

It is clear that $B \in \mathrm{Strip}(\omega)$ if and only if $-B \in \mathrm{Strip}(\omega)$ with $L(B, \omega') = L(-B, \omega')$ for each $\omega' > \omega$. We call

$$\omega_{st}(B) := \min\{\omega \geq 0 \mid B \in \mathrm{Strip}(\omega)\}$$

the **spectral height** of B. The following picture illustrates the notion of a strip type operator:



Let us say that an operator B is a **strong strip type operator** of height ω if for each $\varphi > \omega$ there is L_φ such that

$$\|R(\lambda, B)\| \leq \frac{L_\varphi}{|\mathrm{Im}\, \lambda| - \varphi}$$

for all $|\mathrm{Im}\, \lambda| > \varphi$. Such operators obviously have the property that for each $\varphi > \omega$ both operators $\varphi - iB$ and $\varphi + iB$ are sectorial of angle $\frac{\pi}{2}$.

Examples 3.1. We describe three classes of strip type operators which arise in a natural manner.

1) Let $0 \leq \omega < \pi$ and $A \in \mathrm{Sect}(\omega)$ be injective. Then $B := \log A$ is a strong strip type operator of height ω, as we learn from Nollau's theorem (Proposition 2.27). We will prove below (see §3) that in fact $\omega_A = \omega_{st}(\log A)$.
2) Let iB generates a C_0-group T. Then B is a strong strip type operator of height $\theta(T)$, where $\theta(T)$ is the group type of T. In general, it can occur that $\omega_{st}(B) < \theta(T)$. However, Gearhart's Theorem B.24 implies that $\omega_{st}(B) = \theta(T)$ if X is a Hilbert space.
3) Let H be a Hilbert space and B a selfadjoint operator on H. Then $\sigma(B) \subset \mathbb{R}$ and
$$\|R(\lambda, B)\| \leq \frac{1}{|\mathrm{Im}\, \lambda|} \quad (\lambda \notin \mathbb{R})$$
(see Proposition B.10). In particular, B is a strong strip type operator of height 0. Of course, this is a special case of 2) since iB generates a unitary group on H, by Stone's Theorem B.22.

Remark 3.2. As a matter of fact, instead of dealing with *horizontal* strips we could have defined all notions for *vertical* strips. (The stripe type operators in the horizontal and vertical case correspond to each other via the mapping

$(B \mapsto iB)$.) This in fact seems more natural for generators of groups and therefore was done in our papers [Haa02] and [Haa01]. Since the logarithm of a sectorial operator is our guiding example, we will stick to horizontal strips in this exposition.

§ 2 The Natural Functional Calculus

We define the functional calculus for a strip type operator $B \in \mathrm{Strip}(\omega)$ analogously to the sectorial case. Since we do not want to give a systematic treatment here, we can give more *ad hoc* definitions.

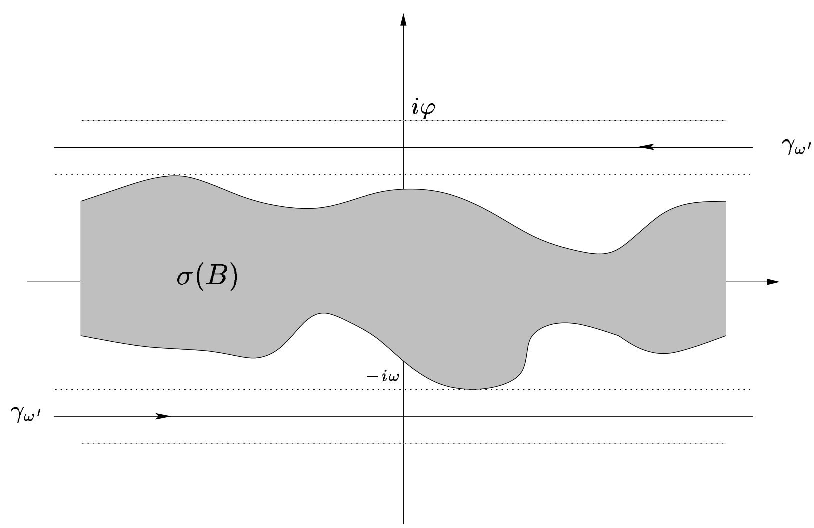
For $\varphi > 0$ we let

$$\mathcal{F}(H_\varphi) := \{f \in \mathcal{O}(H_\varphi) \mid f(z) = O(|\mathrm{Re}\, z|^{-2}) \text{ for } |z| \to \infty\}$$

and define $\gamma_\varphi := \partial H_\varphi$ (oriented in the positive sense).
Given $B \in \mathrm{Strip}(\omega)$, $\omega < \omega' < \varphi$, and $f \in \mathcal{F}(H_\varphi)$, the Cauchy integral

$$f(B) := \frac{1}{2\pi i} \int_{\gamma_{\omega'}} f(z) R(z, B)\, dz \in \mathcal{L}(X)$$

exists, since $R(\cdot, B)$ is bounded on the path $\gamma_{\omega'}$. Here is an illustration:



By Cauchy's theorem, the definition of $f(B)$ is independent of the actual choice of ω'. The following result is not surprising, its proof being analogous to the proof of Proposition 1.7.

Proposition 3.3. *a) The mapping* $(f \longmapsto f(B)) : \mathcal{F}(H_\varphi) \longrightarrow \mathcal{L}(X)$ *is a homomorphism of algebras.*

b) For $\lambda, \mu \notin \overline{H_\varphi}$ *the identity* $\left(\frac{1}{(\lambda - z)(\mu - z)}\right)(B) = R(\lambda, B) R(\mu, B)$ *holds.*

c) If $f \in \mathcal{F}(H_\varphi)$ *and* $\lambda \notin \overline{H_\varphi}$, *then* $(f(z)(\lambda - z)^{-1})(B) = R(\lambda, B) f(B) = f(B) R(\lambda, B)$.

d) *If C is a closed operator commuting with the resolvents of B, then C also commutes with $f(B)$, where $f \in \mathcal{F}(H_\varphi)$. In particular, $f(B)$ commutes with B.*

Choose $\nu > \varphi$. We extend the functional calculus to the class

$$\mathcal{G}(H_\varphi) := \{f : H_\varphi \longrightarrow \mathbb{C} \mid \exists n \in \mathbb{N} : f(z)(\nu^2 + z^2)^{-n} \in \mathcal{F}(H_\varphi)\}$$

in defining

$$f(A) := (\nu^2 + B^2)^n \left(\frac{f(z)}{(\nu^2 + z^2)^n} \right) (B)$$

for $f(\nu^2 + z^2)^{-n} \in \mathcal{F}$. This definition as well as the class $\mathcal{G}(H_\varphi)$ does not depend on the particular choice of $\nu > \varphi$ and $n \in \mathbb{N}$. Note that the class $\mathcal{G}(H_\varphi)$ contains all rational functions r with poles outside of $\overline{H_\varphi}$ as well as the class $H^\infty(H_\varphi)$ of all bounded holomorphic functions on H_φ.

We obtain the characteristic properties of a functional calculus.

Proposition 3.4. *Let $f \in \mathcal{G}(H_\varphi)$. Then the following assertions hold.*

a) *The operator $f(B)$ is closed. It is bounded, if B is. In this case the mapping $f \longmapsto f(B)$ coincides with the usual Dunford calculus.*

b) *If $T \in \mathcal{L}(X)$ commutes with B, then it commutes with $f(B)$. If $f(B)$ is bounded, it commutes with B.*

c) *Let $g \in \mathcal{G}(H_\varphi)$. Then*

$$f(B) + g(B) \subset (f + g)(B) \quad \textit{and} \quad f(B)g(B) \subset (fg)(B)$$

with $\mathcal{D}((fg)(B)) \cap \mathcal{D}(g(B)) = \mathcal{D}(f(B)g(B))$. In particular, if $g(B) \in \mathcal{L}(X)$, one has equality in both formulas.

d) *If also $f^{-1} \in \mathcal{G}(S_\varphi)$, then $f(B)^{-1} = f^{-1}(B)$. In particular, $f(B)$ is injective.*

e) *Let $\lambda \notin \overline{f(H_\varphi)}$. Then we have $(\lambda - f(B))^{-1} = (\lambda - f(z))^{-1}(B)$. We have $\lambda \in \varrho(f(B))$ if and only if $(\lambda - f(z))^{-1}(B) \in \mathcal{L}(X)$.*

f) *If $f, h \in \mathcal{G}$, $f(B) \in \mathcal{L}(X)$ and $f(B)$ is injective, then $f(A)^{-1}h(B)f(B) = h(B)$ in case that either $\varrho(h(B)) \neq 0$ or $f^{-1} \in \mathcal{G}$.*

g) *If $f = r$ is a rational function with poles off $\overline{H_\varphi}$, then $f(B) = r(B)$, where $r(B)$ is understood in the sense of Appendix A, Section A.5.*

Proof. The assertions are proved in exactly the same way as the corresponding statements in Proposition 1.9 and Corollary 1.11. (See also Proposition 1.16 and Corollary 1.18.) □

Let us turn to the standard convergence result.

Proposition 3.5. (Convergence Lemma)
Let X be a Banach space and $B \in \mathrm{Strip}(\omega)$ on X. Let $\varphi > \omega$ and $(f_\alpha)_\alpha$ a net of holomorphic functions on H_φ converging pointwise to a function f on H_φ.

a) *If*

$$\sup_\alpha \sup_{z \in H_\varphi} |f(z)| \, (1 + |\mathrm{Re}\, z|^2) < \infty,$$

then $f \in \mathcal{F}(H_\varphi)$ and $f_\alpha(B) \to f(B)$ in norm.

b) If $f_\alpha \in H^\infty(H_\varphi)$, $f(B) \in \mathcal{L}(X)$ *for all* α, *and*

$$\sup_\alpha \sup_{z \in H_\varphi} |f(z)| < \infty \quad \textit{as well as} \quad \sup_\alpha \|f(B)\| < \infty,$$

then $f_\alpha(B)x \to f(B)x$ *for all* $x \in \overline{\mathcal{D}(B^2)}$.

Proof. The proof is anlogous to the proof of Proposition 1.26 □

One should note that if $B \in \mathrm{Strip}(\omega)$ is densely defined, then $\overline{\mathcal{D}(B^2)} = \overline{\mathcal{D}(B)} = X$.

As in the situation of sectorial operators we have to deal with some instances of the **composition rule**. We do not enumerate all cases but confine ourselves to the most important ones (with a view to logarithms!).

Proposition 3.6. *Let* X *be a Banach space and* B *a closed operator on* X. *Let* $0 \leq \omega < \varphi$ *and* $0 \leq \omega' < \varphi'$. *Then the composition rule*

$$(f \circ g)(B) = f(g(B))$$

holds in the following cases.

1) $\varphi < \pi$, $B \in \mathrm{Sect}(\omega)$ *injective,* $g \in \mathcal{B}(S_\varphi)$, $g(S_\varphi) \subset \overline{H_{\omega'}}$, $g(B) \in \mathrm{Strip}(H_{\omega'})$, $f \in \mathcal{G}(H_{\varphi'})$, $f \circ g \in \mathcal{B}(S_\varphi)$.
2) $B \in \mathrm{Strip}(\omega')$, $g \in \mathcal{G}(H_{\varphi'})$, $g(H_{\varphi'}) \subset \overline{S_\omega}$, $\varphi < \pi$, $g(B) \in \mathrm{Sect}(\omega)$ *injective,* $f \in \mathcal{B}(S_\varphi)$, $f \circ g \in \mathcal{G}(H_{\varphi'})$.
3) $B \in \mathrm{Strip}(\omega')$, $g \in \mathcal{G}(H_{\varphi'})$, $g(H_{\varphi'}) \subset \overline{H_\omega}$, $g(B) \in \mathrm{Strip}(\omega)$, $f \in \mathcal{G}(H_\varphi)$, $f \circ g \in \mathcal{G}(H_{\varphi'})$.

Proof. The proofs are similar to those proofs of instances of the composition rule we have already seen. We therefore restrict ourselves to show the claim in case *1)* for $f \in H^\infty(H_{\varphi'})$. Then clearly $f \circ g \in H^\infty(S_\varphi)$. We choose $\omega < \omega_1 < \varphi$ and $\omega' < \omega_1' < \varphi'$ and let $\Gamma_1 := \partial S_{\omega_1}$ and $\Gamma_2 := \partial H_{\omega_1'}$. Recall that $\Lambda_B^{-1} = B(1+B)^{-2}$. We choose $\nu > \varphi'$ and define $\Lambda_{g(B)} := \nu^2 + g(B)^2$. Then

$$\begin{aligned}
\Lambda_B^{-1} f(g(B)) \Lambda_{g(B)}^{-1} &= \frac{1}{2\pi i} \int_{\Gamma_2} \frac{f(\lambda)}{\nu^2 + \lambda^2} \left(\frac{1}{2\pi i} \int_{\Gamma_1} \frac{z}{(1+z)^2(\lambda - g(z))} R(z,B)\, dz \right) d\lambda \\
&= \frac{1}{2\pi i} \int_{\Gamma_1} \frac{z}{(1+z)^2} \left(\frac{1}{2\pi i} \int_{\Gamma_2} \frac{f(\lambda)}{(\nu^2+\lambda^2)(\lambda - g(z))}\, d\lambda \right) R(z,B)\, dz \\
&= \frac{1}{2\pi i} \int_{\Gamma_1} \frac{z f(g(z))}{(1+z)^2(\nu^2 + g(z)^2)}\, R(z,B)\, dz \\
&= \left(\frac{(f \circ g)(z) z}{(1+z)^2} \right)(B) \Lambda_{g(B)}^{-1}.
\end{aligned}$$

If we multiply both sides by $\Lambda_{g(B)}\Lambda_B$ we obtain $f(g(B)) = \Lambda_{g(B)}(f \circ g)(B)\Lambda_{g(B)}^{-1} = (f \circ g)(B)$ by *f)* of Proposition 3.4. □

Corollary 3.7. *Let* $0 \leq \omega < \pi$ *and* $A \in \mathrm{Sect}(\omega)$ *be injective. Then*

$$f(\log A) = (f(\log z))(A)$$

for all $f \in H^\infty(S_\varphi)$ and all $\omega < \varphi \le \pi$. In particular we have

$$\left(\frac{1}{\lambda - e^z}\right)(\log A) = R(\lambda, A) \qquad \text{and}$$

$$\left(\frac{e^z}{(\lambda - e^z)(\mu - e^z)}\right)(\log A) = AR(\lambda, A)R(\mu, A)$$

for all $\lambda, \mu \notin \overline{S_\omega}$.

Corollary 3.8. *An injective sectorial operator on a Banach space X is uniquely determined by its logarithm, i.e., if A and B are injective sectorial operators on X with $\log A = \log B$, then $A = B$.*

Proof. By Corollary 3.7, we have

$$R(\lambda, A) = (\lambda - e^z)^{-1}(\log A) = (\lambda - e^z)^{-1}(\log B) = R(\lambda, B)$$

for $\lambda < 0$. This implies $A = B$. □

§3 The Spectral Height of the Logarithm

The purpose of this section is to prove the following

Theorem 3.9. *Let X be a Banach space and $A \in \mathrm{Sect}(\omega)$ be injective. If $\log A \in \mathrm{Strip}(\omega')$ for some $\omega' \ge 0$, then $A \in \mathrm{Sect}(\omega')$. In particular, we have*

$$\omega_{st}(\log A) = \omega_A.$$

We let $B := \log A$. Without restriction we can assume $\omega' < \omega$. Let us consider the operator family

$$T(\lambda, \mu) := \left(\frac{e^z}{(\lambda - e^z)(\mu - e^z)}\right)(B) \in \mathcal{L}(X)$$

defined for $\lambda, \mu \notin \overline{S_{\omega'}}$. (Note that $(e^z(\lambda - e^z)^{-1}(\mu - e^z)^{-1}) \in \mathcal{F}(H_\varphi)$ for $\omega' < \varphi < |\arg \lambda|, |\arg \mu|$.) From Corollary 3.7 we learn that

$$T(\lambda, \mu) = AR(\lambda, A)R(\mu, A) \tag{3.1}$$

for $\omega < |\arg \lambda|, |\arg \mu|$.

Lemma 3.10. *Fix μ with $|\arg \mu| > \omega'$. Then the mapping*

$$(\lambda \longmapsto T(\lambda, \mu)) : \mathbb{C} \setminus \overline{S_{\omega'}} \longrightarrow \mathcal{L}(X)$$

is holomorphic.

Proof. It suffices to show that the function

$$\frac{e^z}{(\lambda - e^z)(\mu - e^z)} R(z, B)$$

is integrable on horizontal lines uniformly in $\lambda \in K$, for each compact $K \subset \mathbb{C} \setminus \overline{S_{\omega'}}$.
Choose $\omega' < \varphi < |\arg \mu|$. Define $C_\nu := \mathrm{dist}(\nu, \overline{S_\varphi})$ for $|\arg \nu| > \varphi$. Let $|\arg \lambda| > \varphi$ and define $f_\lambda(z) := e^z(\lambda - e^z)^{-1}(\mu - e^z)^{-1}$. Then

$$|f_\lambda(z)| \le \frac{e^{\mathrm{Re}\, z}}{C_\mu C_\lambda} \quad \text{and} \quad |f_\lambda(z)| \le \frac{e^{-\mathrm{Re}\, z}}{|\lambda|\,|\mu|\, C_{\frac{1}{\lambda}} C_{\frac{1}{\mu}}}$$

for $z \in H_\varphi$. Hence there is $C(\lambda)$, locally bounded in λ, such that $|f_\lambda(z)| \le C(\lambda)e^{-|\mathrm{Re}\, z|}$. This proves the claim. □

We need another Lemma.

Lemma 3.11. *Assume $\omega' < \varphi < \pi$. Then the set*

$$\{\|(\mu-\lambda)T(\lambda,\mu)\| \;\mid\; |\arg\lambda| \ge \varphi,\; \mu = -|\lambda|\}$$

is bounded.

Proof. Choose $\omega' < \omega_1 < \varphi$. We write $(\mu-\lambda)T(\lambda,\mu)$ as a Cauchy integral along the path $\gamma := \partial H_{\omega_1}$. We obtain

$$\|(\mu-\lambda)T(\lambda,\mu)\| = \left\|\frac{1}{2\pi i}\int_\gamma \frac{(\mu-\lambda)e^z}{(\lambda-e^z)(\mu-e^z)} R(z,B)\,dz\right\| \le \frac{L(B,\omega_1)}{2\pi}\int_\gamma \frac{|(\mu-\lambda)e^z|\;|dz|}{|\lambda-e^z|\,|(\mu-e^z)|}$$

We estimate the integral over the path $\mathbb{R}+i\omega_1$ by

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{|\mu-\lambda|\,|e^{r+i\omega_1}|}{|\lambda-e^{r+i\omega_1}|\,|\mu-e^{r+i\omega_1}|}\,dr \overset{t=e^r}{=} \int_0^\infty \frac{|\mu-\lambda|}{|\lambda-te^{i\omega_1}|\,|\mu-te^{i\omega_1}|}\,dt \\
&= \int_0^\infty \frac{|\mu-\lambda|\,|\lambda|}{|\lambda-|\lambda|\,te^{i\omega_1}|\,|\mu-|\lambda|\,te^{i\omega_1}|}\,dt \overset{\mu=-|\lambda|}{=} \int_0^\infty \frac{\left|1+\frac{\lambda}{|\lambda|}\right|}{\left|\frac{\lambda}{|\lambda|}-te^{i\omega_1}\right|\,|1+te^{i\omega_1}|}\,dt \\
&\le \int_0^\infty \frac{2}{\left|\frac{\lambda}{|\lambda|}-te^{i\omega_1}\right|\,|1+te^{i\omega_1}|}\,dt.
\end{aligned}$$

The last term is uniformly bounded, since $\varphi \le |\arg\lambda| \le \pi$. Needless to say that the second integral can be treated analogously. □

We are now able to *complete the proof of Theorem 3.9.* By elementary calculations we obtain from (3.1) the identity

$$\lambda R(\lambda,A) = (\mu-\lambda)T(\lambda,\mu) + \mu R(\mu,A) \tag{3.2}$$

which holds for all λ,μ with $|\arg\lambda|, |\arg\mu| > \omega$. Keeping μ fixed we see that the right hand side of this equation is defined even for $|\arg\lambda| > \omega'$, and is in fact holomorphic as a function of λ (Lemma 3.10). Since the norm of the resolvent blows up if one approaches a spectral value, we see that no λ with $|\arg\lambda| > \omega'$ can belong to $\sigma(A)$. Furthermore, if we choose $\omega' < \varphi < \pi$ and let $\mu = \mu_\lambda := -|\lambda|$ in (3.2), we arrive at

$$\|\lambda R(\lambda,A)\| \le \|(\mu_\lambda-\lambda)T(\lambda,\mu_\lambda)\| + \left\||\lambda|\,(|\lambda|+A)^{-1}\right\|.$$

This is bounded uniformly for $|\arg\lambda| \ge \varphi$, by Lemma 3.11 and the sectoriality of A. □

We state two important corollaries. Recall that, if iB generates a group T on a Banach space X, one always has $\theta(T) \ge \omega_{st}(B)$ by the Hille -Yosida Theorem A.32. If $X = H$ is a Hilbert space, even $\theta(T) = \omega_{st}(B)$ holds, by Gearhart's Theorem B.24. Now, if A is sectorial and $A \in \mathrm{BIP}$, we know that $i\log A$ generates the group $(A^{is})_{s\in\mathbb{R}}$. Employing the identity $\omega_A = \omega_{st}(\log A)$ we obtain the following two results.

Corollary 3.12. (Prüss-Sohr I)
Let A be an injective and sectorial operator on a Banach space X, having dense domain and dense range. Assume that $A \in \mathrm{BIP}(X)$. Then $\omega_A \le \theta_A$, i.e., the group type of the group $(A^{is})_{s\in\mathbb{R}}$ is always larger than the spectral angle of A.

Corollary 3.13. *Let H be a Hilbert space and A an injective sectorial operator on H which has bounded imaginary powers. Then $\theta_A = \omega_A$, i.e., the group type of $(A^{is})_{s\in\mathbb{R}}$ equals the spectral angle of A.*

We will see later that even more is true in the Hilbert space situation (see §6 below).

§4 A Theorem of Prüss and Sohr

Recall that, by Proposition 2.30, one has

$$A^{is} \in \mathcal{L}(X) \quad \Rightarrow \quad (A+\varepsilon)^{is} \in \mathcal{L}(X)$$

for each injective sectorial operator A and all $s \in \mathbb{R}, \varepsilon > 0$. The reverse implication is valid if A is invertible. Unfortunately, this perturbation result does not say anything about the group types θ_A and $\theta_{A+\varepsilon}$ (in case $\mathcal{D}(A)$ and $\mathcal{R}(A)$ are dense). In this section we want to prove the following theorem.

Theorem 3.14. *Let A be an injective sectorial operator on the Banach space X (no density assumptions!). Assume there is a constant K such that $\left\|A^{is}\right\| \le Ke^{\theta|s|}$ ($s \in \mathbb{R}$) for some $\theta > \omega_A$. Then there is K' such that*

$$\left\|(A+\varepsilon)^{is}\right\| \le K'e^{s|\theta|} \quad (s \in \mathbb{R})$$

uniformly in $\varepsilon > 0$.

We base the proof of Theorem 3.14 on the following perturbation result which certainly is interesting in its own right.

Proposition 3.15. *Let $A \in \mathrm{Sect}(\omega)$ be injective and $\omega < \varphi < \pi$. Assume that $T \in \mathcal{L}(X)$ such that $A + T \in \mathrm{Sect}(\omega)$ and $A + T$ is invertible. Then there are constants K_1, K_2 such that*

$$\|f(A+T)\| \le K_1 \|f(A)\| + K_2 \|f\|_\varphi$$

for all $f \in H^\infty(S_\varphi)$ such that $f(A) \in \mathcal{L}(X)$.

Proof. Take $f \in H^\infty(S_\varphi)$ such that $f(A) \in \mathcal{L}(X)$ and a contour $\Gamma = \Gamma_{\omega'}$ with $\omega < \omega' < \varphi$. By definition,

$$f(A+T) = (2 + (A+T) + (A+T)^{-1})\frac{1}{2\pi i}\int_\Gamma \frac{f(z)z}{(1+z)^2} R(z, A+T)\, dz.$$

Writing $R(z, A+T) = R(z,A)TR(z,A+T) + R(z,A)$ within the integral, we have to estimate the two summands

$$(1) \qquad (2 + A + T + (A+T)^{-1})\frac{1}{2\pi i}\int_\Gamma \frac{f(z)z}{(1+z)^2} R(z,A)\, dz \qquad \text{and}$$

$$(2) \qquad (2 + A + T + (A+T)^{-1})\frac{1}{2\pi i}\int_\Gamma \frac{f(z)z}{(1+z)^2} R(z,A)TR(z,A+T)\, dz.$$

Since $f(A)$ is known to be a bounded operator we can write the first term as $(2 + A + T + (A+T)^{-1})A(1+A)^{-2}f(A)$ and estimate

$$\left\|(2 + A + T + (A+T)^{-1})A(1+A)^{-2}\right\| \le M + 1 + \left\|1 + T + (A+T)^{-1}\right\| M(M+1),$$

where $M = M(A)$.
The second summand splits up in two parts. The first one (discarding A from the factor in front of the integral) can be estimated by

$$\left\|2 + T + (A+T)^{-1}\right\| M(A,\omega') \left[\frac{1}{2\pi}\int_\Gamma \frac{1}{|1+z|^2} \|TR(z, A+T)\| \, |dz|\right] \|f\|_\varphi .$$

The second part (i.e., A times the integral) can be estimated by

$$(M(A,\omega')+1)\left[\frac{1}{2\pi}\int_\Gamma \frac{|z|}{|1+z|^2} \|TR(z, A+T)\| \, |dz|\right] \|f\|_\varphi .$$

Note that both integrals are finite since $A + T$ is invertible (behaviour in 0) and sectorial (behaviour in ∞). □

Remark 3.16. The statement is false without the assumption that $A + T$ is invertible. This follows from the fact that there is a bounded, injective operator A (on a Hilbert space) with unbounded natural H^∞-calculus (see Chapter 1, §6). However, A can be written as $A = (A+1) + (-1)$, and $A + 1$ does have a bounded H^∞-calculus, since it is bounded and invertible.

In passing by, we note the following corollary.

Corollary 3.17. *Let A be an injective operator on a Banach space X and $T \in \mathcal{L}(X)$. Assume that $A, A+T \in \mathrm{Sect}(\omega)$ and $0 \in \varrho(A+T)$. If, for some $\omega < \varphi < \pi$, the natural $H^\infty(S_\varphi)$-calculus for A is bounded, then this also holds for $A + T$.*

Turning back to our principal aim, we consider a special case of the previous proposition.

Corollary 3.18. *Let $A \in \mathrm{Sect}(\omega)$ be injective. Let $\omega < \omega' < \varphi$ and $f \in H^\infty(S_\varphi)$. If $f(A) \in \mathcal{L}(X)$ then $f(A+1) \in \mathcal{L}(X)$ and*

$$\|f(A+1)\| \le K_1 \|f(A)\| + K_2 \|f\|_\varphi$$

where $K_1 = 1 + M + M(M+1)(M+2)$ and

$$K_2 = (M+3)M(A,\omega')^2 C_1(\omega') + M(A,\omega')(M(A,\omega')+1)C_2(\omega')$$

where $C_1(\omega'), C_2(\omega')$ only depend on ω'.

Proof. Let $T = 1$ and look into the proof of the previous proposition. Since $\left\|(A+1)^{-1}\right\| \le M$, the statement concerning K_1 is trivial. For $z \in \Gamma$, i.e. $|\arg z| = \omega'$ we have

$$\|TR(z, A+T)\| = \|R(z, A+1)\| = \|R(z-1, A\| \le \frac{M(A,\omega')}{|z-1|}.$$

If we plug this into the two formulas in the previous proof, we obtain the statement for K_2 with

$$C_1(\omega') = \left[\frac{1}{2\pi}\int_\Gamma \frac{|dz|}{|1+z|^2\,|z-1|}\right] \quad \text{and} \quad C_2(\omega') = \left[\frac{1}{2\pi}\int_\Gamma \frac{|z|\,|dz|}{|1+z|^2\,|z-1|}\right]$$

□

We will now *prove Theorem 3.14*. Assume A is sectorial and injective and $\theta > \omega_A$. Choose $\omega < \varphi < \min(\theta, \pi)$ and suppose $\left\|A^{is}\right\| \leq Ke^{\theta|s|}$ $(s \in \mathbb{R})$. Given $\varepsilon > 0$ and $s \in \mathbb{R}$ we apply Corollary 3.18 to the operator $\varepsilon^{-1}A$ and the function $f(z) = z^{is}$ and obtain

$$\left\|(\varepsilon^{-1}A + 1)^{is}\right\| \leq K_1 \left\|(\varepsilon^{-1}A)^{is}\right\| + K_2 \left\|z^{is}\right\|_{\varphi}$$

for all $s \in \mathbb{R}$. A closer look on the shape of the constants K_1, K_2 in Corollary 3.18 reveals that they do *not* depend on ε (due to *i)* of Proposition 1.1). Now $\left|\varepsilon^{-is}\right| = 1$, $\left\|z^{is}\right\|_{\varphi} = e^{\varphi|s|}$, and $(\varepsilon^{-1}A + 1)^{is} = (\varepsilon^{-1}(A + \varepsilon))^{is} = \varepsilon^{-is}(A + \varepsilon)^{is}$ by a (trivial) application of the composition rule. Similarly, $(\varepsilon^{-1}A)^{is} = \varepsilon^{-1}A^{-1}$. Altogether this yields

$$\left\|(A + \varepsilon)^{is}\right\| \leq K_1 \left\|A^{is}\right\| + K_2 e^{\varphi|s|} \leq (K_1 K + K_2) e^{\theta|s|} \quad (s \in \mathbb{R})$$

uniformly in $\varepsilon \geq 0$, whence Theorem 3.14 is proved. □

We state a corollary which summarizes our considerations.

Corollary 3.19. (Prüss-Sohr II)
Let A be sectorial with dense domain and dense range and let $\theta \geq 0$. If $A \in \mathrm{BIP}(X, \theta)$ then $A + \varepsilon \in \mathrm{BIP}(X, \theta)$ for all $\varepsilon > 0$. In fact, if $\theta > \omega_A$, there is K' such that $\left\|(A + \varepsilon)^{is}\right\| \leq K' e^{\theta|s|}$ $(s \in \mathbb{R})$ uniformly in $\varepsilon \geq 0$.

Remark 3.20. We have labelled the last corollary "Prüss-Sohr II", since the statement very much resembles a theorem of PRÜSS and SOHR from [PS90]. However, this is only resemblance and not complete conformity. First, we do not have to assume neither $\theta > 0$ nor $\theta < \pi$, and in so far Corollary 3.12 is a generalization of the original result. On the other hand, "$A \in \mathrm{BIP}(X, \theta)$" in [PS90] means that there is K such that $\left\|A^{is}\right\| \leq Ke^{\theta|s|}$ $(s \in \mathbb{R})$. (Compare our definition on page 74.) Therefore, in case one knows that $\left\|A^{is}\right\| \leq Ke^{\theta|s|}$ with $\theta = \omega_A$, our result Theorem 3.14 is weaker than the original Prüss-Sohr theorem.

§ 5 A Counterexample

In this section we want to point out that Theorem 3.9 sheds light on a long-standing problem in the field.

Question: Is there a Banach space X and a sectorial operator $A \in \mathrm{BIP}(X)$ such that $\theta_A \geq \pi$?

To answer this question we need a remarkable result of MONNIAUX.

Theorem 3.21. (Monniaux) [Mon99, Theorem 4.3]
Let X be a Banach space with the UMD *property and let $U = (U(s))_{s \in \mathbb{R}}$ be a C_0-group on X such that the group type satifies $\theta(U) < \pi$. Then there is a sectorial operator $A \in \mathrm{BIP}(X)$ such that $A^{is} = U(s)$ for all $s \in \mathbb{R}$.*

(See [ABHN01, Section 3.12] for the definition of UMD spaces.) With the help of our Theorem 3.9 we can generalize Monniaux's result.

Theorem 3.22. *Let X be a Banach space with the* UMD *property and let iB be the generator of a C_0-group $U = (U(s))_{s\in\mathbb{R}}$ on X with any group type. If B is strip type of height $\omega < \pi$ then there is a sectorial operator $A \in \mathrm{BIP}(X)$ such that $A^{is} = U(s)$ for all $s \in \mathbb{R}$.*

Proof. Let θ denote the group type of U. Find $0 < \alpha < 1$ such that $\theta\alpha < \pi$. Then $i\alpha B$ generates the group $(U(\alpha s)_{s\in\mathbb{R}}$ which has group type $\alpha\theta$. Furthermore, $\alpha B \in \mathrm{Strip}(\alpha\omega)$. Applying Monniaux's Theorem 3.21 we can find an injective sectorial operator C on X such that $C^{is} = U(\alpha s)$ for all $s \in \mathbb{R}$. Theorem 3.9 yields that $\omega_C \le \alpha\omega$. If we define $A := C^{1/\alpha}$ we know from Proposition 2.2 and Proposition 2.18 that A is also an injective sectorial operator and $A^{is} = U(s)$ for all $s \in \mathbb{R}$. □

On UMD spaces, the question stated above is intimately connected with the failing of Gearhart's theorem (Theorem B.24) for C_0-groups. We state this as a proposition.

Proposition 3.23. *Let X be a Banach space with the* UMD *property. Then the following assertions are equivalent.*

(i) There is a sectorial operator $A \in \mathrm{BIP}(X)$ such that $\theta_A \ge \pi$.

(ii) There is a sectorial operator $A \in \mathrm{BIP}(X)$ such that $\omega_A < \theta_A$.

(iii) There is an operator B on X which generates a C_0-semigroup T such that $s_0(B) < \omega_0(T)$ and T is a group.

Recall the definition (B.7) of $s_0(B)$ on page 164.

Proof. The implication $(i) \Rightarrow (ii)$ is trivial. To prove $(ii) \Rightarrow (i)$ simply consider a scaled operator A^α for som $\alpha > 1$ in case that $\theta_A < \pi$. We prove $(ii) \Rightarrow (iii)$. Assume that $A \in \mathrm{BIP}(X)$ with $\omega_A < \theta_A$. Then $i \log A$ generates $T_1 := (A^{is})_{s\ge 0}$ and $-i\log A$ generates $T_2 := (A^{-is})_{s\ge 0}$. But

$$\max\{s_0(i\log A), s_0(-i\log A)\} = \omega_{st}(\log A) = \omega_A < \theta_A = \max\{\omega_0(T_1), \omega_0(T_2)\}$$

Hence either $B := i\log A$ or $B := -i\log A$ will do the job.
Finally we prove the implication $(iii) \Rightarrow (ii)$.
Assume that $(U(s))_{s\in\mathbb{R}}$ is a C_0-group on X, the operator B generates the semigroup $T := (U(s))_{s\ge 0}$, and $s_0(B) < \omega_0(T)$. Changing B to $\alpha(B + \lambda)$ with suitable $\lambda, \alpha > 0$ we can assume that $\sup_{s\le 0} \|U(s)\| < \infty$ and $0 \le s_0(B) < \omega_0(T) < \pi$. By Monniaux's theorem 3.21 there is a sectorial operator $A \in \mathrm{BIP}(X)$ such that $A^{is} = U(s)$ for all $s \in \mathbb{R}$. Hence $i\log A = B$ by the uniqueness of generators. Obviously,

$$\omega_{st}(\log A) = s_0(B) < \omega_0(T) = \theta(U) = \theta_A.$$

Since $\omega_A = \omega_{st}(\log A)$ by Theorem 3.9, assertion (ii) readily follows. □

We are now going to give an example of a Banach space X with the UMD property such that (iii) of Proposition 3.23 holds.

Let $1 < p < 2 < q < \infty$ and $a > q$ such that $\frac{2}{p} < \frac{a}{q}$. We define the weight $w : \mathbb{R} \to [0, \infty)$ by

$$w(x) := \begin{cases} e^{ax} & x \le 0 \\ 1 & x \ge 0 \end{cases}$$

Now we let $X := \mathbf{L}^p(\mathbb{R}, e^{2x}\,dx) \cap \mathbf{L}^q(\mathbb{R}, w(x)dx)$ with the norm $\|f\|_X := \|f\|_p + \|f\|_q$, where

$$\|f\|_p = \left(\int_{-\infty}^{\infty} |f(x)|^p \, e^{2x}\, dx\right)^{\frac{1}{p}} \quad \text{and} \quad \|f\|_q = \left(\int_{-\infty}^{\infty} |f(x)|^q \, w(x)\, dx\right)^{\frac{1}{q}}.$$

It can be shown that X has in fact the UMD property. The space $\mathbf{C_c}(\mathbb{R})$ of compactly supported continuous functions is dense in X.

[Let $f \in X$. Then $\mathbf{1}_{[-n,n]}f \to f$ in X for $n \to \infty$. Hence we can assume that f is compactly supported. Now we approximate f in $\mathbf{L}^q(\mathbb{R})$ by functions $f_n \in \mathbf{C_c}(\mathbb{R})$ such that $\operatorname{supp}(f_n) \subset [a,b]$ where $a < b$ do not depend on n. Since

$$\mathbf{L}^q((a,b), w(x)dx) \cong \mathbf{L}^q(a,b) \hookrightarrow \mathbf{L}^p(a,b) \cong \mathbf{L}^p((a,b), e^{2x}dx)$$

the sequence f_n tends to f even in the norm of X.]

On X we consider the left shift group $(T(t))_{t\in\mathbb{R}}$ defined by

$$[T(t)f](x) := f(x+t) \qquad (x \in \mathbb{R},\ t \in \mathbb{R})$$

We have the norm inequalities

$$\begin{aligned} \|T(t)f\|_X &\le e^{-\frac{2}{p}t}\,\|f\|_p + \|f\|_q \le \|f\|_X\,, \\ \|T(-t)f\|_X &\le e^{\frac{2}{p}t}\,\|f\|_p + e^{\frac{a}{q}t}\,\|f\|_q \end{aligned}$$

for $f \in X$ and $t \ge 0$.

[Obviously, $\|T(t)f\|_p = e^{-\frac{2}{p}t}\,\|f\|_p$ for $t \in \mathbb{R}$. If $t \ge 0$, we have

$$\begin{aligned} \|T(t)f\|_q^q &= \int_{-\infty}^{\infty} |f(x)|^q \, w(x-t)\, dt \\ &= e^{-at}\int_{-\infty} |f(x)|^q\, e^{ax}\, dx + e^{-at}\int_0^t |f(x)|^q\, e^{ax}\, dx + \int_t^{\infty} |f(x)|^q \; dx \\ &\le e^{-at}\int_{-\infty} |f(x)|^q\, e^{ax}\, dx + \int_0^{\infty} |f(x)|^q \; dx \le \|f\|_q^q\,. \end{aligned}$$

The computation for $\|T(-t)f\|_q^q$ is similar.]

In particular, it follows that $\|T(t)\| \le 1$ for all $t \ge 0$ and that T is a group. Since $\mathbf{C_c}(\mathbb{R})$ is dense in X and $(t \longmapsto T(t)f) : \mathbb{R} \longrightarrow X$ is continuous for each $f \in \mathbf{C_c}(\mathbb{R})$, we conclude that T is in fact a C_0-group.

Claim. We have $\|T(t)\| = 1$ for all $t \ge 1$.

[Let $t_0 \ge 0$. Choose $t_1 > t_0$ arbitrary. Since $\mathbf{L}^p((t_0,t_1), e^{2x}dx) \cong \mathbf{L}^p(t_0,t_1) \not\hookrightarrow \mathbf{L}^q(t_0,t_1)$, there is no inequality of the form $\|f\|_q \le C\,\|f\|_p$ with $f \in \mathbf{C_c}(t_0,t_1)$. Hence there is a sequence $g_n \in \mathbf{C_c}(t_0,t_1)$ such that $\|g_n\|_q \ge n\,\|g_n\|_p$ for all n. Letting $f_n := g_n/\,\|g_n\|_q$ we have

$$\|f_n\|_q = 1, \quad \|f_n\| \le \frac{1}{n}, \quad \text{and} \quad f_n \equiv 0 \ \text{ on } (-\infty, t_0]$$

Thus, $\|T(t_0)f_n\|_X = e^{-\frac{2}{p}t_0}\,\|f_n\|_p + \|f_n\|_q \ge 1$ and $\|f_n\|_X \le \frac{1}{n} + 1$. Hence, $\|T(t_0)\| \ge \frac{1}{1+\frac{1}{n}}$ for all $n \in \mathbb{N}$. This proves the claim.]

We now determine the generator A of T. We claim that

$$\mathcal{D}(A) = \{f \in X \mid f' \in X\} \quad \text{and} \quad Af = f',$$

where f' denotes the (distributional) derivative of f.

[Let us denote the derivative operator on distributions by ∂. If $f \in \mathcal{D}(A)$ there is a $g \in X$ such that $\frac{1}{t}(T(t)f - f) \to g$ in X for $t \searrow 0$. Since $X \hookrightarrow \mathcal{D}(\mathbb{R})'$ we see that $g = f'$. Hence, $\mathcal{D}(A) \subset \mathcal{D} := \{f \in X \mid f' \in X\}$ and $Af = f'$ for $f \in \mathcal{D}(A)$. Now, $I - A$ is bijective (since T is a contraction semigroup) and $I - \partial : \mathcal{D} \longrightarrow X$ is injective (since $f' = f \in X$ implies that f is more or less the exponential function). This implies the claim.]

Claim. We have $s(A) := \sup\{\operatorname{Re}\lambda \mid \lambda \in \sigma(A)\} \le -\frac{2}{p}$.

[Let $X_p := \mathbf{L}^p(\mathbb{R}, e^{2x}dx)$ with the norm $\|f\|_p$ defined above, and denote by A_p the generator of the left shift group on X_p. Since $\|T(t)f\|_p = e^{-\frac{2}{p}t}\|f\|_p$ for all $t \in \mathbb{R}$ and all $f \in X_p$, we have $s_0(A_p) \le -\frac{2}{p}$. By the same reasoning as above one can show that $\mathcal{D}(A_p) = \{f \in X_p \mid f' \in X_p\}$ with $A_p f = f'$ for $f \in A_p$. Next, we claim that $\mathcal{D}(A_p) \hookrightarrow X$. Actually, by the Closed Graph Theorem, only inclusion has to be shown. If $f \in \mathcal{D}(A_p)$ we let $g := e^{\frac{2}{p}\cdot} f$ and note that $g \in \mathbf{L}^p(\mathbb{R}, dx)$ and $g' = \frac{2}{p}g + e^{\frac{2}{p}\cdot}f' \in \mathbf{L}^p(\mathbb{R}, dx)$. Hence $g \in \mathbf{W}^{1,p}(\mathbb{R})$ and since $\mathbf{W}^{1,p}(\mathbb{R}) \hookrightarrow \mathbf{C}_0(\mathbb{R})$ there is a constant $C > 0$ such that $|f(x)| \le Ce^{-\frac{2}{p}x}$ for $x \in \mathbb{R}$. This immediately implies that $f \in \mathbf{L}^q((0,\infty), dx)$. Moreover, we have $|f(x)|\, e^{\frac{a}{q}x} \le Ce^{\left(\frac{a}{q} - \frac{2}{p}\right)x}$ for all $x \in \mathbb{R}$, and since $\frac{a}{q} > \frac{2}{p}$ we conclude that $f \in \mathbf{L}^q((-\infty,0), e^{ax}dx)$. Altogether we obtain $f \in X$.

Now take $\lambda \in \mathbb{C}$ with $-\frac{2}{p} < \operatorname{Re}\lambda$. Since $(\lambda - A)f = (\lambda - A_p)f$ for $f \in \mathcal{D}(A) \subset \mathcal{D}(A_p)$ we see that $\lambda - A$ is injective. If $f \in X$ then $f \in X_p$ and $g := R(\lambda, A_p)f \in \mathcal{D}(A_p) \subset X$. But this implies that $g \in \mathcal{D}(A)$, since $g' = \lambda g - f \in X$. Hence $\lambda - A$ is also surjective. This proves that $\lambda \in \varrho(A)$.]

We take the last step. Obviously, the space X is not only a Banach space but even a *Banach lattice* and the semigroup T is *positive*, i.e., $T(t)f \ge 0$ for all $0 \le f \in X$ and all $t \ge 0$. By [ABHN01, Theorem 5.3.1] we conclude that $s_0(A) = s(A) \le -\frac{2}{p} < 0 = \omega_0(T)$, whence we are done.

Corollary 3.24. *There is a Banach space X with the* UMD *property and an injective sectorial operator A on X such that $A \in \operatorname{BIP}(X)$ and $\theta_A > \pi$.*

§ 6 The H^∞-calculus for Strip Type Operators on Hilbert Spaces and the McIntosh-Yagi Theorem

In this section we consider a strip type operator $B \in \operatorname{Strip}(\omega)$ on a *Hilbert space H*. We will give a characterization of the fact that $A := iB$ generates a C_0-group on H in terms of the functional calculus for B.

Let $\varphi_1, \varphi_2 \in \mathbb{R} \setminus [-\omega, \omega]$. Then we have

$$(t \mapsto R(t{+}i\varphi_1, B)x) \in \mathbf{L}^2(\mathbb{R}, H) \quad \Longleftrightarrow \quad (t \mapsto R(t{+}i\varphi_2, B)x) \in \mathbf{L}^2(\mathbb{R}, H).$$

for each $x \in H$. (Use the resolvent identity and the fact, that the resolvent is uniformly bounded on the horizontal lines $\mathbb{R} + i\varphi_1$ and $\mathbb{R} + i\varphi_2$).

We say that B allows **quadratic estimates**, if for every $\varphi \in \mathbb{R} \setminus [-\omega, \omega]$ there is $c = c(B, \varphi)$ such that

$$\int_{\mathbb{R}} \|R(t + i\varphi, B)x\|^2 \, dt \le c(B, \varphi)\, \|x\|^2 \qquad (x \in H).$$

From the remarks above and the Closed Graph Theorem it follows that B allows quadratic estimates if and only if there exists $\varphi \in \mathbb{R} \setminus [-\omega, \omega]$ such that $R(\cdot + i\varphi, B)x \in \mathbf{L}^2(\mathbb{R}, H)$ for every $x \in H$. If B allows quadratic estimates then also $-B$ and $B + \lambda$ do, for each $\lambda \in \mathbb{C}$.

Example 3.25. Let $B \in \mathrm{Strip}(\omega)$ and assume that $A = iB$ is the generator of a C_0-semigroup T on H. We claim that B allows quadratic estimates. In fact, we can find constants M, ω_0 such that $\|T(t)\| \leq Me^{\omega_0 t}$ for all $t \geq 0$. Then

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x \, dt$$

for all $\mathrm{Re}\,\lambda > \omega_0$. An application of the Plancherel Theorem now yields

$$\begin{aligned}\int_{\mathbb{R}} \|R(t - i\varphi, B)x\|^2 \, dt = \int_{\mathbb{R}} \|R(it, A - \varphi)x\|^2 \, dt &= 2\pi \int_0^\infty \left\|e^{-\varphi s} T(s)x\right\|^2 \, ds \\ &\leq \frac{\pi M^2}{\varphi - \omega_0} \|x\|^2\end{aligned}$$

for all $\varphi > \omega, \omega_0$ and all $x \in H$.
Since also $i(-B^*) = A^*$ generates a C_0-semigroup, the operator B^* also allows quadratic estimates.

We can now state the main result.

Theorem 3.26. *Let H be a Hilbert space and $B \in \mathrm{Strip}(\omega)$ a strip type operator on B, where $\omega \geq 0$. Assume that B is densely defined. Then the following assertions are equivalent.*

(i) The natural $H^\infty(H_\alpha)$-calculus for B is bounded, for one / all $\alpha > \omega$.

(ii) The operator iB generates a C_0-group.

(iii) The operator iB generates a C_0-semigroup.

(iv) The operators B and B^* both allow quadratic estimates.

If iB generates the C_0-group T, then $\omega(T) \leq \omega$.

Note that "Gearhart's theorem for groups" is implicit in Theorem 3.26.

We turn to the *Proof of Theorem 3.26*. The implication $(ii) \Rightarrow (iii)$ is obvious, and $(iii) \Rightarrow (iv)$ is in Example 3.25. To prove $(i) \Rightarrow (ii)$ one only has to note that the boundedness of the H^∞-functional calculus for B on some horizontal strip H_α implies the Hille-Yosida conditions for iB and $-iB$ (see Theorem A.32). Hence iB generates a C_0-group T. Futhermore, we see that in this case the group type of T is as most as large as α.

To establish the implication $(iv) \Rightarrow (i)$ needs a little more effort. We fix $\omega < \alpha < \mu$. We will need the auxiliary function ψ defined by

$$\psi(z) := \frac{c}{(\mu^2 + z^2)^2} \quad (z \in H_\alpha)$$

where c is chosen such that

$$\int_{\mathbb{R}} \psi(t) \, dt = \int_{\mathbb{R}} \frac{c}{(\mu^2 + t^2)^2} \, dt = 1.$$

(One can easily compute $c = (4\mu^3)/\pi$). For a given $f \in H^\infty(H_\alpha)$ we now define the approximants f_n by

$$f_n(z) := \int_{-n}^{n} (f\psi_t)(z)\,dt = f(z)\int_{-n}^{n} \psi(z+t)\,dt \quad (z \in H_\alpha), \tag{3.3}$$

where here and in the following for a function g on the strip H_α we will denote by g_t the function $g_t := (z \longmapsto g(t+z)) : H_\alpha \longrightarrow \mathbb{C}$. The next lemma collects the properties of these approximants.

Lemma 3.27. *Let $f \in H^\infty(S_\alpha)$ and let the sequence $(f_n)_n$ be defined by (3.3). Then the following holds.*

a) $f_n \in \mathcal{F}(H_\alpha)$ *for all* n.

b) $\sup_n \|f_n\|_\infty < \infty$.

c) $f_n \to f$ *pointwise on* S_α.

d) The function $(t \mapsto (f\psi_t)(A)) : \mathbb{R} \longrightarrow \mathcal{L}(H)$ *is continuous and*

$$f_n(A) = \int_{-n}^{n} (f\psi_t)(B)\,dt \in \mathcal{L}(H).$$

e) $\sup_n \|f_n(B)\| < \infty$.

Proof. Ad *a)*. By elementary Complex Analysis it is clear that f_n is holomorphic on H_α for each n. We can choose $d > 0$ such that

$$|\psi(z)| = \frac{c}{|\mu^2 + z^2|^2} \le \frac{d}{(1 + |\mathrm{Re}\, z|^2)^2} \quad (z \in H_\alpha).$$

For fixed $n \in \mathbb{N}$ one can find $d_n > 0$ such that

$$\frac{1}{1 + |\mathrm{Re}\, z + t|^2} \le \frac{d_n}{1 + |\mathrm{Re}\, z|^2} \quad (z \in H_\alpha,\ |t| \le n).$$

With the help of this we can compute

$$\begin{aligned} |f_n(z)| \le \|f\|_\infty \int_{-n}^{n} |\psi(z+it)|\,dt &\le \|f\|_\infty \int_{-n}^{n} \frac{d}{(1 + |\mathrm{Re}\, z + t|^2)^2}\,dt \\ &\le \|f\|_\infty \frac{d_n}{1 + |\mathrm{Re}\, z|^2}\left(\int_{\mathbb{R}} \frac{d}{1+t^2}\,dt\right) = \|f\|_\infty \frac{d d_n \pi}{1 + |\mathrm{Re}\, z|^2} \end{aligned}$$

for $z \in H_\alpha$. This proves *a)*.

Ad *b)*. Let $n \in \mathbb{N}$ and $z \in H_\alpha$. Then

$$\begin{aligned} |f_n(z)| &\le \|f\|_\infty \int_{\mathbb{R}} |\psi(z+t)|\,dt \\ &\le \|f\|_\infty \int_{\mathbb{R}} \frac{d}{(1 + |\mathrm{Re}\, z + t|^2)^2}\,dt = \|f\|_\infty \int_{\mathbb{R}} \frac{d}{(1+t^2)^2}\,dt. \end{aligned}$$

Ad *c)*. By b) and Vitali's Theorem is suffices to show that $f_n(z) \to f(z)$ for all $z \in \mathbb{R}$. But this is obvious from (3.3).

Ad *d)*. This is immediate from the Convergence Lemma (Proposition 3.5).

Ad *e)*. We let $\eta(z) := 1/(\mu^2 + z^2)$. Hence, we have $\psi = c\eta^2$. Choose $\omega < \omega_1 < \alpha$ and let $\gamma := \gamma_{\omega_1} = \partial H_{\omega_1}$. Now we fix $t \in \mathbb{R}$ and compute

$$\begin{aligned}
\int_\gamma \frac{1}{|\mu^2 + (z+t)^2|}\, d|z| &= 2\int_{\mathbb{R}} \frac{ds}{|\mu^2 + (s+i\omega_1)^2|} \\
&= 2\int_{\mathbb{R}} \frac{ds}{|(s+i\omega_1+i\mu)(s+i\omega_1-i\mu)|} \\
&\le 2\left(\int_{\mathbb{R}} \frac{ds}{|s+i\omega_1+i\mu|^2}\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \frac{ds}{|s+i\omega_1-i\mu|^2}\right)^{\frac{1}{2}} \\
&= 2\left(\int_{\mathbb{R}} \frac{ds}{(\mu+\omega_1)^2+s^2}\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \frac{ds}{(\mu-\omega_1)^2+s^2}\right)^{\frac{1}{2}} \\
&= \frac{2\pi}{\sqrt{\mu^2-\omega_1^2}}.
\end{aligned}$$

Using this we can estimate $\|(f\eta_t)(B)\|$ for each $t \in \mathbb{R}$ by

$$\begin{aligned}
\|(f\eta_t)(B)\| &= \left\| \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{\mu^2 + (z+t)^2}\, R(z,B)\, dz \right\| \\
&\le \frac{L(B,\omega_1)}{2\pi}\, \|f\|_\infty \int_\gamma \frac{1}{|\mu^2+(z+t)^2|}\, d|z| \\
&\le \frac{L(B,\omega_1)}{2\pi}\, \|f\|_\infty\, \frac{2\pi}{\sqrt{\mu^2-\omega_1^2}} \\
&= \frac{L(B,\omega_1)}{\sqrt{\mu^2-\omega_1^2}}\, \|f\|_\infty\, .
\end{aligned}$$

Thus, for arbitrary $x, y \in H$ one has

$$\begin{aligned}
|(f_n(B)x\,|\,y)| &= \left| \int_{-n}^{n} \left((f\psi_t)(B)x\,|\,y\right)\, dt \right| \\
&= c\left| \int_{-n}^{n} \left(\eta_t(B)(f\eta_t)(B)x\,|\,y\right)\, dt \right| \\
&= c\left| \int_{-n}^{n} \left((f\eta_t)(B)R(i\mu-t,B)x\,|\,R(i\mu-t,B^*)y\right)\, dt \right| \\
&\le \frac{c\,L(B,\omega_1)}{\sqrt{\mu^2-\omega_1^2}}\, \|f\|_\infty \int_{\mathbb{R}} \|R(t+i\mu,B)x\|\, \|R(t+i\mu,B^*)y\|\, dt \\
&\le \frac{c}{\sqrt{\mu^2-\omega_1^2}}\, L(B,\omega_1)\, \|f\|_\infty \\
&\qquad \left(\int_{\mathbb{R}} \|R(t+i\mu,B)x\|^2\, dt\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \|R(t+i\mu,B^*)y\|^2\, dt\right)^{\frac{1}{2}} \\
&\le \frac{c}{\sqrt{\mu^2-\omega_1^2}} L(B,\omega_1)\, c(B,\mu)\, c(B^*,\mu)\, \|f\|_\infty\, \|x\|\, \|y\|\, .
\end{aligned}$$

In particular, this shows that

$$\|f_n(B)\| \le \|f\|_\infty \frac{c}{\sqrt{\mu^2-\omega_1^2}} L(B,\omega_1)\, c(B,\mu)\, c(B^*,\mu)$$

for each n. Thus *e)* is completely proved. □

It is now easy to complete the proof of Theorem 3.26. We simply apply the Convergence Lemma (Proposition 3.5) to the sequence $(f_n(A))_{n\in\mathbb{N}}$. Hence, we obtain the boundedness of the functional calculus and, more explicitly,

$$\|f(A)\| \le \|f\|_\infty \frac{4\mu^3}{\pi\sqrt{\mu^2-\omega_1^2}} L(B,\omega_1)\, c(B,\mu)\, c(B^*,\mu) \tag{3.4}$$

for all $f \in H^\infty(H_\alpha)$.
As a first corollary to Theorem 3.26 we obtain a result which is due to LIU.

Corollary 3.28. (Liu)
Let A be the generator of a C_0-semigroup T on the Hilbert space H. If the resolvent of A exists and is uniformly bounded on a left halfplane, then T is a group.

The equivalence $(ii) \Leftrightarrow (i)$ in Theorem 3.26 is originally due to BOYADZHIEV and DELAUBENFELS.

Corollary 3.29. (Boyadzhiev-deLaubenfels)
Let iB be the generator of a C_0-group of group type ω on a Hilbert space H. Then the natural $H^\infty(H_\alpha)$-calculus for B is bounded for every $\alpha > \omega$.

Another corollary is a special case of Monniaux's Theorem 3.21.

Corollary 3.30. (Monniaux)
Let T be a C_0-group on the Hilbert space H. If $\theta(T) < \pi$ then there is an injective sectorial operator A on H such that $A^{is} = T(s)$ for all $s \in \mathbb{R}$.

Proof. Let iB be the generator of T. Define $\omega := \theta(T)$ and $R(\lambda) := (\lambda - e^z)^{-1}(B)$ for $\lambda \notin \overline{S_\omega}$. Then $R(\lambda) \in \mathcal{L}(H)$ by Theorem 3.26. It is easy to see that $R(\cdot)$ is a pseudo-resolvent. Hence by Proposition A.8 there is a uniquely determined m.v. operator A such that $R(\lambda, A) = R(\lambda)$ for all $\lambda \notin \overline{S_\omega}$. Again by the boundedness of the natural $H^\infty(H_\alpha)$-calculi for B, we see that $A \in \operatorname{Sect}(\omega)$. Let $x \in A0$ and recall that $A0 = \mathcal{N}(R(\lambda))$ for each λ. Then $\lambda R(\lambda)(B^2+\pi^2)^{-1}x = 0$ and by the Convergence Lemma (Proposition 3.5) we obtain $(B^2 + \pi^2)^{-1}x = 0$ for $\lambda \to \infty$. This yields $x = 0$, whence A is single-valued. To show that A is injective, define $M(\lambda) = (\lambda - e^z)^{-1}(-B) = (\lambda - e^{-z})(B)$ for $\lambda \notin \overline{S_\omega}$. The above arguments show that $M(\lambda) = R(\lambda, A_1)$ for some sectorial and single-valued operator $A_1 \in \operatorname{Sect}(\omega)$. Now,

$$I - (1+A)^{-1} = (1 - (1+e^z)^{-1})(B) = \frac{e^z}{1+e^z}(B) = (1+e^{-z})(B) = (1+A_1)^{-1}$$

by functional calculus. Hence $A_1 = A^{-1}$ as the fundamental identity (1.1) shows. Since A_1 is single-valued, A is injective. By the composition rule (Proposition 3.6) we conclude that $B = \log A$ and since iB generates T, we must have $T(s) = A^{is}$ for $s \in \mathbb{R}$ by Corollary 2.32. ☐

We turn once again to sectorial operators. The next result is a slight improvement of a result by MCINTOSH and YAGI, hence we name it after them.

Corollary 3.31. (McIntosh-Yagi)
Let $0 \le \omega < \pi$ and $A \in \operatorname{Sect}(\omega)$ be an injective sectorial operator on the Hilbert space H. Assume that $A^{is} \in \mathcal{L}(H)$ for all s in some small interval $[0, \varepsilon]$ and

$$\sup_{0 \le s \le \varepsilon} \left\| A^{is} \right\| < \infty.$$

(This the case, e.g., if $A \in \operatorname{BIP}(H)$.) Then the natural $H^\infty(S_\varphi)$-calculus is bounded for each $\omega < \varphi < \pi$.

Note that an injective sectorial operator on a Hilbert space is densely defined and has dense range (see Proposition 1.1), and its logarithm is densely defined (see Proposition 2.27).

Proof. Pick $\omega < \varphi < \pi$. Let $s > 0$ and choose $n \in \mathbb{N}$ such that $s/n \leq \varepsilon$. Then $A^{is} = z^{is}(A) = (z^{is/n})^n(A) = (A^{is/n})^n \in \mathcal{L}(H)$. Since $(s \mapsto A^{is}x)$ is continuous for each $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$, the hypothesis implies that $T(s) := A^{is}$ defines a C_0-semigroup. By Proposition 2.31 we see that $i \log A$ generates this semigroup. But we know that $\log A \in \mathrm{Strip}(\omega)$, hence from Theorem 3.26 we learn that the natural H^∞-calculus for $\log A$ on H_φ is bounded. The composition rule in Proposition 3.6 now shows that $f(A) = (f \circ e^z)(\log A) \in \mathcal{L}(X)$ for every $f \in H^\infty(S_\varphi)$. □

§ 7 Comments

§1/§2 Strip Type Operators and their Natural Functional Calculus. The natural functional calculus for strip type operators appears first in [Bad53]. It is used in [Bd94] and companion papers. However, the composition rule proved here in its full generality is seemingly new. Suppose one is given an operator B with spectrum in a horizontal strip H_ω and such that the resolvent is bounded on some horizontal lines $\mathbb{R} \pm i\varphi$ with $\varphi > \omega$. Then one can construct the "functional calculus" like we did in §2, since the basic Cauchy integrals converge. However, it can happen that this "functional calculus" is the zero mapping. If one requires that the calculus behaves well for rational functions, i.e., is really a functional calculus for B, then B has to be strip type, as it was shown in [Haa02].

§3/§4 The Spectral Height of the Logarithm and the Prüss-Sohr Theorem. Theorem 3.9 is new, as far as we know. The Prüss-Sohr I result (Corollary 3.12) is the first part of a celebrated theorem of PRÜSS and SOHR given in [PS90, Theorem 3.3]. Its original proof rests on the Mellin transform calculus for C_0-groups, cf. also [Uit98, Proposition 3.19], [Uit00], and [MCSA01, Chapter 9]. We included the Prüss-Sohr II result (Corollary 3.19) for the reason of "completeness". It is the second part of [PS90, Theorem 3.3] and gave us the possibility to present a strikingly simple perturbation argument, namely the proof of Proposition 3.15. The idea goes back to the paper [ABH01]. A closer examination of the proof shows that the assumptions on the perturbing operator T can be weakened, cf. [ABH01, Remark 2.5 b]. A similar, but weaker perturbation result was obtained by UITERDIJK in [Uit98, Theorem 2.3.3], see also [Uit98, Theorem 2.3.7]. Another result with further references can be found in [MCSA01, Theorem 8.2.6].

§5 A Counterexample. For the proof of Theorem 3.21 in [Mon99], MONNIAUX utilizes the theory of analytic generators of C_0-groups. She also showed that the conclusion fails when one discards the UMD assumption. (A counterexample is $X = \mathbf{L}^1(\mathbb{R})$ with $(U(s))_{s \in \mathbb{R}}$ being the left shift group.)
The term "UMD" is an abbreviation for "Unconditional Martingale Differences" and was introduced by BURKHOLDER. It was shown by BURKHOLDER, MCCONNEL and BOURGAIN that the UMD property is equivalent to the boundedness of the Hilbert transform, see [MCSA01, Section 8.4] and the references therein.
The example of a group with differing growth bound and abszissa of uniform boundedness of the resolvent is an adaptation of an example given by WOLFF in [Wol81]. We are indebted to Charles Batty for showing this result to us.

A Problem for Further Research. The question stated in the beginning of §5 has, as far as we know, never been explicitly formulated in the literature. Nevertheless the question "jumps into the face" when reading the papers of PRÜSS and SOHR, UITERDIJK, and MONNIAUX. The monograph [MCSA01] says, after giving the definition of "$A \in \mathrm{BIP}(X, \theta_A)$",

"The most interesting case is when $\theta_A \in [0, \pi[$."

Unfortunately, the authors do not entangle in further arguing in favour of this daring thesis. We believe that the reason for this statement lies in the fact that the methods used up to now in the literature — namely the Mellin transform functional calculus and analytic generators of C_0-groups — *require* this property to be applicable.
At least, the remarkably simple proof of Theorem 3.9 suggests that the focus on operators with bounded imaginary powers and analytic generators of C_0-groups which is prevalent in the literature concealed some of the underlying concepts. Moreover, the generalization of Monniaux's theorem in Theorem 3.22 which of course was proved by *using* Monniaux's theorem brings up the following question:

> Let B be a strong strip type operator of height $\omega < \pi$ on the Banach space X. Which additional assumptions are sufficient to assure the existence of a sectorial operator A such that $B = \log A$?

We call this problem the *inversion problem for logarithms*. Arguing in the same way as in the proof of Corollary 3.30, the boundedness of the H^∞-calculus for B on some horizontal strip is a sufficient condition. In general, using the natural functional calculus for strip type operators developed in §2 one can construct a "canonical" candidate for $A =: e^B$ which has the following properties:

1) A is an injective closed operator.
2) For $\lambda \notin \overline{S_\omega}$ the operator $(\lambda - A)$ is injective with dense range.
3) $\lambda \in \varrho(A)$ for *some* $\lambda \notin \overline{S_\omega}$, then this is the case for *all* of them.
4) *If* A is sectorial, then $B = \log A$.

This is all we could achieve till now. One can hope that proceeding in this direction one can find a direct proof of the generalized Monniaux Theorem and even new results.

§6 H^∞-calculus for strip type operators. This section is based on our paper [Haa02], where we worked with vertical instead of horizontal strips. The idea is to somehow adapt MCINTOSH's methods (see §4 in Chapter 4) for sectorial operators to strip type operators. In the sectorial case, the multiplicative group of positive real numbers acts on the sector by dilation, in the strip case the additive group $\mathbb{R}$ acts on the strip by translation. Combining this structural property with the right *quadratic estimates* yields a proof which is quite analogous to MCINTOSH's method in the sectorial case.
Corollary 3.28 is [Liu00, Theorem 1]. Actually, LIU has an additional assumption which however is easily seen to be redundant. Another characterization of

C_0-group generators appearing as Theorem 2 in [Liu00] could also be reproved by functional calculus methods, see [Haa02, Corollary 5.4]. In [Haa02, Corollary 5.3] we gave a short argument proving ZWART's generalization [Zwa01, Theorem 2.2] of LIU's first result.
Corollary 3.29 is [Bd94, Theorem 3.2]. Their proof proceeds in two steps. First, assuming $\omega < \pi/2$ without loss of generality, they construct the operator e^B (see above; their construction however relies on the theory of regularized semigroups). Then they show that this operator is sectorial and has a bounded H^∞-functional calculus on a sector. The statement follows by means of some special case of the composition rule. We will give a different — and even much simpler — proof of Corollary 3.29 in §3 of Chapter 5, making use of a similarity result.
Corollary 3.31 is one part of a celebrated theorem of MCINTOSH and YAGI appearing as the equivalence $(a) \Leftrightarrow (d)$ in [McI86, Section 8]. Its original proof rests on methods of complex interpolation (see also §5 of Chapter 2), whereas our proof only needs the natural functional calculus and the Plancherel Theorem. We will present other parts of this theorem in §4 of Chapter 4.

Fourth Chapter
Similarity Results for Sectorial Operators

In §1 we introduce variational operators on Hilbert spaces as an abstraction of elliptic differential operators on $\mathbf{L^2}$. Since these operators are characterized by a numerical range condition, the problem is posed to characterize them *modulo similarity*. In §2 we provide the necessary facts about the functional calculus on Hilbert spaces, including the compatibility of the NFCSO with the Borel functional calculus for positive operators. In §3 fundamental facts on fractional powers of m-accretive operators are proved. We cite Kato's theorem and state the square root problem. In §4 we develop the results of MCINTOSH and YAGI and their co-workers concerning the connection of bounded H^∞-calculus to quadratic estimates/equivalent Hilbert norms. In §5 we present the similarity theorem generalizing a theorem of FRANKS and LEMERDY, with a different proof. The solution to the similarity problem (posed in §1) is given and a dilation theorem for groups is proved. In §6 we construct an example of a generator A of a bounded C_0-semigroup T such that $A \in \mathrm{BIP}(H)$ and T is not similar to a quasi-contraction semigroup.

§ 1 The Similarity Problem for Variational Operators

We briefly review the construction of operators by means of elliptic forms. The scheme is an abstraction of the standard way for obtaining $\mathbf{L^2}$-realiziations of elliptic differential operators in divergence form, see [Eva98, Chapter 6].

Let H be a Hilbert space, $V \subset H$ a dense subspace and $a \in \mathrm{Ses}(V)$ a sesquilinear form on V. For $\lambda \in \mathbb{C}$ we define the form $a_\lambda \in \mathrm{Ses}(V)$ by

$$a_\lambda(u,v) := a(u,v) + \lambda\,(u\,|\,v)_H$$

for $u, v \in V$. The form a is called **elliptic** if there is $\lambda_0 \geq 0$ such that for $\lambda = \lambda_0$ the following two conditions hold.

1) The form $\mathrm{Re}\, a_\lambda = (\mathrm{Re}\, a) + \lambda\,(\,.\,|\,.\,)_H$ is a scalar product on V which turns V into a Hilbert space such that the inclusion mapping $V \subset H$ is continuous.
2) The form a is continuous with respect to this scalar product on V.

With an elliptic form a on $V \subset H$, we associate an operator A in the following way. We define a norm on V by

$$\|u\|_V^2 := \operatorname{Re} a(u) + \lambda_0 \|u\|_H^2$$

for $u \in V$. Condition 1 then implies that V is a Hilbert space. Since V is continuously embedded in H (also by Condition 1) there is a continuous embedding of H^* into V^* (injectivity follows since V is dense in H). If we identify H with its antidual by means of the scalar product $(\cdot\,|\,\cdot)_H$ of H (Riesz-Fréchet Theorem), we obtain a sequence of continuous embeddings

$$V \subset H(\cong H^*) \subset V^*.$$

In doing this, H becomes a *dense* subspace of V^*. Now we define the mapping $\mathcal{A} : V \longrightarrow V^*$ by

$$\mathcal{A} := (u \longmapsto a(u, .)) : V \longrightarrow V^*$$

Using the identifications as above yields $(\mathcal{A} + \lambda_0)(u) = a(u, .) + \lambda_0\,(u\,|\,.)_H$. The operator $\mathcal{B} := \mathcal{A} + \lambda_0 : V \longrightarrow V^*$ is an isomorphism.

[Consider the form a_{λ_0}. Then $\operatorname{Re} a_{\lambda_0}$ *is* the scalar product of V, hence a is coercive (see (B.5) on page 162). By Condition 1, a is also continuous, hence satisfies the hypotheses of the Lax-Milgram Theorem B.17.]

Using the embedding $H \subset V^*$ we define

$$\begin{aligned} \mathcal{D}(A) &:= \{u \in V \mid \text{there is } y \in H \text{ s.t. } a(u, \cdot) = (y\,|\,\cdot)_H\} \\ &= \{u \in V \mid \mathcal{A}u \in H\} \end{aligned}$$

and $Au := \mathcal{A}u$ regarded as an element of H. Then the two fundamental identities

$$a(u, v) = \langle \mathcal{A}u, v\rangle \quad (u \in V) \tag{4.1}$$

$$a(u, v) = (Au\,|\,v)_H \quad (u \in \mathcal{D}(A)) \tag{4.2}$$

hold for $v \in V$. The operator A is called **associated** with the form a (notation: $a \sim A$). (Note that the definition of A is actually independent of the chosen λ_0.) An operator A on H is called **variational** if there is a dense subspace $V \subset H$ and an elliptic form $a \in \operatorname{Ses}(V)$ on V such that $a \sim A$. Clearly, if A is variational, then $A + \lambda$ is variational for each $\lambda \in \mathbb{C}$. Furthermore, also A^* is variational (if $A \sim a$ then $A^* \sim \overline{a}$).

Remark 4.1. Very often the starting point is a bit different in that V already carries a (natural) Hilbert space structure such that V is densely and continuously embedded in H. We denote this given scalar product on V by $(\cdot\,|\,\cdot)_V$. One can then replace the Conditions 1) and 2) by

1) $\operatorname{Re} a_\lambda(u) \geq \delta \|u\|_V^2$ for some $\delta > 0$ and all $u \in V$.
2) $|a(u, v)| \leq M \|u\|_V \|v\|_V$ for some $M \geq 0$ and all $u, v \in V$.

Here, Condition 2 says that a is continuous on V and this together with Condition 1 implies that $\operatorname{Re} a_\lambda$ is an equivalent scalar product on V.

It is important to realize that the construction of the operator A by means of (V, a) relies heavily on the particular scalar product of H. In fact, a result of MATOLCSI says that if A is not a bounded operator, one *always* can find a scalar product on H such that A is not variational (not even quasi-m-accretive, see [Mat03]). On the other hand, an operator which is not variational *may well be* variational with respect to some other scalar product H. Thus we are lead to the following problem.

Similarity Problem. Let A be an operator on the Hilbert space H. Which are necessary and sufficient conditions for A to be variational with respect to *some* (equivalent) scalar product on H?

We will give a complete answer to this question in § 5. To obtain *necessary* conditions we first characterize the operators which are variational with respect to the given (fixed) scalar product. For this purpose we have to introduce a new notion.

Let $0 \le \omega \le \frac{\pi}{2}$. An operator A on a Hilbert space H is called **ω-accretive** if $W(A) \subset \overline{S_\omega}$, i.e.

$$|\mathrm{Im}\,(Au \,|\, u)| \le (\tan \omega)\ \mathrm{Re}\,(Au \,|\, u) \qquad (u \in \mathcal{D}(A)).$$

The operator A is called **m-ω-accretive** if it is ω-accretive and $\mathcal{R}(A+1)$ is dense in H. Hence, an operator is (m-)accretive if and only if it is (m-)$\frac{\pi}{2}$-accretive. Furthermore, each m-ω-accretive operator is m-accretive. A 0-accretive operator is symmetric. An operator is positive if and only if it is m-0-accretive. The following proposition gives a useful characterization.

Proposition 4.2. *Let A be an operator on the Hilbert space H and let $0 < \omega < \frac{\pi}{2}$ and $\theta := \frac{\pi}{2} - \omega$. The following assertions are equivalent.*

(i) The operator A is m-ω-accretive.

(ii) The operators $e^{\pm i\theta} A$ are m-accretive.

(iii) The operator $-A$ generates a holomorphic C_0-semigroup $(T(z))_{z \in S_\theta}$ on S_θ such that $\|T(z)\| \le 1$ for all $z \in S_\theta$.

Note that (ii) does not imply (i) if $\omega = 0$. For the precise meaning of (iii) see § 3 of Chapter 2.

Proof. The equivalence of (i) and (ii) is clear from Corollary B.8. Assume (ii). From Proposition B.7 we infer that $A \in \mathrm{Sect}(\omega)$. Since $\omega < \frac{\pi}{2}$ we conclude that $-A$ generates a bounded holomorphic semigroup $T : S_\theta \longrightarrow \mathcal{L}(H)$. For each $\varphi \in (-\theta, \theta)$ the operator $-e^{i\varphi}A$ generates the semigroup $(T(e^{i\varphi}t))_{t>0}$ (see Remark 2.25). But clearly $e^{i\varphi}A$ is m-accretive, whence $\|T(e^{i\varphi}t)\| \le 1$ for all $t > 0$. This proves (iii).
Assume (iii). As above, the operator $-e^{i\varphi}A$ generates the semigroup $(T(e^{i\varphi}t))_{t>0}$, for each $\varphi \in (-\theta, \theta)$. Since this is a contraction semigroup we conclude that $e^{i\varphi}A$ is m-accretive. Letting φ tend to $\pm\theta$ yields that $e^{\varphi\theta}A$ is m-accretive. □

We return to the original problem. Assume that a is an elliptic form on $V \subset H$, $A \sim a$, and λ_0 is chosen as above.

The operator $B := A + \lambda_0 : \mathcal{D}(A) \longrightarrow H$ is bijective and B^{-1} is a bounded operator on H, since B is the restriction of $\mathcal{B} : V \longrightarrow V^*$ to the range H. Because H is dense in V^*, $\mathcal{D}(A) = \mathcal{D}(B)$ is dense in V, hence a forteriori in H. This means that B is a densely defined closed operator in H with $0 \in \varrho(B)$. Furthermore, the form a_{λ_0} is continuous with respect to the scalar product $\operatorname{Re} a_{\lambda_0}$.

[Since the embedding $V \subset H$ is continuous, there is a constant M_1 such that $\|u\|_H^2 \le M_1 \|u\|_V^2$ for all $u \in V$. The continuity of the form a (Condition 2) yields the existence of a constant M_2 such that $|a(u,v)| \le M_2 \|u\|_V^2$. Putting the two inequalities together we obtain $|a_{\lambda_0}(u,v)| \le (M_2 + \lambda_0 M_1) \|u\|_V^2$.]

Now a short glance on Proposition B.3 shows that the form a_{λ_0} is sectorial (of some angle ω which depends on the continuity constants of a_{λ_0} with respect to $\operatorname{Re} a_{\lambda_0}$). Since $(Bu \,|\, u)_H = a_{\lambda_0}(u,u)$ for $u \in \mathcal{D}(B)$ the operator B is m-ω-accretive. Hence we have shown that if A is a variational operator, the operator $A + \lambda$ is m-ω-accretive for some λ and some ω. On the other hand, each m-ω-accretive operator is variational, as the following proposition shows.

Proposition 4.3. *Let $0 \le \omega < \frac{\pi}{2}$ and A be a m-ω-accretive operator on the Hilbert space H. Then A is variational. More precisely, there is a dense subspace $V \subset H$ and an elliptic form $a \in \operatorname{Ses}(V)$ such that $A \sim a$ and $\operatorname{Re} a$ is positive.*

Proof. On $\mathcal{D}(A)$ we define the sesquilinear form a by

$$a(u,v) := (Au \,|\, v)_H$$

and a scalar product (!) by

$$(u \,|\, v)_V := (\operatorname{Re} a)(u,v) + (u \,|\, v)_H .$$

The form a is continuous with respect to this scalar product. In fact, since a is sectorial of angle ω by Proposition B.3 we have

$$|a(u,v)| \le (1 + \tan\omega)\sqrt{\operatorname{Re} a(u)}\sqrt{\operatorname{Re} a(v)} \le (1 + \tan\omega) \|u\|_V \|v\|_V$$

for all $u, v \in \mathcal{D}(A)$. Obviously, the embedding $(\mathcal{D}(A), \|.\|_V) \subset H$ is continuous. Hence it has a continuous extension $\iota : V \longrightarrow H$, where V is the (abstract) completion of $\mathcal{D}(A)$ with respect to $\|.\|_V$ The mapping ι is injective.

[*Proof.* Let $x \in V$ and $\iota x = 0$. This means that there is $(u_n)_n \subset \mathcal{D}(A)$ such that $\|u_n - u_m\|_V \to 0$ and $\|u_n\|_H \to 0$. Hence for all $n, m \in \mathbb{N}$ we have

$$\begin{aligned}
\operatorname{Re} a(u_n) &= \operatorname{Re}(Au_n \,|\, u_n)_H = \operatorname{Re}(Au_n \,|\, u_n - u_m)_H + \operatorname{Re}(Au_n \,|\, u_m)_H \\
&= \operatorname{Re}[a(u_n, u_n - u_m)] + \operatorname{Re}(Au_n \,|\, u_m)_H \\
&\le |a(u_n, u_n - u_m)| + \left|(Au_n \,|\, u_m)_H\right| \\
&\le M \|u_n\|_V \|u_n - u_m\|_V + \|Au_n\|_H \|u_m\|_H .
\end{aligned}$$

Since (u_n) is V-Cauchy, $C := \sup_n \|u_n\|_V < \infty$. Hence

$$\operatorname{Re} a(u_n) \le MC \limsup_m \|u_n - u_m\|_V \to 0 \quad \text{as } n \to \infty.$$

This shows that $\|u_n\|_V \to 0$, whence $x = 0$.]

We therefore can regard V as being continuously and densely embedded into H. The form a has a unique extension to a continuous sesquilinear form on V (again denoted by a). Clearly, the form a is elliptic (with $\lambda_0 = 1$). Hence it remains to show that A is associated with a.

We denote by B the operator which is associated with a and choose $u \in \mathcal{D}(A)$. The equation $a(u,v) = (Au \,|\, v)_H$ holds for all $v \in V$ since it holds for all $v \in \mathcal{D}(A)$, $\mathcal{D}(A)$ is dense in V, a is continuous on V, and V is continuously embedded in H. This shows $u \in \mathcal{D}(B)$ and $Bu = Au$, whence $A \subset B$. On the other hand, we have $-1 \in \varrho(B)$ by construction and $-1 \in \varrho(A)$ since A is m-ω-accretive. This implies that $A = B$. $\square$

Corollary 4.4. *An operator A on H is variational if and only if there is $\lambda \in \mathbb{R}$ such that $A + \lambda$ is m-ω-accretive for some ω if and only if $-A$ generates a holomorphic semigroup T on some sector S_θ such that*

$$\|T(z)\| \le e^{\lambda \operatorname{Re} z} \qquad (z \in S_\theta)$$

for some $\lambda \in \mathbb{R}$.

These characterizations are not invariant with respect to changing the scalar product on H. To solve the similarity problem we therefore have to look for other conditions. At this point the functional calculus enters the scene.

§ 2 The Functional Calculus on Hilbert Spaces

This section is to provide some background information which is necessary for the things to come. We will deal with adjoints of operators (first), selfadjoint operators (next), and m-accretive operators (last).

The reader may have noticed that in Chapter 1 where we treated the functional calculus for sectorial operators we successfully avoided talking about adjoint operators. However, in the Hilbert space setting we cannot do this anymore. Let $0 < \varphi \le \pi$ and $f : S_\varphi \longrightarrow \mathbb{C}$ be holomorphic. The function f^*, defined by

$$f^* := \left(z \longmapsto \overline{f(\overline{z})} \right) : S_\varphi \longrightarrow \mathbb{C}$$

is called the **conjugate** of the function f. Obviously, all the function spaces $\mathcal{DR}(S_\varphi), \mathcal{DR}_0(S_\varphi), H^\infty(S_\varphi), \mathcal{A}(S_\varphi), \mathcal{B}(S_\varphi), \mathcal{C}(S_\varphi), \dots$ considered in Chapter 1 are *invariant* with respect to coniugation. Hence, if $f(A)$ is defined for some sectorial operator A, also $f^*(A)$ is.

Proposition 4.5. *Let $A \in \operatorname{Sect}(\omega)$ be a sectorial operator on a Hilbert space H. Then the following assertions hold.*

- *a) The operator A is densely defined and $H = \mathcal{N}(A) \oplus \overline{\mathcal{R}(A)}$. In particular, A is injective if and only if $\mathcal{R}(A)$ is dense in H.*
- *b) The operator A^* is also sectorial of angle ω with $M(A, \omega') = M(A^*, \omega')$ for all $\omega < \omega' \le \pi$. In particular, we have $\omega_A = \omega_{A^*}$.*
- *c) The operator A is injective/invertible/bounded if and only if A^* is injective/invertible/bounded.*
- *d) Let $\omega < \varphi \le \pi$ and $f : S_\varphi \longrightarrow \mathbb{C}$ holomorphic. If $f(A)$ is defined within the natural functional calculus for sectorial operators, then also $f(A^*)$ is, and we have $f(A^*) = [f^*(A)]^*$.*
- *e) The identity*
$$(A^\alpha)^* = (A^*)^{\overline{\alpha}} \tag{4.3}$$
is true for all $\operatorname{Re} \alpha > 0$, and even for all $\alpha \in \mathbb{C}$ in case that A is injective.
- *f) If $A \in \operatorname{BIP}(H)$ then also $A^* \in \operatorname{BIP}(H)$ with $\theta_A = \theta_{A^*}$.*

g) Let $\varphi > \omega$. If the natural $\mathcal{DR}(S_\varphi)$-calculus for A is bounded, the same is true for A^* and the bounds are the same.

Proof. Ad *a)*. This follows from Proposition 1.1, since a Hilbert space is reflexive.
Ad *b)*. This follows from Corollary B.5.
Ad *c)* If A is injective, then $\mathcal{R}(A)$ is dense. By part *e)* of Proposition B.4, this implies $\mathcal{N}(A^*) = 0$. The reverse implication follows from $A^{**} = A$.
Ad *d)*. Let $f \in \mathcal{DR}(S_\varphi)$ or $f \in \mathcal{DR}_0(S_\varphi)$. Then

$$\begin{aligned}[f^*(A)]^* &= \left(\frac{1}{2\pi i}\int_\Gamma f^*(z)R(z,A)\,dz\right)^* = \frac{-1}{2\pi i}\int_{\overline{\Gamma}} f(z)R(\overline{z},A)^*\,dz \\ &= \frac{-1}{2\pi i}\int_{\overline{\Gamma}} f(z)R(z,A^*)\,dz \overset{!}{=} \frac{1}{2\pi i}\int_\Gamma f(z)R(z,A^*)\,dz = f(A^*),\end{aligned}$$

since Γ is such that $(z \longmapsto \overline{z})$ just reverses the orientation of Γ. The same proof applies if $0 \in \varrho(A)$ and f is regularly decaying at ∞. To cover the other cases, note that $g^* = g$ if $g(z) = (1+z)^{-1}$ or $g(z) = z(1+z)^{-2}$. Thus if $f \in \mathcal{A}$ such that $F(z) := (1+z)^{-n}f(z) \in \mathcal{DR} + \mathcal{DR}_0$ we have

$$f(A^*) = (1+A^*)^n F(A^*) \overset{(1)}{=} [(1+A)^n]^*[F^*(A)]^* \overset{(2)}{=} [F^*(A)(1+A)^n]^* \overset{(3)}{=} [f^*(A)]^*$$

Here we have used Proposition B.6 in (1) and Proposition B.4 in (2). Equation (3) is justified by Proposition B.4 and the fact that $\mathcal{D}(A^n)$ is a core for $f^*(A)$ by *d)* of Proposition 1.9.
The proof of the statement in case that A is injective and $f \in \mathcal{B}$ is similar. One has to use the identity $[\Lambda_A^n]^* = \Lambda_{A^*}^n$ which holds for each n. The same applies to the other cases.
The statement *e), f)* and *g)* are consequences of *d)*. ☐

Since selfadjoint operators possess a nice functional calculus (see Section C.6 in Appendix C) we have to prove a coherence result.

Proposition 4.6. *Let (Ω, μ) be a standard measure space and let $a : \Omega \to \mathbb{C}$ be a continuous function with $a(\Omega) \subset \overline{S_\omega}$, where $0 \le \omega < \pi$. Denote by $A := M_a$ the multiplication operator on $H := \mathbf{L^2}(\Omega, \mu)$. Then A is m-ω-accretive and $f(A) = M_{f\circ a}$ whenever $f(A)$ is defined by the NFCSO.*

Proof. Let $\psi \in \mathcal{D}(A)$. Then

$$(A\psi \,|\, \psi)_{\mathbf{L^2}} = \int_\Omega a\psi\overline{\psi}\,d\mu = \int_\Omega a\,|\psi|^2\,du.$$

Now $a(s)\,|\psi(s)|^2 \in \overline{S_\omega}$ for every $s \in \Omega$ by hypothesis and $\overline{S_\omega}$ is closed and convex. Hence also $\int_\Omega a\,|\psi| \in \overline{S_\omega}$. This shows that $W(A) \subset \overline{S_\omega}$.
To prove the second assertion, asssume first that $f \in \mathcal{DR}[S_\omega]$. Then $f(A) = \frac{1}{2\pi i}\int_\Gamma f(z)R(z,A)\,dz$ and the integral converges in the operator norm topology. By Proposition C.1, $R(z,A) = M_{(z-a)^{-1}}$ for each $z \notin \overline{S_\omega}$. Moreover, the mapping $(g \longmapsto M_g) : \mathbf{C^b}(\Omega) \longrightarrow \mathbf{L^2}(\Omega,\mu)$ is an isometric embedding. Therefore, $f(A) = M_g$ for some bounded and continuous function on Ω. Since evaluation at a point $s \in \Omega$ is a continuous functional on $\mathbf{C^b}(\Omega)$, we obtain

$$g(s) = \left[\frac{1}{2\pi i}\int_\Gamma f(z)(z-a(\cdot))^{-1}\,dz\right](s) = \frac{1}{2\pi i}\int_\Gamma f(z)(z-a(s))^{-1}\,dz = f(a(s))$$

by Cauchy's theorem. This shows $f(A) = M_{f\circ a}$. The same proof (with a different Γ) works in the case $f \in \mathcal{DR}_0[S_\omega]$. So we know the statement for $f \in \mathcal{DR}_{\text{ext}}[S_\omega]$. If $f \in \mathcal{A}$, there is n such that $g := f(z)(1+z)^{-n} \in \mathcal{DR}_{\text{ext}}$. Then $f(A) = (1+A)^n g(A) = M_{(1+a)^n}M_{g\circ a} = M_{f\circ a}$. (Note that $(1+a)^{-n}$ is bounded.)
Assume that A is injective and $f \in \mathcal{B}$. Then there is n such that $\eta f \in \mathcal{DR}$, where we set $\eta(z) = z^n(1+z)^{-2n}$. By what we have proved so far combined with part *h)* of Proposition C.1 we obtain

$$(f\eta)(A) = (\eta f)(A) = M_{(\eta\circ a)(f\circ a)} = M_{f\circ a}M_{\eta\circ a} = M_{f\circ a}\eta(A).$$

Hence, $f(A) = \eta(A)^{-1} M_{f\circ a}\eta(A)$. Moreover, $M_{f\circ a}\eta(A) = M_{(\eta\circ a)(f\circ a)} \supset M_{\eta\circ a}M_{f\circ a} = \eta(A)$ $M_{f\circ a}$. So $f(A) \supset M_{f\circ a}$. Let $\psi \in \mathcal{D}(f(A))$. Then from above we know that $(f\circ a)(\eta\circ a)\psi \in \mathcal{D}(\eta(A)^{-1})$. Hence there is $g \in \mathbf{L}^2(\Omega)$ such that $(f\circ a)(\eta\circ a)\psi = (\eta\circ a)g$. Since M_a is injective the set $\Omega_0 := \{a = 0\}$ is locally μ-null. (See part *e)* of Proposition C.1.) Thus, $(f\circ a)\psi = g$ on $\Omega \setminus \Omega_0$. This implies that $(f\circ a)\psi \in \mathbf{L}^2$, whence $\psi \in \mathcal{D}(M_{f\circ a})$.
The proofs in the other cases (e.g., $0 \in \varrho(A)$ and $f \in \mathcal{C}$), are similar. □

Corollary 4.7. *Let A be a selfadjoint operator on a Hilbert space H.*

a) If $A \geq 0$, then $A \in \mathrm{Sect}(0)$ and for each $\varphi > 0$ the natural $\mathcal{DR}(S_\varphi)$-calculus is bounded. In fact $\|f(A)\| \leq \|f\|_{(0,\infty)}$ for all $f \in \mathcal{DR}_{\mathrm{ext}}[S_0]$.

b) We have $iA \in \mathrm{Sect}(\frac{\pi}{2})$ and $\|f(iA)\| \leq \|f\|_{i\mathbb{R}}$ for all $f \in \mathcal{DR}_{\mathrm{ext}}[S_{\frac{\pi}{2}}]$.

Proof. By the spectral theorem C.11 there is a standard measure space (Ω, μ) and a continuous, real-valued function $a \in \mathbf{C}(\Omega)$ such that (H, A) is unitarily equivalent to $(\mathbf{L}^2(\Omega, \mu), M_a)$. Now the assertions follow from Proposition 4.6 and Proposition C.1. □

Let us turn to *m-accretive* operators (see Section B.6 for definitions and fundamental results). By Proposition B.20 and Theorem B.21 the following assertions are equivalent for an operator A on H.

(i) A is m-accretive.

(ii) $-A$ generates a C_0-contraction semigroup.

(iii) $\{\mathrm{Re}\, z < 0\} \subset \varrho(A)$ and $\left\|(\lambda + A)^{-1}\right\| \leq (\mathrm{Re}\,\lambda)^{-1}$ for all $\mathrm{Re}\,\lambda > 0$.

In particular, $A \in \mathrm{Sect}(\frac{\pi}{2})$, so we have the functional calculus for sectorial operators at hand.
The next result, stated without proof, sometimes may help to reduce a problem on m-accretive operators to selfadjoint opertors.

Theorem 4.8. (Szökefalvi-Nagy)
Let $(T(t))_{t\geq 0}$ be a contraction semigroup on a Hilbert space H. Then there exists a Hilbert space K, an isometric embedding $\iota : H \longrightarrow K$ and a unitary C_0-group $(U(t))_{t\in\mathbb{R}}$ on K such that

$$P \circ U(t) \circ \iota = \iota \circ T(t)$$

for all $t \geq 0$. Here, $P : K \longrightarrow \iota(H)$ denotes the othogonal projection on the closed subspace $\iota(A)$ of K.

The triple (K, U, ι) is called a **dilation** of the contraction semigroup T. For a proof see, e.g., [Dav80, Chapter 6, Section 3].

The characterization of m-accretive operators given above shows that m-accretivity has a lot to do with contractivity. The next result is in this vein.

Proposition 4.9. *Let $A \in \mathrm{Sect}(\frac{\pi}{2})$ on the Hilbert space H. Then A is m-accretive if and only if*

$$\|f(A)\| \leq \|f\|_{\frac{\pi}{2}} \tag{4.4}$$

for all $f \in \mathcal{DR}_{\mathrm{ext}}[S_{\frac{\pi}{2}}]$.

Proof. If we plug in $f(z) = \frac{z-1}{z+1}$ in (4.4) we obtain $\left\|(A-1)(A+1)^{-1}\right\| \leq 1$. Then we apply (iv) of Proposition B.20 to conclude that A is m-accretive.
The converse is proved by means of the Sz.-Nagy Theorem. Assume that A is m-accretive. Then $-A$ generates a contraction semigroup T. By Theorem 4.8 there is a dilation (K, U, ι) of T. By Stone's theorem B.22 the generator of U is of the form $-iB$ where B is a selfadjoint operator on K. We claim that $PR(\lambda, iB)\iota = \iota R(\lambda, A)$ for each $\operatorname{Re}\lambda < 0$. In fact,

$$\begin{aligned} PR(\lambda, iB)\iota &= -P(-\lambda, -iB)\iota = -\int_0^\infty e^{\lambda s} PU(s)\iota\, ds = -\int_0^\infty e^{\lambda s}\iota T(s)\, ds \\ &= -\iota R(-\lambda, -A) = \iota R(\lambda, A) \end{aligned}$$

Now choose $\frac{\pi}{2} < \varphi \leq \pi$ and $f \in \mathcal{DR}(S_\varphi)$. Then

$$\begin{aligned} Pf(iB)\iota &= \frac{1}{2\pi i}\int_\Gamma f(z) PR(z, iB)\iota\, dz = \frac{1}{2\pi i}\int_\Gamma f(z)\iota\, R(z, A)\, dz \\ &= \iota\, \frac{1}{2\pi i}\int_\Gamma f(z) R(z, A)\, dz = \iota f(A) \end{aligned}$$

The same arguments apply if $f \in \mathcal{DR}_0$. Hence we obtain $Pf(iB)\iota = \iota f(A)$ for all $f \in \mathcal{DR}_{\text{ext}}$. Since ι is isometric, by *b)* of Corollary 4.7 we have

$$\|f(A)x\|_H = \|\iota f(A)x\|_K = \|Pf(iB)\iota x\|_K \leq \|f(iB)\iota x\|_K \leq \|f\|_{\frac{\pi}{2}} \|\iota x\|_K = \|f\|_{\frac{\pi}{2}} \|x\|_H$$

for every $x \in H$. This finishes the proof. □

Remark 4.10. Let us sketch another proof of Proposition 4.9. Since A is m-accretive, the Cayley transform $T := (A-1)(A+1)^{-1}$ satisfies $\|T\| \leq 1$. Now the **von Neumann inequality** says that $\|p(T)\| \leq \|p\|_\infty$ for every polynomial $p \in \mathbb{C}[z]$, where $\|p\|_\infty$ is the uniform norm of p on the unit disc. This readily yields $\|r(T)\| \leq \|r\|_\infty$ for every rational function r with poles ouside of $\{z \mid |z| \leq 1\}$. Hence we can conclude that the natural $\mathcal{R}^\infty(S_{\frac{\pi}{2}})$-calculus for A is bounded with bound 1. Now apply Proposition 1.34.

Corollary 4.11. *Let A be an injective m-accretive operator on the Hilbert space H. Then the natural $H^\infty(S_\varphi)$-calculus for A is contractive for each $\frac{\pi}{2} < \varphi \leq \pi$. In particular, $A \in \mathrm{BIP}(A)$ with $\left\|A^{is}\right\| \leq e^{\frac{\pi}{2}|s|}$ for all $s \in \mathbb{R}$.*

Proof. Apply Proposition 4.9 together with Proposition 1.32. □

What does Proposition 4.9 tell us concerning the Similarity Problem? Recall that a variational operator A is more or less m-ω-accretive for some $\omega < \frac{\pi}{2}$. (In fact, it is after after shifting.) By Corollaries 4.4 and 4.11, we must have $A + \lambda \in \mathrm{BIP}(H) \cap \mathrm{Sect}(\omega)$ for some $\omega < \frac{\pi}{2}$ if λ is large. This is a property which does not depend on the particular scalar product of H (and in fact will give the desired characterization).

§ 3 Fractional Powers of m-Accretive Operators and the Square Root Problem

We turn to fractional powers of m-accretive operators. The first results are so to speak "mapping theorems for the numerical range".

Proposition 4.12. *Let $\delta > 0$ and $A - \delta$ be m-accretive. Then $A^\alpha - \delta^\alpha$ is m-accretive for each $0 < \alpha < 1$.*

Proof. See [Tan79, Lemma 2.3.6]. □

Proposition 4.13. *Let A be m-accretive and let $0 \leq \alpha \leq 1$. Then the operator A^α is m-$\alpha\frac{\pi}{2}$-accretive, i.e.,*

$$|\mathrm{Im}\,(A^\alpha x \,|\, x)| \leq \tan \frac{\alpha\pi}{2}\, \mathrm{Re}\,(A^\alpha x \,|\, x)$$

for all $x \in \mathcal{D}(A)$.

Proof. Consider the function

$$f(z) := \frac{e^{i\frac{\pi}{2}(1-\alpha)} z^\alpha - 1}{e^{i\frac{\pi}{2}(1-\alpha)} z^\alpha + 1}$$

Note that this function is holomorphic on S_π. It is not difficult to see that in fact $f \in \mathcal{DR}_{\mathrm{ext}}$ (cf. the proof of Proposition 2.2). Since the mapping $z \longmapsto e^{i\frac{\pi}{2}(1-\alpha)} z^\alpha$ maps $S_{\frac{\pi}{2}}$ to itself, we have $\|f\|_{\frac{\pi}{2}} \leq 1$. Applying Proposition 4.9 and the composition rule we obtain

$$\left\|(e^{i\frac{\pi}{2}(1-\alpha)} A^\alpha - 1)(e^{i\frac{\pi}{2}(1-\alpha)} A^\alpha + 1)^{-1}\right\| = \|f(A)\| \leq \|f\|_{\frac{\pi}{2}} \leq 1.$$

By Proposition B.20 this implies that $e^{i\frac{\pi}{2}(1-\alpha)} A^\alpha$ is m-accretive. Obviously the same reasoning applies to $e^{-i\frac{\pi}{2}(1-\alpha)} A^\alpha$. Now it follows from Proposition 4.2 that A^α is m-ω-accretive, where $\omega = \frac{\pi}{2} - \frac{\pi}{2}(1-\alpha) = \frac{\alpha\pi}{2}$. □

For completeness, we prove another mapping theorem for the numerical range.

Proposition 4.14. *Let A be an injective m-ω-accretive operator for some $0 \leq \omega \leq \frac{\pi}{2}$. Then $W(\log A) \subset \overline{H_\omega}$.*

Proof. Define $\theta := \frac{\pi}{2} - \omega$. Then

$$\omega + i \log A = \frac{\pi}{2} - \theta + i \log A = \frac{\pi}{2} + i(i\theta + \log A) = \frac{\pi}{2} + i \log(e^{i\theta} A)$$

by the composition rule. Since $g(z) := \frac{\pi}{2} + i \log z$ maps $S_{\frac{\pi}{2}}$ into itself, the function $f(z) := \frac{g(z)-1}{g(z)+1}$ satisfies $\|f\|_{\frac{\pi}{2}} \leq 1$. Since $e^{i\theta A}$ is m-accretive, $\left\|f(e^{i\theta} A)\right\| \leq 1$. But $f(e^{i\theta} A)$ is the Cayley transform of $g(e^{i\theta} A)$, whence $g(e^{i\theta} A)$ is m-accretive. This shows that $\omega + i \log A$ is m-accretive. Similarly one proves that $\omega - i \log A$ is m-accretive. Combining both statements yields $W(\log A) \subset \overline{H_\omega}$. □

The next theorem is a miracle.

Theorem 4.15. (Kato)
Let A be m-accretive and $0 \leq \alpha < \frac{1}{2}$. Then the following assertions hold:

1) $\mathcal{D}(A^\alpha) = \mathcal{D}(A^{*\alpha}) =: \mathcal{D}_\alpha$.
2) $\|A^{*\alpha} u\| \leq \tan \frac{\pi(1+2\alpha)}{4} \|A^\alpha u\|$ *for all $u \in \mathcal{D}_\alpha$.*
3) $\mathrm{Re}\,(A^\alpha u \,|\, A^{*\alpha} u) \geq (\cos \pi\alpha) \|A^\alpha u\| \, \|A^{*\alpha} u\|$ *for all $u \in \mathcal{D}_\alpha$.*

Proof. See [Kat61a, Theorem 1.1]. The proof proceeds in two steps. First the statements are proved under the additional hypothesis of A being bounded with $\mathrm{Re}\,(Au \,|\, u) \geq \delta \|u\|^2$ for all $u \in H$ and some $\delta > 0$. (This is the difficult part of the proof.). The second (easy) step uses "sectorial approximation" (in our language) and Proposition 2.9. □

Remark 4.16. Kato's theorem is remarkable, also in that it fails in the case $\alpha = \frac{1}{2}$. KATO did not know this when he wrote the article [Kat61a] but only shortly afterwards, LIONS in [Lio62] produced a counterexample. For some time it had been an open question if at least for m-ω-accretive operators A it is true that

$$\mathcal{D}(A^{\frac{1}{2}}) = \mathcal{D}(A^{\frac{1}{2}*}). \tag{4.5}$$

If A is associated with the form $a \in \mathrm{Ses}(V)$ and, say, we have $\lambda_0 = 0$, then (4.5) is equivalent to $V = \mathcal{D}(A^{\frac{1}{2}})$, as LIONS in [Lio62] and KATO in [Kat62] have shown. Finally, MCINTOSH gave a counterexample in [McI72]. However, this was not the end of the story. In fact, even if the statement is false *in general* it might be true for *particular operators* such as second order elliptic operators on $\mathbf{L}^2(\mathbb{R}^n)$ in divergence form. For these operators, the problem became famous under the name **Kato's Square Root Problem** and has stimulated a considerable amount of research which lead to the discovery of deep results connecting Operator Theory and Harmonic Analysis (see also the comments in §7).

Let us call an operator A on the Hilbert space H **square root regular** if $A + \lambda$ is sectorial and $\mathcal{D}((A+\lambda)^{\frac{1}{2}}) = \mathcal{D}((A+\lambda)^{\frac{1}{2}*})$ for large $\lambda \in \mathbb{R}$. We pose a second problem.

Second Similarity Problem. Assume A is variational. Is there an equivalent scalar product on H such that A is variational *and* square root regular with respect to the new scalar product?

We will give solutions to both similarity problems in §5

§4 McIntosh-Yagi Theory

We will now show how the boundedness of the natural $\mathcal{DR}(S_\varphi)$-calculus for an operator A on a Hilbert space H is related to similarity. The connection is given by the following proposition.

Proposition 4.17. *Let $A \in \mathrm{Sect}(\omega)$ be a sectorial operator on a Hilbert space H. Assume that the natural $\mathcal{DR}(S_\varphi)$-calculus is bounded for some $\omega < \varphi < \pi$. If $0 \neq f \in \mathcal{DR}(S_\varphi)$ then by*

$$(x \,|\, y)_f := \int_0^\infty (f(tA)x \,|\, f(tA)y)_H \,\frac{dt}{t} \qquad (x, y \in H)$$

a semi-scalar product is defined on H which is 0 on $\mathcal{N}(A)$ and is an equivalent scalar product on $\overline{\mathcal{R}(A)}$.

The statement includes the existence of the integral. Since $H = \mathcal{N}(A) \oplus \overline{\mathcal{R}(A)}$ and $f(tA)x = 0$ for $x \in \mathcal{N}(A), f \in \mathcal{DR}[S_\omega], t > 0$, we essentially have to establish an inequality of the form

$$C_1 \|x\|^2 \leq \int_0^\infty \|f(tA)x\|^2 \,\frac{dt}{t} \leq C_2 \|x\|^2$$

for some constants $C_1, C_2 > 0$ and all $x \in \overline{\mathcal{R}(A)}$. To do this we need a new concept.

The **Cantor group** is the set $G := \{-1,1\}^{\mathbb{Z}}$, i.e., the $\mathbb{Z}$-fold direct product of the multiplicative discrete group $Z_2 \cong \{-1,1\}$. By Tychonoff's theorem, G is a compact topological group. We denote by μ the normalized Haar measure on G. The projections

$$\varepsilon_k := ((g_n)_n \longmapsto g_k) : G \longrightarrow \{-1,1\} \quad (k \in \mathbb{Z})$$

are called **Rademacher functions**. As the Rademachers obviously are continuous characters of the compact group G, they form an orthonormal set in $\mathbf{L^2}(G,\mu)$, i.e.,

$$\int_G \varepsilon_n \varepsilon_m \, d\mu = \delta_{nm}$$

for all $n,m \in \mathbb{Z}$. (One can show that the set of Rademachers actually generates the character group of G.) Given any Banach space X, the space $\mathrm{Rad}(X)$ is defined by

$$\mathrm{Rad}(X) := \mathrm{span}\{\varepsilon_n \otimes x \mid n \in \mathbb{Z}\ x \in X\} \subset \mathbf{L^2}(G,X)$$

with the norm induced by the norm of $\mathbf{L^2}(G,X)$[1]. If $X = H$ is a Hilbert space, we have the important identity

$$\sum\nolimits_n \|x_n\|_H^2 = \Big\| \sum\nolimits_n \varepsilon_n \otimes x_n \Big\|_{\mathrm{Rad}(H)}^2 \tag{4.6}$$

for every *finite* two-sided sequence $(x_n)_{n\in\mathbb{Z}} \subset H$. In fact,

$$\begin{aligned} \Big\| \sum\nolimits_n \varepsilon_n \otimes x_n \Big\|_{\mathrm{Rad}(H)}^2 &= \int_G \Big\| \sum\nolimits_n \varepsilon_n(g) x_n \Big\|_H^2 \ \mu(dg) \\ &= \sum\nolimits_{n,m} \int_G \varepsilon_n(g)\varepsilon_m(g) \, (\, x_n \mid x_m \,) \ \mu(dg) \\ &= \sum\nolimits_{n,m} \delta_{nm} \, (\, x_n \mid x_m \,) \ = \sum\nolimits_n \|x_n\|_H^2 . \end{aligned}$$

We can now return to the main theme.

Proof of Proposition 4.17. Let C be such that

$$\|g(A)\| \leq C \, \|g\|_{S_\varphi}$$

for all $g \in \mathcal{DR}(S_\varphi)$ and let $f \in \mathcal{DR}(S_\varphi)$. We have

$$\begin{aligned} \int_{2^{-N}}^{2^N} \|f(tA)x\|^2 \, \frac{dt}{t} &= \sum_{k=-N}^{N} \int_{2^k}^{2^{k+1}} \|f(tA)x\|^2 \, \frac{dt}{t} = \sum_{k=-N}^{N} \int_1^2 \left\| f(t2^k A)x \right\|^2 \frac{dt}{t} \\ &= \int_1^2 \sum_{k=-N}^{N} \left\| f(t2^k A)x \right\|^2 \frac{dt}{t} = \int_1^2 \Big\| \sum\nolimits_{-N}^{N} \varepsilon_k \otimes f(t2^k A)x \Big\|_{\mathrm{Rad}(H)}^2 \frac{dt}{t} \\ &\leq \int_1^2 \Big\| \sum\nolimits_{-N}^{N} \varepsilon_k \otimes f(t2^k A) \Big\|_{\mathrm{Rad}(\mathcal{L}(H))}^2 \frac{dt}{t} \, \|x\|^2 . \end{aligned}$$

[1] Actually, the **Khintchine-Kahane inequality** asserts, that the norms on $\mathrm{Rad}(X)$ induced by the different embeddings $\mathrm{Rad}(X) \subset \mathbf{L^p}(G,X)$ for $1 \leq p < \infty$ are all equivalent, see [LT96, Part I, Theorem 1.e.13].

At this point we use the Kalton-Weis Lemma 1.39 to conclude that

$$\left\| \sum_{-N}^{N} \varepsilon_k(g) f(t2^k A) \right\|_{\mathcal{L}(H)} \leq DC$$

for all $g \in G$, $t > 0$, and $N \in \mathbb{N}$, where D is a constant which does only depend on f and φ. This implies

$$\int_0^\infty \|f(tA)x\|^2 \frac{dt}{t} \leq C^2 D^2 (\log 2) \|x\|^2$$

for all $x \in H$. So we are left to show the second inequality. For this we observe that by Proposition 4.5 the operator A^* also satisfies the hypothesis of the proposition, even with the same constant C. By what we have shown,

$$\int_0^\infty \|f(tA^*)x\|^2 \frac{dt}{t} \leq (\log 2) C^2 D^2 \|x\|^2$$

for all $x \in H$. Consider the function $g := f^* f$. Obviously, $g \in \mathcal{DR}(S_\varphi)$ and $g(t) := |f(t)|^2$ for $t > 0$. Since $f \neq 0$ and f is holomorphic, we have

$$\alpha := \int_0^\infty |f(t)|^2 \frac{dt}{t} > 0.$$

By *c)* of Proposition 1.29 we can compute

$$\begin{aligned}
\alpha \|x\|^2 &= (\alpha x \,|\, x) = \left(\int_0^\infty g(tA)x \frac{dt}{t} \,\middle|\, x \right) = \int_0^\infty (f^*(tA) f(tA) x \,|\, x) \frac{dt}{t} \\
&= \int_0^\infty (f(tA)x \,|\, f(tA^*)x) \frac{dt}{t} \leq \left(\int_0^\infty \|f(tA)x\|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \left(\int_0^\infty \|f(tA^*)x\|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \\
&\leq CD(\log 2)^{\frac{1}{2}} \|x\| \left(\int_0^\infty \|f(tA)x\|^2 \frac{dt}{t} \right)^{\frac{1}{2}}
\end{aligned}$$

for all $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$. We can divide by $(\log 2)^{\frac{1}{2}} CD \|x\|$ and approximate an arbitrary element of $\overline{\mathcal{R}(A)}$ by elements from $\mathcal{D}(A) \cap \mathcal{R}(A)$ to finish the proof. □

Fix $A \in \mathrm{Sect}(\omega)$ on H and assume for simplicity that A is injective (i.e., H has dense domain and dense range). For $\varphi > \omega$ and $0 \neq \psi \in \mathcal{DR}(S_\varphi)$ we define

$$H_\psi := \left\{ x \in H \;\middle|\; \int_0^\infty \|\psi(tA)x\|^2 \frac{dt}{t} < \infty \right\} \quad \text{and}$$

$$\|x\|_\psi := \left(\int_0^\infty \|\psi(tA)x\|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \quad \text{for } x \in H_\psi.$$

Lemma 4.18. *Let A, ω, φ, ψ as above. Then the following assertions hold.*

a) The norm $\|\cdot\| + \|\cdot\|_\psi$ is complete.

b) The space $\mathcal{D}(A) \cap \mathcal{R}(A)$ is contained in H_ψ and is dense in H_ψ with respect to $\|\cdot\| + \|\cdot\|_\psi$.

c) The seminorm $\|\cdot\|_\psi$ is in fact a norm on H_ψ.

Proof. Ad *a)*. Let $(x_n)_n \subset H_\psi$ be a Cauchy sequence with respect to $\|\cdot\| + \|\cdot\|_\psi$. Then $x_n \to x$ in H for some $x \in H$ and $(\psi(\cdot A)x_n)_n$ is Cauchy in $\mathbf{L^2}((0,\infty), \frac{dt}{t}; H)$. Hence there is $f \in \mathbf{L^2}((0,\infty), \frac{dt}{t}; H)$ such that $\|\psi(\cdot A)x_n - f\|_{\mathbf{L^2}} \to 0$. Extracting a subsequence we can assume that $\psi(tA)x_n \to f(t)$ for almost all t. This implies that $f(t) = \psi(tA)x$ for almost all t. Hence we have $x \in H_\psi$ and $\|x_n - x\|_\psi \to 0$.

Ad *b)*. Let $x \in H_\psi$ and define $T_n := (n+A)^{-1}A(\frac{1}{n}+A)^{-1}$. Then $\|T_n\| \le M(A)^2$ for all n and $T_n x \to x$, see Proposition 1.1. Since $\|\psi(tA)T_n x\| \le M(A)^2 \, \|\psi(tA)x\|$ we have $T_n x \in H_\psi$ and $\|T_n x - x\|_\psi \to 0$ by the Dominated Convergence Theorem.
Let $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$. Then $x = A(1+A)^{-2}y$ for some $y \in H$. Choose $0 < \alpha < 1$ such that

$$\int_\Gamma \frac{|\psi(z)|}{|z|^{1+\alpha}} \, |dz| \,, \quad \int_\Gamma \frac{|\psi(z)|}{|z|^{1-\alpha}} \, |dz| \; < \infty$$

where Γ is the boundary of a sector lying between S_φ and S_ω. Now

$$\begin{aligned} \left\|\psi(tA)A(A+1)^{-2}\right\| &\le C \int_\Gamma |\psi(tz)| \, \frac{|z|}{|1+z|^2} \, \frac{|dz|}{|z|} = C \int_\Gamma |\psi(z)| \, \frac{t}{|t+z|^2} \, |dz| \\ &\le \begin{cases} Ct^\alpha \left(\sup_{t>0,\, z\in\Gamma} \frac{|z|}{|t+z|} \frac{t^{1-\alpha}|z|^\alpha}{|t+z|}\right) \int_\Gamma \frac{|\psi(z)|}{|z|^{1+\alpha}} \, |dz| \\ Ct^{-\alpha} \left(\sup_{t>0,\, z\in\Gamma} \frac{t}{|t+z|} \frac{t^\alpha |z|^{1-\alpha}}{|t+z|}\right) \int_\Gamma \frac{|\psi(z)|}{|z|^{1-\alpha}} \, |dz| \end{cases} \end{aligned}$$

This shows that $\int_0^\infty \|\psi(tA)x\|^2 \, dt/t = \int_0^\infty \left\|\psi(tA)A(A+1)^{-2}y\right\|^2 \, dt/t < \infty$.
Ad *c)*. Let $\theta := \psi\psi^*$. Then $c := \int_0^\infty \theta(t) \, dt/t = \int_0^\infty |\psi(t)|^2 \, dt/t \neq 0$, since we have assumed ψ to be different from 0. By Proposition 1.29 we know that $\int_a^b \theta(tA)x \, dt/t \to cx$ for $(a,b) \to (0,\infty)$ and $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$. Now suppose there is $x \in H$ such that $\|x\|_\psi = 0$. This implies $\psi(tA)x = 0$ for all $t > 0$. Hence we have $\int_a^b \psi(tA)x \, dt/t = 0$ for all $0 < a < b$. Therefore,

$$0 = A(1+A)^{-2} \int_a^b \psi(tA)x \, dt/t = \int_a^b \psi(tA)A(1+A)^{-2}x \, dt/t \to cA(1+A)^{-2}x,$$

whence $x = 0$ (recall that $A(A+1)^{-2}$ is injective and $c \neq 0$). □

Proposition 4.19. *Let $A \in \operatorname{Sect}(\omega)$ with dense domain and dense range. Assume $0 \neq \psi, \theta \in \mathcal{DR}(S_\varphi)$ where $\omega < \varphi < \pi$. Then, for all $f \in H^\infty(S_\varphi)$,*

$$\begin{aligned} \|\gamma_{a,b}(A)f(A)x\|_\psi \le &\left[\sup_{s>0} \|(f(\psi^*\theta^*)_s)(A)\|\right] \\ &\times \left[\sup_{t>0} \int_0^\infty \|\psi(sA)\psi(tA)\| \, \frac{ds}{s}\right] \left(\int_a^b \|\theta(sA)x\|^2 \, \frac{ds}{s}\right)^{\frac{1}{2}} \end{aligned}$$

for $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$ and $0 < a < b < \infty$. Here, $\gamma := \psi\psi^*\theta\theta^*$ and $\gamma_{a,b}(A) = \int_a^b \gamma(sA) \, ds/s$.

Proof. Let $\beta := \psi^*\theta^*$ and define $E := \sup_{s>0} \|(f\beta_s)(A)\|$ and $F := \sup_{t>0} \int_0^\infty \|\psi(tA)\psi(sA)\| \, \frac{ds}{s}$. Note that $E, F < \infty$ by Proposition 1.29. Then

$$\begin{aligned} \|\gamma_{a,b}(A)f(A)x\|_\psi &= \left[\int_0^\infty \left\|\psi(tA) \int_a^b \gamma(sA)f(A)x \, \frac{ds}{s}\right\|^2 \frac{dt}{t}\right]^{\frac{1}{2}} \\ &\le \left[\int_0^\infty \left(\int_a^b \|(f\beta_s)(A)\psi(tA)\psi(sA)\theta(sA)x\| \, \frac{ds}{s}\right)^2 \frac{dt}{t}\right]^{\frac{1}{2}} \\ &\le E \left[\int_0^\infty \left(\int_a^b \|\psi(tA)\psi(sA)\|^{\frac{1}{2}} \, \|\psi(tA)\psi(sA)\|^{\frac{1}{2}} \, \|\theta(sA)x\| \, \frac{ds}{s}\right)^2 \frac{dt}{t}\right]^{\frac{1}{2}} \\ &\le E \left[\int_0^\infty \left(\int_a^b \|\psi(tA)\psi(sA)\| \, \frac{ds}{s}\right) \left(\int_a^b \|\psi(tA)\psi(sA)\| \, \|\theta(sA)x\|^2 \, \frac{ds}{s}\right) \frac{dt}{t}\right]^{\frac{1}{2}} \\ &\le EF^{\frac{1}{2}} \left[\int_a^b \int_0^\infty \|\psi(tA)\psi(sA)\| \, \|\theta(sA)x\|^2 \, \frac{dt}{t} \, \frac{ds}{s}\right]^{\frac{1}{2}} \le EF \left(\int_a^b \|\theta(sA)x\|^2 \, \frac{ds}{s}\right)^{\frac{1}{2}} \end{aligned}$$

for all $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$ and $0 < a < b < \infty$. □

The previous result has important consequences.

Corollary 4.20. *Let $0 \neq \psi, \theta \in \mathcal{DR}[S_\omega]$. Then $H_\psi = H_\theta$ and the norms $\|\cdot\|_\psi, \|\cdot\|_\theta$ are equivalent. Furthermore,*

$$\lim_{(a,b)\to(0,\infty)} \left\| c^{-1}\gamma_{a,b}(A)x - x \right\|_\psi = 0$$

for each $x \in H_\psi$, where $c := \int_0^\infty \gamma(t)\, dt/t > 0$ and γ is as in Proposition 4.19.

Proof. Let $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$. Applying Proposition 4.19 with $f := \mathbf{1}$ we obtain

$$\|\gamma_{a,b}(A)x\|_\psi \leq C \left(\int_a^b \|\theta(sA)x\|^2 \, \frac{ds}{s} \right)^{\frac{1}{2}} \leq C \|x\|_\theta$$

for all $0 < a < b < \infty$ and some constant $C \geq 0$. By *b)* of Lemma 4.18, this shows that $\gamma_{a,b}(A)x$ is a Cauchy net with respect to $\|\cdot\|_\psi$. Since $\gamma_{a,b}x \to cx$ in H by Proposition 1.29 and H_ψ is complete in the norm $\|\cdot\|_\psi + \|\cdot\|$ we conclude that $\gamma_{a,b}(A)x \to cx$ in $\|\cdot\|_\psi$. This gives $\|x\|_\psi \leq Cc^{-1}\|x\|_\theta$ for all $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$. Again from *b)* of Lemma 4.18 it follows that $H_\psi = H_\theta$ with $\|\cdot\|_\psi \sim \|\cdot\|_\theta$. Furthermore, by letting $\theta = \psi$ we see that H_ψ is invariant under each $\gamma_{a,b}(A)$, with the family $(\gamma_{a,b}(A))_{a,b}$ being uniformly bounded as operators on $(H_\psi, \|\cdot\|_\psi)$. It follows that $\|\gamma_{a,b}(A)x - cx\|_\psi \to 0$ for all $x \in H_\psi$. □

Corollary 4.21. *Let $A \in \mathrm{Sect}(\omega)$ with dense domain and dense range. Assume $\omega < \varphi < \pi$ and there is $0 \neq \psi \in \mathcal{DR}(S_\varphi)$ such that $H = H_\psi$ with $\|\cdot\|_\psi$ being equivalent to the original norm on H. Then the natural $H^\infty(S_\varphi)$-calculus for A is bounded.*

Proof. Take $f \in \mathcal{DR}(S_\varphi)$ and $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$. Then $f(A)x \in \mathcal{D}(A) \cap \mathcal{R}(A)$ again and we obtain

$$\|\gamma_{a,b}(A)f(A)x\|_\psi \leq C \left[\sup_{s>0} \|(f(\psi^*\psi^*)_s)(A)\| \right] \|x\|_\psi$$

from Proposition 4.19 (with $\psi = \theta$). Here $C := \left[\sup_{t>0} \int_0^\infty \|\psi(sA)\psi(tA)\| \, \frac{ds}{s} \right]$ which is finite by Proposition 1.29. Applying Corollary 4.20 we can let $(a, b) \to (0, \infty)$ to obtain

$$\|f(A)x\|_\psi \leq c^{-1}CD \|f\|_\infty \|x\|_\psi ,$$

where D is a constant depending neither on x nor on f. (See part *d)* of Proposition 1.29.) By assumption, $\|\cdot\|_\psi \sim \|\cdot\|$, whence $\|f(A)x\| \leq \tilde{C}\|f\|_\infty \|x\|$ for some constant $\tilde{C}$ and all $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$. Applying Proposition 1.32 proves the claim. □

Remark 4.22. We remark that in proving Proposition 4.17 up to Corollary 4.21, we did not make use of Lemma 1.30 or of Proposition 1.31. The weaker form (part *d)* of Proposition 1.29) of McIntosh's approximation technique was sufficient, cf. the comments in §7 of Chapter1.

Let us summarize our results.

Theorem 4.23. (McIntosh-Yagi, extended version)
Let A be a sectorial operator on the Hilbert space H. Denote by B the injective part of A, i.e., B is the part of A in $\overline{\mathcal{R}(A)}$. The following statements are equivalent.

(i) *The natural $\mathcal{DR}(S_\varphi)$-calculus for A is bounded, for some/each $\varphi > \omega_A$.*

(ii) *For some/each* $0 \neq f \in \mathcal{DR}[S_{\omega_A}]$ *there exist constants* $C_1(f), C_2(f) > 0$ *such that*

$$C_1(f)\,\|x\|^2 \leq \int_0^\infty \|f(tA)x\|^2 \,\frac{dt}{t} \leq C_2(f)\,\|x\|^2 \tag{4.7}$$

for all $x \in \overline{\mathcal{R}(A)}$.

(iii) *One has* $B \in \mathrm{BIP}(\overline{\mathcal{R}(A)})$.

(iv) *The natural* $H^\infty(S_\varphi)$*-calculus for* B *is bounded for some/each* $\varphi > \omega_B$.

Proof. The equivalence $(iii) \Leftrightarrow (iv)$ is Theorem 3.31. If $\varphi > \omega_A$ then the natural $\mathcal{DR}(S_\varphi)$-calculus for A is bounded if and only if the natural $\mathcal{DR}(S_\varphi)$-calculus for B is bounded. This proves $(i) \Rightarrow (iii)$ and $(iv) \Rightarrow (i)$. The implication $(i) \Rightarrow (ii)$ is Proposition 4.17.
Assume that $\omega < \varphi$ and (4.7) holds for some $0 \neq f \in \mathcal{DR}(S_\varphi)$. Without restriction we can assume that A is injective, i.e., $A = B$. By Corollary 4.21, the natural $H^\infty(S_\varphi)$-calculus for A is bounded. Hence (iv) holds. □

Remark 4.24. One should note that in the results from Lemma 4.18 up to Corollary 4.21 the assumption that H is a Hilbert space is in fact not needed. Everything remains valid if we just assume that $H = X$ is an arbitrary Banach space. But more is true. We could have even defined

$$\|x\|_\psi := \|x\|_{\psi,p} := \left(\int_0^\infty \|\psi(tA)x\|^p \,\frac{dt}{t}\right)^{\frac{1}{p}}$$

with fixed $1 < p < \infty$. So we obtain a collection of sufficient conditions for the H^∞-calculus to be bounded on an arbitrary Banach space. Unfortunately, the proof for necessity, based on the Rademacher functions, does not work in general. Moreover, the norms $\|\cdot\|_{\psi,p}$ are in general not equivalent for different p, even if A admits a bounded H^∞-calculus. This follows from an example from [CDMY96, last paragraph] together with the fact, that we have an inequality $\|\cdot\|_{\psi,p} \lesssim \|\cdot\|$ if the Banach space has Rademacher cotype p.

§ 5 The Similarity Theorem

We use the McIntosh-Yagi theory developed in the last section to give solutions to the two Similarity Problems posed in §1 and §3. We start with a result which is interesting in its own right.

Corollary 4.25. (Callier-Grabowski-LeMerdy)
Let $-A$ *be the generator of a bounded holomorphic* C_0*-semigroup* T *on the Hilbert space* H*. Then* T *is similar to a contraction semigroup if and only if* $B \in \mathrm{BIP}(\overline{\mathcal{R}(A)})$*, where* B *is the injective part of* A*.*

Proof. For abbreviation we define $K := \overline{\mathcal{R}(A)}$. Assume T is similar to a contraction semigroup. Changing the scalar product we can assume that A is m-accretive. Hence $B \in \mathrm{BIP}(K)$ by Proposition 4.9 and Theorem 4.23.
To prove the converse, assume that $B \in \mathrm{BIP}(K)$. Since $-A$ generates a bounded holomorphic semigroup, A is sectorial with $\omega_A < \frac{\pi}{2}$. By Theorem 4.23 we can change the norm on K to

$(\int_0^\infty \|f(tA)x\|^2 \, dt/t)^{\frac{1}{2}}$ where $f \in \mathcal{DR}[S_{\omega_A}]$ is arbitrary. If we choose $f(z) := z^{\frac{1}{2}} e^{-z}$ we obtain the new norm

$$\|x\|_{\text{new}}^2 = \int_0^\infty \left\|(tA)^{\frac{1}{2}} e^{-tA} x\right\|^2 \frac{dt}{t} = \int_0^\infty \left\|A^{\frac{1}{2}} T(t) x\right\|^2 \, dt$$

on K. Employing the semigroup property it is easy to see that each $T(t)$ is contractive on K with respect to this new norm.
However, on $\mathcal{N}(A)$ each operator $T(t)$ acts like the identity. Since $H = \mathcal{N}(A) \oplus K$ we can choose the new scalar product on H in such a way that it is the old one on $\mathcal{N}(A)$, the one just constructed on K and $\mathcal{N}(A) \perp K$. With respect to this new scalar product, the semigroup T is contractive. $\square$

Using Corollary 4.25 we state our main result.

Theorem 4.26. (Similarity Theorem)
Let A be a sectorial operator on the Hilbert space H satisfying $\omega_A < \frac{\pi}{2}$. Assume that A satisfies the equivalent properties (i) to (iv) of Theorem 4.23. Then, for each $\omega_A < \varphi < \frac{\pi}{2}$ there is an equivalent scalar product $(\cdot\,|\,\cdot)_\circ$ on H with the following properties.

1) *$\mathcal{N}(A) \perp \overline{\mathcal{R}(A)}$ with respect to $(\cdot\,|\,\cdot)_\circ$.*
2) *The operator A is m-φ-accretive with respect to $(\cdot\,|\,\cdot)_\circ$.*
3) *One has $\mathcal{D}(A^\alpha) = \mathcal{D}(A^{\circ\alpha})$ for $0 \le \alpha \le \frac{\pi}{4\varphi}$. Here, A° denotes the adjoint of A with respect to $(\cdot\,|\,\cdot)_\circ$.*
4) *One has $\|f(A)\|_\circ \le \|f\|_\varphi$ for all $f \in \mathcal{DR}_{\text{ext}}(S_\varphi) \cap H^\infty(S_\varphi)$.*

Note that $\frac{\pi}{4\varphi} > \frac{1}{2}$. Hence in particular $\mathcal{D}(A^{\frac{1}{2}}) = \mathcal{D}(A^{\circ\frac{1}{2}})$.

Proof. We choose $\omega_A =: \omega < \varphi' < \varphi$ and define $\beta := \frac{\pi}{2\varphi'}$ and $B := A^\beta$. Then $B \in \mathrm{Sect}(\omega')$, where $\omega' = \beta\omega = \frac{\omega}{\varphi'}\frac{\pi}{2} < \frac{\pi}{2}$. Hence $-B$ generates a bounded holomorphic C_0-semigroup. By hypothesis, A satisfies the condition (i) of Theorem 4.23. Applying Proposition 2.4 we see that B has the same property. By Corollary 4.25 there is an equivalent scalar product $(\cdot\,|\,\cdot)_\circ$ which makes B m-accretive and such that $\mathcal{N}(A) \perp \overline{\mathcal{R}(A)}$. Now, $A = B^{\frac{1}{\beta}}$, whence by Proposition 4.13, A is m-$\frac{\pi}{2\beta}$-accretive with respect to the new scalar product. Moreover, Kato's theorem 4.15 says that $\mathcal{D}(A^{\alpha\beta}) = \mathcal{D}(B^\alpha) = \mathcal{D}(B^{\circ\alpha}) = \mathcal{D}(A^{\circ\alpha\beta})$ for all $0 \le \alpha < \frac{1}{2}$. Thus, Assertion 3) follows since $0 < \alpha\beta < \frac{\beta}{2} = \frac{\pi}{4\varphi'}$ and $\frac{\pi}{4\varphi'} > \frac{\pi}{4\varphi}$. If we take a bounded $f \in \mathcal{DR}(S_\varphi) + \mathcal{DR}_0(S_\varphi) + \mathbb{C}\mathbf{1}$ then

$$\|f(A)\|_\circ = \left\|f(z^{\frac{1}{\beta}})(B)\right\|_\circ \le \left\|f(z^{\frac{1}{\beta}})\right\|_{S_{\frac{\pi}{2}}} = \|f\|_{S_{\frac{\pi}{2\beta}}} = \|f\|_{S_{\varphi'}} \le \|f\|_{S_\varphi}$$

by Proposition 2.4 and Proposition 4.9. $\square$

As a corollary, we obtain a simultaneous solution to the two similarity problems posed on page 101 and 108.

Corollary 4.27. *Let A be a closed operator on a Hilbert space. Then A is variational with respect to some equivalent scalar product if and only if $-A$ generates a holomorphic C_0-semigroup and $A + \lambda \in \mathrm{BIP}(H)$ for large $\lambda \in \mathbb{R}$. In this case, the scalar product can be chosen such that A is variational <u>and</u> square root regular.*

We state as another corollary a theorem which (historically) was the starting point for our investigations.

Corollary 4.28. (Franks-LeMerdy)
Let A be an injective sectorial operator on the Hilbert space H. Assume that $A \in \mathrm{BIP}(H)$. Then for each $\varphi > \omega_A$ there is an equivalent scalar product on H with respect to which the natural $H^\infty(S_\varphi)$-calculus for A is contractive.

Proof. If $\omega_A < \frac{\pi}{2}$ one can just apply the Similarity Theorem 4.26 in combination with Proposition 1.32. In case $\omega_A \geq \frac{\pi}{2}$ we apply this to the operator $A^{\frac{1}{2}}$ and use Proposition 2.4. □

Finally we give an application of the Similarity Theorem to derive a dilation theorem for groups on Hilbert spaces.

Corollary 4.29. (Dilation theorem for groups)
Let T be a C_0-group on the Hilbert space H. For each $\omega > \theta(T)$ there is a Hilbert space K, a (not necessarily isometric) embedding $\iota : H \longrightarrow K$ onto a closed subspace of K, and a normal C_0-group U on K with $\|U(s)\| \leq e^{\omega|s|}$ for all $s \in \mathbb{R}$ such that

$$P \circ U(s) \circ \iota = \iota \circ T(s)$$

for all $s \in \mathbb{R}$. Here, $P : K \longrightarrow \iota(H)$ denotes the orthogonal projection of K onto $\iota(H)$.

Proof. Choose $\alpha > 0$ such that $\alpha\omega = \pi/2$. We consider the group $T(\alpha\cdot)$ which has group type $\alpha\theta(T)$. By Monniaux's Theorem (Corollary 3.30) we find an injective sectorial operator A on H such that $A^{is} = T(\alpha s)$ for all $s \in \mathbb{R}$. Then $\omega_A = \alpha\theta(T) < \pi/2$ by Gearhart's Theorem B.24 and Theorem 3.9. By the Callier-Grabowski-LeMerdy Theorem (Corollary 4.25) we can change the scalar product on H in order to have $-A$ generating a contraction semigroup $(e^{-tA})_{t\geq 0}$. The Sz.-Nagy Theorem 4.8 yields a new Hilbert space K and an isomorphic embedding ι : $H \longrightarrow K$ (which is isometric with respect to the new scalar product) and a unitary group $(W(t))_{t\in\mathbb{R}}$ on K such that $PW(t)\iota = \iota e^{-tA}$ for all $t \geq 0$. Let $-B$ be the generator of W. In the proof of Proposition 4.9 we showed that $Pf(B)\iota = \iota f(A)$ for all $f \in \mathcal{DR}_{\text{ext}}[S_{\frac{\pi}{2}}]$. By applying the Convergence Lemma (Proposition 1.26) to the sequence $f_n(z) := \frac{z^{is}zn}{(z+\frac{1}{n})(n+z)}$ we obtain $PB^{is}\iota = \iota A^{is} = \iota T(\alpha s)$ for $s \in \mathbb{R}$. Thus we have arrived at the desired dilation when we define $U(s) := B^{is/\alpha}$. Since $\|B^{is}\| \leq e^{|s|\frac{\pi}{2}}$ and $\omega = \frac{\pi}{2\alpha}$ we clearly have $\|U(s)\| \leq e^{\omega|s|}$ for all $s \in \mathbb{R}$. □

Remark 4.30. Corollary 4.29 has the following consequence. If T is a C_0-group on the Hilbert space H and $\omega > \theta(T)$ then there is an equivalent Hilbert norm $\|\cdot\|_\circ$ on H such that $\|T(s)\|_\circ \leq e^{\omega|s|}$ for all $s \in \mathbb{R}$. We will give a different (and quite simple) proof of this result in §2 of Chapter 5.

§ 6 A Counterexample

In this section we want to prove the following theorem.

Theorem 4.31. *There exists a Hilbert space H and an operator A on H such that the following conditions hold.*

1) *The operator $-A$ generates a bounded C_0-semigroup T on H.*
2) *The operator A is invertible and $A \in \mathrm{BIP}(H)$.*
3) *The semigroup T is not similar to a quasi-contractive semigroup.*

Here, a C_0-semigroup T on H is called **quasi-contractive**, if there is $\omega \in \mathbb{R}$ such that

$$\|T(t)\| \leq e^{\omega t}$$

for all $t \geq 0$. We will combine a counterexample given by LEMERDY with a technique due to CHERNOFF.

Based on an example given by PISIER in [Pis97], LEMERDY showed in [LM98b, Proposition 4.8] that there exists a Hilbert space H, a bounded C_0-semigroup T on H with injective generator $-A \in \mathrm{BIP}(H)$ such that T is not similar to a contraction semigroup. Let $\alpha = (\alpha_n)_n$ be a scalar sequence with $\alpha_n > 0$ for all n. We consider the space $\mathcal{H} := \ell^2(H)$ and the operator $\mathcal{A}$ defined on $\mathcal{H}$ by

$$\mathcal{D}(\mathcal{A}) := \{x = (x_n)_n \in \mathcal{H} \mid x_n \in \mathcal{D}(A)\ \forall n \in \mathbb{N}\} \qquad \mathcal{A}x := (\alpha_n A x_n)_n.$$

Denote by $\|\!|\cdot|\!\|$ the (Hilbert)-norm on $\mathcal{H}$.

Lemma 4.32. *The following assertions hold.*

a) The operator $-\mathcal{A}$ generates the bounded C_0-semigroup $\mathcal{T}$ defined by

$$\mathcal{T}(t)x := (T(\alpha_n t))_n \qquad (x = (x_n)_n \in \mathcal{H},\ t \geq 0).$$

b) We have

$$\varrho(\mathcal{A}) = \{\lambda \mid \lambda\alpha_n^{-1} \in \varrho(A) \text{ and } \sup_n \left\|(\lambda - \alpha_n A)^{-1}\right\| < \infty\},$$

with $R(\lambda, \mathcal{A})x = ((\lambda - \alpha_n A)^{-1}x_n)_n$ for $x = (x_n)_n \in \mathcal{H}$.

c) The operator $\mathcal{A}$ is injective.

d) Let $\varphi > \frac{\pi}{2}$ and $f \in \mathcal{DR}(S_\varphi)$. Then $f(\mathcal{A})x = (f(\alpha_n A)x_n)_n$ for each $x = (x_n) \in \mathcal{H}$. Moreover $\|\!|f(\mathcal{A})|\!\| \leq \sup_n \|f(\alpha_n A)\|$

Proof. Since T is a bounded semigroup, all operators $\mathcal{T}(t)$ are well defined bounded operators on $\mathcal{H}$, and even uniformly bounded in t. Obviously, the semigroup law holds. Since the space of finite H-sequences is dense in $\mathcal{H}$ and each $T(\alpha_n\cdot)$ is strongly continuous on H, $\mathcal{T}$ is a C_0-semigroup on $\mathcal{H}$. Denote its generator by $\mathcal{B}$ and let $x = (x_n)_n \in \mathcal{D}(\mathcal{B})$. Then $\lim_{t\searrow 0} \frac{1}{t}(\mathcal{T}(t)x - x) \to \mathcal{B}x$. This implies $\lim_{t\searrow 0} \frac{1}{t}(T(\alpha_n t)x - x) \to [\mathcal{B}x]_n$ for each n, whence $x_n \in \mathcal{D}(A)$ and $[\mathcal{B}x]_n = -\alpha_n A x_n$ for each n. Hence we have $\mathcal{B} \subset -\mathcal{A}$. Therefore, by a resolvent argument, *a)* will be proved as soon as we will have established *b)*.
Obviously, the inclusion "$\supset$" holds in *b)*. Assume $\lambda \in \varrho(\mathcal{A})$. Then

$$((x_n) \longmapsto ((\lambda - \alpha_n A)x_n)) : \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{H}$$

is bijective. By composing this mapping with suitable injections and projections we see that $(\lambda - \alpha_n A) : \mathcal{D}(A) \longrightarrow H$ is bijective for each n and that $\sup_n \left\|(\lambda - \alpha_n A)^{-1}\right\| < \infty$.
The assertion *c)* follows from the fact that A is injective and each $\alpha_n > 0$. Part *d)* is immediate from *b)* and the representation of $f(A)$ as a Cauchy integral. □

Since $A \in \mathrm{BIP}(H)$ we know from the McIntosh-Yagi Theorem 3.31 that the natural $H^\infty(S_\varphi)$-calculus is bounded for each $\varphi > \frac{\pi}{2}$. Fix $\varphi > \frac{\pi}{2}$ and $C \geq 0$ such that $\|f(A)\| \leq C\|f\|_\varphi$ for all $f \in H^\infty(S_\varphi)$. If $f \in \mathcal{DR}(S_\varphi)$, part *d)* of Lemma 4.32 yields $\|f(\mathcal{A})\|_{\mathcal{L}(\mathcal{H})} \leq C \sup_n \|f(\alpha_n\cdot)\|_\varphi = C\|f\|_\varphi$. Hence, also $\mathcal{A}$ has a bounded $H^\infty(S_\varphi)$-calculus (see Proposition 1.32), whence $\mathcal{A} \in \mathrm{BIP}(\mathcal{H})$.
We turn to CHERNOFF's observation. From now on we specialize $\alpha_n := \frac{1}{n}$ in the above construction.

Lemma 4.33. (Chernoff)
If $\mathcal{T}$ is similar to a quasi-contractive semigroup, then T is similar to a contractive semigroup.

Proof. We denote by $(\,\cdot\,|\,\cdot\,)$ and $\langle\cdot,\cdot\rangle$ the scalar products on H and $\mathcal{H}$, respectively. Assume that there is an equivalent scalar product $\langle\cdot,\cdot\rangle_{\text{new}}$ on $\mathcal{H}$ and a scalar $\omega \geq 0$ such that $\|\mathcal{T}(t)x\|_{\text{new}} \leq e^{\omega t}\|x\|_{\text{new}}$ for all $x \in \mathcal{H}$ and all $t \geq 0$. Define $(\,x\,|\,y\,)_n := \langle \iota_n x, \iota_n y\rangle_{\text{new}}$ for $x, y \in H$ and $n \in \mathbb{N}$, where $\iota_n : H \longrightarrow \mathcal{H}$ is the natural inclusion mapping onto the n-th coordinate. Obviously, each $(\,\cdot\,|\,\cdot\,)_n$ is an equivalent scalar product on H. More precisely, if $c > 0$ such that $c^{-1}\|x\|^2 \leq \|x\|_{\text{new}} \leq c\|x\|$ for all $x \in \mathcal{H}$ then $c^{-1}\|x\|^2 \leq (\,x\,|\,x\,)_n \leq c\|x\|^2$ for all $n \in \mathbb{N}$ and all $x \in H$. In particular, for each pair of vectors $x, y \in H$, the sequence $((\,x\,|\,y\,)_n)_n$ is bounded by $c\|x\|\,\|y\|$. Now, choose some free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and define

$$(\,x\,|\,y\,)_{\text{new}} := \mathcal{U} - \lim\,(\,x\,|\,y\,)_n$$

for $x, y \in H$. Obviously, this yields a positive sesquilinear form on H. Moreover, we have

$$c^{-1}\|x\|^2 \leq (\,x\,|\,x\,)_{\text{new}} \leq c\|x\|^2$$

for all $x \in H$, whence $(\,\cdot\,|\,\cdot\,)_{\text{new}}$ is an equivalent scalar product on H. Since

$$\|T(t)x\|_n^2 = \|\iota_n T(t)x\|_{\text{new}}^2 = \|\mathcal{T}(t/n)\iota_n x\|_{\text{new}}^2 \leq e^{2\omega\frac{t}{n}}\|\iota_n\|_{\text{new}}^2 = e^{2\omega\frac{t}{n}}\|x\|_n^2\,,$$

we obtain $\|T(t)x\|_{\text{new}}^2 \leq \|x\|_{\text{new}}^2$ for all $x \in H$, $t \geq 0$. Hence T is contractive with respect to $(\,\cdot\,|\,\cdot\,)_{\text{new}}$. □

Since we started with a bounded semigroup which is not similar to a contraction semigroup, $\mathcal{T}$ is not similar to a quasi-contraction semigroup. Obviously this remains true also for each rescaled semigroup $(e^{-\varepsilon t}\mathcal{T}(t))_{t\geq 0}$. Hence for each $\varepsilon > 0$ the operator $\mathcal{A} + \varepsilon$ on the Hilbert space $\mathcal{H}$ does the job in Theorem 4.31.

Remark 4.34. Actually a statement slightly stronger than Theorem 4.31 is valid. In LEMERDY's example the operator A even has a bounded $\mathcal{R}^\infty(S_{\frac{\pi}{2}})$-calculus. By construction, this immediately carries over to the operator $\mathcal{A}$, and this is strictly stronger than to say that $\mathcal{A} \in \mathrm{BIP}(\mathcal{H})$.

§ 7 Comments

§1 Variational Operators. The classical reference for operators constructed via sesquilinear forms is KATO's early paper [Kat61a] as well as his book [Kat95]. Proposition 4.2 essentially is KATO's "First Representation Theorem" [Kat95, Chapter VI, Theorem 2.1]. It is also included as Theorem 1.2 in [ABH01]. The applications to PDE in the literature are numerous and we omit further references. In [ABH01, Example 3.2] an example of two similar operators on a Hilbert space is given, one variational but not the other.

§2 Functional Calculus on Hilbert spaces. The material (including the proofs) is standard. The classical reference for the dilation theorem is the book [SNF70] by SZ.-NAGY and FOIAS. The proof of Proposition 4.9 via the dilation theorem can also be found in [LM98a, Theorem 4.5]. A different proof, attributed to FRANKS, is in [ADM96]. This proof is in the spirit of Bernard and François DELYON's proof of the von Neumann inequality in [DD99]. Other proofs of this famous result can be found in [Pau86, Corollary 2.7] and [Pis01, Chapter 1].

§3 Fractional Powers of m-Accretive Operators and the Square Root Problem. KATO's seminal papers [Kat61a] and [Kat62] still seem to be a good (not

to say: *the*) reference on the matter. Though the book [Tan79] contains parts of the results of these papers, the proofs are a mere paraphrase of the original ones and do not contain new aspects. Proposition 4.13 is [Kat61a, Theorem 2.4] and [Tan79, Lemma 2.3.6], but as already said, our proof is different. By the same method, one can also give a proof for Proposition 4.12. We could not find any reference for Proposition 4.14, although its simple proof makes it quite probable that there is one.
It would be interesting if there is a proof for Kato's Theorem 4.15 which differs essentially from KATO's original proof. Apart from [Tan79, Lemma 2.3.8] which only copies KATO's arguments, we do not know of any other account. The Square Root Problem (cf. Remark 4.16) has been solved only recently by AUSCHER, HOFFMAN, LACEY, LEWIS, MCINTOSH, and TCHAMITCHIAN in [AHL$^+$01]. Surveys and a deeper introduction to these matters as well as the connection to Calderon's conjecture for Cauchy integrals on Lipschitz curves are in [McI90] [McI84], [McI85], and [AT98]. A counterexample to KATO's original question is presented in [AT98, Section 0, Theorem 6].

§4 McIntosh-Yagi Theory. The *name* of this section is somewhat misleading. In fact, we do not know which people exactly contributed to the results presented here. Our choice of name reflects the history of the ideas, originating in the 1984 paper [Yag84] by YAGI and subsequently developed by MCINTOSH in [McI86] and both of them in [MY90]. Our main reference is the Lecture Notes [ADM96, Section E]. However, we modified and systematized the proofs or set up proofs for facts which were posed as exercises. The proof for Proposition 4.17 via the Kalton-Weis Lemma is taken from [LM01, Theorem 4.2], although it seems probable that it has been known before. Let us point out that in the main Theorem 4.23 the equivalence of (iii) and the other statements could be proved without making use of interpolation theory. This certainly is a new aspect. The connection of the spaces H_ψ to interpolation spaces are examined in [AMN97]. The observation in Remark 4.24 seems not to have been stated before.
Although there are no really new *results* in this section, it seemed desirable to have a fully worked-out account of this "McIntosh-Yagi Theory".

§5/§6 The Similarity Theorem and the Counterexample (after a little detour). Similarity problems have a long tradition in operator theory. In 1947 SZ.-NAGY had observed in [dS47] that a bounded and invertible operator T on a Hilbert space H is similar to a unitary operator if and only if the discrete group $(T^n)_{n\in\mathbb{Z}}$ is uniformly bounded. His question was whether the same is true if one discards the invertibility of T. In [SN59] he showed that the answer is yes if T is compact, but FOGUEL disproved the general conjecture by giving a counterexample in [Fog64]. Von Neumann's inequality shows that if T is a contraction, then T is not only power-bounded, but polynomially bounded, i.e.,

$$\sup\{\|p(T)\| \mid p \in \mathbb{C}[z],\ \|p\|_\infty \le 1\} < \infty$$

where $\|p\|_\infty$ denotes the uniform norm on the unit disc. So HALMOS asked in [Hal70] if polynomial boundedness in fact characterizes the bounded operators on H which are similar to a contraction. This question remained open only until recently when PISIER found a counterexample, see [Pis97] and [DP97]. Meanwhile, PAULSEN had shown in [Pau84] that T is similar to a contraction if and only if T is *completely* polynomially bounded. His characterization is a special case of a general similarity result [Pau86, Theorem 8.1] for completely bounded homomorphisms of operator algebras (see [Pau86] for definitions and further results). We will address this result as "Paulsen's theorem" in the following.

One can set up a semigroup analogue of SZ.-NAGY's question, namely if every bounded C_0-semigroup on a Hilbert space is similar to a contraction semigroup. The corresponding result for groups is true as was also proved by SZ.-NAGY in the very same article [dS47]. The question is a bit more special than the original one, since not *every* power-bounded operator is the Cayley transform of a C_0-semigroup generator. It was answered in the negative by PACKEL in [Pac69]. By using this result, CHERNOFF provided an example of a bounded operator with the generated C_0-semigroup not being similar to a contraction one. Furthermore, he constructed a C_0-semigroup which is not even similar to a quasi-contractive semigroup. It is in this construction where Lemma 4.33 appears.[2]

Being probably unaware of the history of this problem, CALLIER and GRABOWSKI in an unpublished research report [GC94] proved our Corollary 4.25 in the case where the semigroup is exponentially stable. They did this using two facts from interpolation theory: first, both the (complex) interpolation space $[H, \mathcal{D}(A)]_{\frac{1}{2}}$ and the (real) interpolation space $(H, \mathcal{D}(A))_{\frac{1}{2},2}$ are equal to $\mathcal{D}(A^{\frac{1}{2}})$ if $A \in \mathrm{BIP}(H)$ (see the comments in §7 of Chapter 1); and second, this real interpolation space is given by

$$(H, \mathcal{D}(A))_{\frac{1}{2},2} = \{x \in H \mid \int_0^\infty \left\| t^{\frac{1}{2}} A e^{-tA} x \right\|^2 \frac{dt}{t} < \infty\}.$$

(See [Lun95, Proposition 2.2.2].) Our proof of Corollary 4.25 uses an observation of LEMERDY in [LM01, Theorem 4.2].
Corollary 4.25 as it stands is also a consequence of Corollary 4.28 and is stated as such in [LM98b, Theorem 4.3]. LEMERDY proves Corollary 4.28 with the help of Paulsen's theorem (see above). The same is done by FRANKS in [Fra97, Section 4]. Seemingly new is our observation that — by using the scaling technique and the results of KATO on accretive operators — the Franks-LeMerdy and the Callier-Grabowski-LeMerdy theorems are actually *equivalent*, and that Paulsen's theorem is *not necessary* to prove the Similarity Theorem 4.26.

It has been noted in [ABH01, Theorem 3.3] that the Franks-LeMerdy result solves the first similarity problem posed on page 101. However, from their proof it is not clear if also the second problem concerning the square roots can be solved. That it actually can sheds a new light on the orginal square root problem. It also complements a result of YAGI in [Yag84, Theorem B] which

[2]Our setting up this historical panorama is mainly based on [LM98b, Introduction]. More on similarity problems and their connection to the theory of operator algebras can be found in [Pis98].

says that a sectorial invertible operator A on a Hilbert space has bounded imaginary powers if $\mathcal{D}(A^\alpha) = \mathcal{D}(A^{*\alpha})$ for all α contained in a small interval $[0, \varepsilon)$. It is consequence of Theorem 4.26 combined with the scaling technique that the converse holds *modulo similarity*.

Finally, let us formulate an open

Problem: Is there an m-accretive operator A on a Hilbert space H such that

$$\mathcal{D}(A^{\frac{1}{2}}) \neq \mathcal{D}(A^{\circ \frac{1}{2}})$$

for *every* equivalent scalar product $(\cdot\,|\,\cdot)_\circ$? (Note that, by Theorem 4.26, such an operator necessarily has to satisfy the relation $\omega_A = \frac{\pi}{2}$.)

Fifth Chapter
A Decomposition Theorem for Group Generators

In §1 we extend LIAPUNOV's classical theorem to holomorphic semigroups and C_0-groups on infinite-dimensional Hilbert spaces. Using this method we establish, for an arbitrary C_0-group generator A, a decomposition $A = B + C$ as a sum of a skew-adjoint operator B and a bounded and selfadjoint operator C, with respect to an equivalent scalar product (§2). In §3 we use the decomposition theorem to obtain a new (and simple) proof of Corollary 3.29. We prove a similarity theorem for generators of cosine functions on Hilbert spaces in §4.

§1 Liapunov's Method for Groups

Recall the classical Liapunov Theorem for linear dynamical systems in $\mathbb{C}^n$.

Theorem 5.1. (Liapunov)
Let $A \in \mathrm{Mat}(n, \mathbb{C})$ with $\sigma(A) \subset \{z \mid \mathrm{Re}\, z > 0\}$. Then there is a Hilbert norm $\|\cdot\|_\circ$ on $\mathbb{C}^n$ and $\varepsilon > 0$ such that

$$\left\|e^{-tA}\right\|_\circ \leq e^{-\varepsilon t}$$

for all $t \geq 0$.

The theorem has two components. *First* it states that the spectral condition

$$s(-A) := \sup\{\mathrm{Re}\, \lambda \mid \lambda \in \sigma(-A)\} < 0$$

for the generator $-A$ of a semigroup $T(t) = e^{-tA}$ on the Hilbert space $\mathbb{C}^n$ implies exponential stability of the semigroup. *Second* it states that an exponentially stable semigroup on $(\mathbb{C}^n, \|\cdot\|_2)$ is similar to a contraction semigroup.

One can wonder about infinite-dimensional analogues. Concerning the first part, it is well-known that a sole condition on the position of the spectrum does not imply exponential stability of the semigroup on a general Hilbert space. (See [ABHN01, Example 5.3.2] or [CZ95, Example 5.1.4].) Concerning the second part, we have seen in §6 of Chapter 4 that there is a Hilbert space H and a C_0-semigroup T on H that is not similar to a quasi-contractive semigroup, i.e., for *no* $\omega \in \mathbb{R}$ one can find an equivalent Hilbert norm $\|\cdot\|_\circ$ on H such that $\|T(t)\|_\circ \leq e^{\omega t}$ for all $t \geq 0$.

So if one looks for infinite dimensional versions of Liapunov's theorem, one has to impose further conditions. In fact, $A \in \mathcal{L}(H)$ is sufficient, but one can weaken this, as the following result shows.

Proposition 5.2. (Arendt, Bu, H.)
Let A be an operator on a Hilbert space H. Assume that $-A$ generates a holomorphic C_0-semigroup T and $s(-A) < 0$. If $A + \lambda \in \mathrm{BIP}(H)$ for some $\lambda > 0$ then, for every $0 < \varepsilon < -s(-A)$ there is an equivalent Hilbert norm $\|\cdot\|_\circ$ on H such that

$$\|T(t)\|_\circ \le e^{-\varepsilon t}$$

for all $t \ge 0$.

Proof. Choose $0 < \varepsilon < \delta := -s(-A)$ and consider the operator $B := A - \varepsilon$. We claim that B is still sectorial with $\omega_B < \frac{\pi}{2}$. For this, it suffices to show that $\{\lambda \mid \operatorname{Re}\lambda < 0\} \subset \varrho(B)$ with $\|\lambda R(\lambda, B)\|$ being uniformly bounded for $\operatorname{Re}\lambda < 0$. Since $0 \in \varrho(B)$ we only have to consider $|\lambda| \ge R$ for some radius $R > 0$. We choose $\omega < \omega' < \frac{\pi}{2}$ and let $R := \sqrt{\varepsilon^2(1 + \tan^2\omega')}$. Now $\lambda R(\lambda, B) = \frac{\lambda}{\lambda+\varepsilon}[(\lambda+\varepsilon)R(\lambda+\varepsilon, A)]$ and the factor $\lambda/(\lambda+\varepsilon)$ is uniformly bounded for $|\lambda| \ge R$. The second factor is uniformly bounded for $\operatorname{Re}\lambda \le -\varepsilon$ hence we have to check that

$$\sup\{\|(\lambda+\varepsilon)R(\lambda+\varepsilon, A)\| \mid -\varepsilon \le \operatorname{Re}\lambda \le 0,\ |\lambda| \ge R\} < \infty.$$

But $|\lambda| \ge R$ together with $-\varepsilon \le \operatorname{Re}\lambda \le 0$ implies $|\operatorname{Im}(\lambda+\varepsilon)| = |\operatorname{Im}\lambda| \ge \sqrt{R^2 - \varepsilon^2} = \varepsilon\tan\omega'$, whence $|\arg(\lambda+\varepsilon)| \ge \omega'$. Since $\omega' > \omega_A$, the claim is proved.
So $-B$ generates a bounded holomorphic C_0-semigroup. Moreover, $B + \lambda \in \mathrm{BIP}(H)$ for some $\lambda > 0$. By *b)* of Proposition 2.30, $B^{is} \in \mathcal{L}(H)$ for all $s \in \mathbb{R}$. Applying Corollary 2.32 we obtain $B \in \mathrm{BIP}(H)$. We can now apply the Callier-Grabowski-LeMerdy Theorem (Corollary 4.25). This yields an equivalent scalar product such that $(e^{\varepsilon t}T(t))_{t\ge 0}$ becomes contractive. □

Recall that one can write down the new scalar product in Proposition 5.2. In fact, it follows from the argument in the proof of Corollary 4.25 that

$$\|x\|_\circ^2 = \int_0^\infty \left\|(A-\varepsilon)^{\frac{1}{2}} e^{\varepsilon t} T(t)x\right\|^2 \, dt$$

for $x \in H$. The function $\|\cdot\|_\circ^2$ is a **Liapunov function** for the dynamical system given by

$$\dot{u} + (A - \varepsilon)u = 0,$$

i.e., it decreases along the orbits $(t \longmapsto e^{\varepsilon t}T(t)x)$. In the finite dimensional situation (or in infinite dimensions with A being bounded) a different Liapunov function is given by

$$\|x\|_\circ^2 := \int_0^\infty \left\|e^{\varepsilon t}T(t)x\right\|^2 \, dt \tag{5.1}$$

for $x \in H$. (Note that $(e^{\varepsilon t}T(t))_{t\ge 0}$ is still exponentially stable.) This is sometimes called **Liapunov's direct method**. The problem is that in general (5.1) does not define an equivalent Hilbert norm (see Corollary 5.8 below). (It always defines a continuous norm, though, as it is easily seen.) However, this norm *is* equivalent if T is a *group*.

Proposition 5.3. *Let A be the generator of an exponentially stable C_0-semigroup T on H. Then the operator Q defined by*

$$Q := \int_0^\infty T(t)^* T(t) \, dt$$

is a bounded, positive, and injective operator on H. By

$$(x \,|\, y)_{\circ} := (Qx \,|\, y) = \int_0^{\infty} (T(t)x \,|\, T(t)y) \; dt$$

a continuous scalar product is defined on H. The semigroup T is contractive with respect to $\|\cdot\|_{\circ}$ and Q satisfies the **Liapunov inclusion**

$$QA \subset -I - A^* Q. \tag{5.2}$$

Equivalently, $Q\mathcal{D}(A) \subset \mathcal{D}(A^*)$ and $QAx + A^*Qx = -x$ for all $x \in \mathcal{D}(A)$. If T is a group then Q is invertible and $(\,\cdot\,|\,\cdot\,)_{\circ}$ is an equivalent scalar product on H. Moreover, one has

$$QA = -I - A^*Q \quad \textit{and} \quad A = -Q^{-1} - A^{\circ}, \tag{5.3}$$

where A° denotes the adjoint of A with respect to the scalar product $(\,\cdot\,|\,\cdot\,)_{\circ}$.

Proof. Boundedness and positivity of Q are clear from the definition. Since we have $(Qx \,|\, x) = \int_0^{\infty} \|T(t)x\|^2 \; dt$, the operator Q is injective and $(\,\cdot\,|\,\cdot\,)_{\circ}$ is in fact a scalar product (not only a semi-scalar product, see Proposition B.15). A simple change of variable yields the formula $\|T(t_0)x\|_{\circ} \le \|x\|_{\circ}$ for all $x \in H$ and $t_0 \ge 0$. If T is a group, we find $M \ge 1$ and $\omega > 0$ such that $\|T(t)\| \le Me^{\omega|t|}$ for all $t \in \mathbb{R}$. Thus,

$$\|x\|_{\circ}^2 = (Qx \,|\, x) = \int_0^{\infty} \|T(t)x\|^2 \; dt \ge \int_0^{\infty} M^{-2} e^{-2\omega t} \, dt \, \|x\|^2 = \frac{1}{2\omega M^2} \, \|x\|^2$$

for all $x \in H$. Hence in this case the new scalar product is equivalent and Q is invertible (Proposition B.15). For the proof of the Liapunov equation note that since T is assumed to be exponentially stable, the resolvent of its generator A is given by

$$-A^{-1} = R(0, A) = \int_0^{\infty} T(s) \, ds$$

(see Proposition A.27). Therefore,

$$\begin{aligned} Q(-A^{-1}) &= \int_0^{\infty} T(t)^* T(t) \int_0^{\infty} T(s) \, ds \, dt = \int_0^{\infty} T^*(t) \int_t^{\infty} T(s) \, ds \, dt \\ &= \int_0^{\infty} \left(\int_0^s T(t)^* \, dt \right) T(s) \, ds. \end{aligned}$$

Similarly,

$$-(A^*)^{-1} Q = \int_0^{\infty} T(t)^* \int_0^{\infty} T(s)^* T(s) \, ds \, dt = \int_0^{\infty} \left(\int_s^{\infty} T(t)^* \, dt \right) T(s) \, ds.$$

Adding the two identities we obtain

$$-(QA^{-1} + (A^*)^{-1} Q) = \int_0^{\infty} \int_0^{\infty} T(t)^* \, dt \, T(s) \, ds = (A^*)^{-1} A^{-1}.$$

This yields $QA^{-1} = -(A^*)^{-1}(A^{-1} + Q)$, whence 5.2 holds. Suppose T is a group. Then Q is invertible and $A^{\circ} = Q^{-1} A^* Q$ is the adjoint with respect to the new scalar product (see Lemma B.16). Multiplying the Liapunov inclusion from the left by Q^{-1} yields $A \subset -A^{\circ} - Q^{-1}$. But $-A^{\circ} - Q^{-1}$ is a bounded perturbation of a C_0-semigroup generator, whence it is also a generator of a C_0-semigroup. This implies readily $A = -A^{\circ} - Q^{-1}$. Multiplying by Q from the left yields (5.3). ∎

Corollary 5.4. (Liapunov's theorem for groups)
Let T be an exponentially stable C_0-semigroup on the Hilbert space H. Assume that T is a group. For each $0 < \varepsilon < -\omega_0(T)$ there is an equivalent Hilbert norm $\|\cdot\|_\circ$ on H such that

$$\|T(t)\|_\circ \le e^{-\varepsilon t}$$

for all $t \ge 0$.

Remark 5.5. Since the "Liapunov inclusion" $QA \subset -I - A^*Q$ is not an equation in general, we do not dare to call it "Liapunov equation" (which is the name in the finite dimensional setting). However, one can reformulate the inclusion as a *system* of equations

$$(\, Ax \,|\, Qy \,) + (\, Qx \,|\, Ay \,) = -\,(\, x \,|\, y \,)$$

with $x, y \in \mathcal{D}(A)$. In [CZ95, p.160 and p.217] this system is called "Liapunov equation".

We now examine, in which cases the "direct method" works.

Lemma 5.6. *Let A be the generator of an exponentially stable C_0-semigroup T on H. Assume that $Q := \int_0^\infty T(t)^*T(t)\, dt$ is invertible. Then, for all $t > 0$, the operators $T(t)$ are injective with closed range. Furthermore, there is $r_0 > 0$ such that the operators $\lambda + A$ are injective with closed range for all λ with $\operatorname{Re}\lambda > r_0$.*

Proof. Since the set of invertible operators is an open subset of $\mathcal{L}(H)$,

$$T(t)^*QT(t) = T(t)^* \int_0^\infty T(s)^*T(s)\, ds\, T(t) = \int_t^\infty T(s)^*T(s)\, ds$$

is invertible for small $t > 0$. Thus, $T(t)^*$ is surjective for small $t > 0$. From the Closed Range Theorem it is immediate that $T(t)$ is injective with closed range for small $t > 0$. To obtain the result for general $t > 0$, simply write $T(t) = T(t/n)^n$ with $n \in \mathbb{N}$ large enough. The Liapunov inclusion (5.2) yields $-A \subset A^\circ + Q^{-1}$. From this it follows that $\mathcal{D}(A)$ is a closed subset of $\mathcal{D}(A^\circ)$, where the norm on $\mathcal{D}(A^\circ)$ is the usual graph norm. Now, $A^\circ + Q^{-1}$ generates a C_0-semigroup on H. This implies that there is $r_0 > 0$ such that $\lambda - (A^\circ + Q^{-1})$ is an isomorphism of $\mathcal{D}(A^\circ)$ onto H for each λ with $\operatorname{Re}\lambda > r_0$. Hence $\lambda + A$ is injective with closed range for each such λ. $\square$

Proposition 5.7. *Let T, A and Q be as in Lemma 5.6. The following assertions are equivalent.*

- *(i) The semigroup T is a group.*
- *(ii) The operator Q is invertible and $T(t)$ has dense range for some $t > 0$.*
- *(iii) The operator Q is invertible and $T^*(t)$ is injective for some $t > 0$.*
- *(iv) Both operators Q and $\tilde{Q} := \int_0^\infty T(t)T^*(t)\, dt$ are invertible.*
- *(v) The operator Q is invertible and no left halfplane is contained in the residual spectrum of A.*

Proof. Assume (i). Then, $T^* = (T(t)^*)_{t\ge 0}$ is also a group and $-A$ is a C_0-semigroup generator. By Proposition 5.3, the assertions (ii), (iii), (iv) and (v) follow (one has to change the roles of T and T^* for the proof of (iv)).
From Lemma 5.6 and the first part of its proof it is clear that each one of the assertions (ii), (iii) and (iv) immediately implies (i). Suppose (v) holds and let r_0 be as in Lemma 5.6. By (v), there is λ with $\operatorname{Re}\lambda > 0$ such that $(\lambda + A) : \mathcal{D}(A) \longrightarrow H$ is bijective. This implies $\mathcal{D}(A) = \mathcal{D}(A^\circ)$, hence $-A = A^\circ + Q^{-1}$ is a C_0-semigroup generator. This proves (i). $\square$

Corollary 5.8. *Let T, A and Q be as in Lemma 5.6 and suppose that Q is invertible.*

a) *If each $T(t)$ is a normal operator, then T is a group.*

b) *If T is a holomorphic semigroup, then A is bounded.*

Proof. If $T(t)$ is normal for each t, then $Q = \tilde{Q}$. If $T(\cdot)$ is holomorphic, then also $T(\cdot)^*$ is holomorphic. This implies that $T(t)^*$ is injective for each t. Apply now Proposition 5.7 to obtain that T is a group. In case T is a holomorphic semigroup, this implies that A is bounded. ☐

Let $(S(t))_{t\geq 0}$ be the right translation semigroup on the Hilbert space $\mathbf{L^2}(0,\infty)$ (see [EN00, I.4.16]) and let $\omega > 0$. Then $T(t) := e^{-\omega t}S(t)$ defines an exponentially stable semigroup with $T(t)^*T(t) = e^{-2\omega t}I$. Hence the associated operator Q is invertible. This shows that the invertibility of Q is not sufficient for having a group.

§ 2 A Decomposition Theorem

Let A be the generator of a C_0-group T on the Hilbert space H. Recall the definition of the *group type* $\theta(T)$ in Appendix A on page 153. We fix $\omega > \theta(T)$ and define

$$\begin{aligned}(x\,|\,y)_\circ &:= \int_{\mathbb{R}} (T(t)x\,|\,T(t)y)\,e^{-2\omega|t|}\,dt \\ &= \int_0^\infty (T(t)x\,|\,T(t)y)\,e^{-2\omega t}\,dt + \int_0^\infty (T(-t)x\,|\,T(-t)y)\,e^{-2\omega t}\,dt \qquad (5.4)\end{aligned}$$

for $x, y \in H$, i.e., we apply the Liapunov method simultaneously to the rescaled 'forward' and 'backward' semigroups obtained from the group T. From Proposition 5.3 it is immediate that $(\cdot\,|\,\cdot)_\circ$ is an equivalent scalar product on H. The following theorem summarizes its properties.

Theorem 5.9. *Let A be the generator of a C_0-group T on a Hilbert space H and let $\omega > \theta(T)$. With respect to the (equivalent) scalar product $(\cdot\,|\,\cdot)_\circ$ defined by* (5.4) *the following assertions hold.*

a) *The operators $A - \omega$ and $-A - \omega$ are both m-dissipative; i.e., $\|T(t)\|_\circ \leq e^{\omega|t|}$ for all $t \in \mathbb{R}$.*

b) *$\mathcal{D}(A) = \mathcal{D}(A^\circ)$ and $A = B + C$ with*

$$B := \frac{1}{2}(A - A^\circ) \quad \textit{and} \quad C := \frac{1}{2}(A + A^\circ).$$

c) *B is skewadjoint with $\mathcal{D}(A) = \mathcal{D}(B)$.*

d) *C has an extension to a bounded and selfadjoint operator (also denoted by C) with $-\omega \leq C \leq \omega$.*

e) *$\mathcal{D}(A)$ is C-invariant, i.e., $C(\mathcal{D}(A)) \subset \mathcal{D}(A)$, and $[B, C] = BC - CB$ has an extension to a bounded and selfadjoint operator on H.*

Proof. We first show *a)*. One has

$$\begin{aligned}
\|T(s)x\|_\circ^2 &= \int_{\mathbb{R}} \|T(t)T(s)x\|^2 \, e^{-2\omega|t|} \, dt = \int_{\mathbb{R}} \|T(t+s)x\|^2 \, e^{-2\omega|t|} \, dt \\
&= \int_{\mathbb{R}} \|T(t)x\|^2 \, e^{-2\omega|t-s|} \, dt = \int_{\mathbb{R}} \|T(t)x\|^2 \, e^{-2\omega|t|} e^{2\omega(|t|-|t-s|)} \, dt \\
&\le e^{2\omega|s|} \, \|x\|_0^2 \quad (s \in \mathbb{R}, x \in H)
\end{aligned}$$

since $|t| - |t-s| \le |s|$ for all $s, t \in \mathbb{R}$ by the triangle inequality. To prove *b)*, let

$$Q_\oplus := \int_0^\infty T(t)^* T(t) e^{-2\omega t} \, dt, \quad Q_\ominus := \int_0^\infty T(-t)^* T(-t) e^{-2\omega t} \, dt,$$

and

$$Q := Q_\oplus + Q_\ominus = \int_{\mathbb{R}} T(t)^* T(t) e^{-2\omega|t|} \, dt.$$

Then $(\, x \,|\, y \,)_\circ = (\, Qx \,|\, y \,)$ for all $x, y \in H$. The Liapunov equations for $Q_\oplus$ and $Q_\ominus$ read

$$Q_\oplus(A - \omega) = -I - (A^* - \omega)Q_\oplus \tag{5.5}$$

$$Q_\ominus(-A - \omega) = -I - (-A^* - \omega)Q_\ominus \tag{5.6}$$

Adding both equations one obtains

$$QA \subset -A^* Q + 2\omega(Q_\oplus - Q_\ominus).$$

If we multiply by Q^{-1} we arrive at $A \subset -A^\circ + 2\omega Q^{-1}(Q_\oplus - Q_\ominus)$. Since an inclusion of C_0-semigroup generators must be an equality, $A = -A^\circ + 2\omega Q^{-1}(Q_\oplus - Q_\ominus)$. In particular, $\mathcal{D}(A^\circ) = \mathcal{D}(A)$. This proves *b)*, and a short computation also yields *d)*.
For the proof of *c)* we note first that, by *d)*, $B = -A + C$ is a bounded perturbation of the generator of a C_0-group. Therefore, B° and $-B$ are C_0-group generators as well. But it is easily seen that $B^\circ \supset -B$, whence it follows that $B^\circ = -B$.
The C-invariance of $\mathcal{D}(A)$ is clear from the formula $C = \omega Q^{-1}(Q_\oplus - Q_\ominus)$ and $\mathcal{D}(A) = \mathcal{D}(A^\circ) = Q^{-1}\mathcal{D}(A^*)$. Furthermore, employing (5.5), (5.6) and the fact that $A^\circ = 2C - A$, we compute

$$\begin{aligned}
CA &= \omega Q^{-1}(Q_\oplus A - Q_\ominus A) \\
&= \omega Q^{-1}(-I + 2\omega Q_\oplus - A^* Q_\oplus - I + 2\omega Q_\ominus + A^* Q_\ominus) \\
&= \omega Q^{-1}(-2I + 2\omega Q - A^*(Q_\oplus - Q_\ominus)) \\
&= -2\omega Q^{-1} + 2\omega^2 I - \omega A^\circ Q^{-1}(Q_\oplus - Q_\ominus) \\
&= -2\omega Q^{-1} + 2\omega^2 I - A^\circ C \\
&= -2\omega Q^{-1} + 2\omega^2 I - (2C - A)C \\
&= -2\omega Q^{-1} + 2\omega^2 I - 2C^2 + AC.
\end{aligned}$$

This shows, that $[B, C] = [A, C]$ has an extension to a bounded operator which is selfadjoint with respect to $(\,\cdot\,|\,\cdot\,)_\circ$. □

Corollary 5.10. *Let A generate a C_0-group T on a Hilbert space H. Then there exists a bounded operator C such that $B := A - C$ generates a bounded C_0-group. Moreover, the operator C can be chosen in such a way that $\mathcal{D}(A)$ is C-invariant and the commutator $[A, C] = AC - CA$ has an extension to a bounded operator on H.*

Remark 5.11. Let $(T(t))_{t \in \mathbb{R}}$ be a C_0-group on H with generator A and let $\omega \ge 0$. Then the following assertions are equivalent (see Section B.6):

(i) $\|T(t)\| \le e^{\omega|t|}$ for all $t \in \mathbb{R}$.

(ii) $\omega \pm A$ are both m-accretive.

(iii) $W(A) \subset \{z \mid \; |\mathrm{Re}\, z| \le \omega\}$, i.e., $|\mathrm{Re}\,(\, Ax \,|\, x \,)| \le \omega \, \|x\|^2$ for $x \in \mathcal{D}(A)$.

However, there is another equivalent characterization:

(iv) $A = B + C$ where B is skewadjoint and C is bounded with $-\omega \le C \le \omega$.

[The implication (iv) ⇒ (iii) is obvious. Assume (iii). Applying the generalized Cauchy-Schwarz Inequality (Proposition B.2) to the form $c(u,v) := \mathrm{Re}[(A \cdot \mid \cdot)](u,v) = \frac{1}{2}\left((Au \mid v) + (u \mid Av)\right)$ on $V := \mathcal{D}(A)$ we obtain

$$|(Au \mid v) + (u \mid Av)| \le 2\omega \, \|u\| \, \|v\|$$

for $u, v \in \mathcal{D}(A)$. Since $\mathcal{D}(A)$ is dense in H, the form c extends to a continuous symmetric form on H. Hence we can find a bounded symmetric operator $C \in \mathcal{L}(H)$ such that $(Cx \mid y) = \frac{1}{2}((Ax \mid y) + (x \mid Ay))$. Obviously, $-\omega \le C \le \omega$. Now we define $B := A - C$. Obviously, $(Bu \mid u) = (Au \mid u) - \mathrm{Re}\,(Au \mid u) = i\,\mathrm{Im}\,(Au \mid u)$ for all $u \in \mathcal{D}(A) = \mathcal{D}(B)$, whence $W(B) \subset i\mathbb{R}$. Since A generates a C_0-group, also B does and this implies that B is skewadjoint, by Stone's Theorem B.22.]

In general however, $\mathcal{D}(A)$ is not C-invariant. In fact, let $H := \mathbf{L^2}(\mathbb{R})$ and $B = d/dt$ the generator of the shift group. Furthermore, let $C := (f \mapsto \omega m f)$ where $m(x) = \operatorname{sgn} x$ is the sign function. Then C is bounded and selfadjoint and $A := B + C$ generates a C_0-group T with $\|T(t)\| \le e^{\omega|t|}$. Obviously, $\mathcal{D}(A) = \mathcal{D}(B) = \mathbf{W^{1,2}}(\mathbb{R})$ is not invariant with respect to multiplication by m. This shows that part *e)* of Theorem 5.9 is not a matter of course and is due to the particular way of renorming.

§ 3 The H^∞-Calculus for Groups Revisited

Let iA be the generator of a C_0-group on the Hilbert space H. In Theorem 3.26 we have shown that the natural $H^\infty(H_\alpha)$-calculus for A is bounded for each $\alpha > \theta(T)$. In this section we will give a second proof of this fact, using the decomposition obtained in Theorem 5.9. Let us reformulate the result.

Theorem 5.12. *Let iA be the generator of a C_0-group T on a Hilbert space H and let let $\alpha > \theta(T)$. Then the natural $H^\infty(H_\alpha)$-calculus for A is bounded.*

We start with a special case.

Proposition 5.13. *Let A be a selfadjoint operator on H, i.e., iA generates a unitary group on H. Then for each $\alpha > 0$ the natural $H^\infty(H_\alpha)$-calculus for A is bounded. In fact,*

$$f(A) = \Psi(f|_{\sigma(A)})$$

for all $f \in H^\infty(H_\alpha)$, where $\Psi : \mathbf{B}(\sigma(A)) \longrightarrow \mathcal{L}(H)$ is the bounded Borel functional calculus obtained by the spectral theorem (see Theorem C.13).

Proof. By the spectral theorem C.11 we can assume that $H = \mathbf{L^2}(\Omega, \mu)$ for some standard measure space (Ω, μ) and that A is multiplication M_a by a function $a \in \mathbf{C}(\Omega, \mathbb{R})$. Analogous to the proof of Proposition 4.6 we can show that $f(A) = M_{f \circ a}$ for each $f \in H^\infty(H_\alpha)$. □

Let us now turn to the proof of Theorem 5.12. Since iA generates a group, the operator A allows quadratic estimates (see Example 3.25). This means that the resolvent $R(\cdot, A)$ is strongly square integrable along vertical lines $\mathrm{Im}\, z = \omega$ with $|\omega| > \theta(T)$. Hence we have

$$f(A) = (A + i\nu)\frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z + i\nu} R(z, A)\, dz$$

for $f \in H^\infty(H_\alpha)$, where $\nu > \alpha$ and Γ is the positively oriented boundary of a vertical strip H_ω with $\theta(T) < \omega < \alpha$.

[The integral converges in the strong sense because of the quadratic estimates. Applying the resolvent identity and Cauchy's theorem we obtain

$$\begin{aligned}\frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z^2+\nu^2}R(z,A)\,dz &= (A-iz)^{-1}\frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z+i\nu}R(z,A) - \frac{f(z)}{z^2+\nu^2}\,dz\\ &= (A-iz)^{-1}\frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z+i\nu}R(z,A)\,dz.\end{aligned}$$

This proves the claim.]

With this simplification achieved we can now go *medias in res*.

Proof of Theorem 5.12. We show that $f(A) \in \mathcal{L}(H)$ for every $f \in H^\infty(S_\alpha)$. By Theorem 5.9 we can choose an equivalent scalar product such that there is a selfadjoint operator B with $\mathcal{D}(B) = \mathcal{D}(A)$ and a bounded selfadjoint operator C such that $A = B + iC$.
The quadratic estimates for A yield a constant $c(A)$ such that

$$\int_\mathbb{R} \|R(r \pm i\omega, A)x\|^2\;dr \le c(A)^2\,\|x\|^2$$

for all $x \in H$. The same applies to B instead of A. Finally, we have

$$\int_\Gamma f(z)\,R(z,B)iCR(z,A)\,dz\;\in \mathcal{L}(H) \tag{5.7}$$

since

$$\begin{aligned}\left|\left(\int_\Gamma f(z)\,R(z,B)iCR(z,A)x\,dz\,\middle|\,y\right)\right| &\le \int_\Gamma |f(z)|\,|(\,CR(z,A)x\,|\,R(z,B)^*y)|\;|dz|\\ \le \|f\|_\infty\,\|C\|\int_\Gamma \|R(z,A)x\|\,\|R(\overline{z},B)y\|\;|d\lambda| &\le 2c(B)c(A)\,\|f\|_\infty\,\|C\|\,\|x\|\,\|y\|\end{aligned}$$

for $x, y \in H$. We can now complete the proof. Writing "$F \approx G$" as an abbreviation for "F is bounded if and only if G is bounded" we compute

$$\begin{aligned}(A+i\nu)\int_\Gamma \frac{f(z)}{z+i\nu}R(z,A)\,dz &\overset{(1)}{\approx} (B+i\nu)\int_\Gamma \frac{f(z)}{z+i\nu}R(z,A)\,dz\\ &\overset{(2)}{\approx} (B+i\nu)\int_\Gamma \frac{f(z)}{z+i\nu}(R(z,A)-R(z,B))\,dz\\ &= (B+i\nu)\int_\Gamma \frac{f(z)}{z+i\nu}R(z,B)\,iC\,R(z,A)\,dz\\ &= \int_\Gamma \frac{f(z)}{z+i\nu}[-I+(z+i\nu)R(z,B)]\,iC\,R(z,A)\,dz\\ &\overset{(3)}{\approx} \int_\Gamma f(z)R(\lambda,B)\,iC\,R(\lambda,A)\,d\lambda.\end{aligned}$$

The last operator has already been shown to be bounded. We have used the quadratic estimates for A in (1) and (3) and Proposition 5.13 in (2). □

§4 Cosine Function Generators

A **cosine function** on a Banach space X is a strongly continuous mapping $\mathrm{Cos} : \mathbb{R}_+ \longrightarrow \mathcal{L}(X)$ such that $\mathrm{Cos}(0) = I$,

$$2\;\mathrm{Cos}(t)\;\mathrm{Cos}(s) = \mathrm{Cos}(t+s) + \mathrm{Cos}(t-s) \quad (t \ge s \ge 0)$$

and $\mathrm{Cos}(t)\,\mathrm{Cos}(s) = \mathrm{Cos}(s)\,\mathrm{Cos}(t)$ for all $s, t \geq 0$. In the following we cite some basic results of the theory of cosine functions from [ABHN01, Sections 3.14-3.16].
Given a cosine function, one can take its Laplace transform and define its **generator** B by

$$\lambda\, R(\lambda^2, B)x = \int_0^\infty e^{-\lambda t}\, \mathrm{Cos}(t)x\, dt$$

for $x \in X$ and $\mathrm{Re}\,\lambda$ sufficiently large. Then, for each pair $(x, y) \in X^2$ the function

$$u(t) := \mathrm{Cos}(t)x + \int_0^t \mathrm{Cos}(s)y\, ds$$

is the unique mild solution of the second order abstract Cauchy problem

$$\begin{cases} u''(t) = B\, u(t) \quad (t \geq 0), \\ u(0) = x, \\ u'(0) = y \end{cases}$$

(cf. [ABHN01, Corollary 3.14.8]). If B generates a cosine function, then it also generates an exponentially bounded holomorphic semigroup of angle $\pi/2$ (cf. [ABHN01, Theorem 3.14.17]).

Proposition 5.14. [ABHN01, Theorem 3.14.11]
Let A generate a cosine function on the Banach space X. Let the operator $\mathcal{A}$ on $X \times X$ be defined by

$$\mathcal{D}(\mathcal{A}) := \mathcal{D}(A) \times X, \quad \mathcal{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ Ax \end{pmatrix}.$$

Then there exists a unique Banach space V such that $\mathcal{D}(A) \hookrightarrow V \hookrightarrow X$ and the part $\mathcal{B}$ of $\mathcal{A}$ in $V \times X$ generates a C_0-semigroup.

The space $V \times H$ is called the **phase space** associated with A. If A generates a cosine function and $\lambda \in \mathbb{C}$, then $A + \lambda$ generates a cosine function with the same phase space (cf. [ABHN01, Corollary 3.14.13]).
The connection to the theory of C_0-groups is given by the following: If an operator A generates a C_0-group $(U(t))_{t\in\mathbb{R}}$ on the Banach space X, then A^2 generates a cosine function Cos with phase space $D(A) \times X$, where $\mathrm{Cos}(t) = (U(t) + U(-t))/2$ $(t \geq 0)$ (cf. [ABHN01], Example 3.14.15). Moreover, a remarkable theorem of FATTORINI states the (partial) converse.

Theorem 5.15. (Fattorini)
Let B be the generator of a cosine function on an UMD-space X. If $-B$ is sectorial, then $A := i(-B)^{\frac{1}{2}}$ generates a strongly continuous group and $A^2 = B$.

A proof can be found in [ABHN01, Theorem 3.16.7]. Altogether this suggests to consider squares of group generators.

Theorem 5.16. *Let A generate a strongly continuous group T on the Hilbert space H. Assume that there is $\omega \geq 0$ such that*

$$\|T(t)\| \leq e^{\omega|t|} \quad (t \in \mathbb{R}),$$

i.e., both $\omega - A$ and $\omega + A$ are m-accretive. Then, for every $0 \leq |\varphi| < \pi/2$ the operator

$$e^{i\varphi}\left(\left(\frac{\omega}{\cos\varphi}\right)^2 - A^2\right) \tag{5.8}$$

is m-accretive. The operator A^2 generates a holomorphic semigroup $(S(z))_{\operatorname{Re} z>0}$ of angle $\pi/2$ such that

$$\|S(z)\| \leq e^{\left(\frac{\omega}{\cos\varphi}\right)^2 \operatorname{Re} z} \quad (|\arg z| \leq \varphi < \frac{\pi}{2}).$$

Proof. The case $\omega = 0$ is trivial since then the group is unitary and A is skewadjoint. This implies that A^2 is selfadjoint with $A^2 \leq 0$, and the assertions of the theorem are immediate. Assume $\omega > 0$, let $0 \leq |\varphi| < \pi/2$ and fix $\varepsilon > 0$. Define $\alpha = \omega \tan\varphi$, i.e., $z := \omega - i\alpha = (\omega/\cos\varphi)e^{-i\varphi}$. By assumption and (ii) of Proposition B.20, the operators $z - A$ and $z + A$ are m-accretive. Applying (iv) of Proposition B.20) we obtain

$$\left\|\frac{A-(z-\varepsilon)}{A-(z+\varepsilon)}\right\|, \left\|\frac{A+(z-\varepsilon)}{A+(z+\varepsilon)}\right\| \leq 1.$$

(Here and in the following we write $\frac{A+\lambda}{A+\mu}$ instead of $(A+\lambda)(A+\mu)^{-1}$ to make the computations more perspicious.) Hence,

$$\left\|\frac{A^2-(z-\varepsilon)^2}{A^2-(z+\varepsilon)^2}\right\| = \left\|\left(\frac{A-(z-\varepsilon)}{A-(z+\varepsilon)}\right)\left(\frac{A+(z-\varepsilon)}{A+(z+\varepsilon)}\right)\right\| \leq 1.$$

Now,

$$\frac{A^2-(z-\varepsilon)^2}{A^2-(z+\varepsilon)^2} = \frac{A^2-(z^2+\varepsilon^2)+2z\varepsilon}{A^2-(z^2+\varepsilon^2)-2z\varepsilon} = \frac{e^{i\varphi}[A^2-(z^2+\varepsilon^2)]+2\varepsilon\,|z|}{e^{i\varphi}[A^2-(z^2+\varepsilon^2)]-2\varepsilon\,|z|}.$$

We can apply Proposition B.20 again (note that $2\varepsilon\,|z| > 0$) to conclude that

$$e^{i\varphi}[(z^2+\varepsilon^2)-A^2] = (\frac{\omega}{\cos\varphi})^2 e^{-i\varphi} + \varepsilon^2 e^{i\varphi} - e^{i\varphi}A^2$$

is m-accretive. Letting $\varepsilon \searrow 0$ we see that $(\frac{\omega}{\cos\varphi})^2 e^{-i\varphi} - e^{i\varphi}A^2$ and finally that

$$e^{i\varphi}((\frac{\omega}{\cos\varphi})^2 - A^2) = (\frac{\omega}{\cos\varphi})^2 e^{i\varphi} - e^{i\varphi}A^2$$

is m-accretive. (Note that $e^{i\varphi}$ differs from $e^{-i\varphi}$ only by a purely imaginary number and this does not affect m-accretivity by (ii) of Proposition B.20.) This finishes the proof of the first part of the theorem. The second part follows from standard semigroup theory, cf. [ABHN01, Chapter 3.4 and Chapter 3.9]. ☐

Theorem 5.16 can be seen as a mapping theorem for the numerical range (cf. Remark 4.16), as the following corollary shows.

Corollary 5.17. *Let A be an operator on the Hilbert space H and $\omega \geq 0$. Assume $W(A) \subset V_\omega := \{z \mid |\operatorname{Re} z| \leq \omega\}$ and $\mathbb{C} \setminus V_\omega \subset \varrho(A)$. Then the numerical range of $B := \omega^2 - A^2$ is contained in the vertical parabola*

$$\overline{\Pi_\omega} = \{z \in \mathbb{C} \mid \operatorname{Re} z \geq 0 \text{ and } |\operatorname{Im} z| \leq 2\omega\sqrt{\operatorname{Re} z}\}.$$

This is equivalent to say that B is m-accretive and

$$|\mathrm{Im}\,(Bu\,|\,u)| \le 2\omega\,\mathrm{Re}\,(Bu\,|\,u)^{\frac{1}{2}}\,\|u\| \tag{5.9}$$

for all $u \in \mathcal{D}(B) = \mathcal{D}(A^2)$.

Proof. The hypotheses are equivalent to $\omega \pm A$ both being m-accretive. Applying Theorem 5.16 we obtain that $e^{i\varphi}((\omega/\cos\varphi)^2 - A^2)$ is m-acccretive for every $0 \le |\varphi| < \pi/2$. Specializing $\varphi = 0$ yields that $B = \omega^2 - A^2$ is m-accretive. Now we write

$$e^{i\varphi}(\frac{\omega^2}{\cos^2\varphi} - A^2) = e^{i\varphi}(B + (\frac{\omega^2}{\cos^2\varphi} - \omega^2)) = e^{i\varphi}(B + \omega^2\tan^2\varphi).$$

Since this operator is m-accretive and $\tan^2\varphi = \tan^2(-\varphi)$ we obtain $W(B+\omega^2\tan^2\varphi) \subset \overline{S_{\frac{\pi}{2}-\varphi}}$ by Proposition 4.2, i.e.,

$$|\mathrm{Im}\,(Bu\,|\,u)| \le \tan(\frac{\pi}{2} - \varphi)(\mathrm{Re}\,(Bu\,|\,u) - \omega^2\tan^2\varphi\,\|u\|^2) \qquad (u \in \mathcal{D}(B))$$

for $0 < \varphi < \frac{\pi}{2}$. Now $\tan(\frac{\pi}{2} - \varphi) = (\tan\varphi)^{-1}$ and if we parametrize $\tau := \tan\varphi$ with $0 < \tau < \infty$ we obtain $|\mathrm{Im}\,(Bu\,|\,u)| \le \frac{1}{\tau}\,\mathrm{Re}\,(Bu\,|\,u) - \omega^2\tau\,\|u\|^2$ for every $u \in \mathcal{D}(B)$. The right hand side has $2\sqrt{\mathrm{Re}\,(Bu\,|\,u)\,\omega^2\,\|u\|^2}$ as its minimum value, whence we arrive at (5.9). □

Corollary 5.18. *Let B be the generator of a cosine function on the Hilbert space H. Then, with respect to an equivalent scalar product, $-B$ has numerical range in a horizontal parabola $\lambda + \Pi_\omega$ for some $\lambda \in \mathbb{R}$, $\omega > 0$.*

Proof. Apply Fattorini's theorem 5.15, change the scalar product according to 5.9, then apply Corollary 5.17. □

The next proposition is needed for the proof of Theorem 5.20.

Proposition 5.19. *Let A be as in Theorem 5.16. Then*

$$\mathcal{D}\left(i(\omega^2 - A^2)^{\frac{1}{2}}\right) = \mathcal{D}(A).$$

Proof. First note that the operator $\omega^2 - A^2$ is m-accretive, hence sectorial. Thus the square root is well defined. Since A generates a group, A^2 generates a cosine function with phase space $\mathcal{D}(A) \times H$. By general cosine function theory (see the remarks at the beginning of this section), $A^2 - \omega^2$ also generates a cosine function with the same phase space. Fattorini's Theorem 5.15 implies that $B := i(\omega^2 - A^2)^{\frac{1}{2}}$ generates a group and $B^2 = A^2 - \omega^2$. Then $\mathcal{D}(B) = \mathcal{D}(A)$ follows from the uniqueness of the phase space (see Proposition 5.14). □

Now we are prepared for the final theorem.

Theorem 5.20. *Let B be the generator of a cosine function on a Hilbert space H. Then $-B$ is variational and square root regular with respect to some equivalent scalar product. In particular, $\lambda - B \in \mathrm{BIP}(H)$ for large $\lambda \in \mathbb{R}$.*

Proof. First, one can find β such that $-B+\beta$ is sectorial. Since $B-\beta$ generates a cosine function as well, we can apply Fattorini's Theorem 5.15. Thus, the operator $A := i(\beta - B)^{1/2}$ generates a strongly continuous group T on H. Choose $\omega > \theta(T)$. By Theorem 5.9 we obtain a new scalar product $(\cdot\,|\,\cdot)_\circ$ making $\omega \pm A$ m-accretive and sucht that $\mathcal{D}(A) = \mathcal{D}(A^\circ)$ holds. Apply now Theorem 5.16 together with Corollary 4.4 to conclude that $-A^2 = -B + \beta$ is variational. This implies that $-B$ is variational. Finally, we apply Proposition 5.19 to the operators A and A° and obtain

$$\begin{aligned}\mathcal{D}((\beta + \omega^2 - B)^{\frac{1}{2}}) &= \mathcal{D}((\omega^2 - A^2)^{\frac{1}{2}}) = \mathcal{D}(A) = \mathcal{D}(A^\circ)\\ &= \mathcal{D}((\omega^2 - A^{\circ 2})^{\frac{1}{2}}) = \mathcal{D}((\beta + \omega^2 - B)^{\circ\frac{1}{2}}).\end{aligned}$$

This completes the proof. □

Remark 5.21. There is another proof of Theorem 5.20 relying on the results of Chapter 4. In fact, knowing that $A := i(\beta - B)^{\frac{1}{2}}$ generates a C_0-group, the operator $(\beta - B)^{\frac{1}{2}}$ must have a bounded H^∞-calculus on vertical strips. By a composition rule-type argument one concludes that $\beta - B$ has a bounded H^∞-calculus on a sector (use rational functions first and apply the results from §6 of Chapter 1). Then the desired facts on B follow from Corollary 4.27. However, the approach given in this section is more direct in that we can easily construct the new scalar product by our modified Liapunov method.

§ 5 Comments

This chapter is an adapted version of our article [Haa01].

§**1 Liapunov's Method for Groups.** Proposition 5.2 is [ABH01, Theorem 4.1]. In that paper, Liapunov type theorems are also established for hyperbolic holomorphic semigroups and quasi-compact holomorphic semigroups, and Proposition 5.2 is applied to semilinear equations.

In the case when A is bounded, the Liapunov method is used in Chapter I of the book [DKn74]. There the operator equation $QA + A^*Q = -I$ is directly linked to the problem of finding a Liapunov function for the semigroup (which is sometimes called 'Liapunov's direct method'). For the unbounded case, the relevant facts are included in [CZ95, Theorem 5.1.3], where a characterization of exponential stability of the semigroup is given in terms of the existence of an operator Q satisfying the Liapunov equation. (Extensions of this result can be found in [GN81], [ARS94] and [Alb01].) It is shown in [Zwa01] that this method in fact gives an *equivalent* scalar product if the semigroup is a group.

After completing this section we learned of a paper [LR] of LIU and RUSSELL on exact controllability where nearly the same results (and others) are established.

§**2 A Decomposition Theorem.** The following well-known theorem by SZ.-NAGY from [dS47] can be regarded as the "limit case" in Theorem 5.9: Every generator of a bounded group is similar to a skew-adjoint operator. This result cannot be deduced directly from Theorem 5.9. However, ZWART in [Zwa01] gives a proof using the Liapunov renorming.

In [deL97, Theorem 2.4] it is proved that, given a C_0-group T on a Hilbert space, one has $\|T(t)\|_\circ \le e^{\omega|t|}$ for some equivalent scalar product $(\cdot\,|\,\cdot)_\circ$ and some ω strictly larger than the group type $\theta(T)$. (This is covered by part *a)* of our Theorem 5.9. Compare also Remark 4.30.) While the proof in [deL97] is based on the boundedness of the H^∞-calculus and on Paulsen's theorem (see the comments in §7 of Chapter 4) our approach is more direct and considerably shorter.

One can wonder, if one starts with a group T such that $\|T(s)\| \le Ke^{\theta|s|}$ ($s \in \mathbb{R}$), whether one can always take $\omega = \theta$ in Theorem 5.9. However, in [Sim99] it is shown that this is not possible in general.

§**3 The H^∞-calculus for Groups Revisited.** In the proof of Theorem 5.12 presented in this section we use the same idea as in the proof of the perturbation result Proposition 3.15. Regarding the spectral theorem as known, this proof is considerably shorter and more perspicious than the proof in [Bd94] and even more elegant than the one given in Chapter 3, §6.

§**4 Cosine Function Generators.** The mapping theorem for the numerical range established in Theorem 5.16 and Corollary 5.17 is new, as far as we know. McIntosh has shown in [McI82] that an operator B satisying the conclusion of Corollary 5.17 also has the square root property $\mathcal{D}(B^{\frac{1}{2}}) = \mathcal{D}(B^{*\frac{1}{2}})$. This yields an alternative proof of Proposition 5.19.

Theorem 5.20 is of course not the final word. With an appropriate definition of H^∞-calculus on parabolas one shows, combining Fattorini's theorem and Theorem 5.12, that a cosine function generator on a Hilbert space always has a bounded H^∞-calculus on a parabola. The converse statement is always true, since the boundedness of the H^∞-calculus for B on a parabola warrants the Hille-Yosida conditions for the operator $(\lambda+B)^{\frac{1}{2}}$, for sufficiently large λ. However, there is still one open

Problem. Assume that B is m-accretive and has numerical range in a parabola Π_ω for some $\omega > 0$. Does $-B$ generate a cosine function?

Appendices

Appendix A
Linear Operators

This chapter is supposed to be a "reminder" of some operator theory, including elementary spectral theory and approximation results, rational functional calculus and semigroup theory. There is a slight deviation from the standard literature on operator theory in that we deal with multivalued operators right from the start.

A.1 The Algebra of Multivalued Operators

Let X, Y, Z be Banach spaces. A linear **operator** from X to Y is a linear subspace of the direct sum space $X \oplus Y$.
A linear operator may fail to be the graph of a mapping. The subspace

$$A0 := \{x \in X \mid (0, x) \in A\} \subset X$$

is a measure for this failure. In case $A0 = 0$, the relation $A \subset X \oplus Y$ is *functional*, i.e., the operator A is the graph of a mapping, and it is called **single-valued**. Since in the main text we deal with single valued linear operators almost exclusively (not without significant exceptions, of course), we make the following

Agreement: Unless otherwise stated, the term "operator" always is to be understood as "single-valued linear operator". We call an operator **multivalued** (in short "m.v.") if we want to stress that it is *not necessarily* single-valued (but it may be).

The **image** of a point x under the m.v. operator A is the set

$$Ax := \{y \in Y \mid (x, y) \in A\}.$$

This set is either empty (this means, the m.v. operator A is "undefined" at x) or it is an affine subspace of Y in the "direction" of the space $A0$.

With a m.v. operator $A \subset X \oplus Y$ we associate the spaces

kernel	$\mathcal{N}(A)$	$:=$	$\{x \in X \mid (x, 0) \in A\}$,
domain	$\mathcal{D}(A)$	$:=$	$\{x \in X \mid$ there is $y \in Y$ such that $(x, y) \in A\}$,
range	$\mathcal{R}(A)$	$:=$	$\{y \in Y \mid$ there is $x \in X$ such that $(x, y) \in A\}$.

The m.v. operator A is called **injective** if $\mathcal{N}(A) = 0$ and **surjective** if $\mathcal{R}(A) = Y$. If $\mathcal{D}(A) = X$, then A is called **fully defined**.

Let $A, B \subset X \oplus Y$ and $C \subset Y \oplus Z$ be m.v. operators and $\lambda \in \mathbb{C}$ a scalar. We define the **sum** $A + B$, the **scalar multiple** λA, the **inverse** A^{-1} and the **composite** CA by

$$\begin{aligned} A + B &:= \{(x, y + z) \in X \oplus Y \mid (x, y) \in A,\ (x, z) \in B\}, \\ \lambda A &:= \{(x, \lambda y) \in X \oplus Y \mid (x, y) \in A\}, \\ A^{-1} &:= \{(y, x) \in Y \oplus X \mid (x, y) \in A\}, \\ CA &:= \{(x, z) \in X \oplus Z \mid \exists\, y \in Y : (x, y) \in A \wedge (y, z) \in C\}. \end{aligned}$$

Then we have the following identities:

$$\begin{aligned} \mathcal{D}(A + B) &= \mathcal{D}(A) \cap \mathcal{D}(B), \\ \mathcal{D}(A^{-1}) &= \mathcal{R}(A), \\ \mathcal{D}(\lambda A) &= \mathcal{D}(A), \\ \mathcal{D}(CA) &= \{x \in \mathcal{D}(A) \mid \exists\, y \in \mathcal{D}(C) : (x, y) \in A\}. \end{aligned}$$

The **zero-operator** is $0 := \{(x, 0) \mid x \in X\}$ and $I := \{(x, x) \mid x \in X\}$ is the **identity operator**. (This means that in general we have only $0A \subset 0$ but *not* $0A = 0$.)

Proposition A.1. *Let X be a Banach space and $A, B, C \subset X \oplus X$ m.v. linear operators on X.*

a) *The set of m.v. operators on X is a* semigroup *with respect to composition, i.e., the* **Law of Associativity** *$A(BC) = (AB)C$ holds. The identity operator I is a (the) neutral element in this semigroup. Moreover, the* **Law of Inversion** *$(AB)^{-1} = B^{-1}A^{-1}$ holds.*

b) *The set of m.v. operators on X is an abelian semigroup with respect to sum, with the zero operator 0 as its neutral element.*

c) *For $\lambda \neq 0$ one has*
$$\lambda A = (\lambda I)A = A(\lambda I).$$

d) *The m.v. operators A, B, C satisfy the following* **Laws of Monotonicity***:*
$$\begin{aligned} A \subset B &\implies AC \subset BC,\ CA \subset CB \\ A \subset B &\implies A + C \subset B + C,\ \lambda A \subset \lambda B. \end{aligned}$$

e) *The following* **Distributivity Inclusions** *hold:*
$$\begin{aligned} (A + B)C &\subset AC + BC, && \text{with equality if } C \text{ is single-valued; and} \\ CA + CB &\subset C(A + B), && \text{with equality if } \mathcal{R}(A) \subset \mathcal{D}(C). \end{aligned}$$

In particular, there is equality in both cases if $C \in \mathcal{L}(X)$ (see below).

A m.v. operator $A \subset X \oplus Y$ is called **closed** if it is closed in the natural topology on $X \oplus Y$. If A is closed, then the m.v. operators λA (for $\lambda \neq 0$) and A^{-1} are closed as well. Furthermore, the spaces $\mathcal{N}(A)$ and $A0$ are closed. Sum and composition of closed m.v. operators are *not* necessarily closed.
For each m.v. operator A one can consider its **closure** $\overline{A}$ in $X \oplus Y$. A *single-valued* operator is called **closable** if $\overline{A}$ is again single-valued.
If A is closed, then every subspace $V \subset \mathcal{D}(A)$ such that $\overline{\{(x,y) \in A \mid x \in V\}} = A$ is called a **core** for A.

A single-valued operator A is called **continuous** if there is $c \geq 0$ such that $\|Ax\| \leq c\,\|x\|$ for all $x \in \mathcal{D}(A)$. Every continuous operator is closable. An operator is called **bounded** if it is continuous and fully defined. We let

$$\mathcal{L}(X,Y) := \{A \subset X \oplus Y \mid A \text{ is a bounded operator}\}$$

be the set of all bounded operators from X to Y. If $X = Y$ we write just $\mathcal{L}(X)$ in place of $\mathcal{L}(X,X)$.
For a single-valued operator A there is a natural norm on $\mathcal{D}(A)$, namely the **graph norm**

$$\|x\|_A := \|x\| + \|Ax\| \qquad (x \in \mathcal{D}(A)).$$

Note that A is closed if and only if $(\mathcal{D}(A), \|.\|_A)$ is complete.

Lemma A.2. *Let X and Y be Banach spaces. A single-valued operator $A \subset X \oplus Y$ is continuous if and only if $\mathcal{D}(A)$ is a closed subspace of X.*

Proof. The continuity of the operator is equivalent to the fact that the graph norm is equivalent to the original norm. Closedness of the operator is equivalent to the fact that $\mathcal{D}(A)$ is complete with respect to the graph norm. Therefore, the assertion follows from the open mapping theorem. □

Lemma A.3. *Let A be a closed m.v. operator on the Banach space X, and let $T \in \mathcal{L}(X)$. Then AT is closed.*

Proof. Let $(x_k, y_k) \in AT$ with $x_k \to x$ and $y_k \to y$. Then there is z_k such that $(x_k, z_k) \in T$ and $(z_k, y_k) \in A$. Since $T \in \mathcal{L}(X)$, one has $z_k = Tx_k \to Tx$. The closedness of A implies that $z \in \mathcal{D}(A)$ and $(z, y) \in A$. But this means exactly that $(x, z) \in AT$. □

A m.v. operator $A \subset X \oplus Y$ is called **invertible**, if $A^{-1} \in \mathcal{L}(Y,X)$. We denote by

$$\mathcal{L}(X)^{\times} := \{T \mid T, T^{-1} \in \mathcal{L}(X)\}$$

the set of bounded invertible operators on X.

Lemma A.4. *Let A be a closed m.v. operator on the Banach space X, and let T be an invertible m.v. operator. Then TA is closed.*

Proof. Suppose $(x_n, y_n) \in TA$ with $x_n \to x$ and $y_n \to y$. Then there are z_n such that $(x_n, z_n) \in A$ and $(z_n, y_n) \in T$. Since $T^{-1} \in \mathcal{L}(X)$, $z_n = T^{-1}y_n \to T^{-1}y$. The closedness of A implies $(x, T^{-1}y) \in A$, whence $(x, y) \in TA$. □

Note that the last result in general is false without the assumption that T is invertible.

A.2 Resolvents

In this section A denotes a m.v. linear operator on the Banach space X. The starting point for the spectral theory is the following lemma.

Lemma A.5. *The identity*

$$I - [I + \lambda A^{-1}]^{-1} = \lambda(\lambda + A)^{-1}$$

holds for all $\lambda \in \mathbb{C}$.

Proof. For $\lambda = 0$ the assertion is (almost) trivial. Therefore, let $\lambda \neq 0$. If $x, y \in X$ and $z := (1/\lambda)y$, one obtains

$$\begin{aligned}(x, y) \in \lambda(\lambda + A)^{-1} &\Leftrightarrow (x, z) \in (\lambda + A)^{-1} \Leftrightarrow (z, x) \in (\lambda + A) \Leftrightarrow (z, x - \lambda z) \in A \\ &\Leftrightarrow (x - \lambda z, z) \in A^{-1} \Leftrightarrow (x - \lambda z, \lambda z) \in \lambda A^{-1} \\ &\Leftrightarrow (x - \lambda z, x) \in I + \lambda A^{-1} \Leftrightarrow (x, x - \lambda z) \in [I + \lambda A^{-1}]^{-1} \\ &\Leftrightarrow (x, \lambda z) \in I - [I + \lambda A^{-1}]^{-1} \Leftrightarrow (x, y) \in I - [I + \lambda A^{-1}]^{-1}.\end{aligned}$$

This shows the claim to be true. □

We call the mapping

$$(\lambda \longmapsto R(\lambda, A) := (\lambda - A)^{-1}) : \mathbb{C} \longrightarrow \{\text{m.v. operators on } X\}$$

the **resolvent** of A. The set

$$\varrho(A) := \{\lambda \in \mathbb{C} \mid R(\lambda, A) \in \mathcal{L}(X)\}$$

is called the **resolvent set** and $\sigma(A) := \mathbb{C} \setminus \varrho(A)$ the **spectrum** of A.

Corollary A.6. *For all* $\lambda, \mu \in \mathbb{C}$ *the identity*

$$I - [I + (\lambda - \mu)R(\mu, A)]^{-1} = (\lambda - \mu)R(\lambda, A).$$

holds true.

Proof. Just replace λ by $(\lambda - \mu)$ and A by $\mu - A$ in Lemma A.5. □

Proposition A.7. *Let A be a closed m.v. linear operator on the Banach space X. The resolvent set $\varrho(A)$ is an open subset of $\mathbb{C}$. More precisely, for $\mu \in \varrho(A)$ one has* $\operatorname{dist}(\mu, \sigma(A)) \geq \|R(\mu, A)\|^{-1}$ *and*

$$R(\lambda, A) = \sum_{k=0}^{\infty} (\mu - \lambda)^k R(\mu, A)^{k+1} \quad (|\lambda - \mu| < \|R(\mu, A)\|^{-1}).$$

The resolvent mapping $R(., A) : \varrho(A) \longrightarrow \mathcal{L}(X)$ is holomorphic and the **resolvent identity**

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\mu, A)R(\lambda, A)$$

holds for all $\lambda, \mu \in \varrho(A)$.
In case $A \in \mathcal{L}(X)$ one has $\emptyset \neq \sigma(A) \subset \{z \in \mathbb{C} \mid |z| \leq \|A\|\}$ where $R(\lambda, A) = \sum_{k=o}^{\infty} \lambda^{-(k+1)} A^k$ for all $|\lambda| > \|A\|$.

Proof. As an abbreviation we write $R(\lambda)$ instead of $R(\lambda, A)$. Take $\mu \in \varrho(A)$ and $\lambda \in \mathbb{C}$ such that $|\lambda - \mu| < \|R(\mu)\|^{-1}$. Then, a well known result from the theory of bounded operators states that $I + (\lambda - \mu)R(\mu)$ is invertible with

$$(I + (\lambda - \mu)R(\mu))^{-1} = \sum_{k \geq 0} (\mu - \lambda)^k R(\mu)^k \in \mathcal{L}(X)$$

being its inverse. Combined with Corollary A.6 this gives $R(\lambda) \in \mathcal{L}(X)$ and

$$\begin{aligned} R(\lambda) &= \frac{1}{\lambda - \mu}(I - \sum_{k \geq 0} (\mu - \lambda)^k R(\mu)^k) = \frac{1}{\mu - \lambda} \sum_{k \geq 1} (\mu - \lambda)^k R(\mu)^k \\ &= \sum_{k \geq 0} (\mu - \lambda)^k R(\mu)^{k+1}. \end{aligned}$$

Let $\lambda, \mu \in \varrho(A)$, and $a \in X$. Set $x := R(\lambda)a$. Then, $(x, a) \in (\lambda - A)$, hence $(x, a + (\mu - \lambda)x) \in (\mu - A)$. This implies

$$R(\lambda)a = x = R(\mu)(a + (\mu - \lambda)x) = (R(\mu) + (\mu - \lambda)R(\mu)R(\lambda))a.$$

The proofs of the remaining statements are well known. $\square$

Each mapping $R : \Omega \longrightarrow \mathcal{L}(X)$ with $\emptyset \neq \Omega \subset \mathbb{C}$ such that the resolvent identity

$$R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu) \qquad (\lambda,\ \mu \in \Omega)$$

holds, is called a **pseudo resolvent**. Since we allow m.v. operators, we obtain the following proposition which fails to be true for single-valued operators.

Proposition A.8. *Let $R : \Omega \longrightarrow \mathcal{L}(X)$ be a pseudo resolvent. Then there is one and only one m.v. operator A on X such that $\Omega \subset \varrho(A)$ and $R(\lambda) = R(\lambda, A)$ for all $\lambda \in \Omega$.*

Proof. If $R(\lambda) = R(\lambda, A)$, then $A = \lambda - R(\lambda)^{-1}$. This shows that the operator A is uniquely determined by each single $R(\lambda)$. Thus we define $A_\lambda := \lambda - R(\lambda)^{-1}$. What we have to show is that all the A_λ are equal, i.e., that

$$\lambda + R(\mu)^{-1} = \mu + R(\lambda)^{-1}$$

for all $\lambda, \mu \in \Omega$. By interchanging the roles of μ and λ it is clear that we are done as soon as we know one inclusion. Let $(x, y) \in \mu + R(\lambda)^{-1}$. This means that $x = R(\lambda)a$ where $a := y - \mu x$. Then

$$\begin{aligned} R(\mu)(y - \lambda x) &= R(\mu)(a + (\mu - \lambda)x) \\ &= R(\mu)(I + (\mu - \lambda)R(\lambda))a \\ &= R(\lambda)a = x. \end{aligned}$$

But this gives $(x, y - \lambda x) \in R(\mu)^{-1}$, whence $(x, y) \in \lambda + R(\mu)^{-1}$. $\square$

Corollary A.9. *Let $R_1 : \Omega_1 \longrightarrow \mathcal{L}(X)$ and $R_2 : \Omega_2 \longrightarrow \mathcal{L}(X)$ be pseudo resolvents. If $R_1(z) = R_2(z)$ for some $z \in \Omega_1 \cap \Omega_2$, then $R_1(z) = R_2(z)$ for all such z.*

Let $T \in \mathcal{L}(X)$. We say that the operator T **commutes** with the m.v. operator A, if

$$(x, y) \in A \quad \Longrightarrow \quad (Tx, Ty) \in A$$

for all $x, y \in X$. This is equivalent to the condition $TA \subset AT$. Obviously, T commutes with A if and only if T commutes with A^{-1}. An immediate consequence of this fact is the following proposition.

Proposition A.10. *Let $T \in \mathcal{L}(X)$ and $\varrho(A) \neq \emptyset$. The following assertions are equivalent.*

(i) $[T, R(\lambda, A)] := TR(\lambda, A) - R(\lambda, A)T = 0$ *for some* $\lambda \in \varrho(A)$.

(ii) $[T, R(\lambda, A)] = 0$ *for all* $\lambda \in \varrho(A)$.

(iii) $TA \subset AT$.

Let A, B closed m.v. operators, and suppose $\varrho(A) \neq \emptyset$. We say that B **commutes with the resolvents** of A if B commutes with $R(\lambda, A)$ for *each* $\lambda \in \varrho(A)$. If $\varrho(B) \neq \emptyset$, it is sufficient that this is the case for a single λ. Moreover, it follows that A commutes with the resolvents of B.
One should note that each *single-valued* operator A with $\varrho(A) \neq \emptyset$ commutes with its own resolvents. (This is due to the identity $B^{-1}Bx = x + \mathcal{N}(B)$ for $x \in \mathcal{D}(B)$ which is true for all operators B.)

Proposition A.11. *Let A be a m.v. operator such that $\varrho(A) \neq \emptyset$, and let $T \in \mathcal{L}(X)$ be injective. If T commutes with A, then $T^{-1}AT = A$.*

Proof. Because of $AT \supset TA$ we have $T^{-1}AT \supset T^{-1}TA = A$. The reverse inclusion $T^{-1}AT \subset A$ is equivalent to the following statement: If $(Tx, Ty) \in A$, then $(x, y) \in A$, for all $x, y \in X$.
Let $\lambda \in \varrho(A)$. From $(Tx, Ty) \in A$ it follows that $(Tx, T(y - \lambda x)) = (Tx, Ty - \lambda Tx) \in (A - \lambda)$. This gives $R(\lambda, A)T(\lambda x - y) = Tx$. Now, T commutes with $R(\lambda, A)$, hence $TR(\lambda, A)(\lambda x - y) = Tx$. By injectivity of T we obtain $R(\lambda, A)(\lambda x - y) = x$, whence $(x, y) \in A$. This completes the proof. □

A.3 The Spectral Mapping Theorem for the Resolvent

We define $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ to be the one-point compactification of $\mathbb{C}$ (**Riemann sphere**) with the usual conventions for computation [1]. For a m..v. operator A we call the set

$$\tilde{\sigma}(A) := \begin{cases} \sigma(A) & \text{if} \quad A \in \mathcal{L}(X) \\ \sigma(A) \cup \{\infty\} & \text{if} \quad A \notin \mathcal{L}(X) \end{cases}$$

the **extended spectrum** of A.

Proposition A.12. (Spectral Mapping Theorem)
Let A be a closed m.v. operator on the Banach space X. We have

$$\tilde{\sigma}(\lambda A) = \lambda \tilde{\sigma}(A), \quad \tilde{\sigma}(A + \lambda) = \tilde{\sigma}(A) + \lambda, \quad \textit{and} \quad \tilde{\sigma}(A^{-1}) = \frac{1}{\tilde{\sigma}(A)}$$

for all $\lambda \in \mathbb{C}$. In particular, $\tilde{\sigma}(R(\lambda, A)) = \frac{1}{\lambda - \tilde{\sigma}(A)}$ for all $\lambda \in \mathbb{C}$.

This will follow from Corollary A.14 below. Let $\lambda \in \mathbb{C}_\infty$. The **eigenspace** of A at λ is defined by

$$\mathcal{N}(\lambda, A) := \begin{cases} \mathcal{N}(A - \lambda) & \text{for} \quad \lambda \in \mathbb{C} \\ A0 & \text{for} \quad \lambda = \infty. \end{cases}$$

[1] like, e.g., $\infty + \lambda = \infty$, $\lambda \cdot \infty = \infty$ for $0 \neq \lambda \in \mathbb{C}$, and $0 = 0 \cdot \infty = 1/\infty$.

Similarly, we define the **range space**

$$\mathcal{R}(\lambda, A) := \begin{cases} \mathcal{R}(A-\lambda) & \text{für} \quad \lambda \in \mathbb{C} \\ \mathcal{D}(A) & \text{für} \quad \lambda = \infty \end{cases}$$

of A at λ. Using this notation we define the extended **point spectrum**, **approximate point spectrum**, **residual spectrum**, and **surjectivity spectrum** by

$$\begin{aligned} P\tilde{\sigma}(A) &:= \{\lambda \in \mathbb{C}_\infty \mid \mathcal{N}(\lambda, A) \neq 0\}, \\ A\tilde{\sigma}(A) &:= \{\lambda \in \mathbb{C}_\infty \mid \mathcal{N}(\lambda, A) \neq 0 \text{ or } \mathcal{R}(\lambda, A) \text{ is not closed}\}, \\ R\tilde{\sigma}(A) &:= \{\lambda \in \mathbb{C}_\infty \mid \overline{\mathcal{R}(\lambda, A)} \neq X\}, \\ S\tilde{\sigma}(A) &:= \{\lambda \in \mathbb{C}_\infty \mid \mathcal{R}(\lambda, A) \neq X\}. \end{aligned}$$

Clearly we have $\tilde{\sigma}(A) = P\tilde{\sigma}(A) \cup S\tilde{\sigma}(A) = A\tilde{\sigma}(A) \cup R\tilde{\sigma}(A)$. The classical spectra are obtained by intersecting the extended spectra with the complex plane, i.e.,

$$P\sigma(A) = \mathbb{C} \cap P\tilde{\sigma}(A), \quad A\sigma(A) = \mathbb{C} \cap A\tilde{\sigma}(A), \quad \ldots.$$

By introducing the extended spectra we have obtained, for example, that single-valuedness of an operator is a spectral condition.

Lemma A.13. *Let $\mu \in \mathbb{C}$ and $\lambda \in \mathbb{C}_\infty$. Then*

$$\mathcal{N}(\lambda, \mu - A) = \mathcal{N}(\mu - \lambda, A) \quad \textit{and} \quad \mathcal{N}(\lambda, A^{-1}) = \mathcal{N}(\frac{1}{\lambda}, A), \quad \textit{as well as}$$
$$\mathcal{R}(\lambda, \mu - A) = \mathcal{R}(\mu - \lambda, A) \quad \textit{and} \quad \mathcal{R}(\lambda, A^{-1}) = \mathcal{N}(\frac{1}{\lambda}, A).$$

Proof. We only show the first and the last equality. The other two are proved similarly. Let $y \in X$. We have

$$\begin{aligned} y \in \mathcal{N}(\lambda, \mu - A) \quad &\Leftrightarrow \quad \lambda \in \mathbb{C},\ (y, 0) \in (\mu - A) - \lambda = (\mu - \lambda) - A \\ &\qquad \vee \quad \lambda = \infty,\ (0, y) \in A \\ &\Leftrightarrow \quad (\mu - \lambda) \in \mathbb{C},\ (y, 0) \in A - (\mu - \lambda) \quad \vee \quad (\mu - \lambda) = \infty,\ (0, y) \in A \\ &\Leftrightarrow \quad y \in \mathcal{N}(\mu - \lambda, A), \\ \text{and} \quad y \in \mathcal{R}(\lambda, A^{-1}) \quad &\Leftrightarrow \quad 0 \neq \lambda \in \mathbb{C},\ \exists x : (x, y) \in A^{-1} - \lambda \quad \vee \quad \lambda = 0,\ \exists x : (x, y) \in A^{-1} \\ &\qquad \vee \quad \lambda = \infty,\ \exists x : (y, x) \in A^{-1} \\ &\Leftrightarrow \quad 0 \neq \lambda \in \mathbb{C},\ \exists x : (y + \lambda x, x) \in A \quad \vee \quad \lambda = 0,\ \exists x : (y, x) \in A \\ &\qquad \vee \quad \lambda = \infty,\ \exists x : (x, y) \in A \\ &\Leftrightarrow \quad 0 \neq \lambda^{-1} \in \mathbb{C},\ \exists x : (y + \lambda x, -\lambda^{-1} y) \in A - \lambda^{-1} \\ &\qquad \vee \quad \lambda^{-1} = \infty,\ \exists x : (y, x) \in A \quad \vee \quad \lambda^{-1} = 0,\ \exists x : (x, y) \in A \\ &\Rightarrow \quad y \in \mathcal{R}(\lambda^{-1}, A). \end{aligned}$$

This shows $\mathcal{R}(\lambda, A^{-1}) \subset \mathcal{R}(\lambda^{-1}, A)$. The reverse inclusion follows by symmetry. □

Corollary A.14. *Let $\mu \in \mathbb{C}$ and $\lambda \in \mathbb{C}_\infty$. Then*

$$\mathbf{X}\tilde{\sigma}(\mu - A) = \mu - \mathbf{X}\tilde{\sigma}(A) \quad \textit{und} \quad \mathbf{X}\tilde{\sigma}(A^{-1}) = \frac{1}{\mathbf{X}\tilde{\sigma}(A)}$$

for $\mathbf{X} \in \{P, A, R, S\}$.

Remark A.15. One can characterize the approximate eigenvalues like in the classical theory by the existence of approximate eigenvectors. This leads to a proof of the inclusion $\partial(\sigma(A)) \subset A\sigma(A)$ for m.v. operators A.

A.4 Convergence of Operators

In this section we consider the following situation: Let $(A_n)_n$ be a sequence of m.v. operators on X and $\lambda_0 \in \mathbb{C}$ such that $\lambda_0 \in \bigcap_n \varrho(A_n)$ for all n. Suppose further that the sequence $(R(\lambda_0, A_n))_n$ converges to an operator $R_{\lambda_0} \in \mathcal{L}(X)$ (in norm or in the strong sense). We know from Proposition A.8 that there is a unique m.v. operator A with $R_{\lambda_0} = (\lambda_0 - A)^{-1}$. Since $R_{\lambda_0} \in \mathcal{L}(X)$, we have $\lambda_0 \in \varrho(A)$. We are interested in the question, whether there is convergence of the resolvents at other common resolvent points of the operators A_n. To answer this question, we let

$$\Omega := \{\lambda \in \mathbb{C} \mid \exists N \in \mathbb{N} : \lambda \in \varrho(A_n)\ \forall n \geq N \text{ and } \sup_{n\geq N} \|R(\lambda, A_n)\| < \infty\}$$

It is immediate from the proof of Proposition A.7 that Ω is an open subset of $\mathbb{C}$ and that the mapping $n(\lambda) := \min\{n \in \mathbb{N} \mid \lambda \in \varrho(A_k)\ \forall k \geq n\}$ locally can only decrease on Ω.

Lemma A.16. *Let $\lambda \in \mathbb{C}$.*

a) If $\lambda \in \varrho(A_n)$ for almost all n and if $R(\lambda, A_n)x \to y$, then $(x, y) \in R(\lambda, A)$.

b) If $\lambda \in \Omega$ and $(x, y) \in R(\lambda, A)$, then $R(\lambda, A_n)x \to y$.

Proof. Define $Q_n := (\lambda - \lambda_0)R(\lambda_0, A_n)$ and $Q := (\lambda - \lambda_0)R(\lambda_0, A)$. The resolvent identity implies $(I + Q_n)R(\lambda, A_n) = R(\lambda_0, A_n)$. But $R(\lambda_0, A_n) \to R(\lambda_0, A)$ strongly, hence $Q_n \to Q$ strongly as well. Thus, we have $(I + Q)y = R(\lambda_0, A)x$. This implies readily that $(x, y) \in R(\lambda, A)$.
Let $\lambda \in \Omega$ and $(x, y) \in R(\lambda, A)$. By Corollary A.6 one has $I - (I + Q_n)^{-1} = (\lambda - \lambda_0)R(\lambda, A_n)$. From $\lambda \in \Omega$ it follows that $(I + Q_n)^{-1} \in \mathcal{L}(X)$ $(n \geq n_0)$ and $\sup_{n\geq n_0} \left\|(I + Q_n)^{-1}\right\| < \infty$ for some $n_0 \in \mathbb{N}$. From $(x, y) \in R(\lambda, A)$ it follows that $(I + Q)y = R(\lambda_0, A)x$. Therefore,

$$(I + Q_n)(R(\lambda, A_n)x - y) = R(\lambda_0, A_n)x - (I + Q_n)y \to R(\lambda_0, A)x - (I + Q)y = 0.$$

Applying $(I + Q_n)^{-1}$ yields $R(\lambda, A_n)x - y \to 0$. □

Corollary A.17. *For $\lambda \in \Omega$ the m.v. operator $(\lambda - A)$ is injective and has a closed range, i.e., $\lambda \notin A\sigma(A)$. In particular, $\sigma(A) \cap \Omega$ is an open subset of $\mathbb{C}$. Furthermore, we have*

$$\Omega \cap \varrho(A) = \{\lambda \mid \lambda \in \varrho(A_n) \textit{ almost all } n,\ (R(\lambda, A_n))_n \textit{ strongly convergent}\}.$$

Proof. Let $\lambda \in \Omega$ and $K \geq 0$ such that $\|R(\lambda, A_n)\| \leq K$ for almost all n. By the last lemma, if $(x, y) \in R(\lambda, A)$, then $R(\lambda, A_n)x \to y$. Hence $\|y\| \leq K\|x\|$. This shows that $R(\lambda, A)$ is single-valued and has a closed domain. From Remark A.15 we know that $\partial\sigma(A) \subset A\sigma(A)$. Hence $\partial\sigma(A) \cap \Omega = \emptyset$. For $\lambda \in \Omega$ we have $\lambda \in \varrho(A)$ if and only if $R(\lambda, A)$ is fully defined and this is the case if and only if $(R(\lambda, A_n))_n$ is strongly convergent by Lemma A.16 □

The interesting question now is whether $\Omega = \varrho(A)$ holds. It is true for the norm topology as the following proposition shows.

Proposition A.18. *Let A_n, A and Ω as above. Suppose $R(\lambda_0, A_n) \to R(\lambda_0, A)$ in* norm. *Then $\Omega = \varrho(A)$, and $R(\lambda, A) \to R(\lambda, A)$ in norm for all $\lambda \in \varrho(A)$.*

Proof. Let $\lambda \in \varrho(A)$, and let Q, Q_n be defined as in the proof of Lemma A.16. Then $(I+Q)^{-1} \in \mathcal{L}(X)$. Because of $(I+Q_n) \to (I+Q)$ in norm, we have $(I+Q_n)^{-1} \in \mathcal{L}(X)$ for almost all n and $(I+Q_n)^{-1} \to (I+Q)^{-1}$ in norm. This yields $R(\lambda, A_n) \to R(\lambda, A)$ in norm. In particular, $\lambda \in \Omega$.
Conversely, let $\lambda \in \Omega$. Then, $T_n := I + Q_n$ is invertible for all large n with $K := \sup_{n \geq n_0} \left\| T_n^{-1} \right\| < \infty$. But $T_n \to T := I + Q$ in norm. Thus

$$\left\| T_n^{-1} - T_m^{-1} \right\| \leq \left\| T_n^{-1} \right\| \left\| T_m - T_n \right\| \left\| T_m^{-1} \right\| \leq K^2 \left\| T_m - T_n \right\|,$$

hence $(T_n^{-1})_n$ is a norm-Cauchy sequence. Obviously, this implies that T is invertible, whence $\lambda \in \varrho(A)$. □

A.5 Polynomials and Rational Functions of an Operator

In this section A always denotes a *single-valued* operator on X.

The sequence of **natural powers** $(A^n)_{n \in \mathbb{N}}$ is defined recursively by

$$A^0 := I, \quad A^{n+1} := A^n A \quad (n \geq 0).$$

A simple induction argument shows the validity of the **power law** $A^{n+m} = A^n A^m$ for all $n, m \in \mathbb{N}$.[2] In particular, $A^{n+1} = A\, A^n$, and this shows that the sequence of domains $\mathcal{D}(A^n)$ is decreasing, i.e., $\mathcal{D}(A^{n+1}) \subset \mathcal{D}(A^n)$. Another consequence is the inclusion

$$A^n(\mathcal{D}(A^m)) \subset \mathcal{D}(A^{m-n}) \quad \text{for} \quad m \geq n.$$

Let $p(z) = \sum_{k \geq 0} a_k z^k \in \mathbb{C}[z]$ be a *polynomial*. The operator

$$p(A) := \sum_{k \geq 0} a_k A^k$$

is well-defined (by associativity of operator sums) with domain $\mathcal{D}(p(A)) = \mathcal{D}(A^n)$, where $n = \deg(p)$ in case that $p \neq 0$ and $n = 0$ in case that $p = 0$. (Note that this is not a definition but a conclusion.)

Lemma A.19. *Let $p, q \in \mathbb{C}[z]$. The following statements hold.*

a) If $p \neq 0$, then $p(A)q(A) = (pq)(A)$. In particular, if $p \neq 0$ and $x \in X$:

$$x \in \mathcal{D}(A^{\deg(p)}) \text{ and } p(A)x \in \mathcal{D}(A^n) \iff x \in \mathcal{D}(A^{n+\deg(p)}).$$

b) If $p(A)$ is injective and $q \neq 0$, then $\mathcal{D}(p(A)^{-1}) \cap \mathcal{D}(q(A)) \subset \mathcal{D}(p(A)^{-1}q(A)) \cap \mathcal{D}(q(A)p(A)^{-1})$ and

$$p(A)^{-1}q(A)x = q(A)p(A)^{-1}x$$

for each $x \in \mathcal{D}(p(A)^{-1}) \cap \mathcal{D}(q(A))$.

[2] This can also be seen as follows: The set of all linear operators is a subsemigroup of the semigroup of all binary relations on X, with respect to the usual composition of relations. The power law now is actually true in every semigroup.

c) *One has* $p(A) + q(A) \subset (p+q)(A)$*, and equality holds in case* $\deg(p+q) = \max(\deg(p), \deg(q))$.

d) *If* $T \in \mathcal{L}(X)$ *commutes with* A*, then it also commutes with* $p(A)$.

In particular we have $p(A)q(A) = q(A)p(A)$ *for* $p, q \neq 0$*, and* $p(A)$ *commutes with the resolvents of* A.

Proof. We prove *a)*. The assertion is obviously true, if p or q are just scalars. Hence, by the Fundamental Theorem of Algebra and the associativity of operator multiplication, we can reduce the problem to the case $\deg(p), \deg(q) = 1$. This means that we have to establish the identity

$$A^2 - (\mu + \lambda)A + \mu\lambda = (A - \lambda)(A - \mu)$$

for μ, $\lambda \in \mathbb{C}$. Let $y \in X$ be arbitrary. Because of $\mathcal{D}(A^2) \subset \mathcal{D}(A)$ we have $y \in \mathcal{D}(A)$ und $(A-\mu)y \in \mathcal{D}(A)$ if and only if $y \in \mathcal{D}(A^2)$. Thus, *a)* is proved. From *a)* it follows that $p(A)q(A) = q(A)p(A)$ whenever $p, q \neq 0$. Now, a short argument gives *b)*. The statement in *c)* is trivial. To prove *d)*, one first shows $TA^n \subset A^n T$ for $n \in \mathbb{N}$ by induction (Step: Because of $TA \subset AT$ one has $TA^{n+1} \subset ATA^n \subset AA^nT = A^{n+1}T$). This yields $Tp(A) \subset p(A)T$ almost immediately. □

Proposition A.20. *Let* A *be an operator on a Banach spaces* X *such that* $\varrho(A) \neq \emptyset$*. Then* $p(A)$ *is a closed operator for each polynomial* $p \in \mathbb{C}[z]$*. Furthermore, the* **spectral mapping theorem** $\sigma(p(A)) = p(\sigma(A))$ *holds.*

Proof. We prove the first statement by induction on $n := \deg p$. The case $n = 1$ follows from the fact that the norms $\|x\| + \|Ax\|$ and $\|x\| + \|(A-\mu)x\|$ are equivalent norms on $\mathcal{D}(A)$.
Now let $n \geq 1$, $\deg p = n+1$, and $\lambda \in \varrho(A)$. If one defines $\mu := p(\lambda)$, then there is $r \in \mathbb{C}[z]$ such that $\deg r = n$ and $p = (x - \lambda)r + \mu$. Let $(x_k)_k$ be a sequence in $\mathcal{D}(A^{n+1})$ converging to x (in X) such that $p(A)x_k \to y$ (in X as well). Then we have $r(A)x_k \to (A-\lambda)^{-1}(y - \mu x)$. By the induction hypothesis, $r(A)$ is closed, hence $x \in \mathcal{D}(A^n)$ with $r(A) = (A-\lambda)^{-1}(y-\mu x) \in \mathcal{D}(A)$. From *a)* and *c)* in Lemma A.19 it follows that $x \in \mathcal{D}(A^{n+1})$ and $y - \mu x = (A-\lambda)r(A)x = (p-\mu)(A)x = p(A)x - \mu x$.
To prove the spectral mapping theorem it suffices to show that $p(A)$ is invertible if and only if the roots of p are contained in the resolvent set of A. Therefore, let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, and let

$$T := (A - \lambda_1)(A - \lambda_2)\ldots(A - \lambda_n) : \mathcal{D}(A^n) \longrightarrow X$$

be invertible. Then clearly $A - \lambda_1$ is surjective and $A - \lambda_n$ is injective. But all the $A - \lambda_j$ commute with each other, by part *a)* of Lemma A.19. Hence all $A - \lambda_j$ are bijective. On the other hand, if we assume $\lambda_j \in \varrho(A)$ for all j, then obviously T is invertible. □

We denote by

$$\mathcal{R}_A := \{\frac{p}{q} \mid p, q \in \mathbb{C}[z] \text{ and } \{\lambda : q(\lambda) = 0\} \subset \varrho(A)\}$$

the set of all rational functions having their poles contained in $\varrho(A)$. For $r = p/q \in \mathcal{R}_A$ we define

$$r(A) := p(A)q(A)^{-1}.$$

(This is independent of the special choice of p and q by Lemma A.19.)
Note that there is some arbitrariness in this definition. For example, one could have equally defined $r(A) := q(A)^{-1}p(A)$ (this is essentially the same operator but with a smaller domain). With our definition, the domain of $r(A)$ is

$$\mathcal{D}(r(A)) = \mathcal{D}(A^m) \quad \text{where} \quad m = \begin{cases} \deg(p) - \deg(q) & \text{if } \deg(p) \geq \deg(q), \\ 0 & \text{else.} \end{cases}$$

Proposition A.21. *Let A be an operator on a Banach space X such that $\varrho(A) \neq \emptyset$. For $0 \neq r = p/q$, $\tilde{r} = \tilde{p}/\tilde{q} \in \mathcal{R}_A$ the following assertions hold.*

a) $r(A)$ *is a closed operator.*

b) $r(\tilde{\sigma}(A)) \subset \tilde{\sigma}(r(A))$.

c) $r(A)\tilde{r}(A) \subset (r\tilde{r})(A)$*, and equality holds, e.g., if* $(\deg(p) - \deg(q))(\deg(\tilde{p}) - \deg(\tilde{q}) \geq 0$.

d) $r(A) + \tilde{r}(A) \subset (r + \tilde{r})(A)$*, and equality holds, e.g., if* $\deg(p\tilde{q} + \tilde{p}q) = \max\{\deg(p\tilde{q}), \deg(\tilde{p}q)\}$.

e) If $T \in \mathcal{L}(X)$ *commutes with* A*, then it commutes also with* $r(A)$.

Proof. Assertion *a)* is trivial, and assertions *c)* and *d)* follow from Lemma A.19. To prove *b)* we note first that $r(A) - \lambda I = [(p - \lambda q)/q](A)$ for $\lambda \in \mathbb{C}$. Hence we are left to show that $r(A)$ is invertible, if $\{z : p(z) = 0\} \subset \varrho(A)$. Because of $q(A)^{-1}p(A) \subset r(A) = p(A)q(A)^{-1}$, the operator $r(A)$ is invertible if and only if $p(A)$ is. Assertion *e)* is a consequence of Lemma A.19 and of $Tr(A) = Tp(A)q(A)^{-1} \subset p(A)Tq(A)^{-1} = p(A)q(A)^{-1}T = r(A)T$. □

We end the section with an interesting corollary.

Corollary A.22. *Let A be a closed operator on a Banach space X. Let $\tilde{r} = \tilde{p}/\tilde{q} \in \mathcal{R}_A$ such that $\deg(\tilde{p}) = \deg(\tilde{q})$. Then $r(A)\tilde{r}(A) = \tilde{r}(A)r(A)$ for every $r \in \mathcal{R}_A$.*

Proof. Just apply *c)* of Proposition A.21 twice. Another proof rests on the fact that $\tilde{r}$ can be written as a product of operators of the form $\alpha - \beta R(\lambda, A)$ for some numbers $\lambda \in \varrho(A), \alpha, \beta \in \mathbb{C}$ such that $\alpha \neq 0$. For such operators we have

$$x \in \mathcal{D}(A^n) \quad \Longleftrightarrow \quad (\alpha - \beta R(\lambda, A))x \in \mathcal{D}(A^n)$$

for all n and all $x \in X$. □

A.6 Injective Operators

In this section we consider an *injective* single-valued operator A on a Banach space X. This enables us to extend the insertion mapping $p \longmapsto p(A)$ to the set of polynomials in z and z^{-1}. We begin with a surprising fact.

Lemma A.23. *Let A be injective with $\varrho(A) \neq 0$. Then*

$$p(A^{-1})R(\lambda, A) = R(\lambda, A)p(A^{-1})$$

for all $\lambda \in \varrho(A)$ and all polynomials $p \in \mathbb{C}[z]$.

Proof. Because $R(\lambda, A) \in \mathcal{L}(X)$ both distributivity inclusions (see Proposition A.1) are actually equalities, hence we can reduce the proof to the case $p(z) = z$. We have $R(\lambda, A)A^{-1} \subset A^{-1}R(\lambda, A)$, since $R(\lambda, A)$ commutes with A, hence with A^{-1}. Let $x \in \mathcal{D}(A^{-1}R(\lambda, A))$. Then $R(\lambda, A)x = Ay$ for some $y \in \mathcal{D}(A)$. But $R(\lambda, A)x \in \mathcal{D}(A)$, whence $y \in \mathcal{D}(A^2)$, and we can apply $\lambda - A$ on both sides to obtain $x = (\lambda - A)Ay = A(\lambda - A)y \in \mathcal{R}(A)$. This implies $x \in \mathcal{D}(A^{-1}) = \mathcal{D}(R(\lambda, A)A^{-1})$. □

Let $p(z) = \sum_{k \in \mathbb{Z}} a_k z^k \in \mathbb{C}[z, z^{-1}]$ a *polynomial*. The operator

$$p(A) := \sum_{k \in \mathbb{Z}} a_k A^k$$

is well-defined (associativity for operator sums and injectivity of A). Its domain is

$$\mathcal{D}(p(A)) = \mathcal{D}(A^m) \cap \mathcal{D}(A^n)$$

where $n = \min\{k \mid (z^k p(z))|_{z=0} \neq \infty\}$ and $m = \min\{k \mid (z^{-k} p(z))|_{z=\infty} \neq \infty\}$. One can write $p \in \mathbb{C}[z, z^{-1}]$ in a unique way as

$$p(z) = q(z) z^{-n} \quad n \in \mathbb{Z},\ q \in \mathbb{C}[z],\ q(0) \neq 0.$$

Then we have $p(A) = q(A)A^{-n}$.

[This is clear for $n \leq 0$. In the other case only equality of domains is to be shown. But this is easy.]

Note that a nice product law like in part *a)* of Lemma A.19 for polynomials can not hold in this situation. Simply look at

$$A^{-1}A \subset I \supset AA^{-1}$$

where the inclusions are strict in general. This example also shows that a general law of commutativity can not be expected. However, this is not the end of the story.

Lemma A.24. *Let $p, q \in \mathbb{C}[z]$ with $q(0) \neq 0$. Then*

$$p(A^{-1})q(A) \subset q(A)p(A^{-1}).$$

Proof. We can assume $p(A^{-1}) = A^{-n}$ without restriction. If $x \in \mathcal{D}(q(A))$ with $q(A)x \in \mathcal{D}(A^{-n})$, then there is $y \in \mathcal{D}(A^n)$ such that $q(A)x = A^n y$. Because $q(0) \neq 0$ this implies $x \in \mathcal{R}(A)$, say, $x = Ax_1$. But then $Aq(A)x_1 = A^n y$, and this yields $q(A)x_1 = A^{n-1}y$ by injectivity of A. Inductively, it follows that $x \in \mathcal{R}(A^n)$, say, $x = A^n x_0$. Hence we finally arrive at

$$q(A)A^{-n}x = q(A)x_0 = y = A^{-n}q(A)x.$$

This proves the statement. □

The simple example $A^{-1}(1 + A) \neq (1 + A)A^{-1}$ shows that the inclusion in the last lemma is strict in general.

Corollary A.25. *Let $p, q \in \mathbb{C}[z]$ with $p(0)q(0) \neq 0$. Then*

$$p(A^{-1})q(A) = (p(z^{-1})q(z))(A) = q(A)p(A^{-1}).$$

Proof. The first equality is immediate from Lemma A.24. To prove the second, we can assume $\deg(p) = \deg(q) = 1$, employing the Fundamental Theorem of Algebra and Proposition A.21. The operator identity $(A^{-1} + \mu)(A + \lambda) = (1 + \mu\lambda) + \mu A + \lambda A^{-1}$ then reduces to an almost trivial comparison of domains. □

Proposition A.26. *Let A be an injective operator and $p, q \in \mathbb{C}[z, z^{-1}]$. Then the following inclusions hold.*

$$p(A)q(A) \subset (pq)(A) \text{ and } p(A) + q(A) \subset (p + q)(A).$$

If $T \in \mathcal{L}(X)$ commutes with A, then it also commutes with $p(A)$. If $\varrho(A) \neq \emptyset$, $p(A)$ is a closed operator.

Proof. We leave the proof to the reader. □

A.7 Semigroups and Generators

In this section we review the basic facts on semigroup theory. Our exposition is different to others in that we do not restrict the approach to *strongly continuous* semigroups.

Let X be a Banach space. A **(degenerate) semigroup** on X is a strongly continuous mapping

$$T : (0, \infty) \longrightarrow \mathcal{L}(X)$$

that possesses the **semigroup property**

$$T(t)T(s) = T(t+s) \qquad (t, s > 0).$$

The semigroup T is called **bounded** if

$$\sup_{0<t<\infty} \|T(t)\| < \infty.$$

If the semigroup is just bounded in 0, i.e., if $\sup_{t<1} \|T(t)\| < \infty$, then there are constants $M \geq 1, \omega \in \mathbb{R}$ such that $\|T(t)\| \leq Me^{\omega t}$ for all $t > 0$, compare [EN00, Chapter I, Proposition 5.5]. Hence such a semigroup is called **exponentially bounded**. Given an exponentially bounded semigroup T, the number

$$\omega_0(T) := \inf\{\omega \in \mathbb{R} \mid \exists\, M : \|T(t))\| \leq Me^{\omega t} \;\; (t > 0\}$$

is called the **growth bound** of T. The semigroup T is said to be **exponentially stable** if $\omega_0(T) < 0$. A semigroup T satisfying $\|T(t)\| \leq 1$ for each $t > 0$ is called **contractive** or a **contraction semigroup**. It is called **quasi-contractive** if there is $\omega \geq 0$ such that the semigroup $e^{\omega \cdot}T(\cdot)$ is contractive.
The space

$$\{x \in X \mid \lim_{t \searrow 0} T(t)x = x\}$$

is called the **space of strong continuity** of the semigroup T. If it is the whole space X, the semigroup is called **strongly continuous** or a $\mathbf{C_0}$**-semigroup**.

Let T be an exponentially bounded semigroup and choose constants ω, M such that $\|T(t)\| \leq Me^{\omega t}$ for $t > 0$. Then the Laplace transform of T exists at least in the halfplane $\{\operatorname{Re}\lambda > \omega\}$, i.e.,

$$\hat{T}(\lambda)x := \int_0^\infty e^{-\lambda t}T(t)x\,dt \qquad (x \in X)$$

defines a bounded operator on X for every λ with $\operatorname{Re}\lambda > \omega$. One can show that in fact $\hat{T}(\cdot)$ is a pseudo-resolvent, compare [ABHN01, p.114]. Hence there is a unique m.v. operator A such that

$$(\lambda - A)^{-1} = \int_0^\infty e^{-\lambda t}T(t)\,dt.$$

We call A the **generator** of the semigroup T. By the injectivity of the Laplace transform the semigroup is uniquely determined by its generator. The semigroup T is said to be **non-degenerate** if A is single-valued. We obtain

$$A0 = \bigcap_{t>0} \mathcal{N}(T(t)) = \mathcal{N}(R(\lambda, A)) \quad (\lambda \in \varrho(A)) \tag{A.1}$$

again by the injectivity of the Laplace transform. Obviously, a C_0-semigroup is non-degenerate. If $\mu \in \mathbb{C}$, then $A + \mu$ generates the semigroup $t \longmapsto e^{\mu t}T(t)$. Hence an operator generates a quasi-bounded semigroup if and only if there is $\omega \in \mathbb{R}$ such that $A - \omega$ generates a bounded semigroup.

Proposition A.27. *Let T be a semigroup on the Banach space X satisfying $\|T(t)\| \leq Me^{\omega t}$ for all $t > 0$. Then*

$$R(\lambda, A)^n = \frac{1}{(n-1)!} \int_0^\infty t^{n-1} e^{-\lambda t} T(t)\, dt \quad \textit{and} \tag{A.2}$$

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\operatorname{Re} \lambda - \omega)^n} \tag{A.3}$$

for all $n \in \mathbb{N}$ and all $\operatorname{Re} \lambda > \omega$.

Proof. The proof is the same as in the strongly continuous case, compare [EN00, Chapter I, Corollary 1.11]. □

Note that each operator $T(t)$ commutes with the resolvent of A, whence $\mathcal{D}(A)$ is left invariant by the semigroup T.

Proposition A.28. (Fundamental Identity for Semigroups)
Let T be a quasi-bounded semigroup with generator A on the Banach space X. Define

$$D_t := \frac{1}{t}(T(t) - I) \quad \textit{and} \quad V_t := \frac{1}{t}\int_0^t T(s)\, ds$$

for $t > 0$. Then

$$(\lambda R(\lambda, A) - I)V_\varepsilon = D_\varepsilon R(\lambda, A) \quad \textit{and} \quad (V_\varepsilon x, D_\varepsilon x) \in A$$

for all $\varepsilon > 0$, $\operatorname{Re} \lambda > \omega_0(T)$, and all $x \in X$.

Proof. We compute

$$\begin{aligned}
(\lambda R(\lambda) - I) \int_0^\varepsilon T(s)\, ds &= \int_0^\infty \lambda e^{-\lambda t} T(t) \int_0^\varepsilon T(s)\, ds\, dt - \int_0^\varepsilon T(s)\, ds \\
&= \int_0^\infty \left(\lambda e^{-\lambda t} \int_0^\varepsilon T(t+s)\, ds - \lambda e^{-\lambda t} \int_0^\varepsilon T(s)\, ds \right) dt \\
&= \int_0^\infty \lambda e^{-\lambda t} \left(\int_t^{t+\varepsilon} \cdots - \int_0^\varepsilon \cdots \right) dt \\
&= \int_0^\infty \lambda e^{-\lambda t} \left(\int_\varepsilon^{\varepsilon+t} \cdots - \int_0^t \cdots \right) dt \\
&\overset{I.p.}{=} \int_0^\infty e^{-\lambda t}(T(t+\varepsilon) - T(t))\, dt = (T(\varepsilon) - I)R(\lambda).
\end{aligned}$$

Dividing by ε completes the proof of the first statement. Using this we obtain $V_\varepsilon = R(\lambda)(\lambda V_\varepsilon - D_\varepsilon)$. This shows that $(V_\varepsilon x, \lambda V_\varepsilon x - D_\varepsilon x) \in (\lambda - A)$ for every $x \in X$. □

Corollary A.29. *Let T be a quasi-bounded semigroup on the Banach space X and let $x \in X$. The following assertions are equivalent.*

(i) $x \in \overline{\mathcal{D}(A)}$.

(ii) $T(t)x \to x$ *as* $t \searrow 0$.

(iii) $\frac{1}{\varepsilon}\int_0^\varepsilon T(s)x\,ds \to x$ *as* $\varepsilon \searrow 0$.

(iv) $\lambda R(\lambda, A)x \to x$ *as* $\lambda \to \infty$.

In particular, $\overline{\mathcal{D}(A)}$ *is the space of strong continuity of T. One has the inclusion* $T(t)X \subset \overline{\mathcal{D}(A)}$ *for each* $t > 0$.

Proof. Obviously we have $(iv) \Rightarrow (i)$ and $(ii) \Rightarrow (iii)$. Since $\mathcal{R}(V_\varepsilon) \subset \mathcal{D}(A)$ by Proposition A.28, we also obtain $(iii) \Rightarrow (i)$. Let $x \in \mathcal{D}(A)$ and pick $y \in (\lambda - A)x$ where $\lambda > \omega_0(T)$. Then

$$T(\varepsilon)x - x = (T(\varepsilon) - I)R(\lambda)y = \varepsilon V_\varepsilon(\lambda R(\lambda) - I)y \to 0$$

as $\varepsilon \to 0$. Since $\sup_{t\le 1}\|T(t)\| < \infty$, we obtain $(i) \Rightarrow (ii)$.
To prove the remaining implication $(i) \Rightarrow (iv)$ we note first that $\|\lambda R(\lambda)\|$ is uniformly bounded for $\lambda > \omega$ where $\omega > \omega_0(T)$. This follows from Proposition A.27. Given $x \in \mathcal{D}(A)$ we choose $\mu \in \varrho(A)$ and $y \in (\mu - A)x$ to obtain

$$\lambda R(\lambda)x = \lambda R(\lambda)R(\mu)y = \frac{\lambda}{\lambda - \mu}(R(\mu)y - R(\lambda)y) \to R(\mu)y = x$$

as $\lambda \to \infty$. □

Corollary A.30. *Let $x \in X$. Then we have*

$$x \in \mathcal{D}(A),\ Ax \cap \overline{\mathcal{D}(A)} \neq \emptyset \quad \Leftrightarrow \quad \lim_{t\searrow 0}\frac{1}{t}(T(t)x - x) =: y \quad \textit{exists}$$

and in this case $\{y\} = Ax \cap \overline{\mathcal{D}(A)}$. *In particular,* $A0 \cap \overline{\mathcal{D}(A)} = 0$, *whence* $X_T := A0 \oplus \overline{\mathcal{D}(A)}$ *is a closed subspace of X.*

It follows from Corollary A.30 that the **part** $B := A \cap (\overline{\mathcal{D}(A)} \oplus \overline{\mathcal{D}(A)})$ of A in $\overline{\mathcal{D}(A)}$ is single-valued.

Proposition A.31. *Let T be a quasi-bounded semigroup with generator A on the Banach space X. Let* $Y := \overline{\mathcal{D}(A)}$. *The space Y is left invariant by the semigroup T. The semigroup T restricts to a strongly continuous semigroup on Y which has* $B = A \cap (Y \oplus Y)$ *as its generator.*

The next result is one of the cornerstones of the theory of C_0-semigroups. (See [EN00, Chapter II, Section 3] or [ABHN01, Theorem 3.3.4] for proofs.)

Theorem A.32. (Hille-Yosida)
Let A be a (single-valued) linear operator on the Banach space X. Assume that A has dense domain and there is $M \geq 1$ and $\omega \in \mathbb{R}$ such that $(\omega, A) \subset \varrho(A)$ and

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\lambda - \omega)^n} \tag{A.4}$$

for all $n \in \mathbb{N}$ and all $\lambda > \omega$. Then A generates a C_0-semigroup satisfying $\|T(t)\| \leq Me^{\omega t}$ *for* $t \geq 0$.

Remark A.33. Unfortunately there is no similar characterization for generators of general exponentially bounded semigroups. The resolvent condition (A.3) guarantees that A generates a so-called **integrated semigroup** $S(\cdot)$, see [ABHN01, Theorem 3.3.1]. For $x \in \overline{\mathcal{D}(A)}$ one has $S(t)x = \int_0^t T(s)x\,ds$, where T is the semigroup generated by the part B of A in $\overline{\mathcal{D}(A)}$. Then A generates an exponentially bounded semigroup if and only if $S(\cdot)x \in C^1((0,\infty), X)$ for each $x \in X$. Employing [Are87, Theorem 6.2] one can show that this is always true if the Banach space X has the Radon-Nikodym property.

Finally we deal with the important case of *groups*.

Proposition A.34. *Let T be an exponentially bounded semigroup with generator A. The following assertions are equivalent:*

(i) There exists $t_0 > 0$ such that $T(t_0)$ is invertible.

(ii) Each $T(t)$ is invertible and the mapping $\tilde{T} : \mathbb{R} \to \mathcal{L}(X)^\times$, defined by

$$\tilde{T}(t) := \begin{cases} T(t) & (t > 0) \\ I & (t = 0) \\ T(-t)^{-1} & (t < 0) \end{cases}$$

is a strongly continuous group homomorphism.

(iii) The operator $-A$ generates an exponentially bounded semigroup and A is single-valued.

Proof. $(i) \Rightarrow (ii) \Rightarrow (iii)$. Assume that $T(t_0)$ is invertible. Then the semigroup property readily implies that each $T(t)$ is invertible. Moreover, since $\mathcal{R}(T(t)) \subset \overline{\mathcal{D}(A)}$ for each $t > 0$, we have that A is densely defined. Hence A is single valued and we are in the standard (C_0-) case. So we can refer to [EN00, subsection 3.11], or [Paz83, Section 1.6] for the remaining arguments.
$(iii) \Rightarrow (i)$. We denote by S the semigroup generated by $-A$. Choose $\omega > \omega_0(T), \omega_0(S)$. Let $Y := \overline{\mathcal{D}(A)}$ and let $B := A \cap (Y \oplus Y)$ the part of A in Y. From Proposition A.31 we know that B generates the C_0-semigroup obtained by restricting T to Y. Analogously, $-B$ generates the C_0-semigroup obtained by restricting S to Y. (Note that $\mathcal{D}(-A) = \mathcal{D}(A)$.) The Theorem from [EN00, Section 3.11] now yields that $T(t)S(t)y = S(t)T(t)y = y$ for all $y \in Y$. Let $t_0 > 0$ and suppose that $T(t_0)x = 0$ for some $x \in X$. Then

$$R(\lambda, A)x = \int_0^{t_0} e^{-\lambda t} T(t)x\,dt =: f(\lambda)$$

for $\operatorname{Re}\lambda > \omega$. Obviously, f has a holomorphic continuation to an entire function which is bounded on every right halfplane.

Claim: $f(\lambda) = R(\lambda, A)$ for all $\operatorname{Re}\lambda < -\omega$.

Proof of Claim. Consider the function $F : \mathbb{C} \to X \oplus X/A$ defined by $F(\lambda) := (f(\lambda), \lambda f(\lambda) - x) + A$. Then F is entire and $F(\lambda) = 0$ for $\operatorname{Re}\lambda > \omega$, since $(f(\lambda), x) \in \lambda - A$ for these λ. By the uniqueness theorem for holomorphic functions, $F(\lambda) = 0$ for *all* $\lambda \in \mathbb{C}$. Hence $(f(\lambda), x) \in \lambda - A$ even for $\operatorname{Re}\lambda < -\omega$. However, these λ are contained in the resolvent set of A, whence the claim is proved.

From the claim we follow that f is in fact bounded also on some left halfplane. Hence it is constant. However, $f(\lambda) \to 0$ if $\operatorname{Re}\lambda \to \infty$. Thus, $f(\lambda) = 0$ for all $\lambda \in \mathbb{C}$. This implies $R(\lambda, A)x = 0$ for a lot of λ, whence $x = 0$, since A is assumed to be single-valued.
So we have shown that $T(t_0)$ is injective. We obtain that $T(t_0) : X \longrightarrow Y$ is an isomorphism. But since $T(t_0) : Y \longrightarrow Y$ is also an isomorphism, we must have $X = Y$. □

One usually writes T again instead of $\tilde{T}$ and calls it a **$\mathbf{C_0}$-group**. In general one cannot omit the assumption "A is single-valued" from *(iii)*. Indeed, let S be a C_0-group on a Banach space Y and let $X := Y \oplus \mathbb{C}$ with $T(t)(y, \lambda) := (S(t), 0)$ for all $t \in \mathbb{R}$. If B generates S, then

$$A = \{((y, 0), (By, \lambda)) \mid y \in \mathcal{D}(B),\ \lambda \in \mathbb{C}\}$$

generates $(T(t))_{t \geq 0}$ and $-A$ generates $(T(-t))_{t \geq 0}$.

Given a C_0-group $(T(t))_{t \in \mathbb{R}}$ we call

$$\theta(T) := \inf\{\theta > 0 \mid \exists\, M \geq 1,\ \|T(t)\| \leq Me^{\theta|t|}\ (t \in \mathbb{R})\}$$

the **group type** of T. Let us call the semigroups $T_\oplus$ and $T_\ominus$, defined by

$$T_\oplus(t) := T(t) \quad \text{and} \quad T_\ominus(t) := T(-t) \quad (t \geq 0),$$

the **forward semigroup** and the **backward semigroup** corresponding to the group T. Then we obviously have $\theta(T) = \max\{\omega_0(T_\oplus),\, \omega_0(T_\ominus)\}$.

References

The basic results on m.v. operators, covered by Section A.1 and Section A.2, are contained, e.g., in [FY99, Chapter I] or the monograph [Cro98]. Proposition A.8 and its corollary on pseudo-resolvents is not included in these books and may be new. The same is true for Propositions A.11 and parts of Section A.3. The results on convergence in Section A.4 are generalizations of well-known facts for single-valued operators which can be found, e.g., in [Kat95, Theorem IV.2.23 and Chapter VIII,§1]. Polynomials of operators (Section A.5) are studied in [DS58, Chapter VII.9] including the spectral mapping theorem (Proposition A.20). By [Cro98, Theorem VI.5.4], the spectral mapping theorem for polynomials remains valid for multivalued operators. We provided the results for rational functions and on injective operators (Section A.6) without direct reference, but it is quite likely that these facts have been published somewhere. The results of [Cro98, Chapter 6] let expect that there is a generalization of Sections A.5 and A.6 to multivalued operators. Section A.7 is an adaptation from the standard monographs in semigroup theory, like [Paz83], [EN00], and [ABHN01]. However, if multivalued operators are involved (like in Proposition A.28 and Proposition A.34), we do not know of a direct reference.

A treatment of adjoints of multivalued operators can be found in [Cro98, Chapter III]. See also Section B.2.

Appendix B
Operator Theory on Hilbert Spaces

This chapter provides some facts on linear operators on Hilbert spaces, including adjoints (of multivalued operators), numerical range, symmetric and accretive operators, and the Lax-Milgram Theorem. The main difference to standard texts lies in the fact that we have avoided to employ the spectral theorem while dealing with spectral theory of selfadjoint operators (Proposition B.11 – Corollary B.14).

We take for granted the basic Hilbert space theory as can be found in [Con90, Chapter I], [RS72, Chapter II], or [Rud87, Chapter 4]. During the whole chapter the letter H denotes some *complex* Hilbert space. The scalar product on H is denoted by $(\cdot \,|\, \cdot)$.

B.1 Sesquilinear Forms

Let V be a vector space over the field of complex numbers. We denote by

$$\mathrm{Ses}(V) := \{a \mid a : V \times V \longrightarrow \mathbb{C} \text{ sesquilinear}\}$$

the space of **sesquilinear forms** on V. Given $a \in \mathrm{Ses}(V)$, the **adjoint form** $\overline{a}$ is defined by

$$\overline{a}(u, v) := \overline{a(v, u)} \qquad (u, v \in V).$$

The **real part** and the **imaginary part** of the form $a \in \mathrm{Ses}(V)$ are defined by

$$\mathrm{Re}\, a := \frac{1}{2}(a + \overline{a}) \quad \text{and} \quad \mathrm{Im}\, a := \frac{1}{2i}(a - \overline{a}),$$

respectively. Hence $a = (\mathrm{Re}\, a) + i(\mathrm{Im}\, a)$ for every $a \in \mathrm{Ses}(V)$. Using the shorthand notation

$$a(u) := a(u, u)$$

for $a \in \mathrm{Ses}(V)$ and $u \in V$, we obtain

$$(\mathrm{Re}\, a)(u) = \mathrm{Re}(a(u)) \quad (u \in V).$$

Given $a \in \mathrm{Ses}(V)$ we have

$$\frac{1}{2}(a(u + v) + a(u - v)) = a(u) + a(v) \tag{B.1}$$

$$a(u, v) = \frac{1}{4}(a(u + v) - a(u - v) + i(a(u + iv) - a(u - iv))) \tag{B.2}$$

which equations are called **Parallelogram Law**
and **Polarization Identity**, respectively. (The importance of the Polarization Identity lies in the consequence that each sesquilinear form a is already determined by the associated quadratic form $a(\cdot)$.)
A form a is called **real** if $a(u) \in \mathbb{R}$ for all $u \in V$. It is called **symmetric** if $a = \overline{a}$. Note that $\operatorname{Re} a$ and $\operatorname{Im} a$ are always symmetric forms. A sesquilinear form $a \in \operatorname{Ses}(V)$ is called **positive** (or **monotone**) if $\operatorname{Re} a(u) \geq 0$ for all $u \in V$.

Lemma B.1. *A form $a \in \operatorname{Ses}(V)$ is real if and only if it is symmetric.*

Proof. Obviously, a symmetric form is real. Let a be real. We have

$$a(u,v) + a(v,u) = \frac{1}{2}(a(u+v) - a(u-v)) \in \mathbb{R}$$

for all $u, v \in V$. Replacing u and v by iu and iv, respectively, we obtain also

$$ia(u,v) - ia(v,u) \in \mathbb{R}$$

for all $u, v \in V$. Combining these informations we arrive at $\operatorname{Im}(a(u,v)) = -\operatorname{Im}(a(v,u))$ and $\operatorname{Re}(a(u,v)) = \operatorname{Re}(a(v,u))$. But this is nothing else than $a(u,v) = \overline{a(v,u)}$. □

Proposition B.2. (Generalized Cauchy-Schwarz Inequality)
Let $a, b \in Ses(V)$ be symmetric, and assume there is $c \geq 0$ such that

$$|a(u)| \leq c\, b(u) \qquad (u \in V).$$

(Note that this implies b to be positive.) Then

$$|a(u,v)| \leq c\sqrt{b(u)}\,\sqrt{b(v)}$$

for all $u, v \in V$.

Proof. The simple proof can be found in [Sch71, Chapter XII, Lemma 3.1]. □

A positive form a is sometimes called a **semi-scalar product**. It is called a **scalar product** if it is even **positive definite**, i.e., if it is positive and if $a(u) = 0$ implies $u = 0$ for each $u \in V$. If $a \in \operatorname{Ses}(V)$ is a semi-scalar product on V, then by

$$\|x\|_a := \sqrt{a(u)}$$

a seminorm on V is defined. The form a is continuous with respect to this seminorm. (This follows from the Cauchy-Schwarz Inequality.)

Let $0 \leq \omega < \frac{\pi}{2}$. A form $a \in \operatorname{Ses}(V)$ is called **sectorial of angle** ω if

$$a(u) \neq 0 \;\;\Rightarrow\;\; |\arg a(u)| \leq \omega \qquad (u \in V).$$

The form a is called **sectorial** if it is sectorial of some angle $\omega < \frac{\pi}{2}$. Obviously, if a is sectorial, then $\operatorname{Re} a$ is positive.

Proposition B.3. *Let $a \in \operatorname{Ses}(V)$ such that $\operatorname{Re} a \geq 0$. The following assertions are equivalent.*

(i) The form a is sectorial.

(ii) *The form a is continuous with respect to the seminorm induced by the semi-scalar product* $\operatorname{Re} a$.

More precisely: If $|a(u,v)| \le M \sqrt{\operatorname{Re} a(u)}\sqrt{\operatorname{Re} a(v)}$ *for all* $u, v \in V$*, then a is sectorial of angle* $\arccos M^{-1}$. *Conversely, if a is sectorial of angle ω, then*

$$|a(u,v)| \le (1 + \tan\omega)\sqrt{\operatorname{Re} a(u)}\sqrt{\operatorname{Re} a(v)}$$

for all $u, v \in V$.

Proof. Suppose $|a(u,v)| \le M\sqrt{\operatorname{Re} a(u)}\sqrt{\operatorname{Re} a(v)}$ for all $u, v \in V$. Then, letting $u = v$, we have $|a(u)| \le M \operatorname{Re} a(u)$, whence $M \ge 1$ and $a(u) \ne 0 \;\Rightarrow\; |\arg a(u)| \le \arccos M^{-1}$. for all $u \in V$. Conversely, if a is sectorial of angle $\omega < \frac{\pi}{2}$, then $|\operatorname{Im} a(u)| \le (\tan\omega)\operatorname{Re} a(u)$ for all $u \in V$. The generalized Cauchy-Schwarz Inequality, applied to $\operatorname{Re} a$ and $\operatorname{Im} a$, yields

$$|a(u,v)| \le |(\operatorname{Re} a)(u,v)| + |(\operatorname{Im} a)(u,v)| \le (1+\tan\omega)\sqrt{\operatorname{Re} a(u)}\sqrt{\operatorname{Re} a(v)}$$

for all $u, v \in V$. □

B.2 Adjoint Operators

Let $A \subset H \oplus H$ be a multivalued linear operator on H. The **adjoint** of A, usually denoted by A^*, is defined by

$$(x,y) \in A^* \quad :\Longleftrightarrow \quad (v \,|\, x) = (u \,|\, y) \;\text{ for all } (u,v) \in A. \tag{B.3}$$

If for the moment we define $J := ((u,v) \longmapsto (-v,u)) : H \oplus H \longrightarrow H \oplus H$, then we can write

$$A^* = [JA]^\perp$$

where the orthogonal complement is taken in the Hilbert space $H \oplus H$. Hence A^* is always a closed operator. We enumerate the basic properties.

Proposition B.4. *Let A, B be m.v. linear operators on H. Then the following statements hold.*

a) $A^* = (\overline{A})^*$

b) $(A^{-1})^* = (A^*)^{-1}$.

c) $(\lambda A)^* = \overline{\lambda} A^*$, *for* $0 \ne \lambda \in \mathbb{C}$.

d) $A^{**} = \overline{A}$.

e) $\mathcal{N}(A^*) = \mathcal{R}(A)^\perp$ *and* $\mathcal{N}(\overline{A}) = \mathcal{R}(A^*)^\perp$.

f) $A^*0 = \mathcal{D}(A)^\perp$ *and* $\overline{A}0 = \mathcal{D}(A^*)^\perp$.

g) $\mathcal{D}(A^*) \subset (A0)^\perp$ *and* $\mathcal{R}(A^*) \subset \mathcal{N}(A)^\perp$.

h) *If* $A \in \mathcal{L}(H)$, *then* $A^* \in \mathcal{L}(H)$ *and* $\|A^*\| = \|A\| = \sqrt{\|A^*A\|}$.

i) $A \subset B \quad \Rightarrow \quad B^* \subset A^*$.

j) $A^* + B^* \subset (A+B)^*$ *with equality if* $A \in \mathcal{L}(H)$.

k) $A^*B^* \subset (BA)^*$ *with equality if* $B \in \mathcal{L}(H)$. *If* $A \in \mathcal{L}(H)$ *and B is closed, one has* $\overline{A^*B^*} = (BA)^*$.

Proof. Ad *a)*. Since J is a topological isomorphism, $\overline{A}^* = (J\overline{A})^\perp = \overline{JA}^\perp = (JA)^\perp$.
Ad *b)*. This follows from $(JA)^{-1} = J(A^{-1})$ and $(A^{-1})^\perp = (A^\perp)^{-1}$.
Ad *c)*. We have $(x,y) \in (\lambda A)^* \Leftrightarrow (-\lambda v\,|\,x)+(u\,|\,y) = 0 \;\forall (u,v) \in A \Leftrightarrow \big(-v\,\big|\,\overline{\lambda}x\big)+(u\,|\,y) = 0 \;\forall(u,v) \in A \Leftrightarrow (\overline{\lambda}x, y) \in A^* \Leftrightarrow (\overline{\lambda}x, \overline{\lambda}y) \in \overline{\lambda}A^* \Leftrightarrow (x,y) \in A^*$.
Ad *d)*. $A^{**} = (J(JA)^\perp)^\perp = (JJA)^{\perp\perp} = A^{\perp\perp} = \overline{A}$.
Ad *e)*. We have $x \in \mathcal{N}(A^*) \Leftrightarrow (x,0) \in A^* \Leftrightarrow (-v\,|\,x) = 0 \;\forall v \in \mathcal{R}(A) \Leftrightarrow x \in \mathcal{R}(A)^\perp$. The second satement follows from the first together with *d)*.
Ad *f)*. We have $y \in A^*0 \Leftrightarrow (0,y) \in A^* \Leftrightarrow (u\,|\,y) = 0 \;\forall u \in \mathcal{D}(A) \Leftrightarrow y \in \mathcal{D}(A)^\perp$. The second statement follows from the first together with *d)*.
Ad *g)*. If $(x,y) \in A^*$, $v \in A0$, and $u \in \mathcal{N}(A)$, then $(-v\,|\,x) = 0$ and $(u\,|\,y) = 0$ by (B.3).
Ad *h)*. Let $A \in \mathcal{L}(H)$. Then A^* is closed and single-valued by *f)*. We show that $\mathcal{D}(A^*) = H$. In fact, let $x \in H$. Then $(u \longmapsto (Au\,|\,x))$ is a continuous linear functional on H. By the Riesz-Fréchet Theorem [Rud87, Theorem 4.12] there is $y \in H$ such that $(Au\,|\,x) = (u\,|\,y)$ for all $u \in H$. But this means exactly that $(x,y) \in A^*$.
The equation $\|A\| = \|A^*\|$ is easily proved by using the identity $\|T\| = \sup\{|(Tu\,|\,v)| \mid \|u\| = \|v\| = 1\}$, which holds for every bounded operator on H. This implies $\|A^*A\| \leq \|A^*\|\,\|A\| = \|A\|^2$. But if $x \in H$ is arbitrary, we have $\|Ax\|^2 = (Ax\,|\,Ax) = (A^*Ax\,|\,x) \leq \|A^*A\|\,\|x\|^2$ by the Cauchy-Schwarz inequality. Hence $\|A\|^2 \leq \|A^*A\|$.
Ad *i)*. We have $A \subset B \Rightarrow JA \subset JB \Rightarrow (JB)^\perp \subset (JA)^\perp$.
Ad *j)*. Let $(x,y) \in A^*$, $(x,z) \in B^*$. The generic element of $J(A+B)$ is $(-v-w,u)$, where $(u,v) \in A$ and $(u,w) \in B$. So $(x,y) \perp (-v,u)$ and $(x,z) \perp (-w,u)$, hence $(x,y+z) \perp (-v-w,u)$.
If $A \in \mathcal{L}(X)$, we write $B = (A+B) - A$ and note that $A^* \in \mathcal{L}(H)$ by *h)*.
Ad *k)*. Let $(x,y) \in A^*B^*$. Then there is z such that $(x,z) \in B^*$ and $(z,y) \in A^*$. If $(u,v) \in BA$, one has $(u,w) \in A$ and $(w,v) \in B$ for some w. Hence $(v\,|\,x) = (w\,|\,z) = (u\,|\,y)$. Since $(u,v) \in BA$ was arbitrary, we conclude that $(x,y) \in (BA)^*$.
Assume now $B \in \mathcal{L}(H)$ and $(x,y) \in (BA)^*$. Define $z := B^*x$. It suffices to show that $(z,y) \in A^*$. Take $(u,w) \in A$ and define $v := Bw$. Hence, $(u,v) \in BA$. Therefore

$$(u\,|\,y) = (v\,|\,x) = (Bw\,|\,x) = (w\,|\,B^*x) = (w\,|\,z),$$

whence $(z,y) \in A^*$ by (B.3). Finally, assume that B is closed and $A \in \mathcal{L}(H)$. The assertions already proved yield $(A^*B^*)^* = B^{**}A^{**} = (\overline{B})(\overline{A}) = BA$. Hence $\overline{A^*B^*} = (BA)^*$. ☐

Corollary B.5. *Let A be a m.v. linear operator on H. Then*

$$(\lambda - A)^* = (\overline{\lambda} - A^*) \quad \textit{and} \quad R(\lambda, A)^* = R(\overline{\lambda}, A^*)$$

for every $\lambda \in \mathbb{C}$. In particular, we have $\varrho(A^*) = \{\overline{\lambda} \mid \lambda \in \varrho(A)\}$.

Let A be a *single-valued* operator on H. From part *f)* of Proposition B.4 we see that A^* is densely defined if and only if A is *closable*, i.e., $\overline{A}$ is still single-valued; and A^* is single-valued if and only if A is densely defined.

Proposition B.6. *Let A be a densely defined (single-valued) operator on H with $\varrho(A) \neq \emptyset$. For a polynomial $p \in \mathbb{C}[z]$ define $p^*(z) := \overline{p(\overline{z})}$. (Hence, p^* is obtained from p by conjugating all coefficients.) Then we have*

$$[p^*(A)]^* = p(A^*).$$

The same statement holds if p is a rational function with poles inside $\overline{\varrho(A)} = \varrho(A^*)$.

Proof. Let $r = p/q$ be a rational function with poles inside the set $\varrho(A^*)$. Assume first that $\deg p \leq \deg q$. Hence, $r(A^*)$ and $r^*(A)$ are bounded operators. The function r can be written as a product $r = \prod_j r_j$ where each r_j is either of the form $\alpha(\lambda - z)^{-1}$ or of the form $\alpha(\lambda - z)^{-1} + \beta$. Now the claimed formula $[r^*(A)]^* = r(A^*)$ follows from part *k)* of Proposition B.4

and Corollary B.5. From this and part *b)* of Proposition B.4 we can infer that $p^*(A)^* = p(A^*)$ holds for all polynomials p having their roots inside $\varrho(A^*)$. Now suppose $\deg p > \deg q$. Since $\varrho(A) \neq \emptyset$ we can find a polynomial q_1 having its roots inside $\varrho(A^*)$ with $\deg q + \deg q_1 = \deg p$. Define $\tilde{r} := p/(q_1 q)$. Then

$$r(A^*) = q_1(A^*)\tilde{r}(A^*) = [q_1^*(A)]^*[\tilde{r}^*(A)]^* \stackrel{(1)}{=} [\tilde{r}^*(A)q_1^*(A)]^* \stackrel{(2)}{=} [q_1^*(A)\tilde{r}^*(A)]^* = [r^*(A)]^*,$$

where we have used *k)* from Proposition B.4 in (1) and Corollary A.21 in (2). □

B.3 The Numerical Range

From now on, all considered operators are single-valued. We therefore will follow the general terminological agreement made on page 137.

Given an operator A on H we call

$$W(A) := \{(Au \,|\, u) \ \mid u \in \mathcal{D}(A),\ \|u\| = 1\} \subset \mathbb{C}$$

the **numerical range** of A. By [Sch71, Chapter XII, Theorem 5.2] the numerical range $W(A)$ is always a convex subset of the plane.

Proposition B.7. *Let A be a closed operator. Then $P\sigma(A) \subset W(A)$ and $A\sigma(A) \subset \overline{W(A)}$. Furthermore, one has*

$$\|R(\lambda, A)\| \leq \frac{1}{\operatorname{dist}(\lambda, \overline{W(A)})}$$

for $\lambda \in \varrho(A) \setminus \overline{W(A)}$. If $A \in \mathcal{L}(H)$, we have $\sigma(A) \subset \overline{W(A)}$.

Proof. If $\lambda \in P\sigma(A)$, there is $u \in \mathcal{D}(A)$, $\|u\| = 1$ such that $Au = \lambda u$. This gives $(Au \,|\, u) = (\lambda u \,|\, u) = \lambda \|u\|^2 = \lambda$. Hence $\lambda \in W(A)$.
Assume $\lambda \notin \overline{W(A)}$ and define $\delta := \operatorname{dist}(\lambda, \overline{W(A)})$. By definition, $|(Ax \,|\, x) - \lambda| \geq \delta$ for all $x \in \mathcal{D}(A)$ with $\|x\| = 1$. Hence,

$$|((A - \lambda)\, x \,|\, x)| = \left|(Ax \,|\, x) - \lambda \|x\|^2\right| \geq \delta\, \|x\|^2$$

for *all* $x \in \mathcal{D}(A)$. But $|((A - \lambda)x \,|\, x)| \leq \|(A - \lambda)x\|\, \|x\|$, whence

$$\|(\lambda - A)\, x\| \geq \delta\, \|x\|$$

for all $x \in \mathcal{D}(A)$. Since A is closed, this implies that $(\lambda - A)$ is injective and has closed range, i.e., $\lambda \notin A\sigma(A)$. Moreover, it shows that $\|R(\lambda, A)\| \leq \delta^{-1}$ if $\lambda \in \varrho(A)$.
If $A \in \mathcal{L}(X)$, then $W(A^*) = \{\overline{\lambda} \mid \lambda \in W(A)\}$. Now, if $\lambda \in R\sigma(A)$, then clearly $\overline{\lambda} \in P\sigma(A^*)$, whence $\lambda \in W(A)$. □

Corollary B.8. *Let A be a single-valued closed operator on H. Let $U \subset \mathbb{C} \setminus \overline{W(A)}$ be open and connected. If $U \cap \varrho(A) \neq 0$, then $U \subset \varrho(A)$.*

Proof. The statement follows from Proposition B.7 and the fact that $\|R(\lambda, A)\|$ blows up if λ approaches a spectral value (cf. Proposition A.7). □

B.4 Symmetric Operators

We begin with a lemma.

Lemma B.9. *For an operator A on H the following assertions are equivalent:*

(i) $W(A) \subset \mathbb{R}$.

(ii) $(Au \mid v) = (u \mid Av)$ *for all* $u, v \in \mathcal{D}(A)$.

(iii) $A \subset A^*$.

If this is the case, the operator A is called **symmetric**.

Proof. Define the form a on $V := \mathcal{D}(A)$ by $a(u,v) := (Au \mid v)$. The proof is now an easy consequence of Lemma B.1 and the definition of the adjoint (see (B.3)). □

An operator A on H is called **selfadjoint** if $A^* = A$. If A is symmetric/selfadjoint and injective, then A^{-1} is symmetric/selfadjoint, by part *b)* of Proposition B.4.

Proposition B.10. *Let A be an operator on H. Then A is self-adjoint if and only if A is symmetric, closed, and densely defined, and $\mathcal{R}(A \pm i)$ is dense. In this case we have $\sigma(A) \subset \mathbb{R}$ and*

$$\|R(\lambda, A)\| \le \frac{1}{|\operatorname{Im} \lambda|}$$

for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Proof. Assume that $A = A^*$. Then A is closed since A^* is. Morover, A^* is single-valued, since A is, and this implies that A is densely defined by part *f)* of Proposition B.4. Since $A = A^*$ we have $W(A^*) = W(A) \subset \mathbb{R}$. Hence $\mathcal{N}(A^* \pm i) = 0$ by Proposition B.7. This gives $\overline{\mathcal{R}(A \pm i)} = H$ by part *e)* of Proposition B.4.
Now suppose that A is symmetric, closed, and densely defined with $\mathcal{R}(A \pm i)$ being dense. By Proposition B.7 and its corollary we conclude that $\sigma(A) \subset \mathbb{R}$ and $\mathcal{N}(A^* \pm i) = 0$. But $A - i : \mathcal{D}(A) \longrightarrow H$ is bijective and $A - i \subset A^* - i$, whence $A - i = A^* - i$. This proves $A = A^*$. The norm inequality for the resolvent follows from Proposition B.7. □

Let A be an operator on H and $\alpha \in \mathbb{R}$. We write $\alpha \le A$ if A is selfadjoint and $W(A) \subset [\alpha, \infty)$, and we write $A \le \alpha$ if $-\alpha \le -A$. The operator A is called **positive** if $0 \le A$. We obtain the following characterization.

Proposition B.11. *A closed and densely defined operator A on H is positive if and only if $W(A) \subset [0, \infty)$ and $A + \lambda$ is surjective for some/each $\lambda > 0$. In this case, $\{\operatorname{Re} \lambda < 0\} \subset \varrho(A)$ and*

$$\left\|(\lambda + A)^{-1}\right\| \le \frac{1}{\operatorname{Re} \lambda}$$

for all $\operatorname{Re} \lambda > 0$. *Moreover,* $0 \le t(t+A)^{-1} \le 1$ *and* $0 \le A(t+A)^{-1} \le 1$ *for all* $t > 0$.

Proof. Assume that A is single-valued, closed and densely defined with $W(A) \subset [0, \infty)$. Since $\mathbb{C} \setminus [0, \infty)$ is open and connected, the stated equivalence is a consequence of Proposition B.7 and its corollary. Moreover it follows that in this case $\{\operatorname{Re} \lambda < 0\} \subset \varrho(A)$ and the stated norm inequality holds. Since $t + A$ is selfadjoint, we know that $(t+A)^{-1}$ also is. Furthermore $0 \le A(t+A)^{-1} \le 1 \quad \Leftrightarrow \quad 0 \le t(t+A)^{-1} \le 1 \quad \Leftrightarrow \quad (tu \mid (t+A)u) \le ((t+A)u \mid (t+A)u) \ \ \forall u \in \mathcal{D}(A)$. However, this is true if and only if $\|Au\|^2 + t\,(Au \mid u) \ge 0$ which is always the case. □

Lemma B.12. *Let A be a closed and densely defined operator on H.*

a) If $-\alpha \le A \le \alpha$ for some $\alpha \ge 0$, then $A \in \mathcal{L}(H)$ and $\|A\| \le \alpha$.

b) If $A \in \mathcal{L}(H)$ is selfadjoint, then $\|A\| = \sup\{|\lambda| \mid \lambda \in W(A)\}$. In particular, $W(A)$ is a bounded subset of $\mathbb{R}$.

c) If $0 \le A \le 1$ then $0 \le A^2 \le A$.

Proof. Ad *a)*. Define $a(u,v) := (\,Au\,|\,v\,)$ and $b(u,v) := (\,u\,|\,v\,)$ on $V := \mathcal{D}(A)$. The hypothesis implies that $|a(u)| \le \alpha b(u)$ for all $u \in H$. Moreover, a is symmetric. An application of the generalized Cauchy-Schwarz inequality (Proposition B.2) yields $|(\,Au\,|\,v\,)| \le \alpha \|u\| \|v\|$ for all $u, v \in \mathcal{D}(A)$. Since A is densely defined, this inequality holds for all $v \in H$, whence we have $\|Au\| \le \alpha \|u\|$ for all $u \in \mathcal{D}(A)$. Since A is closed, this implies that $\mathcal{D}(A)$ is closed. Hence $A \in \mathcal{L}(H)$ and $\|A\| \le \alpha$.
Ad *b)*. Let $A \in \mathcal{L}(H)$ be selfadjoint. Then $|(\,Au\,|\,u\,)| \le \|A\| \|u\|^2$ for all $u \in H$. Hence $\sup |W(A)| \le \|A\|$. If $\sup |W(A)| \le \alpha$, then $|(\,Au\,|\,u\,)| \le \alpha \|u\|^2$ for all u. Since $W(A) \subset \mathbb{R}$, this is equivalent to $-\alpha \le A \le \alpha$. Now *a)* implies that $\|A\| \le \alpha$.
Ad *c)*. We have $A - A^2 = A^2(1-A) + A(1-A)^2$, and both summands are positive since A and $1 - A$ are. □

The following result is usually proved with the help of the spectral theorem.

Proposition B.13. *Assume $A \ge \alpha$ for some $\alpha \in \mathbb{R}$ and define $\alpha_0 := \inf W(A)$. Then $\alpha_0 \in \sigma(A)$. If $\alpha_0 \in W(A)$, then even $\alpha_0 \in P\sigma(A)$.*

Proof. Without restriction we can assume $\alpha_0 = 0$. Hence $A \ge 0$. Let $Q := A(1+A)^{-1}$. Then $0 \le Q \le 1$. Moreover, $Q \le A$ in the obvious sense, since for $x \in \mathcal{D}(A)$ we have

$$\begin{aligned}\big(\,A(A+1)^{-1}x \,\big|\, x\,\big) - (\,Ax\,|\,x\,) &= \big(\,A((A+1)^{-1} - I)x \,\big|\, x\,\big)\\ &= -\big(\,A(A+1)^{-1}Ax \,\big|\, x\,\big) = -\big(\,(A+1)^{-1}Ax \,\big|\, Ax\,\big) \le 0\end{aligned}$$

Applying *c)* of Lemma B.12 we obtain

$$0 \le \|Qx\|^2 \le (\,Qx\,|\,x\,) \le (\,Ax\,|\,x\,)$$

for all $x \in \mathcal{D}(A)$. Now if there is $x \in \mathcal{D}(A)$ such that $\|x\| = 1$ and $(\,Ax\,|\,x\,) = 0$ it follows that $0 = Qx = A(A+1)^{-1}x$. But this implies $0 \ne (A+1)^{-1}x \in \mathcal{N}(A)$, whence $0 \in P\sigma(A)$. Similarly, if $x_n \in \mathcal{D}(A)$ such that $\|x_n\| = 1$ and $(\,Ax_n\,|\,x_n\,) \to 0$, then $Qx_n \to 0$. Hence $(x_n)_n$ is a generalized eigenvector for Q and $0 \in \sigma(Q)$. But this immediately implies $0 \in \sigma(A)$. □

Corollary B.14. *Let A be a bounded and selfadjoint operator on A. Then*

$$\sup W(A), \inf W(A) \in \sigma(A).$$

In particular, $\|A\| = r(A)$, where $r(A)$ denotes the spectral radius of the operator A.

Proof. Let $\alpha := \inf W(A)$ and $\beta := \sup W(A)$. Then $0 = \inf W(A - \alpha) = \inf W(\beta - A)$, whence $0 \in \sigma(A - \alpha) \cap \sigma(\beta - A)$ by Proposition B.13. From *b)* of Lemma B.12 we know that $r(A) \le \|A\| = \max |\alpha|, |\beta| \le r(A)$. □

Note that positive operators are special cases of m-accretive operators (see Section B.6 below).

B.5 Equivalent Scalar Products and the Lax-Milgram Theorem

Let H be a Hilbert space. We denote by H^* the space of continuous *conjugate-linear* functionals on H, endowed with the norm

$$\|\varphi\|_{H^*} := \sup\{|\varphi(x)| \mid x \in H,\ \|x\| = 1\}$$

for $\varphi \in H^*$. One sometimes writes $\langle \varphi, x\rangle$ instead of $\varphi(x)$, where $x \in H, \varphi \in H^*$.
Let a be a *continuous* sesquilinear form on H. Then we have an induced linear mapping

$$L_a := (u \longmapsto a(u,\cdot)) : H \longrightarrow H^* \tag{B.4}$$

which is continuous. The Riesz-Fréchet Theorem [Rud87, Theorem 4.12] implies that for each $u \in H$ there is a unique $Qu \in H$ such that

$$a(u,\cdot) = L_a(u) = (\,Qu \,|\, \cdot\,)$$

Obviously, Q is a linear and bounded operator ($\|Q\| = \|L_a\|$) and a is uniquely determined by Q. On the other hand, given $Q \in \mathcal{L}(H)$, the form a_Q defined by $a_Q(u,v) := (\,Qu \,|\, v\,)$ is sesquilinear and continuous. Hence the mapping

$$(Q \longmapsto a_Q) : \mathcal{L}(H) \longrightarrow \{\text{continuous, sesquilinear forms on } H\}$$

is an isomorphism. We have $\overline{a_Q} = a_{Q^*}$, whence the form a is symmetric if and only if Q is selfadjoint, and the form is positive if and only if $Q \geq 0$.

Proposition B.15. *Let $Q \in \mathcal{L}(H)$. Then the form a_Q is a scalar product on H if and only if Q is positive and injective. The norm induced by this scalar product is equivalent to the original one if and only if Q is invertible.*

Proof. The form a_Q is positive semi-definite if and only if $Q \geq 0$. This is clear from the definitions. If Q is not injective, then obviously a_Q is not definite. Let Q be injective and positive. By Proposition B.13 we conclude that $0 \notin W(Q)$. But this means that a_Q is definite. Since a_Q is continuous, equivalence of the induced norm is the same as the existence of a $\delta > 0$ such that $\delta \leq Q$. But this is equivalent to Q being invertible by Proposition B.13. □

A scalar product on H which induces a norm equivalent to the original one is simply called an **equivalent scalar product**.

Lemma B.16. *Let A be a m.v. operator on H and let $(\,\cdot\,|\,\cdot\,)_\circ := a_Q$ be an equivalent scalar product on H. Denote by A° the adjoint of A with respect to the new scalar product. Then*

$$(x,y) \in A^\circ \quad \Longleftrightarrow \quad (Qx, Qy) \in A^*$$

for all $x, y \in H$. In particular, $A^\circ = Q^{-1}A^*Q$ if A^* is single-valued.

Proof. Fix $x, y \in H$. Since Q is positive, $(x,y) \in A^\circ \Leftrightarrow (\,v\,|\,x\,)_\circ = (\,u\,|\,y\,)_\circ\, \forall (u,v) \in A \Leftrightarrow (\,v\,|\,Qx\,) = (\,u\,|\,Qy\,)\, \forall (u,v) \in A \Leftrightarrow (Qx, Qy) \in A^*$. The rest is straightforward. □

The next theorem gives a sufficient condition for the mapping L_a to be an isomorphism.

Theorem B.17. (Lax-Milgram)
Let a be a continuous sesquilinear form on H. Assume that there is a $\delta > 0$ such that

$$\operatorname{Re} a(u) \geq \delta \|u\|^2 \tag{B.5}$$

for all $u \in H$. Then the mapping $L_a : H \longrightarrow H^*$ defined by (B.4) is an isomorphism.

The inequality (B.5) is called a **coercivity condition** and a form which satisfies (B.5) for some $\delta > 0$ is called **coercive**.

Proof. Take $u \in H$ with $\|u\| = 1$. Then

$$\|L_a(u)\| \geq |L_a(u)(u)| = |a(u)| \geq \operatorname{Re} a(u) \geq \delta.$$

Hence $\delta \|u\| \leq \|L_a(u)\|$ for all u, whence L_a is injective and has a closed range. So it remains to show that $E := \mathcal{R}(L_a)$ is dense in H^*. The Riesz-Fréchet Theorem implies that

$$\Phi := (x \longmapsto (\, x \,|\, \cdot \,)) : H \longrightarrow H^*$$

is an isomorphism. Hence E is dense in H^* if and only if $\Phi^{-1}(E)$ is dense in H if and only if $\Phi^{-1}(E)^\perp = 0$. Now

$$y \in \Phi^{-1}(E)^\perp \;\Leftrightarrow\; (\, x \,|\, y \,) = 0 \;\forall x \in \Phi^{-1}(E) \;\Leftrightarrow\; \varphi(y) = 0 \;\forall \varphi \in E \;\Leftrightarrow\; a(x, y) = 0 \;\forall x \in H$$

for each $y \in H$. In particular we have $y \in \Phi^{-1}(E)^\perp \Rightarrow a(y, y) = 0$, but this implies $y = 0$ by coercivity. Thus, the proposition is proved. $\square$

Remark B.18. The coercivity condition in the Lax-Milgram Theorem can be weakened to $|a(u)| \geq \delta \|u\|^2$ for all $u \in H$. This is easily seen from the proof.

B.6 Accretive Operators

Here are the defining properties of accretive operators.

Lemma B.19. *Let A be an operator on the Hilbert space H and let $\mu > 0$. The following assertions are equivalent.*

(i) $\operatorname{Re}(\, Au \,|\, u \,) \geq 0$ *for all $u \in \mathcal{D}(A)$, i.e., $W(A) \subset \{\operatorname{Re} z \geq 0\}$.*

(ii) $\|(A + \mu)u\| \geq \|(A - \mu)u\|$ *for all $u \in \mathcal{D}(A)$.*

(iii) $\|(A + \lambda)u\| \geq (\operatorname{Re} \lambda) \|u\|$ *for all $u \in \mathcal{D}(A)$, $\operatorname{Re} \lambda \geq 0$.*

(iv) $\|(A + \lambda)u\| \geq \lambda \|u\|$ *for all $u \in \mathcal{D}(A)$ and all $\lambda \geq 0$.*

An operator A which satisfies the equivalent conditions $(i) - (iv)$ is called **accretive**. An operator A is called **dissipative** if $-A$ is accretive.

Proof. We have $\|(A + \mu)u\|^2 - \|(A - \mu)u\|^2 = 4 \operatorname{Re}(\, Au \,|\, u \,)$ for all $u \in \mathcal{D}(A)$. This gives $(i) \Leftrightarrow (ii)$. For $\lambda > 0$ we have $\|(A + \lambda)u\|^2 - \lambda^2 \|u\|^2 = \|Au\|^2 + 2\lambda (\, Au \,|\, u \,)$ for all $u \in \mathcal{D}(A)$. This shows $(i) \Rightarrow (iv)$; dividing by λ and letting $\lambda \to \infty$ gives the reverse implication. The implication $(iii) \Rightarrow (iv)$ is obvious. Assume (i) and let $\operatorname{Re} \lambda \geq 0$. Let $\alpha := \operatorname{Im} \lambda$. Then (i) holds with A replaced by $A + i\alpha$. Since we have already proved $(i) \Rightarrow (iv)$, we know that $\|((A + i\alpha) + \operatorname{Re} \lambda)u\| \geq (\operatorname{Re} \lambda) \|u\|$ for all $u \in \mathcal{D}(A)$. But this is (iii). $\square$

Note that an operator A is symmetric if and only if $\pm iA$ both are accretive.
An operator A is called **m-accretive** if A is accretive and closed and $\mathcal{R}(A + 1)$ is dense in H.

Proposition B.20. *Let A be an operator on H, $\alpha \in \mathbb{R}$ and $\lambda > 0$. The following assertions are equivalent.*

(i) A *is m-accretive.*

(ii) $A + i\alpha$ *is m-accretive.*

(iii) $A + \varepsilon$ *is m-accretive for all* $\varepsilon > 0$.

(iv) $-\lambda \in \varrho(A)$ *and* $\left\|(A-\lambda)(A+\lambda)^{-1}\right\| \leq 1$.

(v) $\{\mathrm{Re}\,\lambda < 0\} \subset \varrho(A)$ *and*

$$\|R(\lambda, A)\| \leq \frac{1}{|\mathrm{Re}\,\lambda|} \qquad (\mathrm{Re}\,\lambda < 0).$$

(vi) $(-\infty, 0) \subset \varrho(A)$ *and* $\sup_{t>0} \left\|t(t+A)^{-1}\right\| \leq 1$.

(vii) A *is closed and densely defined, and* A^* *is m-accretive.*

The operator $(A-1)(A+1)^{-1}$ is called the **Cayley transform** of A.

Proof. Let B be a closed accretive operator on H. By Proposition B.7 and its corollary, if $\mathcal{R}(B+\mu)$ is dense for some μ with $\mathrm{Re}\,\mu > 0$ then $\{\mathrm{Re}\,\mu < 0\} \subset \varrho(B)$. This consideration is of fundamental importance and is used several times in the sequel. In particular, it shows $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$.
The equivalence $(iv) \Leftrightarrow (i)$ holds by (ii) of Lemma B.19. Similarly, $(v) \Leftrightarrow (i)$ and $(vi) \Leftrightarrow (i)$ hold by (iii) and (iv) of Lemma B.19.
Part (vi) shows in particular, that an m-accretive operator is *sectorial* (see §1 of Chapter1), and as such is densely defined, since H is reflexive (see *h)* of Proposition 1.1). Moreover (vi) implies (vi) with A replaced by A^*, whence $(vi) \Rightarrow (vii)$ follows.
Finally, assume (vii). Since we already have proved $(i) \Rightarrow (vii)$, we conclude that $A = \overline{A} = A^{**}$ is m-accretive. □

Theorem B.21. (Lumer-Phillips)
An operator A is m-accretive if and only if $-A$ generates a strongly continuous contraction semigroup.

Proof. If A is m-accretive, parts (vi) and (vii) of Proposition B.20 show that the Hille-Yosida Theorem A.32 (with $\omega = 0$ and $M = 1$) is applicable to the operator $-A$. Conversely, suppose that $-A$ generates the C_0-semigroup T such that $\|T(t)\| \leq 1$ for all $t \geq 0$. Take $x \in \mathcal{D}(A)$. Then the function $t \longmapsto T(t)x$ is differentiable with derivative $t \longmapsto -AT(t)x$. Thus, $t \longrightarrow \|T(t)x\|^2$ is differentiable with

$$\frac{d}{dt}|_{t=0} \|T(t)x\|^2 = (\,T(0)x \,|\, -AT(0)x\,) = (\,-AT(0)x \,|\, T(0)x\,) = -2\,\mathrm{Re}\,(\,Ax \,|\, x\,)\,.$$

Since T is a contraction semigroup, the mapping $t \longmapsto \|T(t)x\|^2$ is decreasing. This implies $d/dt|_{t=0} \|T(t)x\|^2 \leq 0$. Hence, $\mathrm{Re}\,(\,Ax \,|\, x\,) \geq 0$, i.e., A is accretive. Since $1 \in \varrho(-A)$ we conclude that A is in fact m-accretive. □

Theorem B.22. (Stone)
An operator B on the Hilbert space H generates a C_0 − group $(U(t))_{t\in\mathbb{R}}$ of unitary operators if and only if $B = iA$ for some selfadjoint operator A.

Proof. If $B = iA$ then B and $-B$ are both m-accretive. By the Lumer-Phillips Theorem both operators generate C_0-contraction semigroups. Hence B generates a unitary C_0-group. This proof works also in the reverse direction. □

B.7 The Theorems of Plancherel and Gearhart

We state without proof two theorems which are "responsible" for the fact that life is so much more comfortable in Hilbert spaces.

Theorem B.23. (Plancherel)
Let $f \in \mathbf{L^1}(\mathbb{R}, H) \cap \mathbf{L^2}(\mathbb{R}, H)$ and define $\mathcal{F}(f) : \mathbb{R} \longrightarrow H$ by

$$\mathcal{F}(f)(t) := \int_{\mathbb{R}} f(s) e^{-ist}\, ds \qquad (t \in \mathbb{R})$$

Then $\mathcal{F}(f) \in \mathbf{C_0}(\mathbb{R}, H) \cap \mathbf{L^2}(\mathbb{R}, H)$ with $\|\mathcal{F}(f)\|_{\mathbf{L^2}} = \sqrt{2\pi}\, \|f\|_{\mathbf{L^2}}$.

A proof can bee found in [ABHN01, page 46].

Let X be a Banach space and T a C_0-semigroup on X with generator A. From Proposition A.27 we know that

$$\{\operatorname{Re} z \geq \omega\} \subset \varrho(A) \quad \text{and} \quad \sup\{\|R(z, A)\| \mid \operatorname{Re} z \geq \omega\} \tag{B.6}$$

for every $\omega > \omega_0(T)$. (Recall that $\omega_0(T)$ is the growth bound of T, see Appendix A, Section A.7.) When we define

$$s_0(A) := \inf\{\omega \in \mathbb{R} \mid \text{(B.6) holds}\}, \tag{B.7}$$

this is equivalent to the statement "$s_0(A) \leq \omega_0(T)$". The number $s_0(A)$ is called the **abszissa of uniform boundedness** of the generator A. The next theorem states that there is in fact equality if $X = H$ is a Hilbert space.

Theorem B.24. (Gearhart)
Let A be the generator of a C_0-semigroup on a Hilbert space H. Then $s_0(A) = \omega_0(T)$.

One of the many proofs of this theorem uses the Plancherel Theorem and Datko's theorem which states that $\omega_0(T) < 0$ if and only if $T(\cdot)x \in \mathbf{L^2}(\mathbb{R}_+, H)$ for every $x \in H$. It can be found in [ABHN01, page 347].

References

The following books underly our presentation: [Kat95] and [Sch71, Chapter XII] for sesquilinear forms and numerical range, [Cro98, Chapter III] for adjoints, [Con90, Chapter X, §2] and [RS72, Chapter VIII] for symmetric and selfadjoint operators, [Tan79, Chapter 2] for accretive operators and the Lax-Milgram theorem, and [EN00, Chapter II.3.b] for the Lumer-Phillips Theorem and Stone's theorem.

The idea how to achieve part *c)* of Lemma B.12 came from [Lan93, Chapter XVIII, §4].

Appendix C
The Spectral Theorem

The spectral theorem for normal operators appears in many versions. Basically one can distinguish the "spectral measure approach" and the "multiplicator approach". Following HALMOS's article [Hal63] (and R. NAGEL's preaching), we will give a consequent "multiplicator" account of the subject matter.
In this form, the spectral theorem essentially says that a given selfadjoint operator on a Hilbert space acts "like" the multiplication of a real function on an $\mathbf{L^2}$-space. In contrast to usual expositions we will stress that the underlying measure space can be chosen locally compact and the real function can be chosen continuous.

C.1 Multiplication Operators

A **Radon measure space** is a pair (Ω, μ) where Ω is a locally compact Hausdorff space and μ is a positive functional on $\mathbf{C_c}(\Omega)$. By the Riesz Representation Theorem [Rud87, Theorem 2.14] we can identify μ with a σ-regular Borel measure on Ω. If the measure μ has the property

$$\varphi \in \mathbf{C_c}(\Omega),\ 0 \leq \varphi \neq 0 \quad \Rightarrow \quad \int \varphi \, d\mu > 0, \tag{C.1}$$

then the Radon measure space is called a **standard measure space**. Property C.1 is equivalent to the fact that a nonempty open subset of Ω has positive μ-measure. As a consequence of this we obtain that the natural mapping $\mathbf{C}(\Omega) \longrightarrow \mathbf{L^1_{\mathit{loc}}}(\Omega, \mu)$ is injective.
Let $f \in \mathbf{C}(\Omega)$ be a continuous function on Ω. The **multiplication operator** M_f on $\mathbf{L^2}(\Omega, \mu)$ is defined by

$$\mathcal{D}(M_f) := \{g \in \mathbf{L^2}(\Omega, \mu) \mid gf \in \mathbf{L^2}(\Omega, \mu)\} \quad \text{and} \quad M_f g := fg \ (f \in \mathcal{D}(M_f)).$$

The following proposition summarizes the properties of this operator.

Proposition C.1. *Let $f \in \mathbf{C}(\Omega)$ where (Ω, μ) is a standard measure space. The following assertions hold.*

a) The operator $(M_f, \mathcal{D}(M_f))$ is closed.

b) The space $\mathbf{C_c}(\Omega)$ is a core for M_f.

c) One has $(M_f)^* = M_{\overline{f}}$.

d) *One has $M_f \in \mathcal{L}(\mathbf{L^2})$ if and only if f is bounded. In this case $\|M_f\|_{\mathcal{L}(\mathbf{L^2})} = \|f\|_\infty$.*

e) *The operator M_f is injective if and only if $\mu(\{f = 0\} \cap K) = 0$ for every $K \subset \Omega$ compact, i.e., $\{f = 0\}$ is locally μ-null.*

f) *One has $\sigma(M_f) = \overline{f(\Omega)}$. Furthermore, $R(\lambda, M_f) = M_{(\lambda - f)^{-1}}$ for $\lambda \in \varrho(M_f)$.*

g) *If $M_f \subset 0$ then $f = 0$.*

h) *Let also $g \in \mathbf{C}(\Omega)$. Then $M_f M_g \subset M_{fg}$. One has equality if g is bounded or if M_f is invertible.*

Proof. Ad *a)*. Assume $g_n \in \mathcal{D}(M_f)$ such that $g_n \to g$ and $fg_n \to h$ in $\mathbf{L^2}$. For $\varphi \in \mathbf{C_c}(\Omega)$ we have $\varphi g_n \to \varphi g$ and $\varphi f g_n \to \varphi h$ in $\mathbf{L^2}$, whence $\varphi f g = \varphi h$. Since φ was arbitrary it follows that $fg = h$.
Ad *b)*. Let $g \in \mathcal{D}(M_f)$. Since $g \in \mathbf{L^2}$, it vanishes outside a set $L = \bigcup_n K_n$, where $K_n \subset K_{n+1}$ and each K_n is compact. Then $\mathbf{1}_{K_n} g \to g$ and $\mathbf{1}_{K_n} fg \to fg$. Hence we can assume without restriction that g vanishes outside a compact set K. By Urysohn's lemma [Rud87, Lemma 2.12] one an find a function $\psi \in \mathbf{C_c}(\Omega)$ such that $\mathbf{1}_K \le \psi \le \mathbf{1}$. Since $\mathbf{C_c}(\Omega)$ is dense in $\mathbf{L^2}$, there is a sequence $\varphi_n \in \mathbf{C_c}(\Omega)$ such that $\varphi_n \to g$ in $\mathbf{L^2}$. Then $\varphi_n \psi \to \psi g = g$ and $f \varphi_n \psi g \to \psi f g = fg$.
Ad *c)*. Let $x, y \in \mathbf{L^2}$. Then

$$(x, y) \in (M_f)^* \quad \Leftrightarrow \quad \int_\Omega u\overline{y}\, d\mu = \int_\Omega u\overline{\overline{f}x}\, d\mu \ \ \forall\, u \in \mathcal{D}(M_f).$$

Since $\mathbf{C_c}(\Omega) \subset \mathcal{D}(M_f)$, this is the case if and only if $\overline{f}x \in \mathbf{L^2}$.
Ad *d)*. Let $f \in \mathbf{C^b}(\Omega)$. Then $\|fg\|_2 \le \|f\|_\infty \|g\|_2$ for each $g \in \mathbf{L^2}$. Hence $M_f \in \mathcal{L}(\mathbf{L^2})$ and $\|M_f\| \le \|f\|_\infty$. Assume that M_f is bounded and choose $\omega \in \Omega$. For every neighborhood U of ω we define $\varphi_U := \mu(U)^{-\frac{1}{2}} \mathbf{1}_U$. (Note that $\mu(U) \neq 0$ since (Ω, μ) is *standard*.) Then

$$\left| \mu(U)^{-1} \int_U f\, d\mu \right| = \left| (\, f\varphi_U \mid \varphi_U \,)_{\mathbf{L^2}} \right| \le \|M_f\| \, \|\varphi_U\|_2^2 = \|M_f\| \, .$$

Since f is continuous, $\mu(U)^{-1} \int_U f\, d\mu \to f(\omega)$ if U shrinks to $\{\omega\}$. Thus, f is bounded and $\|f\|_\infty \le \|M_f\|$.
Ad *e)*. Suppose there is K such that $\mu(A) > 0$ where $A := \{f = 0\} \cap K$. Then $0 \neq \mathbf{1}_A \in \mathbf{L^2}$ and $M_f(\mathbf{1}_A) = f\mathbf{1}_A = 0$. Hence M_f is not injective. Now assume that $\mu(\{f = 0\} \cap K) = 0$ holds for every compact K. Let $g \in \mathbf{L^2}$ such that $fg = 0$ and choose K compact. Then $fg\mathbf{1}_K = 0$, whence $K = (\{f = 0\} \cap K) \cup (\{g = 0\} \cap K)$. From the hypothesis it follows that $\mu(\{g = 0\} \cap K) = \mu(K)$. Since K was arbitrary and $g \in \mathbf{L^2}$, we conclude that $g = 0$ μ-a.e..
Ad *f)*. Obviously, we have $\lambda - M_f = M_{\lambda - f}$ for every $\lambda \in \mathbb{C}$. Thus it suffices to consider the case $\lambda = 0$. If $0 \notin \overline{f(\Omega)}$ then $f^{-1} \in \mathbf{C^b}(\Omega)$ and it is easy to see that in this case $(M_f)^{-1} = M_{f^{-1}}$. Let $0 \in f(\Omega)$, say $f(\omega) = 0$. Consider the functions φ_U defined as in the proof of *d)*. Then $\|\varphi_U\|_2 = 1$ and $\|M_f \varphi_U\|_2 = \|f\varphi_U\| = \mu(U)^{-1} \int_U |f|^2 \to |f(\omega)|^2 = 0$ if U shrinks to $\{\omega\}$. Hence $(\varphi_U)_U$ is an approximate eigenvector for 0.
Ad *g)*. Assume $M_f \subset 0$. Since $\mathbf{C_c}(\Omega)$ is a core for M_f, we conclude that $M_f = 0$. Hence we have $f = 0$ by *d)*.
A2 *h)*. It is straightforward to prove $M_f M_g \subset M_{fg}$. Assume that g is bounded. Then $M_f M_g$ is closed by Lemma A.3. Since obviously $\mathbf{C_c}(\Omega) \subset \mathcal{D}(M_f M_g)$ and $\mathbf{C_c}(\Omega)$ is a core for M_{fg} we obtain $M_{fg} \subset M_f M_g$. If M_f is invertible then f is injective and f^{-1} is bounded (by *f)*). If $\psi \in \mathcal{D}(M_{fg})$, i.e., $fg\psi \in \mathbf{L^2}$, we have also $g\psi = f^{-1} fg\psi \in \mathbf{L^2}$, whence $\psi \in \mathcal{D}(M_f M_g)$. ☐

Corollary C.2. *a)* *M_f is symmetric if and only if M_f is selfadjoint if and only if f is real valued.*

b) *M_f is accretive if and only if M_f is m-accretive if and only if* $\operatorname{Re} f \ge 0$.

c) *M_f is positive if and only if $f(\Omega) \subset [0, \infty)$.*

Proof. Just apply the definitions and Proposition C.1. $\square$

In this section we have considered only a very special situation of multiplication operators, namely those who are induced by continuous functions on a standard measure space. Of course one can define M_f on $\mathbf{L^2}(\Omega, \mu)$ also if f is just a locally integrable measurable function and if the measure μ is not standard. Proposition C.1 and Corollary C.2 remain true after some small changes. This will be of some interest in Section C.6.

C.2 Commutative C^*-Algebras. The Cyclic Case

In this section we start with a Hilbert space H and a commutative sub-C^*-algebra with unit $\mathcal{A}$ of $\mathcal{L}(H)$. We assume the reader to be familiar with the basic notions and results of Gelfand theory, as can be found for example in [Rud91, Chapter 11] or [Dou98, Chapters 3 and 4].

Let H' be another Hilbert space and $\mathcal{A}'$ be a commutative sub-C^*-algebra with unit of $\mathcal{L}(H')$. We say that $(\mathcal{A}, H)$ and $(\mathcal{A}', H')$ are **unitarily equivalent**, if there is an unitary isomorphism $U : H \longrightarrow H'$ such that the mapping $(T \longmapsto UTU^{-1}) : \mathcal{A} \longrightarrow \mathcal{A}'$ is bijective.

Suppose there is a **cyclic vector** v, i.e., we have $\overline{\{Tv \mid T \in \mathcal{A}\}} = H$. This implies readily that the mapping $(T \longmapsto Tv) : \mathcal{A} \longrightarrow H$ is injective with dense range $\mathcal{A}\, v := \{Tv \mid T \in \mathcal{A}\}$.

[Injectivity is seen as follows. Let $Tv = 0$ for some $T \in \mathcal{A}$. Then, $TSv = STv = S0 = 0$ for every $S \in \mathcal{A}$. Hence, $T = 0$ on the dense subspace $\mathcal{A}\, v$.]

Let Ω denote the spectrum (Gelfand space) of $\mathcal{A}$. Then Ω is compact. We want to find a Radon measure μ on X which turns (X, μ) into a standard measure space, and a sub-C^*-algebra $\mathcal{B}$ of $\mathbf{C}(X)$ such that $(\mathcal{A}, H)$ is unitarily equivalent to $(\mathcal{B}, \mathbf{L^2}(X, \mu))$.

By the Gelfand-Naimark Theorem [Rud91, Theorem 11.18], the canonical embedding $\Phi : \mathcal{A} \longrightarrow C(\Omega)$ is an isomorphism of C^*-algebras. Define the functional μ on $\mathbf{C}(\Omega)$ by

$$\mu(f) = \int f \, d\mu := \left(\Phi^{-1}(f)v \,\middle|\, v\right)_H \qquad (f \in \mathbf{C}(\Omega)).$$

If $f \geq 0$, there is a g such that $g^*g = f$. This implies $\mu(f) = \mu(g^*g) = \left(\Phi^{-1}(g)v \,\middle|\, \Phi^{-1}(g)v\right)_H = \left\|\Phi^{-1}(g)v\right\|^2 \geq 0$. Hence μ is positive. If in addition $\mu(f) = 0$, we must have $\Phi^{-1}(g)v = 0$. But v is cyclic, so $\Phi^{-1}(g) = 0$. This implies $g = 0$, whence $f = 0$. Thus we have shown that (Ω, μ) is a standard measure space.

We now construct the unitary operator $U : H \longrightarrow \mathbf{L^2}(\Omega, \mu)$ as follows. For $w = Tv \in \mathcal{A}\, v$ we define

$$U(w) = U(Tv) := \Phi(T) \in \mathbf{C}(\Omega) \subset \mathbf{L^2}(\Omega, \mu)$$

Because v is a cyclic vector, U is well-defined and of course linear. The computation

$$\begin{aligned}(Tv \mid Sv)_H = (S^*Tv \mid v) = \int \Phi(S^*T)\,d\mu = \int \overline{\Phi(S)}\Phi(T)\,d\mu \\ = (U(Tv) \mid U(Sv))_{\mathbf{L^2}(\Omega,\mu)}\,,\end{aligned}$$

where $T, S \in \mathcal{A}$, shows that U is isometric. Note that the range of U is all of $\mathbf{C}(\Omega)$ which is a dense subspace of $\mathbf{L^2}(\Omega,\mu)$. Hence, U has a unique extension to an isometric isomorphism from H to $\mathbf{L^2}(\Omega,\mu)$.

We are left to show that in fact U induces an unitary equivalence of $(\mathcal{A}, H)$ and $(\mathbf{C}(\Omega), \mathbf{L^2}(\Omega,\mu))$. Let $T \in \mathcal{A}$. It suffices to show that $UTU^{-1} = M_{\Phi(T)}$. To prove this it is enough to check the action of both operators on the dense subspace $\mathbf{C}(\Omega)$ of $\mathbf{L^2}(\Omega,\mu)$. If $f \in \mathbf{C}(\Omega)$ there is a unique $S \in \mathcal{A}$ such that $\Phi(S) = f$. Hence we have

$$UTU^{-1}(f) = UTU^{-1}\Phi(S) = U(TSv) = \Phi(TS) = \Phi(T)\Phi(S) = M_{\Phi(T)}(f).$$

We thus have proved the following theorem.

Proposition C.3. *Let H be a Hilbert space and $\mathcal{A}$ a commutative sub-C^*-algebra with unit of $\mathcal{L}(H)$. Suppose that H has a cyclic vector with respect to $\mathcal{A}$. Let Ω be the Gelfand space of $\mathcal{A}$. Then there is a standard Radon measure μ on Ω such that $(\mathcal{A}, H)$ and $(\mathbf{C}(\Omega), \mathbf{L^2}(\Omega,\mu))$ are unitarily equivalent.*

Remark C.4. If H is separable and $\mathcal{A}$ is a maximal commutative sub-W^*-algebra of $\mathcal{L}(H)$, then there is a cyclic vector, see [Dou98, Theorem 4.65]. Of course, by an application of Zorn's lemma one can show that each commutative selfadjoint subalgebra of $\mathcal{L}(H)$ is contained in a maximal commutative one (which a forteriori must be a W^*-algebra). Hence a bounded normal operator on a separable Hilbert space is unitarily equivalent to multiplication by a continuous function on an $\mathbf{L^2}$-space over a compact space. This is one version of the spectral theorem (cf. Corollary C.7).

C.3 Commutative C^*-Algebras. The General Case

Suppose we are given a Hilbert space H and a commutative sub-C^*-algebra with unit $\mathcal{A}$ of $\mathcal{L}(H)$, but there is no cyclic vector. We choose any vector $0 \neq v \in H$ Then $H_v := \overline{\mathcal{A}v}$ is a closed subspace of H that **reduces** $\mathcal{A}$, i.e., it is $\mathcal{A}$-invariant (clear) and even its orthogonal complement $H_v^{\perp}$ is $\mathcal{A}$-invariant.
[Let $w \perp H_v$, $S \in T$, and $x \in H_v$. Then $(Sw \mid x) = (w \mid S^*x) = 0$, because $S^*x \in H_v$ again.]

If we restrict the operators from $\mathcal{A}$ to the space H_v we obtain a selfadjoint subalgebra with unit $\mathcal{A}_v$ of $\mathcal{L}(H_v)$. Moreover, v is a cyclic vector with respect to $\mathcal{A}$. Clearly, the whole procedure can be repeated on the Hilbert space $H_v^{\perp}$. Therefore, a standard application of Zorn's lemma yields the following lemma.

Lemma C.5. *Let H be a Hilbert space, and $\mathcal{A} \subset \mathcal{L}(H)$ a commutative sub-C^*-algebra with unit. Then there is a decomposition $H = \bigoplus_{\alpha\in I} H_\alpha$ as a Hilbert space direct sum such that each H_α is $\mathcal{A}$-invariant and has a cyclic vector with respect to $\mathcal{A}$.*

Note that if H is separable, the decomposition in Lemma C.5 is actually countable.

Theorem C.6. *Let H be a Hilbert space and $\mathcal{A} \subset \mathcal{L}(H)$ a commutative selfadjoint subalgebra. Then there is a standard measure space (Ω, μ) and a subalgebra $\mathcal{B}$ of $\mathbf{C^b}(\Omega)$ such that $(\mathcal{A}, H)$ is unitarily equivalent to $(\mathcal{B}, \mathbf{L^2}(\Omega, \mu))$.*

Proof. Without restriction we can assume that $\mathcal{A}$ is a sub-C^*-algebra with unit of $\mathcal{L}(H)$. By Lemma C.5 we can decompose $H = \bigoplus_{\alpha \in I} H_\alpha$, where the H_α are $\mathcal{A}$-invariant and have cyclic vectors, say, v_α. We let $\mathcal{A}_\alpha := \overline{\mathcal{A}|_{H_\alpha}} \subset \mathcal{L}(H_\alpha)$, and define Ω_α to be the spectrum of $\mathcal{A}_\alpha$. Proposition C.3 (cyclic case) yields an unitary isomorphism

$$U_\alpha : (\mathcal{A}_\alpha, H_\alpha) \longrightarrow (\mathbf{C}(\Omega_\alpha), \mathbf{L^2}(\Omega_\alpha, \mu_\alpha))$$

where μ_α is a standard measure on Ω_α. In fact, $U_\alpha(T_\alpha v_\alpha) = \Phi_\alpha(T_\alpha)$ for each $T_\alpha \in \mathcal{A}_\alpha$, where $\Phi_\alpha : \mathcal{A}_\alpha \longrightarrow \mathbf{C}(\Omega_\alpha)$ is the Gelfand isomorphism (cf. the proof of Proposition C.3).
We now let $\Omega := \bigcup \Omega_\alpha$ the (disjoint) topological direct sum of the Ω_α. Clearly, Ω is a locally compact Hausdorff space. Furthermore, each Ω_α is an open subset of Ω.
If $f \in \mathbf{C}(\Omega_{\alpha_0})$ for some particular α_0, we can extend f continuously to the whole of Ω by letting $f|_{X_\alpha} = 0$ for every other α. We can therefore identify the continuous functions on Ω_{α_0} with the continuous functions on Ω having support contained in Ω_{α_0}.
Note that for each $\varphi \in \mathbf{C_c}(\Omega)$ there are only finitely many α such that $\varphi|_{\Omega_\alpha} \neq 0$. Hence, by

$$\mu(\varphi) = \int_\Omega \varphi \, d\mu := \sum_\alpha \mu_\alpha(\varphi|_{\Omega_\alpha})$$

a positive functional on $\mathbf{C_c}(\Omega)$ is defined. Obviously, (Ω, μ) is a standard measure space (each $(\Omega_\alpha, \mu_\alpha)$ is!).
We now construct the unitary operator U. The subspace $H_0 := \operatorname{span}\{T_\alpha v_\alpha \mid \alpha \in I,\ T_\alpha \in \mathcal{A}_\alpha\}$ is dense in H. We define

$$U := (\sum_{\alpha \in F} T_\alpha v_\alpha) \longmapsto \sum_{\alpha \in F} \Phi_\alpha(T_\alpha) : H_0 \longrightarrow \mathbf{C_c}(\Omega)$$

where $F \subset I$ is a finite subset, and $T_\alpha \in \mathcal{A}_\alpha$ for all α. Note that $\Phi_\alpha(T_\alpha)$ is a continuous function on Ω_α, hence can be viewed as a continuous function on Ω. It is clear that U is linear and surjective. The computation

$$\begin{aligned}
(x \,|\, y)_H &= \left(\sum_{\alpha \in F} T_\alpha v_\alpha \,\middle|\, \sum_{\beta \in G} S_\beta v_\beta \right) = \sum_{\alpha \in F \cap G} (T_\alpha V_\alpha \,|\, S_\alpha V_\alpha) \\
&= \sum_{\alpha \in F \cap G} \int_{X_\alpha} \Phi(T_\alpha) \overline{\Phi(S_\alpha)} \, d\mu_\alpha = \int_\Omega \left(\sum_{\alpha \in F} \Phi_\alpha(T_\alpha) \right) \left(\overline{\sum_{\beta \in G} \Phi_\beta(S_\beta)} \right) d\mu \\
&= (Ux \,|\, Uy)_{\mathbf{L^2}(\Omega, \mu)},
\end{aligned}$$

where $x = \sum_{\alpha \in F} T_\alpha v_\alpha, y = \sum_{\beta \in G} S_\beta v_\beta \in H_0$, shows that U is even isometric. Hence, U extends to a unitary isomorphism $U : H \longrightarrow \mathbf{L^2}(\Omega, \mu)$. To conclude the proof we show

$$UTU^{-1} = M_f \quad (T \in \mathcal{A}), \text{ where } f \in \mathbf{C}(\Omega) \text{ is defined by } f|_{X_\alpha} = \Phi_\alpha(T|_{H_\alpha}) \in \mathbf{C}(\Omega_\alpha).$$

Note that, if this is true, it follows that $f \in \mathbf{C^b}(\Omega)$ because the operator M_f is bounded on $\mathbf{L^2}(\Omega, \mu)$ (cf. part *d)* of Proposition C.1). To show the claim we only have to check the action of both operators on the dense subspace $\mathbf{C_c}(\Omega)$ of $\mathbf{L^2}(\Omega, \mu)$. Therefore, let $\varphi \in \mathbf{C_c}(\Omega)$ and define $\varphi_\alpha := \varphi|_{\Omega_\alpha}$. Then, $\varphi = \sum_\alpha \varphi_\alpha$ where the sum is actually finite. We let $S_\alpha := \Phi^{-1}(\varphi_\alpha)$. Then we have $\varphi = Ux$ with $x = \sum_\alpha S_\alpha v_\alpha \in H_0$. Now,

$$\begin{aligned}
UTU^{-1}(\varphi) &= UTx = UT(\sum_\alpha S_\alpha v_\alpha) = U(\sum_\alpha T_\alpha S_\alpha v_\alpha) \\
&= \sum_\alpha \Phi_\alpha(T_\alpha S_\alpha) = \sum_\alpha \Phi_\alpha(T_\alpha)\Phi_\alpha(S_\alpha) = \sum_\alpha f|_{X_\alpha} \Phi_\alpha(S_\alpha) \\
&= f(\sum_\alpha \Phi_\alpha(S_\alpha)) = f\varphi = M_f(\varphi).
\end{aligned}$$

This completes the proof. □

Corollary C.7. (Spectral Theorem I)
Let H be a Hilbert space and $(T_j)_{j\in J}$ a family of commuting bounded normal operators on H. Then there is a standard measure space (Ω,μ) and a family of bounded continuous functions $(f_j)_{j\in J}$ on Ω such that $((T_j)_{j\in J}, H)$ is unitarily equivalent to $((f_j)_{j\in J}, \mathbf{L}^2(\Omega,\mu))$.

Proof. By Fugledge's theorem [Con90, Chapter IX, Theorem 6.7] $T_k^* T_j = T_j T_k^*$ for all $j,k\in J$. Hence the $*$-algebra $\mathcal{A}$ generated by $(T_j)_{j\in J}$ is commutative. Now we can apply Theorem C.6. □

Remark C.8. One should note that in the case of a single operator T, Fugledge's theorem is not needed for the proof of Corollary C.7 (see also Section C.4 below. On the other hand one can *deduce* Fugledge's theorem from the spectral theorem for a single operator, as HALMOS in [Hal63] points out.

C.4 The Spectral Theorem: Bounded Normal Operators

Let H be a Hilbert space and $T\in\mathcal{L}(H)$ a bounded normal operator on H. Denote by $\mathcal{A}$ the sub-C^*-algebra of $\mathcal{L}(H)$ which is generated by T. Because T is normal, $\mathcal{A}$ is commutative. We have $\sigma(\mathcal{A})=\sigma(T)$ and $\hat{T}$ (the Gelfand transform of T, being a continuous function on $\sigma(\mathcal{A})$) is the coordinate function $(z\longmapsto z)$. Moreover, $R(\lambda,T)\in\mathcal{A}$ for every $\lambda\in\varrho(T)$.

[From elementary Gelfand theory it follows that $\hat{T}:\sigma(\mathcal{A})\longrightarrow\sigma(T)$ is surjective. But it is also injective, since T generates $\mathcal{A}$. Hence $\hat{T}$ is a homeomorphism, identifying $\sigma(\mathcal{A})$ and $\sigma(T)$. With this identification $\hat{T}$ becomes just the coordinate function $(z\longmapsto z)$. If $\lambda\in\varrho(T)$, $r_\lambda:=(\lambda-z)^{-1}$ is a continuous function on $\sigma(T)$, and the Gelfand-Naimark Theorem implies that there is an operator $R_\lambda\in\mathcal{A}$ which corresponds to r_λ. But obviously we have $R_\lambda=R(\lambda,T)$.]

In the following we will review the construction of the proof of Theorem C.6. For this we need a lemma whose proof is straightforward.

Lemma C.9. *Let H_0 be a closed subspace of H, with orthogonal projection $P:H\longrightarrow H_0$. The subspace H_0 is $\mathcal{A}$-invariant, if and only if $TP=PT$. In this case one has $\sigma(T|_{H_0})\subset\sigma(T)$ and $R(\lambda,T|_{H_0})=R(\lambda,T)|_{H_0}$ for each $\lambda\in\varrho(T)$. The C^*-closure $\mathcal{A}_0$ of $\{S|_{H_0}\mid S\in\mathcal{A}\}$ is generated by $T|_{H_0}$ and $\sigma(\mathcal{A}_0)=\sigma(T|_{H_0})$.*

Proceeding along the lines of the proof of Theorem C.6, we decompose the Hilbert space $H=\bigoplus_{\alpha\in I}H_\alpha$ into $\mathcal{A}$-invariant and cyclic subspaces H_α. Let $T_\alpha:=T|_{H_\alpha}$ and $\Omega_\alpha:=\sigma(\mathcal{A}_\alpha)=\sigma(T_\alpha)$. Each Ω_α carries a standard measure μ_α such that (T_α,H_α) is unitarily equivalent to $(M_z,\mathbf{L}^2(\Omega_\alpha,\mu_\alpha))$, where M_z is multiplication by the coordinate function (see above). The locally compact Hausdorff space Ω is the disjoint union of the $\sigma(T_\alpha)$, and the standard measure μ on Ω is defined by $\mu|_{\Omega_\alpha}=\mu_\alpha$. Finally, (T,H) and $(M_f,\mathbf{L}^2(\Omega,\mu))$ are unitarily equivalent, where $f\in\mathbf{C}^{\mathbf{b}}(\Omega)$ is the coordinate function on each $\sigma(T_\alpha)$.

By Lemma C.9, $\sigma(T_\alpha)$ is a closed subset of $\sigma(T)$ for each α. Therefore, Ω can be viewed as a closed subset of $\sigma(T)\times I$. The measure μ on Ω extends canonically to a measure on $\sigma(T)\times I$ with support Ω which we denote again by μ. Note that this extension in general is not a standard measure any more. But obviously, there is a unitary equivalence of $(M_f, \mathbf{L^2}(\Omega,\mu))$ and $(M_h, \mathbf{L^2}(\sigma(T) \times I, \mu))$, where $f : \sigma(T) \times I \longrightarrow \mathbb{C}$ is defined by $h(z,\alpha) = z$ for $(z,\alpha) \in \sigma(T) \times I$.

Thus, we have proved the following theorem.

Theorem C.10. (Spectral Theorem II)
Let H be a Hilbert space and $T \in \mathcal{L}(H)$ a bounded normal operator on H. Then there is a discrete set I and a (not necessarily standard) Radon measure μ on $\sigma(T) \times I$ such that (T, H) is unitarily equivalent to $(M_h, \mathbf{L^2}(\sigma(T) \times I, \mu))$, where h is defined by

$$h = ((z,\alpha) \longmapsto z) : \sigma(T) \times I \longrightarrow \mathbb{C}.$$

If H is separable, $I = \mathbb{N}$.

C.5 The Spectral Theorem: Unbounded Selfadjoint Operators

Let H be a Hilbert space and A a (not necessarily bounded) selfadjoint operator on H. Then $\sigma(A) \subset \mathbb{R}$. The spectral theorem for A states that one can find a standard measure space (Ω,μ) and a real-valued continuous function f on Ω such that (A, H) and $(M_f, \mathbf{L^2}(\Omega,\mu))$ are **unitarily equivalent**. This means that there is a unitary isomorphism $U : H \longrightarrow \mathbf{L^2}(\Omega,\mu)$ which takes the graph of A bijectively onto the graph of M_f, i.e., it holds

$$(x, y) \in A \quad \text{if and only if} \quad (Ux, Uy) \in M_f.$$

The situation can be reduced to the bounded case by taking resolvents. Let $T := (i - A)^{-1}$. Then clearly T is a bounded normal operator on H. By Corollary C.7 we find a standard measure space (Ω_1, μ_1) and a bounded continuous function g_1 on Ω such that (T, H) and $(M_{g_1}, \mathbf{L^2}(\Omega_1, \mu_1))$ are unitarily equivalent. Now T is injective, hence $(g_1 = 0)$ is a closed subset of Ω_1 which is locally μ_1-null (see part *e)* of Proposition C.1). If we set $\Omega := \Omega_1 \setminus (g_1 = 0)$ and $\mu := \mu_1|_\Omega$, we obtain a standard measure space (Ω,μ). It is easy to see that $(M_{g_1}, \mathbf{L^2}(\Omega_1,\mu_1))$ is unitarily equivalent to $(M_g, \mathbf{L^2}(\Omega,\mu))$, where $g := g_1|_\Omega$. But g does not vanish, hence $f := i - g^{-1}$ defines a continuous function on Ω. Clearly, (A, H) is unitarily equivalent to $(M_f, \mathbf{L^2}(\Omega,\mu))$.

If we use Theorem C.10 instead of Corollary C.7, we obtain a discrete set I and a Radon measure ν on $\sigma(T) \times I$ such that (T, H) is unitarily equivalent to $(h, \mathbf{L^2}(\sigma(T) \times I, \nu))$, where $h : \sigma(T) \times I \longrightarrow \mathbb{C}$ is the projection onto the first coordinate. Thanks to the spectral mapping theorem for resolvents (Proposition A.12) we have $\sigma(T) = \varphi(\sigma(A))$, with $\varphi(w) = (i - w)^{-1}$. We define the Radon measure μ on $\sigma(A) \times I$ by

$$\int_{\sigma(A)\times I} f(w,\alpha)\, d\mu(w,\alpha) := \int_{\sigma(T)\times I} f(\varphi^{-1}(z),\alpha)\, d\nu(z,\alpha)$$

for $f \in \mathbf{C_c}(\sigma(A) \times I)$. Then it is immediate that (A, H) is unitarily equivalent to $(M_f, \mathbf{L^2}(\sigma(A) \times I, \mu))$ where $f : \sigma(A) \times I \longrightarrow \mathbb{R}$ is the projection onto the first coordinate.

We summarize our considerations in the next theorem.

Theorem C.11. (Spectral Theorem III)
Let H be a Hilbert space and A a selfadjoint operator on H.

a) *There is a standard measure space (Ω, μ) and a function $f \in \mathbf{C}(\Omega, \mathbb{R})$ such that (A, H) is unitarily equivalent to $(M_f, \mathbf{L^2}(\Omega, \mu))$.*

b) *There is a discrete set I and a positive Radon measure μ on $\sigma(A) \times I$ such that (A, H) is unitarily equivalent to $(M_f, \mathbf{L^2}(\sigma(A) \times I, \mu))$, where f is given by*

$$f = ((z, \alpha) \longmapsto z) : \sigma(A) \times I \longrightarrow \mathbb{R}.$$

If H is separable, $I = \mathbb{N}$.

Remark C.12. We focused our considerations on selfadjoint operators but with the same proofs can obtain a similar result for unbounded, normal operators.

C.6 The Functional Calculus

The spectral theorem(s) allow(s) us to define a functional calculus for a normal operator on a Hilbert space. Let Ω be a locally compact space, μ a positive Radon measure on Ω, and $f \in \mathbf{C}(\Omega)$ a continuous function on Ω. We let $X := \overline{f(\Omega)} \subset \mathbb{C}$. Denote by $\mathbf{B}(X)$ the bounded Borel measurable functions on X. If $g \in \mathbf{B}(X)$, $g \circ f \in \mathbf{B}(\Omega)$, hence $M_{g \circ f}$ is a bounded operator on $\mathbf{L^2}(\Omega, \mu)$ satisfying $\|M_{g \circ f}\|_{\mathcal{L}(\mathbf{L^2})} \leq \|g \circ f\|_\infty \leq \|g\|_\infty$. Obviously, the mapping

$$(g \longmapsto g(M_f) := M_{g \circ f}) : \mathbf{B}(X) \longrightarrow \mathcal{L}(\mathbf{L^2}(\Omega, \mu))$$

is a homomorphism of C^*-algebras. Moreover, if g_n is unifomly bounded sequence in $\mathbf{B}(X)$ converging pointwise to $g \in \mathbf{B}(X)$, then Lebesgue's Dominated Convercence theorem yields that $g_n(M_f) \to g(M_f)$ strongly. If we put this together with the spectral theorem(s), we obtain the following result.

Theorem C.13. *Let A be a selfadjoint operator on a Hilbert space H. Then there exists a unique mapping $\Psi : \mathbf{B}(\sigma(A)) \longrightarrow \mathcal{L}(H)$ with the following properties.*

1) *Ψ is a $*$-homomorphism.*
2) *$\Psi((\lambda - z)^{-1}) = R(\lambda, A)$ for all $\lambda \notin \mathbb{R}$.*
3) *If $(g_n)_n \subset \mathbf{B}(\sigma(A))$ is uniformly bounded and $g_n \to g$ pointwise, then $\Psi(g_n) \to \Psi(g)$ strongly.*

One usually writes $g(A)$ instead of $\Psi(g)$.

Proof. Existence is clear from the remarks at the beginning of the section and the spectral theorem C.11. We show uniqueness. Observe that the (selfadjoint) algebra which is generated by the set $\{(\lambda - z)^{-1} \mid \lambda \notin \mathbb{R}\}$ is uniformly dense in $\mathbf{C_0}(\mathbb{R})$ by the Stone-Weierstrass theorem. The sequence of functions $g_n(z) := in(in - z)^{-1}$ is uniformly bounded on $\mathbb{R}$ and converges pointwise to the constant $\mathbf{1}$. On the other hand is is clear that $inR(in, A) \to I$ strongly. Hence $\Psi(\mathbf{1}) = I$. Moreover, we see from this that Ψ is determined on $\mathbf{C^b}(\mathbb{R})$. By the Tietze Extension theorem we know that each bounded continuous function on $\sigma(A)$ is the restriction of a bounded continuous function on $\mathbb{R}$. Hence Ψ is determined on $\mathbf{C^b}(\sigma(A))$. From this follows that Ψ is determined on the smallest class $\mathcal{M}$ of functions which contains the bounded continuous ones and is closed under bounded and pointwise convergence.
Now, $\sigma(A)$ is a metric, separable, locally compact space, whence by Urysohn's lemma [Rud87, Lemma 2.12] the characteristic functions of compact sets are contained in $\mathcal{M}$. The class $\mathcal{A} :=$

$\{M \subset \sigma(A) \mid \mathbf{1}_M \in \mathcal{M}\}$ is easily seen to be a σ-algebra on $\sigma(A)$ which contains the compact subsets. Hence $\mathcal{A} = \mathfrak{B}(\sigma(A))$, the Borel σ-algebra on $\sigma(A)$. But a standard result from measure theory says that each bounded Borel measurable function can be approximated uniformly by a sequence of Borel simple functions. Altogether this implies $\mathbf{B}(\sigma(A)) = \mathcal{M}$. □

Remark C.14. The spectral theorem allows to define $g(A)$ for a Borel measurable function $g : \sigma(A) \longrightarrow \mathbb{C}$ which is *not* bounded any more. This is done in the same way as for bounded functions g. For example, if $A \geq 0$, i.e., $\sigma(A) \subset [0, \infty)$ one can define the fractional powers A^α as $(z^\alpha)(A)$ in this way.

References

A highly readable account of the spectral theorem including also the "spectral measure version" is [RS72, Chaper VII and Section VIII.3]. One may also profit from [Rud91, p.321 and p. 368] and [Con90, Chapter IX and ChapterX], where the stress is on the spectral measures. We have based our exposition mainly on the book [SK78], especially its Chapter IX, from which we learned about standard measure spaces and the "continuous multiplicator version" of the spectral theorem. Theorem C.10 in the separable case is [Dav96, Theorem 2.5.1], but it is proved with entirely different methods.

For some historical remarks on the spectral theorem see [Ric99].

Appendix D
Approximation by Rational Functions

In this appendix we provide some results from approximation theory. The objective is to approximate a given continuous function f on a compact subset K of the Riemann sphere $\mathbb{C}_\infty$ by rational functions in some sense. Unfortunately, there is not enough room to develop the necessary complex function theory. Hence we have to refer to the literature. However, we could not find any account of the topic which served our purposes perfectly. Therefore, we will take two results from the book [Gam69] of GAMELIN as a starting point and modify them according to our needs.

Note that a subset $K \subset \mathbb{C}_\infty$ is called **finitely connected** if $\mathbb{C}_\infty \setminus K$ has a finite number of connected components. If $K \subset \mathbb{C}_\infty$ is compact, we consider

$$A(K) := \{f \in \mathbf{C}(K) \mid f \text{ is holomorphic on } \mathring{K}\}.$$

The set of *all* rational functions is denoted by $\mathbb{C}(z)$. We view a rational function $r \in \mathbb{C}(z)$ as a continuous (or holomorphic) function from $\mathbb{C}_\infty$ to $\mathbb{C}_\infty$. A point $\lambda \in \mathbb{C}_\infty$ is called a **pole** of r, if $r(\lambda) = \infty$. Given *any* subset $K \subset \mathbb{C}_\infty$ we define

$$\mathcal{R}(K) := \{r \in \mathbb{C}(z) \mid r(K) \subset \mathbb{C}\}$$

to be the set of rational functions with poles lying outside of M. If K is compact, we denote by $R(K)$ the closure of $\mathcal{R}(K)$ in $\mathbf{C}(K)$. Then, $A(K)$ is a closed subalgebra of $\mathbf{C}(K)$ with $R(K) \subset A(K)$.

Proposition D.1. [Gam69, Chapter II, Theorem 10.4]
Let $K \subset \mathbb{C}$ be compact and finitely connected. Then $A(K) = R(K)$, i.e. each function $f \in \mathbf{C}(K)$ which is holomorphic on $\mathring{K}$ can be approximated uniformly on K by rational functions r_n which have poles outside of K.

The other result we will need is concerned with pointwise bounded approximation. We say that a sequence f_n of functions on a set $\Omega \subset \mathbb{C}_\infty$ converges **pointwise boundedly** on Ω to a function f, if $\sup_n \sup_{z\in\Omega} |f_n(z)| < \infty$ and $f_n(z) \to f(z)$ for all $z \in \Omega$. For $\Omega \subset \mathbb{C}_\infty$ open we let

$$H^\infty(\Omega) := \{f : \Omega \to \mathbb{C} \mid f \text{ is bounded and holomorophic}\}$$

be the Banach algebra of bounded holomorphic functions on Ω. We will occasionally write $\|f\|_\Omega = \|f\|_{\infty,\Omega}$ to denote the supremum norm of $f \in H^\infty(\Omega)$.

Proposition D.2. [Gam69, Chapter VI, Theorem 5.3]
Let $K \subset \mathbb{C}$ be compact and finitely connected. Then, for every $f \in H^\infty(\mathring{K})$ there is a sequence of rational functions r_n with poles outside K such that $\|r_n\|_K \leq \|f\|_{\mathring{K}}$ and $r_n \to f$ pointwise on $\mathring{K}$. In particular, $r_n \to f$ pointwise boundedly.

Propositions D.1 and D.2 refer only to subsets K of the plane $\mathbb{C}$. But one can easily extend these results to (strict) subsets $K \subset \mathbb{C}_\infty$ of the Riemann sphere by a rational change of coordinates, a so-called **Möbius transformation**. These are the mappings

$$m(a,b,c,d) := \left(z \longmapsto \frac{az+b}{cz+d}\right) : \mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty$$

with complex numbers $a, b, c, d \in \mathbb{C}$, $ad - bc \neq 0$. It is well known that each Möbius transformation is invertible, its inverse being again a Möbius tranformation. In particular, they are homeomorphisms of $\mathbb{C}_\infty$.

Let $K \subset \mathbb{C}_\infty$ be compact. If $\infty \in K \subset \mathbb{C}_\infty$ and K is not the whole sphere, there is some $p \in \mathbb{C} \setminus K$. The transformation $\varphi := m(0,1,1,-p)$ has the property that $\varphi(K)$ is a compact subset of $\mathbb{C}$. Because φ is a homeomorphism, $\varphi(\mathring{K}) = \varphi(K)^\circ$. Moreover, K is finitely connected if and only if $\varphi(K)$ is. If r is a rational function with poles outside of $\varphi(K)$, $r \circ \varphi$ is a rational function with poles outside of K, and one has $\|r\|_{\varphi(K)} = \|r \circ \varphi\|_K$. Finally, if $f : K \to \mathbb{C}$, then $f \in A(K)$ if and only if $f \circ \varphi^{-1} \in A(\varphi(K))$ and $f \in H^\infty(\mathring{K})$ if and only if $f \circ \varphi^{-1} \in H^\infty(\varphi(K)^\circ)$.
These considerations show that Propositions D.1 and D.2 remain true for subsets $K \subset \mathbb{C}_\infty$ with $K \neq \mathbb{C}_\infty$.

We now deal with some special sets K. Let $\Omega \subset \mathbb{C}$ be open. We denote by K the closure of Ω in $\mathbb{C}_\infty$, while we keep the notation $\overline{\Omega}$ for the closure of Ω in $\mathbb{C}$. We assume

$$K \neq \mathbb{C}_\infty, \quad \infty \in K, \quad \Omega = \mathring{K}, \quad \text{and} \quad K \text{ is finitely connected.}$$

For example, Ω can be anything from the list

- $S_\omega = \{z \mid z \neq 0,\ |\arg z| < \omega\}$ the sector of angle 2ω symmetric about the positive real line, where $\omega < \pi$.
- $H_\omega = \{z \mid\ |\mathrm{Im}\, z| < \omega\}$ the horizontal strip of height 2ω, symmetric about the real line, where $\omega > 0$ is arbitrary.
- $\Sigma_\omega = S_\omega \cup -S_\omega$ a double sector, where $\omega < \frac{\pi}{2}$.
- $\Pi_\omega = \{z \mid (\mathrm{Im}\, z)^2 < 4\omega^2 \,\mathrm{Re}\, z\}$ a horizontal parabola, where $\omega > 0$ is arbitrary.

We define $\mathcal{R}^\infty(\Omega) := \mathcal{R}(\Omega) \cap H^\infty(\Omega)$ and $\mathcal{R}_0^\infty(\Omega) := \{r \in \mathcal{R}^\infty(\Omega) \mid r(\infty) = 0\}$. Then it is clear that $\mathcal{R}(K) = \mathcal{R}^\infty(\Omega)$ and

$$\begin{aligned}
\mathcal{R}_0^\infty(\Omega) &= \mathcal{R}^\infty(\Omega) \cap \mathbf{C_0}(\overline{\Omega}) \subset \mathcal{R}^\infty(\Omega) \\
&\subset A(K) = \{f \in H^\infty(\Omega) \cap \mathbf{C}(\overline{\Omega}) \mid \lim_{z\to\infty} f(z) \text{ ex.}\} \\
&\subset \{f \in \mathbf{C}(\overline{\Omega}) \mid \lim_{z\to\infty} f(z) \text{ ex.}\} = \mathbf{C}(K).
\end{aligned}$$

Proposition D.3. *We have*

$$\mathcal{R}^{\infty}(\Omega) = \{\frac{p}{q} \mid p, q \in \mathbb{C}[z],\ \ (q = 0) \cap \overline{\Omega} = \emptyset,\ \deg(p) \leq \deg(q)\} \tag{D.1}$$

and

$$\mathcal{R}_0^{\infty}(\Omega) = \{\frac{p}{q} \mid p, q \in \mathbb{C}[z],\ \ (q = 0) \cap \overline{\Omega} = \emptyset,\ \deg(p) < \deg(q)\}. \tag{D.2}$$

The algebra $\mathcal{R}_0^{\infty}(\Omega)$ is generated by the elementary rationals $(\lambda - z)^{-1}$ ($\lambda \notin \overline{\Omega}$). The algebra $\mathcal{R}^{\infty}(\Omega)$ is generated by the elementary rationals $(\lambda - z)^{-1}$ ($\lambda \notin \overline{\Omega}$) together with the constant $\mathbf{1}$ function. The closure of $\mathcal{R}_0^{\infty}(\Omega)$ with respect to $\|\cdot\|_{\Omega}$ is $H^{\infty}(\Omega) \cap \mathbf{C_0}(\overline{\Omega})$. The closure of $\mathcal{R}^{\infty}(\Omega)$ with respect to $\|\cdot\|_{\Omega}$ is $A(K)$.

Proof. Let $r = p/q \in C(z)$ be a rational function. Then r is bounded on Ω if and only if it is bounded on K (view r as a continuous function from C_{∞} to itself). So its poles lie outside of K and $r(\infty) \in \mathbb{C}$. This implies $\deg p \leq \deg q$. If $r(\infty) = 0$ it follows that $\deg p < \deg q$. The other inclusions are clear.
Obviously, every elementary rational $(\lambda - z)^{-1}$ with $\lambda \notin \overline{\Omega}$ is contained in $\mathcal{R}_0^{\infty}(\Omega)$. Since we can write

$$\alpha\frac{\mu - z}{\lambda - z} = \alpha(\frac{\mu - \lambda}{\lambda - z} + \mathbf{1}) \quad (\alpha, \mu, \lambda \in \mathbb{C},\ \lambda \notin \overline{\Omega})$$

it follows from (D.2) and the Fundamental Theorem of Algebra that the elementary rationals generate $\mathcal{R}_0^{\infty}(\Omega)$. From (D.1) it is clear that $\mathcal{R}^{\infty}(\Omega) = \mathcal{R}_0^{\infty}(\Omega) \oplus \mathbf{1}$.
From Proposition D.1 we know that $R(K) = A(K)$. Let $f \in A(K)$ suchthat $f(\infty) = 0$. We can find $r_n \in \mathcal{R}^{\infty}(\Omega)$ suchthat $\|r_n - f\|_{\Omega} \to 0$. Since ∞ is in the closure of Ω in $\mathbb{C}_{\infty}$, this implies $r_n(\infty) \to f(\infty) = 0$. Hence $\|(r_n - r_n(\infty)) - f\|_{\Omega} \to 0$ and $r_n - r_n(\infty) \in \mathcal{R}_0^{\infty}(\Omega)$. □

Proposition D.4. *Let Ω and K be as above and $f \in H^{\infty}(\Omega)$. Then there is a sequence of rational functions $r_n \in \mathcal{R}^{\infty}(\Omega)$ such that $\|r_n\|_{\Omega} \leq \|f\|_{\Omega}$ for all n and $r_n \to f$ pointwise on Ω.*

Proof. The statement is just a reformulation of Proposition D.2 combined with the remarks immediately after it. □

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Index

Notation

Sets and Topological Spaces

$\mathbb{C}_\infty$	Riemann sphere, page 142
H_ω	the horizontal strip of height 2ω, symmetric about the real line, page 175
Π_ω	horizontal parabola, page 175
Σ_ω	double sector $S_\omega \cup -S_\omega$, page 175
S_ω	sector of angle 2ω symmetric about the positive real line, page 175
$S_{\omega'}(0, R)$	the set $S_{\omega'} \cap B_R(0)$, page 38
$S_{\omega'}(\varepsilon', \infty)$	the set $S_{\omega'} \setminus \overline{B_{\varepsilon'}(0)}$, page 37

Spaces of Continuous and Integrable Functions

$\|\cdot\|_\Omega$	uniform norm on the set Ω, page 174
$\mathbf{B}(\Omega)$	space of bounded Borel measurable functions on the topological space Ω, page 172
$\mathbf{C^b}(\Omega)$	space of bounded continuous functions on the topological space Ω, page 172
$\mathbf{C_c}(\Omega)$	space of continuous function with compact support on the locally compact space Ω, page 165
$\mathbf{C}(\Omega)$	space of all continuous functions on the locally compact space Ω, page 165
$\mathbf{C_0}(\Omega)$	space of continuous functions vanishing at infinity on the locally compact space Ω, page 172
$\mathbf{L^1_{\mathit{loc}}}(\Omega, \mu)$	space of locally integrable functions on the (Radon) measure space (Ω, μ), page 165
$\mathbf{L^2}(\Omega, \mu)$	space of square-integrable functions on the (Radon) measure space (Ω, μ), page 165
$\mathbf{W^{1,2}}(\Omega)$	first Sobolev space in $\mathbf{L^2}(\Omega)$ where Ω is an open subest of $\mathbb{R}^n$, page 127

Spaces of Holomorphic Functions

$\mathcal{O}(\Omega)$ space of all holomorphic functions on the open set $\Omega \subset \mathbb{C}$, page 22

$\mathcal{O}_c(S_\varphi)$ space of holomorphic functions on S_φ which are bounded on each set $S_\varphi \cap \{r \le |z| \le R\}$ for all $0 < r < R < \infty$, page 24

$A(K)$ space of continuous functions on $K \subset \mathbb{C}_\infty$ holomorphic on $\mathring{K}$, page 174

$\mathcal{A}(S_\varphi)$ class of functions suitable for the natural f.c. for sectorial operators, page 26

$\mathcal{A}[S_\omega]$ abbreviation for $\bigcup_{\varphi>\omega} \mathcal{A}(S_\varphi)$, page 26

$\mathcal{E}(S_\varphi)$ class of functions suitable for the natural f.c. for bounded and invertible sectorial operators, page 39

$\mathcal{B}(S_\varphi)$ space of functions suitable for the natural f.c. for injective sectorial operators, page 33

$\mathcal{B}[S_\omega]$ abbreviation for $\bigcup_{\varphi>\omega} \mathcal{B}(S_\varphi)$, page 33

$\mathcal{C}(S_\varphi)$ space of functions suitable for the natural f.c. for invertible sectorial operators, page 36

$\mathcal{C}[S_\omega]$ abbreviation for $\bigcup_{\varphi>\omega} \mathcal{C}(S_\varphi)$, page 36

$\mathcal{DR}(S_\varphi)$ Dunford-Riesz class on the sector S_φ, page 22

$\mathcal{DR}[S_\omega]$ abbreviation for $\bigcup_{\varphi>\omega} \mathcal{DR}(S_\varphi)$, page 24

$\mathcal{DR}_0(S_\varphi)$ space of functions holomorphic on $S_\varphi \cup \{0\}$ and regularly decaying at ∞, page 23

$\mathcal{DR}_{\text{ext}}(S_\varphi)$ extended Dunford-Riesz class on the sector S_φ, page 23

$\mathcal{DR}_{\text{ext}}[S_\omega]$ abbreviation for $\bigcup_{\varphi>\omega} \mathcal{DR}_{\text{ext}}(S_\varphi)$, page 24

$\mathcal{D}(S_\varphi)$ space of functions suitable for the natural f.c. for bounded sectorial operators, page 38

$\mathcal{F}(H_\varphi)$ space of quadratically decreasing holomorphic functions on the strip H_φ, page 81

$\mathcal{G}(H_\varphi)$ space of functions suitable for the natural f.c. for strip type operators, page 82

$H^\infty(\Omega)$ space of bounded holomorphic functions of the open set $\Omega \subset \mathbb{C}_\infty$, page 174

$H^\infty[S_\omega]$ abbreviation for $\bigcup_{\varphi>\omega} H^\infty(S_\varphi)$, page 24

$\mathcal{R}(K)$ space of rational functions with $r(K) \subset \mathbb{C}$, page 174

$\mathcal{R}^\infty(\Omega)$	space of rational functions bounded on Ω, page 175
$\mathcal{R}_0^\infty(\Omega)$	space of rational functions bounded on Ω and vanishing at ∞, page 175
$\mathcal{R}_A$	rational functions with all poles contained in $\varrho(A)$, page 146
$R(K)$	closure of $\mathcal{R}(K)$ in $\mathbf{C}(K)$, page 174

General Operator Theory

$\overline{A}$	closure of the m.v. operator A, page 139
$\lVert\cdot\rVert_A$	graph norm for the single-valued operator A, page 139
$A+B$	sum of the m.v. operators A and B, page 138
A^{-1}	inverse of the m.v. operator A, page 138
Ax	image of the point x under the m.v. operator A, page 137
A^n	natural powers of the m.v. operator A, page 145
$A\tilde{\sigma}(A)$	extended approximate point spectrum of the m.v. operator A, page 143
CA	composite of the m.v. operators A (first) and C (second), page 138
$\mathcal{D}(A)$	domain of the m.v. operator A, page 138
I	identity operator, page 138
$\mathcal{L}(X)$	space of bounded linear operators on the Banach space X, page 139
$\mathcal{L}(X)^\times$	set of bounded invertible operators on the Banach space X, page 139
$\mathcal{L}(X,Y)$	space of bounded linear operators from X to Y, page 139
$\mathcal{N}(A)$	kernel of the m.v. operator A, page 138
$P\tilde{\sigma}(A)$	extended point spectrum of the m.v. operator A, page 143
$\varrho(A)$	resolvent set of the m.v. operator A, page 140
$\mathcal{R}(A)$	range of the m.v. operator A, page 138
$R(\cdot,A)$	resolvent (mapping) of the m.v. operator A, page 140
$R\tilde{\sigma}(A)$	extended residual spectrum of the m.v. operator A, page 143
$\tilde{\sigma}(A)$	extended spectrum of the m.v. operator A, page 142
$\sigma(A)$	spectrum of the m.v. operator A, page 140
$S\tilde{\sigma}(A)$	extended surjectivity spectrum of the m.v. operator A, page 143

Operator Theory on Hilbert Spaces

$\alpha \leq A$	A is selfadjoint and $W(A) \subset [\alpha, \infty)$, page 159
$\overline{a}$	adjoint of the sesquilinear form a, page 154
$(\cdot \mid \cdot)$	scalar product on the Hilbert space H, page 154
$(\cdot \mid \cdot)_\circ$	equivalent scalar product on H, page 161
$a(u)$	shorthand for $a(u, u)$, page 154
$a \sim A$	the operator A is associated with the form a, page 100
A^*	Hilbert space adjoint of the m.v. operator A, page 156
A°	adjoint of A with respect to the equivalent scalar product $(\cdot \mid \cdot)_\circ$, page 161
a_λ	the sesquilinear form a shifted by λ, page 99
a_Q	sesquilinear form $(Q \cdot \mid \cdot)$ determined by the bounded linear operator $Q \in \mathcal{L}(H)$, page 161
H^*	space of continuous conjugate-linear functionals (= antidual) of H, page 161
$\operatorname{Im} a$	imaginary part of the sesquilinear form a, page 154
$\ell^2(H)$	Hilbert space direct sum of countably many copies of H, page 116
$\operatorname{Re} a$	real part of the sesquilinear form a, page 154
$\operatorname{Ses}(V)$	space of sesquilinear forms on the vector space V, page 154
$W(A)$	numerical range of the operator A, page 158

Particular Notation Related to the Functional Calculus

τ	the function $\frac{z}{(1+z)^2}$, page 33
Λ_A	the operator $\tau(A)^{-1} = (1 + A)A^{-1}(1 + A)$, page 33
$\log A$	the operator logarithm of the injective sectorial operator A, page 70
A^α	fractional power of the sectorial operator A, page 56
A^{is}	purely imaginary power of the injective sectorial operator A, page 71
$e^{-\lambda A}$	holomorphic semigroup generated by the m.v. sectorial operator A, page 66
$\operatorname{BIP}(X)$	the space of injective sectorial operators A on X such that $(A^{is})_{s\in\mathbb{R}}$ is a C_0-group, page 74

Γ_φ	positively oriented boundary of the sector S_φ, page 24
$\Gamma_{\varphi,\delta}$	positively oriented boundary of $S_\varphi \cup B_\delta(0)$, page 24
$C(f,\omega')$	a characteristic constant, determined by $f \in \mathcal{DR}_{\text{ext}}(S_\varphi)$ and $\omega' < \varphi$, page 40
θ_A	the group type of the C_0-group $(A^{is})_{s\in\mathbb{R}}$, page 74
A_ε	(standard) sectorial approximation for A, page 21
$\|\cdot\|_\psi$	the norm on H_ψ, page 110
$H(A)$	space of functions f such that $f(A)$ is a bounded operator, page 29
H_ψ	a subspace of the Hilbert space H defined by boundedness of certain square integrals, page 110
ω_A	spectral angle of the sectorial operator A, page 18
$\omega_{st}(A)$	spectral height of the strip type operator A, page 80
ψ_t	dilation of the function ψ by the positive factor t, page 43
$\psi_{a,b}$	functions used in McIntosh's approximation technique, page 43
$\mathrm{Sect}(\omega)$	class of all sectorial operators of angle ω on the Banach space X, page 17
$\mathrm{Strip}(\omega)$	set of strip type operators on X of height ω, page 79

Semigroup Theory

Cos	cosine function, page 129
$\theta(T)$	group type of the C_0-group T, page 153
$\omega_0(T)$	growth bound of the semigroup T, page 149
$s_0(A)$	abszissa of uniform boundedness, page 164
$T_\ominus$	backward semigroup of the group T, page 153
$T_\oplus$	forward semigroup of the group T, page 153

Others

ε_k	the k-th Rademacher function, page 109
$\mathcal{F}(f)$	Fourier transform of the function f, page 164
f^*	the conjugate function of f, page 103
$\mathrm{Rad}(X)$	the vecor space (algebraically) generated by the Rademacher functions, page 109

Zusammenfassung in deutscher Sprache

Die Dissertation gliedert sich in zwei Teile. Im ersten Teil ("Organon") wird die Theorie des natürlichen Funktionalkalküls für sektorielle Operatoren systematisch entwickelt (Kapitel 1) und anschließend auf die Theorie der fraktionären Potenzen, der holomorphen Halbgruppen und der Operatorlogarithmen angewandt (Kapitel 2). Einerseits hat dies vorbereitenden Charakter, da die vorgestellte Theorie grundlegend für die Kapitel des zweiten Teils sind. Andererseits ist diese Darstellung neu in ihrer Systematik, ebenso wie darin, dass konsequent auf Dichtheitsannahmen (über den Definitionsbereich und den Wertebereich des betrachteten Operators) verzichtet wird. Auch die Injektivität des Operators wird nicht verlangt, vielmehr wird der Funktionalkalkül entsprechend zusätzlichen Spektraleigenschaften des Operators (Injektivität, Invertierbarkeit, Beschränktheit) auf unterschiedliche Funktionenklassen erweitert. Dabei liegt ein besonderes Gewicht auf der sogenannten Kompositionsregel $(f \circ g)(A) = f(g(A))$, die bisher in der Literatur kaum Beachtung gefunden hat. Die "Brauchbarkeit" des Funktionalkalküls wird anschließend durch die stringente (und elegante) Entwicklung der fraktionären Potenzen, holomorphen Halbgruppen und Operatorlogarithmen belegt.

Der zweite Teil der Arbeit ("Problemata"), widmet sich Einzelresultaten. In Kapitel 3 wird mithilfe eines Funktionalkalküls für Operatoren vom Streifentyp das Resultat von NOLLAU über das Spektrum des Operatorlogarithmus ergänzt (Theorem 3.9). Als Konsequenz ergibt sich ein neuer Beweis und eine Verallgemeinerung eines gefeierten Satzes von PRÜSS und SOHR (Corollary 3.12 und Corollary 3.19) sowie ein Beispiel eines injektiven sektoriellen Operators A auf einem UMD Raum, der beschränkte imaginäre Potenzen hat, dessen BIP-Gruppe $(A^{is})_{s\in\mathbb{R}}$ aber einen Gruppentyp hat, der *größer als* π, und damit vom Sektorialitätswinkel von A verschieden ist (Corollary 3.24). Zuletzt wird eine Charakterisierung von Gruppengeneratoren auf Hilberträumen angegeben (Theorem 3.26), die einerseits ein Resultat von LIU verallgemeinert, andererseits ein Theorem von BOYADZHIEV und DELAUBENFELS über die Beschränktheit des H^∞-Kalküls auf horizontalen Streifen umfasst (Corollary 3.29). Als ein weiteres Korollar dieses Satzes erscheint das wichtige Resultat von MCINTOSH, dass für einen sektoriellen Operator auf einem Hilbertraum die Beschränktheit der imaginären Potenzen und die Beschränktheit des H^∞-Kalküls äquivalent sind (Corollary 3.31).

Das Kapitel 4 beginnt mit dem Problem der Charakterisierung von variationellen Operatoren modulo Ähnlichkeit (sog. "variationell-ähnlichen" Operatoren). Dabei heißt ein Operator auf einem Hilbertraum variationell, wenn er mittels einer elliptischen Form konstruiert werden kann. Die Lösung dieses Problems liefert ein Satz von FRANKS und LEMERDY der ursprünglich mithilfe eines tiefen Satzes von PAULSEN bewiesen wurde. Wir beweisen ein Resultat (Theorem 4.26), das es gestattet, den Franks-LeMerdy'schen Satz ohne Rekurs auf den Satz von PAULSEN zu gewinnen (Corollary 4.28). Außerdem wird gezeigt, dass man die Quadratwurzeleigenschaft eines variationellen Operators nach Wahl eines äquivalenten Skalarproduktes immer realisieren kann (Corollary 4.27).

In Kapitel 5 wird die "direkte Methode" von LIAPUNOV auf C_0-Gruppen auf Hilberträumen angewandt. Mit dieser Technik wird gezeigt, dass ein Gruppengenerator A sich immer in der Form $A = B + iC$ schreiben lässt, wobei B, C beide selbstadjungiert (bzgl. eines äquivalenten Skalarproduktes) sind und sowohl B als auch der Kommutator $[B, C]$ beschränkt ist (Theorem 5.9). Dieses Resultat ermöglicht einen neuen (sehr eleganten) Beweis des schon erwähnten Boyadzhiev-deLaubenfels'schen Resultats über die Beschränktheit des H^∞-Kalküls für Gruppengeneratoren (Theorem 5.12). Abschließend wird mithilfe des zu Beginn gezeigten Zerlegungssatzes und eines Theorems von FATTORINI folgendes Resultat bewiesen: Jeder Generator einer Kosinusfunktion auf einem Hilbertraum hat bezüglich eines äquivalenten Skalarproduktes seinen numerischen Wertebereich in einer horizontalen Parabel. Insbesondere ist er (bzgl. des neuen Skalarproduktes) variationell und besitzt die Quadratwurzeleigenschaft (Corollary 5.18 and Theorem 5.20).

Curriculum Vitae

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