Asymptotics, Regularity and Well-Posedness of First- and Second-Order Differential Equations on the Line

DISSERTATION
zur Erlangung des Doktorgrades Dr. rer. nat.
der Fakultät für Mathematik und Wirtschaftswissenschaften der Universität Ulm

von
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aus Stuttgart - Bad Cannstatt

2000

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Tag der mündlichen Prüfung: 17. Juli 2000

Contents

Introduction

The theory of differential equations in Banach spaces plays an important role in mathematics. In fact, frequently a partial differential equation can be transformed into an ordinary differential equation with values in an infinite dimensional space. But the theory has also direct application in Probability Theory, Mathematical Physics and Biology, Economics and other areas.

One important subject concerns the Cauchy problem. While the first-order initial value problem leads to the theory of C_0-semigroups (see in the monographs of Davies [46], Engel and Nagel [53], Fattorini [56], Goldstein [59], Hille and Phillips [63], van Neerven [92], Pazy [95] and the references therein), the second-order initial value problem corresponds to cosine families (see in Fattorini [57], Kisynski [69], Travis and Webb [115], Vasil'ev, Krein and Piskarev [118] and many others).

In contrast to this, we do not consider initial value problems, but study first- and second-order differential equations on the whole real line. Concerning the first-order case we consider the following equation

$$\text{(I)} \qquad u'(t) = Au(t) + f(t), \qquad t \in \mathbb{R},$$

where A is a closed linear operator in a Banach space E. Here, the given inhomogeneity f and the solution u are E-valued functions defined on the whole real line $\mathbb{R}$. The second-order differential equation on the line is defined by

$$\text{(II)} \qquad u''(t) = Au(t) + f(t), \qquad t \in \mathbb{R},$$

where A is again a closed linear operator in a Banach space E and the given function f and the solution u are vector-valued functions defined on $\mathbb{R}$.

This thesis consists of two parts, where in Part I the first- and in Part II the second-order differential equation on the line is considered. In each part, we first examine the most relevant operators for the considered equation. Those are bisectorial operators in Part I and sectorial operators in Part II. Then, we study in each case when the inhomogeneity f and the solution u are bounded uniformly continuous E-valued functions on $\mathbb{R}$, i.e. $u, f \in BUC(\mathbb{R}, E)$. Finally, we study in each part the situation when u and f belong to $L^p(\mathbb{R}, E)$, the space of all p-integrable function on $\mathbb{R}$ with values in E.

Thus, the structure of this thesis can be presented by the following diagram:

	PART I: First-Order Equations	**PART II:** Second-Order Equations
1. Relevant Operators	**Chapter 1:** Bisectorial Operators - definition - analytic semigroups - functional calculus & spectral projections - spectral decomposition	**Chapter 4:** Sectorial Operators - definition - functional calculus - fractional powers
2. Solutions in $BUC(\mathbb{R}, E)$	**Chapter 2:** - uniqueness - well-posedness - discrete spectrum - asymptotic behaviour	**Chapter 5:** - uniqueness - well-posedness - reduction to a 1^{st}-order system - the case: $A = B^2$ - asymptotic behaviour
3. Solutions in $L^p(\mathbb{R}, E)$	**Chapter 3:** - mild solutions - maximal regularity	**Chapter 6:** - mild solutions - maximal regularity

The content of each part is precisely described at the beginning of each part. Therefore, we mainly want to explain here the mathematical context and motivation of the thesis.

Given a differential equation, it is desirable to study well-posedness as well as properties of the solutions depending on the operator A and the inhomogeneity f. It turns out that especially sectorial, respectively bisectorial, operators have nice properties in this respect.

The class of sectorial operators plays independently an important role in functional analysis as well as in differential equations. McIntosh developed in [86] a functional calculus for such operators on Hilbert spaces which was extended to include more general Banach spaces by Cowling, Doust, McIntosh and Yagi [42]. There exists also a natural generalisation of this functional calculus for bisectorial operators which was done by Albrecht, Duong and McIntosh in [1] and by McIntosh and Yagi in [87]. Further work on functional calculi is given by Auscher, McIntosh and Nahmod [17, 18], Boyadzhiev and deLaubenfels [33], Duong and Robinson [52], Le Merdy [74] and Uiterdijk [117]. This calculus provides a possibility to define fractional powers, exponentials, logarithms, imaginary powers and other functions of these operators that can be applied in the theory of differential equations. We give the definition and state the basic properties of sectorial operators in Chapter 4.

In the case of bisectorial operators, one can also define spectral projections with this functional calculus (see for example in [87]). Moreover, Mielke examined bisectorial operators and defined two analytic semigroups corresponding to these operators in [88]. The speciality of these semigroups is that they need not be strongly continuous in 0. In Chapter 1 it is shown how these semigroups are connected to the spectral projections defined via the functional calculus. Here, one must distinguish if the spectral projections are bounded, then the Banach space E splits and there exists a spectral decomposition of the operator A into two generators of corresponding analytic C_0-semigroups. If the spectral projections are unbounded, then E becomes an intermediate space between two Banach spaces where a bounded spectral decomposition exists.

As said in the beginning, differential equations on the line are far less studied than initial value problems. In fact, even well-posedness, i. e. existence and uniqueness of solutions, was not completely understood, so far.

In [107] and [123], Vũ Quôc Phóng and Schüler have shown that the well-posedness of the first-order equation on the line is equivalent to the solvability of the following operator equation

$$AX - XD = -\delta_0$$

where A is the same operator as in Equation (I), D is the generator of the shift group on $BUC(\mathbb{R}, E)$, the space of all bounded uniformly continuous functions on $\mathbb{R}$ with values in E, and $\delta_0(f) = f(0)$. With the help of this equivalence, one can establish well-posedness of Equation (I) if the operator A is the generator of a hyperbolic C_0-semigroup (see [98]) or if A is a densely defined bisectorial operator with $0 \in \rho(A)$ (see Section 2.2 and Section 2.3).

This sort of operator equation is of independent interest and had therefore gone through a far longer development. It was first studied extensively for bounded operators by Daleckii and Krein [43], Lumer and Rosenblum [79] and Rosenblum [103] who also considered in [104] the case of selfadjoint possibly unbounded operators on a Hilbert space. The case when A and D are generators of C_0-semigroups was considered by Vũ Quôc Phóng [120] and Lin and Shaw [76] and by Arendt, Räbiger and Sourour [14].

Later, it was shown independently by Vũ Quôc Phóng and Schüler [108] and by the author [110] that well-posedness of the second-order equation on the line is equivalent to solvability of the operator equation (see also Section 5.2)

$$AX - XD^2 = -\delta_0.$$

This is the case under three different assumptions, namely when A is the generator of a uniformly exponentially stable C_0-semigroup (see [120]), when A is sectorial and invertible (see [123]) or when $A = B^2$ where B is the generator of a uniformly exponentially stable C_0-semigroup (see [110])(see also Section 5.3).

Fattorini [54], Travis and Webb [115] and Vasil'ev, Krein and Piskarev [118] have shown how the second-order initial value problem can be reduced to a first-order equation in the case when A is the generator of a cosine family. We show in Section 5.4 that this can be done also in the corresponding situation on the whole line. Moreover, in Section 5.5 we consider the special case, when $A = B^2$ and the first-order equation is well-posed for the operator B. Then we obtain solvability for the second-order equation for A and more "regularity" for the mild solutions of Equation (II) (see [110]).

Besides well-posedness, the asymptotic behaviour of solutions is an important subject in the theory of differential equations in Banach spaces.

A special kind of asymptotic behaviour is almost periodicity. The theory of scalar-valued almost periodic functions on $\mathbb{R}$ was already created during 1920's by Bohr in [29] and [30]. This theory has then been extended to vector-valued functions and was subject of intensive research (see for example in the monographs of Amerio and Prouse [4], Arendt, Batty, Hieber and Neubrander [11], Corduneanu [40], Levitan and Zhikov [75]). In the framework of one-parameter semigroups, almost periodicity was considered by Arendt and Batty [8, 9], Bart and Goldberg [20], Basit [23], Batty and Chill [25], Batty, van Neerven and Räbiger [26], Chill [37], van Neerven [92], Ruess and Vũ Quôc Phóng [106], Vesentini [119] and Vũ Quôc Phóng [122].

More generally, it is convenient to describe the asymptotic behaviour of solutions by saying u belongs belongs to a special translation-biinvariant subspace of $BUC(\mathbb{R}, E)$. This may be for example the space $C_0(\mathbb{R}, E)$ of all continuous functions vanishing at infinity, the space $W(\mathbb{R}, E)$ of all weakly almost periodic functions or the space $TE(\mathbb{R}, E)$ of all totally uniformly ergodic functions (for some more examples of translation-biinvariant subspaces of $BUC(\mathbb{R}, E)$ see Section 2.5). This is considered by Basit [23], Batty and Chill [25], Chill [36], Kreulich [71], Prüss and Ruess [100] and many others.

In connection with asymptotic behaviour, the spectra of the operator A and the inhomogeneity f play an important role. Many results are known where countability of the spectrum is required. One famous example concerning C_0-semigroups is the well-known Theorem of Arendt-Batty-Lyubich-Phong (see [7] and [82]) where spectral conditions on the operator A determine stability. Another central result in this context is Loomis' Theorem which says that a bounded uniformly continuous vector-valued function on

$\mathbb{R}$ with countable spectrum is almost periodic provided that $c_0 \not\subseteq E$. Loomis proved the scalar version in [78] which was later generalised to vector-valued functions (see for example [8], [11, Chapter 4] or [75, page 92]). The result is false if $c_0 \subseteq E$. After previous work by Baskakov [24], Arendt and Schweiker have shown in [15] that merely the accumulation points in the spectrum are responsible for the failure of Loomis' theorem in c_0. In fact, a bounded uniformly continuous function of $\mathbb{R}$ into a Banach space E with discrete spectrum is almost periodic without any further conditions on the Banach space E. We give a simple proof in Section 2.4 establishing some further properties concerning accumulation points of the spectrum of a function (see also [15]).

The main results concerning the asymptotic behaviour of solutions and the discreteness of the spectra are given in Section 2.5 and Section 5.6 (see also [15]). It is shown that bounded uniformly continuous solutions of the first-order Equation (I) with almost periodic inhomogeneity f are almost periodic provided that $\sigma(A) \cap i\mathbb{R}$ is discrete and $-i\sigma(A)$ contains no accumulation points of the spectrum of f. Similar results hold for the second-order Equation (II) and in the case of other translation-biinvariant subspaces than the space $AP(\mathbb{R}, E)$ of all almost periodic functions.

Next we turn to solutions in $L^p(\mathbb{R}, E)$ for $p \in (1, \infty)$ instead of $BUC(\mathbb{R}, E)$. We define and study mild solutions in $L^p(\mathbb{R}, E)$ which seems to be new (see Section 3.1 and Section 6.1). We establish appropriate spectral properties to obtain uniqueness of mild solutions. Moreover, we show that if the operator A is bisectorial, respectively sectorial, and invertible, then there exist unique mild solutions of Equation (I), respectively Equation (II).

The main interest is, of course, studying strong solutions and maximal L^p-regularity, i. e. the question whether for each inhomogeneity $f \in L^p(\mathbb{R}, E)$ exists a unique strong solution $u \in W^{1,p}(\mathbb{R}, E)$ (respectively $W^{2,p}(\mathbb{R}, E)$) for Equation (I) (respectively (II)).

As for the continuous solutions, also here the initial value problems of first-order was in the centre of mathematical research. The systematic study of this problem goes back to Grisvard [60] who obtained regularity results by replacing E with a suitable interpolation space and to Da Prato and Grisvard [44] who established maximal regularity for the first-order equation in the Sobolev spaces $W^{\theta,p}(\mathbb{R}, E)$ for $\theta \in (0, 1)$ and $p \in (1, \infty]$.

Since then, the theory of maximal regularity for the first-order initial value problem has seen a fast development. Necessary conditions for maximal regularity on $L^p(\mathbb{R}, E)$ were given by Dore in [48]. The p-independence of maximal regularity was first proved for Hilbert spaces by De Simon [47], and then later for general Banach spaces by Cannarsa and Vespri [35] and Coulhon and Lamberton [41].

The most difficult question however was to determine whether a given operator A which is a generator of an analytic semigroup on a Banach space E satisfies maximal L^p-regularity or not. De Simon [47] proved that this is the case if E is a Hilbert space. Dore and Venni [49] obtained the remarkable result that maximal regularity holds provided that E is a UMD space, A is invertible and admits bounded imaginary powers with an appropriate estimate (compare this also with the results from Monniaux [90]). On the other hand, Coulhon and Lamberton [41] found counterexamples on $E = L^2(\mathbb{R}, X)$ whenever X is not an UMD space, Le Merdy [74] gave counterexamples on fundamental spaces like $L^1(\mathbb{T})$, $C(\mathbb{T})$ and $K(l_2)$ and Kalton and Lancien [67] showed that there exist counterexamples to maximal regularity for a large class of Banach spaces which are not Hilbert spaces. Recently, Weis [124] obtained an operator valued version of Mikhlin's Theorem on UMD spaces provided the bounds in the Mikhlin conditions are replaced by $\mathcal{R}$-bounds. Via this result, Weis [124] and Clément and Prüss [39] gave a new characterisation of maximal regularity on UMD spaces.

Compared with the first-order initial value problem, the second-order case is far less examined. In recent time, Clément and Guerre-Delabrière [38] demonstrated the connections between the first-order equation and the second-order equation with A replaced by A^2. Moreover, Weis' multiplier theorem can also be applied to the second-order problem (see [39]).

Even less examined are differential equations on the line. In this context Mielke [88] proved that a necessary condition for maximal regularity in this case is that A is bisectorial with $0 \in \rho(A)$. Moreover, he obtained p-independence for the first-order equation on the line. These results are summarised in Section 3.2 and compared with results about initial value problems. Moreover, we apply the results from Weis also to this kind of equation.

We show in Section 6.2 that a necessary condition for maximal regularity of the second-order equation on the whole real line is that A is sectorial with $0 \in \rho(A)$. We show that the second-order equation is also independent of $p \in (1, \infty)$. Further, we give a characterisation of maximal regularity on UMD spaces via the notion of $\mathcal{R}$-bounds. In the Hilbert spaces case each sectorial operator with $0 \in \rho(A)$ satisfies maximal regularity, but we will also see that this in not the case in general. These results concerning maximal regularity of second-order differential equations on the line can also be found in [111].

Last, I want to point out that the main content of this thesis is published in [15], [110] and [111].

Acknowledgement.

At this point, it is my pleasure to express my thanks to at least some of the people without whom this "DISSERTATION" most probably would not have been finished.

First of all, of course, I would like to thank my advisor Prof. Dr. Wolfgang Arendt for his guidance and support in the last few years and for giving me the possibility to study in Ulm.

From my visit at the "Technische Universiteit Delft" in summer 1998, I profited a lot. I want to thank the "vakgroup algemene wiskunde" for the warm hospitality and, in particular, Dr. Jan van Neerven for his constant help and interest in my work.

Next, I wish to acknowledge the help by Dr. Sylvie Monniaux. The many discussions here in Ulm and during a research visit in Marseille in September 1999 were very helpful.

I do not want to forget the several short stays in Besançon during which I profited from the fruitful discussions with Dr. Florence Lancien and Prof. Dr. Gilles Lancien.

I am indebted to the "Konrad-Adenauer-Stiftung" for their support in the last few years. I profited from the cultural events with the scholar group of Ulm and my responsible local advisor Prof. Dr. Richard Brunner and from the interesting seminars which improved my political education. Evidently, I am most thankful for the financial support without which neither the study in Ulm nor the many research visits in foreign countries would have been possible.

All members of the "Abteilung Angewandte Analysis" helped me by being a vivid, open-minded group of nice people. Special thanks I want to express to Sonja Thomaschewski who did not mind sharing a room with me and for her personal and mathematical advises.

An important role in my mathematical life plays of course the TULKA-group. Especially, I owe thank to Prof. Dr. Frank Räbiger for leading me during my diploma and towards the thesis. Further, I very much appreciated how Prof. Dr. Rainer Nagel created the warm and friendly atmosphere in the TULKA-group. Moreover, I want to thank Prof. Dr. Lutz Weis for several nice discussions and for sending the one relevant page of his not yet published paper.

Finally, I am very thankful for the non-mathematical all-round support by my parents Heinz and Johanna Schweiker. Last, but not least, I thank Doreen for all what she did for me.

Part I

First-Order Differential Equations on the Line

In this part, we examine first-order differential equations on the whole real line $\mathbb{R}$, i.e.

$$\text{(I)} \qquad u'(t) = Au(t) + f(t), \qquad t \in \mathbb{R},$$

where A is a closed, linear operator on a Banach space E and $f : \mathbb{R} \to E$ is a vector-valued function on $\mathbb{R}$. If we want to specify the inhomogeneity f we sometimes write $(\mathrm{I})_f$ instead of (I).

In the following, we will denote by $D(A)$, $\sigma(A)$ and $\rho(A)$ the domain of A, the spectrum of A and the resolvent set of A. For $\lambda \in \rho(A)$ let $R(\lambda, A) = (\lambda - A)^{-1}$.

This part is divided in three chapters:

- Chapter 1: Bisectorial operators
- Chapter 2: Bounded uniformly continuous solutions
- Chapter 3: Solutions in $\mathrm{L}^\mathrm{p}(\mathbb{R},\mathrm{E})$

The bisectorial operators studied in Chapter 1 turn out to be very important in studying first-order equations on the line.

These operators are naturally connected with two analytic semigroups corresponding to the positive and the negative part of the spectrum of the bisectorial operator A. These semigroups need not be strongly continuous in 0, but the limit for $t \to 0$ is a projection on the underlying Banach space E, which however need not be bounded. These semigroups are examined in Section 1.2.

It is well known that for bisectorial operators a functional calculus exists (see [1] or [87]). With this functional calculus it is possible to define spectral projections corresponding to the bisectorial operator A. We show how these spectral projections are related to the projections mentioned above (see Section 1.3).

In section 1.4, we give examples of bisectorial operators for which the corresponding spectral projections defined via the functional calculus are bounded, respectively unbounded.

In Section 1.5, we study the case where the spectral projections of a bisectorial operator are bounded. In this case, the Banach space E can be splitted and there exists a spectral decomposition of the operator A.

But in generally the projections are unbounded. Then we introduce two Banach spaces where the induced spectral projections are bounded such that E becomes an intermediate space between two spaces where a splitting exists (see Section 1.6).

In Chapter 2, we study the case when the solution u and the inhomogeneity f belong to $BUC(\mathbb{R}, E)$, the space of bounded uniformly continuous functions on $\mathbb{R}$, with values in the Banach space E. First, we show that the solutions are unique if the imaginary axis is included in the resolvent set of the underlying operator A (see Section 2.1).

In Section 2.2, we give a necessary condition for existence and uniqueness of mild solutions, i.e. for well-posedness of Equation (I). Furthermore, we recall results by Vũ Quôc Phóng and Schüler ([107] and [120]) who show how Equation (I) is related to a suitable operator equation. Afterwards (in Section 2.3), examples of operators are given such that Equation (I) is well-posed. Bisectorial operators occur in this context, too.

A central result in the theory of asymptotic is Loomis' Theorem (see 2.4.8) which fails if $c_0 \subseteq E$. We will see, though, that merely the accumulation points in the spectrum are responsible for the failure of Loomis' theorem on c_0. In fact, in Section 2.4, we show that a bounded uniformly continuous function of $\mathbb{R}$ into a Banach space E with discrete spectrum is almost periodic without any further conditions on the Banach space E.

The main results about the asymptotic behaviour of mild solutions of Equation (I) is contained in Section 2.5. It is shown that bounded uniformly continuous solutions of first-order differential equations on the line with inhomogeneity f that satisfies a certain asymptotic behaviour have the same asymptotic behaviour provided that $\sigma(A) \cap i\mathbb{R}$ is discrete and $-i\sigma(A)$ contains no accumulation points of the spectrum of f.

In Chapter 3, we examine Equation (I) in the case when the solution u and the inhomogeneity f belong to $L^p(\mathbb{R}, E)$, the space of all E-valued and p-integrable functions on $\mathbb{R}$. We show that mild solutions in $L^p(\mathbb{R}, E)$ are unique if $\sigma(A) \cap i\mathbb{R} = \emptyset$ and give a necessary condition for existence and uniqueness (see Section 3.1).

In Section 3.2, we recall results about maximal $\mathrm{L^p}$-regularity of the first-order equation on the line and compare them with the results about maximal $\mathrm{L^p}$-regularity of first-order initial value problems. We see again that bisectorial operators play an important role. Moreover, we introduce $\mathcal{R}$-bounded families of operators and with the help of an operator valued Mikhlin Theorem (due to Weis [124]) we give a sufficient condition for maximal regularity in UMD spaces.

Chapter 1

Bisectorial operators

In this introductory chapter, we will study a special kind of closed, linear operators - the so called *bisectorial* operators. The name "bisectorial" comes from the fact that the spectrum of these operators is contained in a double sector. We will see later that these operators are very important in studying first-order differential equations on the line.

Throughout this chapter, let A be a closed, linear operator on a Banach space E.

1.1 Definition and basic properties

First of course, we give the definition of bisectorial operators.

Definition 1.1.1 A closed, linear operator A is called *bisectorial* if there exist $\theta \in [0, \frac{\pi}{2})$ and $c > 0$, such that

(i) $\sigma(A) \subseteq S_\theta := \{z \in \mathbb{C} \,:\, |\arg(\pm z)| \leq \theta\} \cup \{0\}$ and

(ii) $\|R(z, A)\| \leq \frac{c}{|z|}$ for all $z \in \mathbb{C} \setminus S_\theta$.

The *spectral angle* ϖ_A of a bisectorial operator is given by

$$\varpi_A := \inf\{\theta \in [0, \frac{\pi}{2}) \,|\, (i) \text{ and } (ii) \text{ hold}\}.$$

We will see later, i.e. in Chapter 2 and Chapter 3, that we are only interested in bisectorial operators such that 0 is included in the resolvent set. Hence, let in the following parts of this chapter A be a bisectorial operator with $0 \in \rho(A)$.

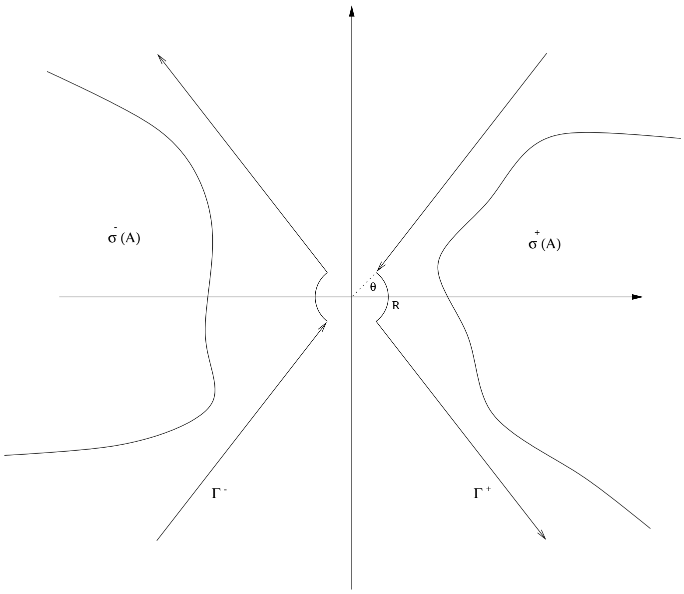
In this case, the spectrum of the bisectorial operator A can be separated in two spectral sets:

$$\begin{aligned} \sigma^-(A) &:= \{\lambda \in \sigma(A) \,:\, \mathcal{R}e(\lambda) < 0\} \\ \sigma^+(A) &:= \{\lambda \in \sigma(A) \,:\, \mathcal{R}e(\lambda) > 0\} \end{aligned} \tag{1.1}$$

Moreover, it is possible to define the curves $\Gamma^+_{\theta,R}$ and $\Gamma^-_{\theta,R}$ by:

$$\Gamma^-_{\theta,R} := \begin{cases} tRe^{\,i\theta}\,, & t \le -1 \\ -Re^{-i\theta t}, & -1 < t < 1 \\ -tRe^{-i\theta}\,, & t \ge 1 \end{cases} \quad \text{and} \quad \Gamma^+_{\theta,R} := \begin{cases} -tRe^{\,i\theta}\,, & t \le -1 \\ Re^{-i\theta t}, & -1 < t < 1 \\ tRe^{-i\theta}\,, & t \ge 1, \end{cases} \tag{1.2}$$

where $\theta > \varpi_A$ and $R > 0$ is chosen such that $B(R,0) = \{\lambda \in \mathbb{C} \,:\, |\lambda| \le R\} \subseteq \rho(A)$. The curves are oriented from $-\infty$ to $+\infty$. So that we are in the situation of the following picture.



From the picture it is easy to see, why these kind of operators are called "bisectorial". We have the following characterisation of bisectorial operators:

Lemma 1.1.2 *For a closed linear operator A, the following are equivalent:*

(i) The operator A is bisectorial with $0 \in \rho(A)$.

(ii) There exists a constant $b \ge 0$ such that $V_b := \{z \in \mathbb{C} \,:\, |\mathcal{R}e(z)| \le \frac{1}{b}(1 + |\Im m(z)|)\} \subseteq \rho(A)$ and $\|R(z,A)\| \le \frac{b}{1+|z|}$ for all $z \in V_b$.

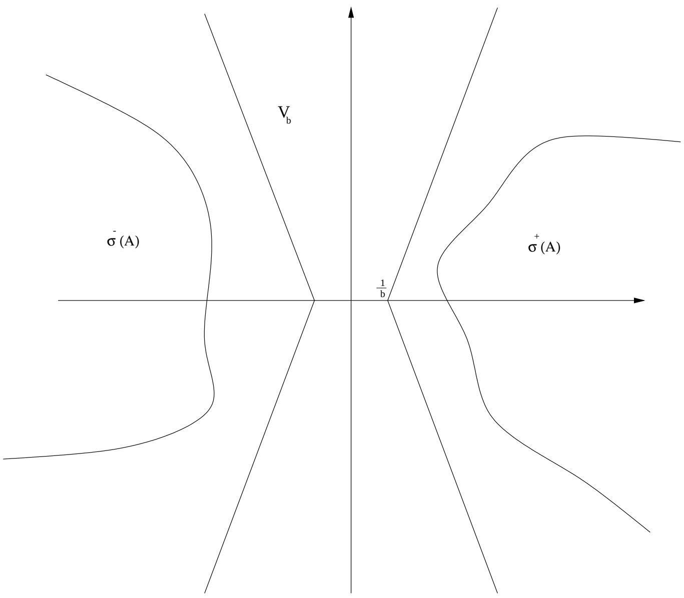
Proof. That, for a suitable choice of b, V_b is included in $\mathbb{C} \setminus S_\theta \cup \{0\}$ is obvious. And similarly, for a given $b \geq 0$ it is easy to find $\theta \in [0, \frac{\pi}{2})$ such that $\mathbb{C} \setminus S_\theta \cup \{0\}$ is included in V_b.

So that we just have to prove the estimates for the resolvent. $(ii) \Rightarrow (i)$ is trivial; for $(i) \Rightarrow (ii)$ consider

$$\begin{aligned} \|R(z,A)\| &= \|AR(z,A)A^{-1}\| \\ &\leq (\|zR(z,A)\| + 1)\|A^{1-}\| \\ &\leq (c+1)\|A^{-1}\|. \end{aligned}$$

Hence $(1 + |z|)\|R(z,A)\| \leq (c+1)\|A^{-1}\| + c =: b.$ $\square$

This setting is described in the following picture



1.2 Analytic semigroups related to bisectorial operators

Let A be again a bisectorial operator with $0 \in \rho(A)$.

With the curves $\Gamma^+_{\theta,R}$ and $\Gamma^-_{\theta,R}$ (see (1.2)), we can define the following operators on E.

$$T^-(t) := \frac{1}{2\pi i} \int_{\Gamma^-_{\theta,R}} e^{\lambda t} R(\lambda, A) d\lambda \qquad t > 0, \tag{1.3}$$

and

$$T^+(t) := \frac{1}{2\pi i} \int_{\Gamma^+_{\theta,R}} e^{-\lambda t} R(\lambda, A) d\lambda \qquad t > 0. \tag{1.4}$$

Since the operator A is bisectorial, these integrals converge in $\mathcal{L}(E)$. By Cauchy's Theorem, the definition of $(T^-(t))_{t>0}$, respectively $(T^+(t))_{t>0}$, is independent of the concrete choice of θ and R. Moreover, we will see that the operator families $(T^-(t))_{t>0}$ and $(T^+(t))_{t>0}$ are analytic semigroups in the following sense.

Definition 1.2.1 A semigroup $(T(t))_{t>0}$ of bounded linear operators on E is called *analytic*, if the mapping

$$(0, \infty) \to X : t \mapsto T(t)x$$

has an analytic extension to a sector $\Sigma_{\theta,0} := \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \theta\}$ for some $\theta > 0$ and for all $x \in X$.

Remark that the semigroup property holds then automatically in the whole sector $\Sigma_{\theta,0}$ and that $(T(z))_{z \in \Sigma_{\theta,0}}$ is strongly continuous. If $z \mapsto T(z)$ is also strongly continuous in $\Sigma_{\theta,0} \cup \{0\}$ and $\lim_{z\to 0} T(z)x = x$ for all $x \in X$ then $(T(t))_{t>0}$ is an analytic C_0-semigroup.

Proposition 1.2.2 *The operator families $(T^-(t))_{t>0}$ and $(T^+(t))_{t>0}$ define analytic semigroups on E.*

Proof. The semigroup-property follows from an application of Cauchy's Theorem, the Resolvent Equation and Fubini's Theorem like in the case of sectorial operators and corresponding analytic C_0-semigroups (see [53, Proposition 4.3]). For the proof of the analyticity of $(T^-(t))_{t>0}$, break up the integrals into three parts γ_1,γ_2 and γ_3 corresponding to $t \le -1$, $-1 < t < 1$ and $t \ge 1$ as in the definition of $\Gamma^-_{\theta,R}$ (see (1.2)). We obtain

$$\begin{aligned} \int_{\gamma_1} \|e^{\lambda t} R(\lambda, A)\| d\lambda &\le \int_{-\infty}^{-R} \|e^{re^{i\theta}t} R(re^{i\theta}, A) e^{i\theta}\| dr \\ &\le \int_R^{\infty} e^{-r\mathcal{R}e(e^{i\theta}t)} \frac{C}{r} dr. \end{aligned}$$

Hence, the integral converges absolutely and uniformly in t in each sector $S_{\mu,\epsilon} := \{t \in \mathbb{C} : |\arg(t)| < \mu \ \& \ |t| > \epsilon\}$, where $\mu < \frac{\pi}{2} - \theta$ and $\epsilon > 0$. Similarly, the same is true for the integral over γ_3 and also for the second integral, since the path of integration is bounded. It follows by a theorem of Weierstrass (see [102, p. 195]), that $(T^-(t))_{t\in S_{\frac{\pi}{2}-\theta,0}}$ is analytic. In a corresponding way, one shows the analyticity of $(T^+(t))_{t>0}$. □

Note, that the semigroups $(T^-(t))_{t>0}$ and $(T^+(t))_{t>0}$ are strongly continuous for $t > 0$, but in general they are not strongly continuous for $t \to 0$. In the next lemma, the behaviour of the semigroups in 0 and ∞ is discussed.

Lemma 1.2.3 *The semigroups $(T^{\mp}(t))_{t>0}$ are integrable, more exactly*

$$\|T^{\mp}(t)\| = O(|\ln(t)|) \qquad \textit{for } t \to 0$$

and $(T^{\mp}(t))_{t>0}$ are exponentially stable for $t \to \infty$.

Proof. By Lemma 1.1.2, there exists $b > 0$ such that $V_b \subseteq \rho(A)$. Let γ^- be the contour of V_b with $\mathcal{R}e(\gamma^-) < 0$, oriented in such a way that $\rho(A)$ lies to the right of γ^-. Then we obtain by Cauchy's Theorem that

$$\|T^-(t)\| = \frac{1}{2\pi}\|\int_{\gamma^-} e^{\lambda t}R(\lambda, A)d\lambda\| \le b' \int_{\frac{1}{b}}^{\infty} e^{-rt}\frac{1}{r}dr = b' \int_t^{\infty} \frac{1}{r}e^{-\frac{r}{b}}dr$$

for a constant $b' > 0$. It follows that $\|T^-(t)\| = O(|\ln(t)|)$ for $t \to 0$.

By substituting λ by $\lambda + c$ for $0 < c < \frac{1}{b}$, one obtains

$$\|T^-(t)\| = \frac{1}{2\pi}\|\int_{\gamma^-} e^{\lambda t}R(\lambda, A)d\lambda\| = \frac{1}{2\pi}e^{-ct}\|\int_{\gamma^-+c} e^{\mu t}R(\mu - c, A)d\mu\| \le Me^{-ct}.$$

Thus $(T^-(t))_{t>0}$ is exponentially bounded for $t \to \infty$.

Naturally, the same is true for $\|T^+(t)\|$. □

In the following, we show some more interesting properties about these semigroups:

Lemma 1.2.4 *For all $x \in D(A)$, the mapping $t \mapsto T^{\mp}(t)x$ is differentiable for all $t > 0$ and*

$$\frac{d}{dt}T^{\mp}(t)x = \pm AT^{\mp}(t)x.$$

Proof. Let $x \in D(A)$ and $t > 0$, then we obtain for $(T^-(t))_{t>0}$

$$\begin{aligned}
\lim_{s\to 0}\frac{T^-(t+s)x - T^-(t)x}{s} &= \lim_{s\to 0}\frac{1}{2\pi i}\int_{\Gamma^-_{\theta,R}} \frac{e^{\lambda(t+s)} - e^{\lambda t}}{s}R(\lambda, A)xd\lambda \\
&= \frac{1}{2\pi i}\int_{\Gamma^-_{\theta,R}} \lambda e^{\lambda t}R(\lambda, A)xd\lambda \\
&= \frac{1}{2\pi i}\int_{\Gamma^-_{\theta,R}} e^{\lambda t}xd\lambda + \frac{1}{2\pi i}A\int_{\Gamma^-_{\theta,R}} e^{\lambda t}R(\lambda, A)xd\lambda.
\end{aligned}$$

By closing the curve $\Gamma^-_{\theta,R}$ by circles with increasing diameter on the left, Cauchy's Theorem implies that $\int_{\Gamma^-_{\theta,R}} e^{\lambda t} x d\lambda = 0$. Thus, we conclude

$$\lim_{s\to 0} \frac{T^-(t+s)x - T^-(t)x}{s} = AT^-(t)x.$$

Similarly, we obtain for $(T^+(t))_{t>0}$

$$\begin{aligned}\lim_{s\to 0} \frac{T^+(t+s)x - T^+(t)x}{s} &= -\frac{1}{2\pi i}\int_{\Gamma^+_{\theta,R}} \lambda e^{-\lambda t} R(\lambda, A)x d\lambda \\ &= -AT^+(t)x.\end{aligned}$$

□

Lemma 1.2.5 *Let A be a bisectorial operator and let $(T^-(t))_{t>0}$, respectively $(T^+(t))_{t>0}$, be the corresponding semigroups. Then $(T^-(t))_{t>0}$ and $(T^+(t))_{t>0}$ commute and*

$$T^-(t)T^+(s) = 0$$

for all $t, s > 0$.

Proof. Let $x \in E$ and $t, s > 0$. Then we obtain by the Resolvent Equation and Fubini's Theorem that

$$\begin{aligned}&T^-(t)T^+(s)x \\ &= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma^-_{\theta,R}} \int_{\Gamma^+_{\theta,R}} e^{\lambda t} e^{zs} R(\lambda, A)R(z, A)x\, dz\, d\lambda \\ &= \frac{1}{-4\pi^2}\left(\int_{\Gamma^+_{\theta,R}} e^{zs} \int_{\Gamma^-_{\theta,R}} \frac{e^{\lambda t}}{\lambda - z} d\lambda\, R(z, A)x\, dz - \int_{\Gamma^-_{\theta,R}} e^{\lambda t} \int_{\Gamma^+_{\theta,R}} \frac{e^{zt}}{\lambda - z} dz\, R(\lambda, A)x\, d\lambda\right) \\ &= 0,\end{aligned}$$

where the last equality follows from Cauchy's Theorem if we close the curves $\Gamma^-_{\theta,R}$ and $\Gamma^+_{\theta,R}$ by circles with increasing diameter. Similarly, one shows that $T^+(s)T^-(t) = 0$ for all $t, s > 0$ □

Corresponding to these analytic semigroups, we can define the following two operators Q^- and Q^+ on E.

Definition 1.2.6 For a bisectorial operator A, define the *initial projections* by $D(Q^-) := \{x \in X : \lim_{t\to 0} T^-(t)x \text{ exists}\}$,

$$Q^-x := \lim_{t\to 0} T^-(t)x \qquad \forall x \in D(Q^-), \tag{1.5}$$

respectively $D(Q^+) := \{x \in X : \lim_{t\to 0} T^+(t)x \text{ exists}\}$ and

$$Q^+x := \lim_{t\to 0} T^+(t)x \qquad \forall x \in D(Q^+). \tag{1.6}$$

In Theorem 1.3.9, we will see that $D(Q^\pm)$ is not empty, in fact $D(A) \subseteq D(Q^\pm)$.

It turns out that these operators are indeed projections on E in the following sense.

Definition 1.2.7 A linear operator P on a Banach space E with domain $D(P)$ is called *projection*, if $D(P) = D(P^2) = \{x \in D(P) | Px \in D(P)\}$ and $Px = P^2x$ for all $x \in D(P)$.

Remark, that a projection P need not be bounded nor closed.

Proposition 1.2.8 *The operators Q^- and Q^+ are projections on E such that*

$$Q^+Q^- = 0 = Q^-Q^+.$$

Proof. From the following calculations

$$\begin{aligned} (Q^-)^2x &= \lim_{t\to 0} T^-(t) \lim_{s\to 0} T^-(s)x \\ &= \lim_{t\to 0} \lim_{s\to 0} T^-(t+s)x \\ &= \lim_{t\to 0} T^-(t)x \;=\; Q^-x, \end{aligned}$$

we obtain that $D((Q^-)^2) = D(Q^-)$ and $(Q^-)^2x = Q^-x$ for all $x \in D(Q^-)$. Similarly, one shows that $(Q^+)^2 = Q^+$. Furthermore, it follows by Lemma 1.2.5 that

$$Q^-Q^+x = \lim_{t\to 0} \lim_{s\to 0} T^-(t)T^+(s)x = 0 = Q^+Q^-x.$$

$\square$

1.3 Spectral projections

In this section, we consider two kinds of projections. First, the spectral projections arising from a functional calculus for bisectorial operators, and second, the initial projections arising from the analytic semigroups defined in the previous section (see Definition 1.2.6).

It is well-known that the more familiar functional calculus for sectorial operators (see Chapter 4) can be generalised to bisectorial operators. This is mentioned in [1, (H)], [17], [18], [86, Section 10] or [87, Section 2]. But since this fact in never described in details, we develop the functional calculus for bisectorial operators with $0 \in \rho(A)$ omitting the proofs which are the same as in the case of sectorial operators.

In the following, let A be a densely defined, bisectorial operator with $0 \in \rho(A)$ and spectral angle ϖ_A. Define the following sectors on the complex plane by

$$\begin{aligned} S^-_{\mu,r} &:= \{z \in \mathbb{C} : |\arg(-z)| < \mu \;\&\; |z| > r\}, \\ S^+_{\mu,r} &:= \{z \in \mathbb{C} : |\arg(z)| < \mu \;\&\; |z| > r\} \quad \text{and} \\ S_{\mu,r} &:= S^-_{\mu,r} \cup S^+_{\mu,r}, \end{aligned}$$

where $\mu \in (\varpi_A, \frac{\pi}{2})$ and $r > 0$ are chosen such that $\sigma(A) \subseteq S_{\mu,r}$. Define the following sets of holomorphic functions on $S_{\mu,r}$:

$$\begin{aligned} H(S_{\mu,r}) &:= \{f : S_{\mu,r} \to \mathbb{C} : f \text{ holomorphic}\}, \\ H^\infty(S_{\mu,r}) &:= \{f \in H(S_{\mu,r}) : \|f\|_\infty < \infty\}, \\ H^\infty_0(S_{\mu,r}) &:= \{f \in H(S_{\mu,r}) : \exists s > 0 : f\varphi^{-s} \in H^\infty(S_{\mu,r})\} \quad \text{and} \\ \mathcal{F}(S_{\mu,r}) &:= \{f \in H(S_{\mu,r}) : \exists s > 0 : f\varphi^{s} \in H^\infty(S_{\mu,r})\}, \end{aligned}$$

where $\varphi(\xi) := \frac{1}{\xi}$. It is easy to see that

$$H^\infty_0(S_{\mu,r}) \subseteq H^\infty(S_{\mu,r}) \subseteq \mathcal{F}(S_{\mu,r}) \subseteq H(S_{\mu,r}).$$

Definition 1.3.1 Let A be a densely defined, bisectorial operator with $0 \in \rho(A)$, $\varpi_A < \theta < \mu < \frac{\pi}{2}$ and $0 < r < R$ such that $B(R,0) \subseteq \rho(A)$. Then define the bounded linear operator $f(A)$ for $f \in H^\infty_0(S_{\mu,r})$ by

$$\begin{aligned} f(A) &:= \frac{1}{2\pi i} \int_{\Gamma_{\theta,R}} f(\lambda) R(\lambda, A) d\lambda \\ &= \frac{1}{2\pi i} \left(\int_{\Gamma^-_{\theta,R}} f(\lambda) R(\lambda, A) d\lambda + \int_{\Gamma^+_{\theta,R}} f(\lambda) R(\lambda, A) d\lambda \right). \end{aligned}$$

The integrals are absolutely norm convergent in $\mathcal{L}(E)$, since $\|f(\lambda)R(\lambda, A)\| \leq \frac{const}{|\lambda|^s(1+|\lambda|)}$ for an $s > 0$. Hence $f(A) \in \mathcal{L}(E)$. By an application of Cauchy's Theorem, one sees that the definition is independent of the choice of $\theta \in (\varpi_A, \mu)$ and $R > r$.

Moreover, $H^\infty_0(S_{\mu,r})$ is an algebra and, as in the case of sectorial operators (see for example in [1], [42] or [117]), it can be shown that the mapping $f \mapsto f(A)$ is an algebra homomorphism such that $\sigma(f(A)) = f(\sigma(A)$. Finally, this definition coincides with the usual Dunford calculus if $A \in \mathcal{L}(E)$.

We can give a Convergence Lemma (compare with [42, Lemma 2.1], [86, Section 4] or [117, Section 2.2.2]) for this functional calculus. Remark that the proof is similar to that for sectorial operators.

Lemma 1.3.2 *Let A be a densely defined, bisectorial operator with $0 \in \rho(A)$. Let $(f_\alpha)_\alpha$ be a uniformly bounded net in $H_0^\infty(S_{\mu,r})$ with $\|f_\alpha\|_\infty \to 0$. Then:*

a) If there exist $c, s > 0$ such that $|f_\alpha(\xi)| \leq \frac{c}{|\xi|^s}$ for all $\xi \in S_{\mu,r}$ and all α, then $\|f_\alpha(A)\| \to 0$.

b) If there exists $M \geq 0$ such that $\|f_\alpha(A)\| \leq M$ for all α, then $f_\alpha(A)u \to 0$ for all $u \in E$.

Proof. Since the integrals $\| \int_{\Gamma_{\theta,R}} f_\alpha(\lambda) R(\lambda, A) d\lambda\| \leq \int_{\Gamma_{\theta,R}} \frac{c}{|\lambda|^s} \frac{c'}{1+|\lambda|} d\lambda$ converge uniformly in α, there exist for each $\epsilon > 0$ an $r > 0$ such that

$$\int_{\Gamma^r} \|f_\alpha(\lambda) R(\lambda, A)\| d\lambda < \epsilon$$

for all α, where $\Gamma^r := \{z \in \Gamma_{\theta,R} : |z| > r\}$. Moreover,

$$\| \int_{\Gamma_{\theta,R} \setminus \Gamma^r} f_\alpha(\lambda) R(\lambda, A) d\lambda\| \leq \|f_\alpha\|_\infty \int_{\Gamma_{\theta,R} \setminus \Gamma^r} \|R(\lambda, A)\| d\lambda \to 0.$$

Hence, $\|f_\alpha(A)\|_\infty \to 0$ which proves a).

For the proof of b), let $g_\alpha(\xi) := \frac{f_\alpha(\xi)}{\xi}$. Then for $c := \sup_\alpha \|f_\alpha\|_\infty > 0$, we obtain $|g_\alpha(\xi)| \leq \frac{c}{|\xi|}$ for all $\xi \in S_{\mu,r}$ and all α. It follows from a) that $\|g_\alpha(A)\|_\infty \to 0$. Now, let $x \in D(A)$. Then there exists $y \in E$ such that $x = R(0, A)y$. We obtain by the Resolvent Equation and Cauchy's Theorem that

$$\begin{aligned} f_\alpha(A)x &= \frac{1}{2\pi i} \int_{\Gamma_{\theta,R}} f_\alpha(\lambda) R(\lambda, A) R(0, A) y d\lambda \\ &= \frac{1}{2\pi i} \int_{\Gamma_{\theta,R}} \frac{f_\alpha(\lambda)}{\lambda} d\lambda\; R(0, A)y - \frac{1}{2\pi i} \int_{\Gamma_{\theta,R}} g_\alpha(\lambda) R(\lambda, A) y d\lambda \\ &= -g_\alpha(A)y \quad \to 0. \end{aligned}$$

Since $D(A)$ is dense and $f_\alpha(A)$ uniformly bounded, it follows that $f_\alpha(A)u \to 0$ for all $u \in E$. $\square$

Next, we will define $f(A)$ for $f \in \mathcal{F}(S_{\mu,r})$. In order to do so, consider again the function $\varphi(\xi) = \frac{1}{\xi}$. Note, that $\varphi \in H_0^\infty(S_{\mu,r})$, hence $\varphi(A) \in \mathcal{L}(E)$ and that $\varphi(A)$ is injective with dense range. Thus, $\varphi(A)^{-1}$ exists and is a closed operator on E.

Definition 1.3.3 For $f \in \mathcal{F}(S_{\mu,r})$ choose $k \in \mathbb{N}$ such that $f\varphi^k \in H_0^\infty(S_{\mu,r})$ and define

$$f(A) := \varphi(A)^{-k}(f\varphi^k)(A),$$

where $D(f(A)) := \{x \in E \mid (f\varphi^k)(A)x \in D(\varphi(A)^{-k})\}$.

Again, as in the case of sectorial operators (see again in [1, (D)], [42, Section 2] or [117, Section 2.2]), we obtain that $f(A)$ is a well-defined, i. e. independent of $k \in \mathbb{N}$, densely defined, closed linear operator. Remark, that $f(A)$ may be unbounded even if f is bounded. If $f \in H_0^\infty(S_{\mu,r})$ then Definition 1.3.1 and Definition 1.3.3 coincide. Moreover, if $f, g \in \mathcal{F}(S_{\mu,r})$ and $\alpha, \beta \in \mathbb{C}$, then $\alpha f(A) + \beta g(A)$ and $f(A)g(A)$ are closable and

$$\begin{aligned} (\alpha f + \beta g)(A) &= \overline{\alpha f(A) + \beta g(A)} \quad \text{and} \\ (fg)(A) &= \overline{f(A)g(A)}. \end{aligned}$$

From this identities, it follows that $\varphi(A) = A^{-1}$. Thus, $D(f(A)) := \{x \in E \mid (f\varphi^k)(A)x \in D(A^k)\}$ and for all $x \in D(A^k)$ we have the representation

$$f(A)x = A^k(f\varphi^k)(A)x = \frac{1}{2\pi i}\int_{\Gamma_{\theta,R}} \frac{f(\lambda)}{\lambda^k} R(\lambda, A)A^k x d\lambda. \tag{1.7}$$

In the case of bisectorial operators, there exist also a Convergence Theorem which is a generalisation of the Convergence Lemma (Lemma 1.3.2) to functions in $H^\infty(S_{\mu,r})$. The proof is again similar to the case of sectorial operators (compare with [1, Theorem D] and [117, 2.2.2]).

Theorem 1.3.4 *Let A be a densely defined, bisectorial operator with $0 \in \rho(A)$. Let $(f_\alpha)_\alpha$ be a uniformly bounded net in $H^\infty(S_{\mu,r})$, such that there exists $M > 0$ with $\|f_\alpha(A)\| \leq M$ for all α. Furthermore, let $f \in H^\infty(S_{\mu,r})$ such that* $\sup\{|f_\alpha(\xi) - f(\xi)| : \xi \in S_{\mu,r}, |\xi| \leq r'\} \to 0$ *for all $r' > 0$. Then $f_\alpha(A)u \to f(A)u$ for all $u \in E$, $f(A) \in \mathcal{L}(E)$ and $\|f(A)\| \leq M$.*

Proof. Let $g_\alpha(\xi) := \frac{f_\alpha(\xi) - f(\xi)}{\xi}$. Then $g_\alpha \in H^\infty(S_{\mu,r})$, $\|g_\alpha\|_\infty \to 0$ and there exists $c > 0$ such that $|g_\alpha(\xi)| \leq \frac{c}{|\xi|}$ for all $\xi \in S_{\mu,r}$ and all α. By Lemma 1.3.2, it follows that $\|g_\alpha(A)\|_\infty \to 0$. Hence, $f_\alpha(A)A^{-1}u = (f_\alpha\varphi)(A) \to (f\varphi)(A)u = f(A)A^{-1}u$. Since $R(A^{-1}) = D(A)$ is dense and $\|f_\alpha(A)\| \leq M$ is uniformly bounded, the statement follows. $\square$

Finally, note that we could choose the function $\varphi(\xi) = \frac{1}{\xi}$ in this manner since $0 \in \rho(A)$. If one replaces the function φ for example by the function $\tilde{\varphi}(\xi) := \frac{\xi}{(i+\xi)^2}$ (note, that $-i \in \rho(A)$), then, one achieves a functional calculus for more general bisectorial operators, i.e. 0 need not be included in the resolvent set.

Next, we apply the functional calculus described above to define two operators corresponding to a bisectorial operator A. Since the operator A is bisectorial, one can separate the spectrum in $\sigma(A) = \sigma^-(A) \cup \sigma^+(A)$ (see (1.1)).

Definition 1.3.5 Let A be a bisectorial operator on E. Then define the *spectral projections corresponding to* A by

$$P^- := \chi^-(A) \qquad \text{and} \qquad P^+ := \chi^+(A), \tag{1.8}$$

where the functions χ^-, respectively χ^+, are defined by

$$\chi^-(\xi) := \begin{cases} 0, & \mathcal{R}e(\xi) > 0 \\ 1, & \mathcal{R}e(\xi) < 0 \end{cases} \quad \text{and} \quad \chi^+(\xi) := \begin{cases} 1, & \mathcal{R}e(\xi) > 0 \\ 0, & \mathcal{R}e(\xi) < 0. \end{cases} \tag{1.9}$$

Clearly, $\chi^-, \chi^+ \in H^\infty(S_{\mu,r}) \subseteq \mathcal{F}(S_{\mu,r})$ for all $\mu \in (0, \frac{\pi}{2})$ and all $r > 0$, thus the definition makes sense for every bisectorial operator A.

It follows from the properties of the functional calculus that for $P^\mp$ the following properties hold (see also [1, (H)] and [87]):

Proposition 1.3.6 *Let A be a densely defined, bisectorial operator with $0 \in \rho(A)$ and $P^\mp$ be the spectral projections corresponding to A as defined above in (1.8). Then $P^\mp$ are closed, densely defined, linear operators on the Banach space E and*

- $D(A) \subseteq D(P^-) = D(P^+) =: D$.
- $P^+ + P^- = I|_D$.
- $(P^-)^2 = P^-$ *and* $(P^+)^2 = P^+$, *i.e.* $P^\mp$ *are projections.*
- $P^+ \cdot P^- = P^+ \cdot P^- = 0|_D$.

Thus, the domain D of $P^\mp$ is given by

$$D = \{x \in E \mid \int_{\Gamma^-_{\theta,R}} \frac{R(\lambda, A)x}{\lambda} d\lambda \in D(A)\} = \{x \in E \mid \int_{\Gamma^+_{\theta,R}} \frac{R(\lambda, A)x}{\lambda} d\lambda \in D(A)\}.$$

Note, that the spectral projections $P^{\mp}$ may be unbounded, although $\chi^{\mp}$ are bounded.

For example, if A is bounded and $\sigma(A) \cap i\mathbb{R} = \emptyset$, then the spectral projections are bounded by the Dunford functional calculus (see [51]). Further, if A is the generator of an analytic semigroup and $\sigma(A) \cap i\mathbb{R} = \emptyset$, then it follows as well that the spectral projections are bounded (see Example 1.4.1). The same is true if $-A$ generates an analytic semigroup (Example 1.4.2). But, there exist also examples for bisectorial operators such that the spectral projections are unbounded (see [87] and Example 1.4.3).

Remark 1.3.7 Grisvard has shown in [61] that the spectral projections $P^{\pm}$ are linear, continuous operators on the interpolation spaces $D_A(\theta, p)$ for every $\theta \in (0,1)$ and all $p \in [1, \infty]$.

In the following example, we will show that, in general, the spectral projections corresponding to a bisectorial operator A do not coincide with the spectral projections $P_{\sigma^{\pm}(A)}$ on bounded spectral sets if the spectrum of A is bounded. Here, $P_{\sigma_1(A)}$ for a bounded spectral set $\sigma_1(A)$ is given by

$$P_{\sigma_1(A)} = \frac{1}{2\pi i} \int_{\gamma} R(\lambda, A) d\lambda, \tag{1.10}$$

where γ is a closed Jordan curve surrounding $\sigma_1(A)$.

Denote by $(J(t))_{\mathcal{R}e(t)>0}$ the Liouville semigroup on $L^p(0,1)$ for one $p \geq 1$ with generator A_J (see [63, p.663]). Recall, that $(J(t))_{\mathcal{R}e(t)>0}$ is a holomorphic semigroup with $\omega(J) = -\infty$, thus $\sigma(A_J) = \emptyset$.

Example 1.3.8 *Let A_J be the generator of the Liouville semigroup on $Y = L^p(0,1)$ for one $p \geq 1$. Further, let B be a bounded operator on a Banach space Z with $\sigma(B) \cap i\mathbb{R} = \emptyset$. Let $X = Y \oplus Z$ be the direct sum and $A = (A_J, B)$ with maximal domain. Then A is bisectorial, but*

$$P^{-} \neq P_{\sigma^{-}(A)}.$$

Proof. Since A_J is sectorial of spectral angle $\omega_{A_J} = 0$, it is easy to see that A_J is bisectorial also with spectral angle $\varpi_{A_J} = 0$ (see Chapter 4, Remark 4.1.2). Since B is bounded with $\sigma(B) \cap i\mathbb{R} = \emptyset$, it follows that B is bisectorial with suitable spectral angle ϖ_B. Thus, A is bisectorial with spectral angle $\varpi_A = \varpi_B$.

Since A_J is an analytic semigroup, it follows that $P^+|_Y = \chi^+(A_J) = 0_Y$ and $P^-|_Y = \chi^-(A_J) = I_Y$. Moreover, let γ^-, be a bounded, closed Jordan curve surrounding $\sigma^-(A) = \sigma^-(B)$. From $\sigma(A_J) = \emptyset$, it follows that

$$\int_{\gamma^-} R(\lambda, A_J)d\lambda = 0_Y.$$

By the boundedness of B it follows that $P^-|_Z = \chi^-(B) = P_{\sigma^-(B)}$. Summarising, we obtain that

$$P^- = \chi^-(A) = (I_Y, P_{\sigma^-(B)}),$$

but

$$P_{\sigma^-(A)} = (0_Y, P_{\sigma^-(B)}),$$

which is different. □

In the remaining part of this section, we show how the projection mappings $Q^\mp$, defined in Section 1.2, and the spectral projections $P^\mp$ are related to each other.

Theorem 1.3.9 *Let A, $Q^\mp$ and $P^\mp$ as above. Then $D(A) \subseteq D(Q^\mp) \subseteq D(P^\mp) = D$, $Q^-x = P^-x$ for all $x \in D(Q^-)$ and $Q^+x = P^+x$ for all $x \in D(Q^+)$. Hence, $Q^\mp$ are closable. Moreover, it holds that*

$$\overline{Q^-} = P^- \qquad \textit{and} \qquad \overline{Q^+} = P^+.$$

Proof. Let $\varpi_A < \theta < \mu$, $0 < r < R$ as usual, and $(f_t)_{t>0} \subseteq H^\infty(S_{\mu,r})$ be a uniformly bounded net defined by

$$f_t(\xi) := \chi^-(\xi)(e^{\xi t} - 1) = \begin{cases} 0 & , \ \mathcal{R}e(\xi) > 0 \\ e^{\xi t} - 1 & , \ \mathcal{R}e(\xi) < 0. \end{cases}$$

Further, let $g_t(\xi) := \frac{f_t(\xi)}{\xi}$. Then $g_t \in H_0^\infty(S_{\mu,r})$, $\|g_t\|_\infty \to 0$ as $t \to 0$ and there exist $c := \sup_{t>0}\{\|f_t\|_\infty\} > 0$ such that $|g_t(\xi)| \leq \frac{c}{|\xi|}$ for all $\xi \in S_{\mu,r}$ and all $t > 0$. It follows from the Convergence Lemma 1.3.2 that $\|g_t(A)\|_\infty \to 0$ as $t \to 0$. Now let $x \in D(A)$ and we obtain

$$\begin{aligned} T^-(t)x - P^-x &= \frac{1}{2\pi i}\int_{\Gamma^-_{\theta,R}} e^{\lambda t}R(\lambda, A)xd\lambda - \frac{1}{2\pi i}A\int_{\Gamma^-_{\theta,R}} \frac{\chi^-(\lambda)}{\lambda}R(\lambda, A)xd\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma^-_{\theta,R}} \frac{e^{\lambda t}}{\lambda}R(\lambda, A)Axd\lambda - \frac{1}{2\pi i}\int_{\Gamma^-_{\theta,R}} \frac{1}{\lambda}R(\lambda, A)Axd\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma^-_{\theta,R}} \frac{e^{\lambda t} - 1}{\lambda}R(\lambda, A)Axd\lambda \\ &= g_t(A)Ax \\ &\longrightarrow 0 \qquad \text{as } t \to 0. \end{aligned}$$

Hence, $\lim_{t\to 0} T^-(t)x$ exists and equals P^-x, thus $D(A) \subseteq D(Q^-)$.

Now, let $x \in D(Q^-)$ and $\mu \in \rho(A)$. With similar considerations as above, we obtain

$$\begin{aligned} &\frac{1}{2\pi i} AR(\mu, A) \int_{\Gamma^-_{\theta,R}} \frac{R(\lambda, A)}{\lambda} x d\lambda \\ &= \frac{1}{2\pi i} R(\mu, A) \int_{\Gamma^-_{\theta,R}} e^{\lambda t} R(\lambda, A) x d\lambda - AR(\mu, A) g_t(A) x \end{aligned}$$

for all $t > 0$. Letting $t \to 0$, we obtain

$$\frac{1}{2\pi i} AR(\mu, A) \int_{\Gamma^-_{\theta,R}} \frac{R(\lambda, A)}{\lambda} x d\lambda = R(\mu, A) Q^- x \in D(A).$$

It follows that $x \in D(P^-) = D$ and $Q^-x = P^-x$ for all $x \in D(Q^-)$, hence, Q^- is closable and $\overline{Q^-} \subseteq P^-$

For $x \in D(P^-) = D$ and $n \in \mathbb{N}$, define $x_n = inR(in, A)x \in D(A) \subseteq D(Q^-)$. It follows for $x \in D(A)$ that

$$\begin{aligned} \|inR(in, A)x - x\| &= \|AR(in, A)x\| \le \|R(in, A)\| \|Ax\| \le \frac{c}{n} \|Ax\| \\ &\longrightarrow 0 \qquad \text{as } n \to \infty. \end{aligned}$$

Since $D(A)$ is dense and $inR(in, A)$ is uniformly bounded, we obtain that $\lim_n \|x_n - x\| = \lim_n \|inR(in, A)x - x\| = 0$. Moreover,

$$\begin{aligned} P^-x_n - P^-x &= \frac{1}{2\pi i} A \int_{\Gamma^-_{\theta,R}} \frac{R(\lambda, A)}{\lambda} (x_n - x) d\lambda \\ &= \frac{1}{2\pi i} A \int_{\Gamma^-_{\theta,R}} \frac{R(\lambda, A)}{\lambda} AR(in, A) x d\lambda \\ &= AR(in, A) P^- x \\ &= inR(in, A) P^- x - P^- x \\ &\longrightarrow 0 \qquad \text{as } n \to \infty. \end{aligned}$$

It follows that $\lim_n \|x_n - x\|_E = 0$ and hence, $\overline{Q^-} = P^-$, which proves the theorem for Q^-. With similar considerations we obtain the results for Q^+, too. $\square$

We will see in Section 1.5 and Section 1.6 that actually $Q^\pm$ and $P^\pm$ are identical.

1.4 Examples of bisectorial operators with bounded and unbounded spectral projections

In this section, we give examples of bisectorial operators such that the corresponding spectral projections defined via the functional calculus described in the previous section are bounded, respectively unbounded.

First, let A be a generator of an analytic C_0-semigroup.

Example 1.4.1 *Let A be the generator of an analytic C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space E such that $\sigma(A)\cap i\mathbb{R} = \emptyset$. Then the spectral projections $P^{\pm}$ are bounded and*

$$P^+ = \frac{1}{2\pi i}\int_\gamma R(\lambda, A)d\lambda =: P,$$

where $\gamma \subseteq \mathbb{C}^+ = \{z \in \mathbb{C} : \mathcal{R}e(z) > 0\}$ is a suitable curve around $\sigma^+(A)$. Moreover, the Banach space E splits into

$$E = E_1 \oplus E_2, \text{ where } E_1 := PE, E_2 := (I - P)E.$$

The splitting induces a decomposition of the operator A:

$$\begin{cases} A_1 & : & E_1 \to E_1 & : & A_1x = Ax \quad \forall x \in E_1, \\ A_2 & : & D(A_2) = D(A) \cap E_2 & : & A_2x = Ax \quad \forall x \in D(A_2). \end{cases}$$

Hereby, A_1 is a bounded operator on E_1, $\sigma(A_1) = \sigma^+(A)$, $\sigma(A_2) = \sigma^-(A)$ and $R(\lambda, A_i) = R(\lambda, A)|_{X_i}$ for all $\lambda \in \rho(A)$ and $i = 1, 2$. Furthermore,

$$\begin{aligned} T^+(t) &= e^{-tA_1}|_{E_1} = e^{-tA_1}P|_{E_1} \\ T^-(t) &= T(t)|_{E_2} = T(t)(I-P)|_{E_2}. \end{aligned}$$

Proof. Assume first $\omega(A) \leq 0$. Then $P^+ = P = 0$ and the rest of the statements of the theorem are trivial. Now, let $\omega(A) > 0$. Since $\sigma(A) \cap i\mathbb{R} = \emptyset$, A is a bisectorial operator for a suitable spectral angle ϖ_A, where $\sigma^+(A)$ is bounded. Let $x \in D(A)$ and choose $\varpi_A < \theta < \mu < \frac{\pi}{2}$ and $0 < r < R$ as in Definition 1.3.1. Define the curve $\gamma_s := \{se^{it} : -\theta \leq t \leq \theta\}$. Then it follows, that

$$\begin{aligned} P^+x &= \frac{1}{2\pi i}\int_{\Gamma^+_{\theta,R}} \frac{R(\lambda, A)}{\lambda}Axd\lambda \\ &= \lim_{s\to\infty}\frac{1}{2\pi i}\int_{\Gamma^+_{\theta,R}\cap B(s,0)} \frac{R(\lambda, A)}{\lambda}Axd\lambda \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2\pi i}\left(\lim_{s\to\infty}\int_{(\Gamma^+_{\theta,R}\cap B(s,0))\cup\gamma_s}\frac{R(\lambda,A)}{\lambda}Axd\lambda - \lim_{s\to\infty}\int_{\gamma_s}\frac{R(\lambda,A)}{\lambda}Axd\lambda\right)\\
&= \frac{1}{2\pi i}\int_\gamma \frac{R(\lambda,A)}{\lambda}(A-\lambda+\lambda)xd\lambda - \frac{1}{2\pi i}\lim_{s\to\infty}\int_{-\theta}^{\theta} R(se^{it},A)dt\\
&= \frac{1}{2\pi i}\int_\gamma R(\lambda,A)d\lambda - \frac{1}{2\pi i}\int_\gamma \frac{1}{\lambda}xd\lambda - 0\\
&= \frac{1}{2\pi i}\int_\gamma R(\lambda,A)d\lambda \;=\; Px
\end{aligned}$$

where $\Gamma^+_{\theta,R}$ is defined as in (1.2). Since P is bounded on E, we obtain that $P^+ = P \in \mathcal{L}(E)$. Hence, by Proposition 1.3.6, also $P^- \in \mathcal{L}(E)$. It follows that the Banach space E splits into

$$E = E_1 \oplus E_2, \text{ where } E_1 := PE, E_2 := (I-P)E$$

and E_1, E_2 are closed subspaces of E. The assertions on A_i, $\sigma(A_i)$ and $R(\lambda, A_i)$ for $i = 1, 2$ follow from the usual functional calculus for analytic semigroups (see for example [80, Appendix A.1]). Moreover, since $\lim_{s\to\infty}\int_{\gamma_s} e^{-\lambda t}R(\lambda,A)d\lambda = 0$, we obtain for $x \in E_1$ and $t > 0$

$$\begin{aligned}
e^{-tA_1}x &= \frac{1}{2\pi i}\int_\gamma e^{-\lambda t}R(\lambda,A_1)Pxd\lambda\\
&= \frac{1}{2\pi i}\int_\gamma e^{-\lambda t}R(\lambda,A)Pxd\lambda\\
&= \frac{1}{2\pi i}\left(\int_{\Gamma^+_{\theta,R}} e^{-\lambda t}R(\lambda,A)xd\lambda - \frac{1}{2\pi i}\int_{\gamma_s} e^{-\lambda t}R(\lambda,A)xd\lambda\right)\\
&= T^+(t).
\end{aligned}$$

And similarly, for $x \in E_2$

$$\begin{aligned}
T(t)|_{X_2}x &= T(t)(I-P)x\\
&= \frac{1}{2\pi i}\int_{\Gamma^-_{\theta,R}} e^{\lambda t}R(\lambda,A_2)(I-P)xd\lambda\\
&= \frac{1}{2\pi i}\int_{\Gamma^-_{\theta,R}} e^{\lambda t}R(\lambda,A)xd\lambda\\
&= T^-(t)x.
\end{aligned}$$

Remark that $T^-(t)$, respectively $T^+(t)$, are defined as in (1.3), respectively (1.4). $\square$

A similar result holds if $-A$ is the generator of an analytic semigroup.

Example 1.4.2 *Let $-A$ be the generator of an analytic C_0-semigroup $(S(t))_{t\geq 0}$ such that $\sigma(A)\cap i\mathbb{R}=\emptyset$. Then the spectral projections $P^{\pm}$ are bounded and*

$$P^- = \frac{1}{2\pi i}\int_\gamma R(\lambda,A)d\lambda =: Q,$$

where $\gamma\subseteq\mathbb{C}^- = \{z\in\mathbb{C} : \mathcal{R}e(z)<0\}$ is a suitable curve around $\sigma^-(A)$. Moreover, the Banach space E splits into

$$E = E_1\oplus E_2,\ \textit{where } E_1 := QE, E_2 := (I-Q)E.$$

The splitting induces a decomposition of the operator A:

$$\begin{cases} A_1 & : & E_1\to E_1 & : & A_1x = Ax \quad \forall x\in E_1,\\ A_2 & : & D(A_2)=D(A)\cap E_2 & : & A_2x = Ax \quad \forall x\in D(A_2). \end{cases}$$

Hereby, A_1 is a bounded operator on E_1, $\sigma(A_1)=\sigma^-(A)$, $\sigma(A_2)=\sigma^+(A)$ and $R(\lambda,A_i)=R(\lambda,A)|_{X_i}$ for all $\lambda\in\rho(A)$ and $i=1,2$. Furthermore,

$$\begin{aligned} T^-(t) &= e^{tA_1}|_{E_1} = e^{tA_1}Q|_{E_1}\\ T^+(t) &= S(t)|_{E_2} = S(t)(I-Q)|_{E_2}. \end{aligned}$$

In the remaining part of this section, we give a construction of a bisectorial operator A defined on a Hilbert space H, hence, A satisfies maximal L^p-regularity (see Theorem 3.2.10), but where the spectral projections $P^{\pm}$ are unbounded (see [87]).

Example 1.4.3 *There exists a bisectorial operator A on a Hilbert space H such that the spectral projections $P^{\pm}$ are unbounded.*

Proof. We will do the construction of this operator in several steps.

Step (1).
For $N\geq 1$ and $\beta>0$ define the following matrices on $\mathbb{C}^{N+1}$:

$$\begin{aligned} D &:= diag(2^j)_{0\leq j\leq N}\\ B &:= (b_{jk})_{0\leq j,k\leq N},\quad \text{where } b_{jk} := \begin{cases} \frac{\beta}{\pi(k-j)}, & k\neq j\\ 0, & k=j \end{cases} \quad \text{and}\\ Z &:= (z_{jk})_{0\leq j,k\leq N},\quad \text{where } z_{jk} := \begin{cases} \frac{2^k\beta}{(2^k+2^j)\pi(k-j)}, & k\neq j\\ 0, & k=j. \end{cases} \end{aligned}$$

Then the following properties hold for D, B and Z:

(i) The operator D is self-adjoint and $\sigma(D) = \{1, ..., 2^N\}$.

(ii) The operator B is skew-adjoint, i.e. $B^* = B^{-1}$ and since B is the Toeplitz-matrix corresponding to the function $\varphi \in L^\infty(\mathbb{T})$ where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ with Fourier-coefficients $\hat{\varphi}(n) = \frac{\beta}{\pi n}$ for $n \neq 0$ (see [50, 7.3]), it follows that $\|B\| \leq \beta$.

(iii) The operator equation $DZ + ZD = BD$ is satisfied.

(iv) $\|Z\| = O(\ln N)$, which can be seen by evaluation with the vector $(1, ..., 1) \in \mathbb{C}^N$.

Step (2).
Now, let $n \in \mathbb{N}$, $\kappa > 1$ and $\beta := \kappa - 1 > 0$. Choose $N = N(n) \in \mathbb{N}$ large enough, such that for the operator Z defined on $\mathbb{C}^{N+1}$ as above, we have $\|Z\| \geq n$. Further, let $H_n := \mathbb{C}^{2N+2} = \mathbb{C}^{N+1} \oplus \mathbb{C}^{N+1}$ and define D and B as before on $\mathbb{C}^{N+1}$. Moreover, define the bounded operators A_n, Q_n^+ and Q_n^- on H_n by

$$A_n := \begin{pmatrix} D & BD \\ 0 & -D \end{pmatrix}, \qquad Q_n^+ := \begin{pmatrix} I & Z \\ 0 & 0 \end{pmatrix} \qquad \& \qquad Q_n^- := \begin{pmatrix} 0 & -Z \\ 0 & I \end{pmatrix}.$$

It follows that

(i) $\sigma(A_n) = \{\pm 1, \pm 2, ..., \pm 2^N\}$.

(ii) $Q_n^+ + Q_n^- = I, Q_n^+ Q_n^- = Q_n^- Q_n^+ = 0, (Q_n^+)^2 = Q_n^+, (Q_n^-)^2 = Q_n^-$.

(iii) $Q_n^+ A_n = A_n Q_n^+ = Q_n^+ \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} Q_n^+$ and $Q_n^- A_n = A_n Q_n^- = Q_n^- \begin{pmatrix} 0 & 0 \\ 0 & -D \end{pmatrix} Q_n^-$.

(iv) $\|Q^\pm\| \geq \|Z\| \geq n$.

(v) $R(\lambda, A_n) = \begin{pmatrix} R(\lambda, D) & R(\lambda, D)BDR(\lambda, -D) \\ 0 & R(\lambda, -D) \end{pmatrix}$ for all $\lambda \notin \mathbb{R}$.

From (v), it follows that $\|R(\lambda, A_n)\| \leq \frac{\kappa}{|\Im m(\lambda)|}$.

Step (3).
Let P_n^+ and P_n^- be the spectral projections corresponding to the operator A_n and the spectral sets $\sigma^+(A_n) := \sigma(A_n) \cap \mathbb{R}^+$ and $\sigma^-(A_n) := \sigma(A_n) \cap \mathbb{R}^-$. Since the operator A_n is bounded, we have $P_n^\pm = \chi^\pm(A_n)$ (as in Section 5). Let Γ_n^+ be a suitable curve surrounding $\sigma^+(A_n)$. Then, it follows

$$P_n^+ = \chi^+(A_n) = \frac{1}{2\pi i} \int_{\Gamma_n^+} \begin{pmatrix} R(\lambda, D) & R(\lambda, D)BDR(\lambda, -D) \\ 0 & R(\lambda, -D) \end{pmatrix} d\lambda,$$

where $\frac{1}{2\pi i}\int_{\Gamma_n^+} R(\lambda, D)d\lambda = I$, $\frac{1}{2\pi i}\int_{\Gamma_n^+} R(\lambda, -D)d\lambda = 0$ and

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\Gamma_n^+} R(\lambda, D)BDR(\lambda, -D)d\lambda \\
&= \frac{1}{2\pi i}\int_{\Gamma_n^+} R(\lambda, D)DZR(\lambda, -D)d\lambda + \frac{1}{2\pi i}\int_{\Gamma_n^+} R(\lambda, D)ZDR(\lambda, -D)d\lambda \\
&= -\frac{1}{2\pi i} Z\int_{\Gamma_n^+} R(\lambda, -D)d\lambda + \frac{1}{2\pi i}\int_{\Gamma_n^+} R(\lambda, D)d\lambda Z \\
&= Z.
\end{aligned}$$

Hence, $P_n^+ = \begin{pmatrix} I & Z \\ 0 & 0 \end{pmatrix} = Q_n^+$. Similarly, one shows that $P_n^- = Q_n^-$.

Step (4).
Finally, define the closed operator A on the Hilbert space $H := \bigoplus_{n\in\mathbb{N}} H_n$ by

$$A := \bigoplus_{n\in\mathbb{N}} A_n.$$

Then, it follows by *(2)* that $\sigma(A) \subseteq (-\infty, -1] \cup [1, \infty)$ and that $\|R(\lambda, A)\| \leq \frac{\kappa}{|\Im m(\lambda)|}$ for all $\lambda \notin \mathbb{R}$. Hence, A is a bisectorial operator on H. Further, let $P^+ = \chi^+(A)$ and $P^- = \chi^-(A)$ be the spectral projections corresponding to A (see Section 1.3 (1.8))). Then, it follows by *(4)* that $P^+ = \bigoplus_{n\in\mathbb{N}} Q_n^+$ and $P^- = \bigoplus_{n\in\mathbb{N}} Q_n^-$. And therefore, by *(3)*, that $\|P^\pm\| \geq \|Q_n^\pm\| \geq n$ for all $n \in \mathbb{N}$. Hence, the spectral projections of the bisectorial operator A are unbounded. □

1.5 Spectral decomposition of bisectorial operators where the corresponding spectral projections are bounded

In this section, we show how the underlying Banach space E can be splitted if the corresponding spectral projections of a densely defined, bisectorial operator A are bounded. This splitting induces then a spectral decomposition of the operator A. Moreover, it follows that the projections $Q^\pm$ defined by the analytic semigroups are closed and hence coincide with the spectral projections $P^\pm$.

Remark first, that if $D(A)$ is not dense in E, one cannot expect a splitting of the underlying Banach space E since the domains of the corresponding initial projections are also not dense. This is shown in the following example.

Example 1.5.1 *Let $E = c$, the space of all convergent sequences, and define*

$$A(x_n)_{n\in\mathbb{N}} \;:=\; \begin{cases} nx_n, & n \in 2\mathbb{N} \\ -nx_n, & n \in 2\mathbb{N}-1 \end{cases}$$

with $D(A) := \{(x_n)_{n\in\mathbb{N}} \in c : A(x_n)_{n\in\mathbb{N}} \in c\}$. Then A is bisectorial, but

$$Q^+(x_n)_{n\in\mathbb{N}} = \begin{cases} x_n, & n \in 2\mathbb{N} \\ 0, & n \in 2\mathbb{N}-1 \end{cases}, \quad \textit{respectively } Q^-(x_n)_{n\in\mathbb{N}} = \begin{cases} 0, & n \in 2\mathbb{N} \\ x_n, & n \in 2\mathbb{N}-1 \end{cases}$$

with $D(Q^\pm) = c_0$, the space of all sequences converging to 0. And $E = c$ cannot be splitted.

Proof. Remark that $\overline{D(A)} = c_0$ which is not dense in c. Moreover, it is easy to check that $\|i\xi R(i\xi, A)\| \leq 1$ for all $\xi \in \mathbb{R}$, thus, A is bisectorial. Moreover, for the corresponding semigroups we obtain

$$T^+(t)(x_n)_{n\in\mathbb{N}} = \begin{cases} e^{-nt}x_n, & n \in 2\mathbb{N} \\ 0, & n \in 2\mathbb{N}-1, \end{cases}$$

respectively

$$T^-(t)(x_n)_{n\in\mathbb{N}} = \begin{cases} 0, & n \in 2\mathbb{N} \\ e^{-nt}x_n, & n \in 2\mathbb{N}-1. \end{cases}$$

From which we obtain the assumptions on the projections $Q^\pm$. Finally, it is easy to see that the sequence $(1, 1, \ldots) \in c$ cannot be splitted. $\square$

Note that if we restrict in the above example the operator A on $\overline{D(A)} = c_0$, then we obtain bounded projections on c_0 and a corresponding spectral decomposition of c_0. Hence, let in the following A be a densely defined, bisectorial operator on a Banach space E.

First, we assume that the spectral projections corresponding to a bisectorial operator A are bounded. In this case, we get a splitting of the Banach space E and a spectral decomposition of the operator A.

Theorem 1.5.2 *Let A be a bisectorial operator on a Banach space E such that $0 \in \rho(A)$. Assume that the spectral projections P^- and P^+ are bounded. Then there exists a splitting of the Banach space E into*

$$E = E_1 \oplus E_2, \quad \textit{where } E_1 = P^-E \textit{ and } E_2 = P^+E.$$

The splitting of E induces a spectral decomposition of the operator A:

$$\begin{cases} A_1 & : & D(A_1) = D(A) \cap E_1 \longrightarrow E_1 & : & A_1 x = Ax \quad \forall x \in D(A_1), \\ A_2 & : & D(A_2) = D(A) \cap E_2 \longrightarrow E_2 & : & A_2 x = Ax \quad \forall x \in D(A_2), \end{cases}$$

such that $\sigma(A_1) = \sigma^-(A)$ and $\sigma(A_2) = \sigma^+(A)$. Moreover, the operators A_1 and $-A_2$ are the generators of the induced analytic C_0-semigroups $(T^-(t)|_{E_1})_{t>0}$ and $(T^+(t)|_{E_2})_{t>0}$ on E_1, respectively E_2 which are now strongly continuous also for $t \to 0$. Finally, $T^-(t)|_{E_2} = 0$, and similarly $T^+(t)|_{E_1} = 0$ for all $t > 0$.

Proof. From the boundedness of the spectral projections P^- and P^+, we obtain the splitting of the Banach space E. Now define $D(A_i) := D(A) \cap E_i$ and $A_i x := Ax$ for all $x \in D(A_i)$ and $i = 1, 2$. Since for $x \in D(A)$, we have $P^- x \in D(A)$ and $AP^- x = P^- Ax$, it follows that $A_i(D(A_i)) \subseteq E_i$ for $i = 1, 2$. Moreover, it is easy to verify that

$$\sigma(A) = \sigma(A_1) \cup \sigma(A_2).$$

Next, we claim that $\sigma(A_1) \subseteq \mathbb{C}^-$. To prove this, let $z \in \overline{\mathbb{C}^+}$ and $\varpi_A < \theta < \mu < \frac{\pi}{2}$ and $0 < r < R$ such that $B(R, 0) \subseteq \rho(A)$. Define $g \in H_0^\infty(S_{\mu,r})$ by

$$g(\lambda) := \begin{cases} \frac{1}{z-\lambda} \ , & \mathcal{R}e(\lambda) < 0 \\ 0 \ , & \mathcal{R}e(\lambda) > 0. \end{cases}$$

Thus, $g(A) = \frac{1}{2\pi i} \int_{\Gamma_{\theta,R}^-} \frac{R(\lambda,A)}{z-\lambda} d\lambda \in \mathcal{L}(E)$. For $x \in D(A)$, we obtain

$$\begin{aligned} (z - A)g(A)x - P^- x &= \frac{1}{2\pi i} \int_{\Gamma_{\theta,R}^-} \frac{(z-A)R(\lambda,A)x}{z-\lambda} - \frac{AR(\lambda,A)x}{\lambda} d\lambda \\ &= \frac{1}{2\pi i} \int_{\Gamma_{\theta,R}^-} \frac{z(\lambda - A)}{\lambda(z-\lambda)} R(\lambda, A) x d\lambda \\ &= \frac{1}{2\pi i} \int_{\Gamma_{\theta,R}^-} \frac{z}{\lambda(z-\lambda)} x d\lambda \\ &= 0. \end{aligned}$$

Since $P^- \in \mathcal{L}(E)$ and $g(A) \in \mathcal{L}(E)$, it follows that $g(A)E \subseteq D(A)$ and $(z - A)g(A) = P^-$. Moreover, since P^- commutes with the operator A, P^- commutes with $g(A)$ and $g(A)E_1 \subseteq E_1$. Hence, $g(A)|_{E_1}(z - A_1) = I_{D(A_1)}$ and $(z - A_1)g(A)|_{E_1} = I_{E_1}$. Thus, $z \in \rho(A_1)$ which proves the claim. Similarly, one shows that $\sigma(A_2) \subseteq \mathbb{C}^+$. From this, it follows that $\sigma(A_1) = \sigma^-(A)$ and $\sigma(A_2) = \sigma^+(A)$.

For $x \in E_1 = P^- E$, we get

$$T^-(t)x = T^-(t)P^- x = P^- T^-(t)x \in E_1,$$

thus, $T^-(t)E_1 \subseteq E_1$. Let $x \in E_2 \cap D(Q^+)$, then it follows by Lemma 1.2.5 that

$$T^-(t)x = T^-(t)P^+x = T^-(t)Q^+x = T^-(t)\lim_{s\to 0} T^+(s)x = \lim_{s\to 0} T^-(t)T^+(s)x = 0.$$

Since $E_2 \cap D(Q^+)$ is dense in E_2, we obtain that $T^-(t)|_{E_2} = 0$. The corresponding results hold of course also for $(T^+(t))_{t>0}$. Thus, $(T^-(t))_{t>0}$ and $(T^+(t))_{t>0}$ are operator families on E_1, respectively E_2.

The problem is now, that we do not have a priori an estimate for the resolvent $R(\lambda, A_i)$ in E_i for $\lambda \in \rho(A_i) \cap \{\lambda \in \mathbb{C} : |\arg(\pm\lambda)| \le \varpi_A\}$ $(i = 1, 2)$.

To obtain such an estimate, we claim first that $R(\lambda, A_1) = \int_0^\infty e^{-\lambda t}T^-(t)dt$ for all $\lambda \in \Sigma_\theta := \{z \in \mathbb{C} : |\arg(z)| \le \theta\} \cup \{0\}$ for $\varpi_A < \theta < \frac{\pi}{2}$. For $x \in E$ and $R > 0$ as in the definition of $T^-(t)$ (see (1.3)) we obtain with Fubini's Theorem

$$\begin{aligned}
\int_0^\infty e^{-\lambda t}T^-(t)x\,dt &= \int_0^\infty e^{-\lambda t}\frac{1}{2\pi i}\int_{\Gamma^-_{\theta,R}} e^{\mu t}R(\mu, A)x\,d\mu\,dt \\
&= \frac{1}{2\pi i}\int_{\Gamma^-_{\theta,R}}\int_0^\infty e^{(-\lambda+\mu)t}dtR(\mu, A)x\,d\mu \\
&= \frac{1}{2\pi i}\int_{\Gamma^-_{\theta,R}} \frac{R(\mu, A)x}{\lambda-\mu}d\mu.
\end{aligned}$$

Note, that the above integral converges by Lemma 1.2.3. Now, let $x \in D(A)$ then

$$\int_{\Gamma^-_{\theta,R}} \frac{R(\mu, A)}{(\lambda-\mu)\mu}Ax\,d\mu = \int_{\Gamma^-_{\theta,R}} \frac{1}{(\lambda-\mu)\mu}x\,d\mu + \int_{\Gamma^-_{\theta,R}} \frac{R(\mu, A)}{\lambda-\mu}x\,d\mu = \int_{\Gamma^-_{\theta,R}} \frac{R(\mu, A)x}{\lambda-\mu}d\mu.$$

Thus, for $x \in D(A_1)$, it follows

$$\begin{aligned}
(\lambda - A)\int_0^\infty e^{-\lambda t}T^-(t)x\,dt &= \int_0^\infty e^{-\lambda t}T^-(t)(\lambda - A)x\,dt \\
&= \frac{1}{2\pi i}\int_{\Gamma^-_{\theta,R}} \frac{R(\mu, A)(\lambda - A)}{(\lambda-\mu)\mu}Ax\,d\mu \\
&= \frac{1}{2\pi i}\int_{\Gamma^-_{\theta,R}} \frac{R(\mu, A)}{\mu}Ax\,d\mu + \frac{1}{2\pi i}\int_{\Gamma^-_{\theta,R}} \frac{1}{(\lambda-\mu)\mu}Ax\,d\mu \\
&= P^-x + 0 \\
&= x.
\end{aligned}$$

Hence, since $\lambda \in \rho(A_1)$ and $D(A_1)$ is dense in E_1 it follows that $R(\lambda, A_1)x = \int_0^\infty e^{-\lambda t}T^-(t)xdt$ for all $x \in E_1$. Similarly, one obtains the corresponding result for $R(\lambda, A_2)$.

Since $(T^-(t))_{t>0}$ is exponentially stable for $t \to \infty$ and integrable around 0 (see Lemma 1.2.3), it follows

$$\|R(\lambda, A_1)\| \leq \| \int_0^\infty e^{-\lambda t} T^-(t)\, dt\| \leq c < \infty$$

for a suitable constant $c \geq 0$. Thus $\lambda R(\lambda, A_1)$ is polynomially bounded in Σ_θ and, since A is bisectorial, $\lambda R(\lambda, A_1)$ is bounded on the rays $\{re^{\pm i\theta} : r \geq 0\}$. With a generalisation of the Phragmen-Lindelöf Theorem (see [12, Theorem 4.2]), it follows that $\lambda R(\lambda, A_1)$ is bounded for all $\lambda \in \Sigma_\theta$. Hence, A_1 is the generator of a bounded holomorphic C_0-semigroup in E_1. Thus, $(T^-(t))_{t>0}$ with generator A_1 is also strongly continuous for $t \to 0$.

The corresponding results follow similarly for A_2 and $(T^+(t))_{t>0}$ on E_2. □

For example, the spectral projections of a bisectorial operator A are bounded if $+A$ or $-A$ is the generator of an analytic C_0-semigroup (see Examples 1.4.1 and 1.4.2). Since $\chi^-, \chi^+ \in H^\infty(\mathbb{C}^- \cup \mathbb{C}^+)$, the spectral projections $P^- = \chi^-(A)$ and $P^+ = \chi^+(A)$ are also bounded if the bisectorial operator A possesses a bounded H^∞-calculus (compare with [1] and [17]).

Moreover, we obtain for the projections Q^- and Q^+ (see (1.5) and (1.6)) the following result.

Corollary 1.5.3 *Let A be a densely defined, bisectorial operator on a Banach space E such that the corresponding spectral projections P^- and P^+ are bounded. Then $D(Q^-) = D(Q^+) = E$ and $Q^-x = P^-x$ and $Q^+x = P^+x$ for all $x \in E$.*

Proof. Let $x = P^-x + P^+x \in E$. It follows by Proposition 1.5.2 that

$$\lim_{t\to 0} T^-(t)x \;=\; \lim_{t\to 0}(T^-(t)P^-x + T^-(t)P^+x) \;=\; P^-x.$$

Thus $x \in D(Q^-)$ and $Q^-x = P^-x$. Similarly, we obtain the result for Q^+. □

But in general, the spectral projections corresponding to a bisectorial operator may be unbounded (see Example 1.4.3).

1.6 Spectral decomposition of bisectorial operators: the general case

In this section, we assume that the corresponding spectral projections of a densely defined, bisectorial operator are unbounded (see Example 1.4.3). We introduce a Banach space D such that the part of A in D is still a bisectorial operator and the corresponding spectral projections are bounded. From this, we obtain that also in this case, $Q^{\pm}$ and $P^{\pm}$ are identical. Moreover, we find a Banach space F such that the extension of A in F is bisectorial and the corresponding projections are also bounded. Hence, the Banach spaces D and F can be splitted and the original Banach space E becomes an intermediate space between D and F.

Since $P^- + P^+ = I_D$, the graph norms of P^- and P^+ on $D = D(P^-) = D(P^+)$ are equivalent. It follows that it does not matter which graph norm we consider. Hence, we denote also by D the Banach space D equipped with the graph norm

$$\|x\|_D := \|x\| + \|P^- x\|$$

for all $x \in D = D(P^-)$. We consider the *part* A_D *of the operator* A *in* D, which is defined by

$$\begin{aligned} D(A_D) &:= \{ x \in D \,:\, Ax \in D \} \\ A_D x &:= Ax \qquad \forall x \in D(A_D). \end{aligned}$$

Lemma 1.6.1 *For the operators A and A_D, it holds that*

$$\sigma(A) = \sigma(A_D).$$

Moreover, A_D is bisectorial, i.e. $\|\lambda R(\lambda, A_D)\|_D \leq c$ for all $\lambda \in i\mathbb{R}$.

Proof. That $\sigma(A) = \sigma(A_D)$ follows from [5, Proposition 1.1] (see also [6]) since $D(A) \subseteq D$ by Theorem 1.3.9.

Moreover, for $y \in D$ and $\lambda \in \rho(A_D)$, it follows that

$$\begin{aligned} \|\lambda R(\lambda, A_D) y\|_D &= \|\lambda P^- R(\lambda, A_D) y\| + \|\lambda R(\lambda, A_D) y\| \\ &\leq \|\lambda R(\lambda, A)\| (\|P^- y\| + \|y\|) \\ &\leq c\|y\|_D \end{aligned}$$

Thus, the analogous estimates for the resolvent of A_D hold as for the resolvent of A and hence, A_D is bisectorial. □

We get the following splitting theorem on D.

Theorem 1.6.2 *Let A be a densely defined, bisectorial operator on a Banach space E with $0 \in \rho(A)$ and A_D be the part of A in D, where D is defined as above. Then the Banach space D splits into*

$$D = D_1 \oplus D_2, \quad \text{where } D_1 = P^- D \text{ and } D_2 = P^+ D.$$

The splitting induces a decomposition of the operator A_D:

$$\begin{cases} A_{D_1} & : \; D(A_{D_1}) = D(A_D) \cap D_1 \longrightarrow D_1 \; : \; A_{D_1}x = A_D x \quad \forall x \in D(A_{D_1}), \\ A_{D_2} & : \; D(A_{D_2}) = D(A_D) \cap D_2 \longrightarrow D_2 \; : \; A_{D_2}x = A_D x \quad \forall x \in D(A_{D_2}), \end{cases}$$

such that $\sigma(A_{D_1}) = \sigma^-(A)$, $\sigma(A_{D_2}) = \sigma^+(A)$. Moreover, A_{D_1} and $-A_{D_2}$ generate the induced analytic C_0-semigroups $(T^-(t)|_{D_1})_{t>0}$ and $(T^+(t)|_{D_2})_{t>0}$ on D_1, respectively D_2. Finally, $T^-(t)|_{D_2} = 0$, and similarly $T^+(t)|_{D_1} = 0$ for all $t > 0$.

Proof. It is obvious that $P_D^- = \chi^-(A_D)$, respectively $P_D^+ = \chi^+(A_D)$. Let $y \in D$, then $P_D^- y = P^- y$, respectively $P_D^+ y = P^+ y$ and

$$\|P_D^- y\|_D = \|P^- y\| + \|(P^-)^2 y\| = 2\|P^- y\| \le 2\|y\|_D,$$

respectively

$$\|P^+ y\|_D = \|P^+ y\| + \|P^- P^+ y\| \le 2\|y\|_D.$$

From this, it follows that the spectral projections corresponding to the bisectorial operator A_D are bounded on D. Since $\sigma(A_D) = \sigma(A)$ (see Lemma 1.6.1), the remaining assertions follow directly from Theorem 1.5.2. □

Compare the following result with Corollary 1.5.3 in the case of bounded spectral projections.

Corollary 1.6.3 *Let A be a densely defined, bisectorial operator on a Banach space E. Then*

$$Q^- = P^- \qquad \text{and} \qquad Q^+ = P^+,$$

with the same domain D.

Proof. From Theorem 1.3.9, we know that $Q^\pm \subseteq P^\pm$, $Q^\pm$ are closable and the closure is equal to $P^\pm$.

Let now $x \in D(P^\pm) = D = D_1 + D_2$. Then it follows with Theorem 1.6.2 that

$$\begin{aligned} \|T^-(t)x - P^- x\| &= \|T^-(t)P^- x - P^- x\| \\ &\le \|T^-(t)|_{D_1} P^- x - P^- x\|_D \\ &\to 0 \qquad \text{for } t \to 0. \end{aligned}$$

Hence, $x \in D(Q^-)$ and $Q^- x = P^- x$. Similarly, one obtains the result for Q^+. $\square$

Next, let again $0 \in \rho(A)$ and define the following norm on E:

$$\|x\|_G := \|R(0, A)x\|.$$

And let G be the completion of the Banach space E with respect to $\|.\|_G$.

The following proposition follows directly from the construction of the Banach space G.

Proposition 1.6.4 *Let A be a bisectorial operator on a Banach space E with $0 \in \rho(A)$ and let $G := (E, \|.\|_G)^c$ as above. Then A is closable in G. Denote by $B = \overline{A}^G$ the closure of A in G. Then the following assertions hold for B:*

(i) $\sigma(B) = \sigma(A)$

(ii) B *is bisectorial*

(iii) $D(B) = E$

(iv) $B_E = A$

(v) $\chi^{\pm}(B) = \overline{P^{\pm}}^G$

Now, we do the same construction in G as we did before in E, i.e. let $F := D(\chi^-(B))$ equipped with the graph norm

$$\|y\|_F := \|y\|_G + \|\chi^-(B)y\|_G \qquad \forall y \in F$$

and consider the part B_F of B in F. Then, as in Lemma 1.6.1, we obtain that B_F is bisectorial and $\sigma(B_F) = \sigma(B) = \sigma(A)$ (by Proposition 1.6.4).

Lemma 1.6.5 *With the same notations as above, we obtain for the domain of B_F that*

$$D(B_F) = D = D(P^-).$$

Proof. Let $x \in D(B_F)$, i.e. $x \in D(B) = E$ and $Bx \in F$. Hence,

$$B \int_{\Gamma_{\theta,R}^-} \frac{R(\lambda, A)}{\lambda} x \, d\lambda = \int_{\Gamma_{\theta,R}^-} \frac{R(\lambda, B)}{\lambda} Bx \, d\lambda \in D(B) = E$$

for suitable $R > 0$ and $\varpi_A < \theta < \frac{\pi}{2}$. It follows that $\int_{\Gamma^-_{\theta,R}} \frac{R(\lambda,A)}{\lambda} x d\lambda \in D(A)$ and by definition $x \in D(P^-) = D$.

Now, let $x \in D \subseteq E = D(B)$. From the definition of $D = D(P^-)$, we obtain

$$\int_{\Gamma^-_{\theta,R}} \frac{R(\lambda, B)}{\lambda} Bx\, d\lambda = A \int_{\Gamma^-_{\theta,R}} \frac{R(\lambda, A)}{\lambda} x\, d\lambda \in E = D(B).$$

Thus, $Bx \in F$ and $x \in D(B_F)$. □

Finally, we obtain the corresponding splitting of the Banach space F.

Theorem 1.6.6 *Let A be a densely defined, bisectorial operator on a Banach space E with $0 \in \rho(A)$. Define the corresponding bisectorial operators B and B_F on the Banach spaces G and F as above. Then F splits into*

$$F = F_1 \oplus F_2 \qquad \text{where } F_1 = \chi^-(B)F \text{ and } F_2 = \chi^+(B)F.$$

The splitting induces a decomposition of the operator B_F:

$$B_{F_i} : D(B_{F_i}) = D \cap F_i = D_i \to F_i : B_{F_i} y = By \ \ \forall y \in D(B_{F_i})$$

for $i = 1, 2$. Again, $\sigma(B_{F_1}) = \sigma^-(A)$ and $\sigma(B_{F_2}) = \sigma^+(A)$. Finally, B_{F_1} and $-B_{F_2}$ are the generators of the analytic C_0-semigroups induced from $(T^-(t))_{t>0}$, respectively $(T^+(t))_{t>0}$.

Proof. The assertions on the splitting of F and the spectral decomposition of the operator B_F follow directly from Theorem 1.6.2.

It remains to show that $D(B_{F_i}) = D_i$ for $i = 1, 2$.

Trivially, we have $D_1 = \chi^-(A)D \subseteq D \cap \chi^-(B)F$. For $x \in D \cap \chi^-(B)F$, there exists $y \in F$ such that $x = \chi^-(B)y = \chi^-(B)^2 y = \chi^-(B)x = P^- x \in D_1$. Thus, $D(B_{F_1}) = D_1$. Similarly, one obtains the result for $i = 2$. □

If we summarise the above results, we have obtained two Banach spaces D and F such that the original Banach space E has become an intermediate space where there exists a splitting on D and F but not on E:

$$D = D_1 \oplus D_2 \hookrightarrow E \hookrightarrow F = F_1 \oplus F_2.$$

If the spectral projections P^- and P^+ are bounded, then $D = E$ and $F = G$.

This results can be visualised in the following diagram:

$$
\begin{tikzcd}
 & & G \\
 & \mathbf{E} \arrow[ur, "B"] \arrow[r, hook, "\iota"] & \mathbf{F} = F_1 \oplus F_2 \arrow[u, hook, "\iota"'] \\
D(A) \arrow[ur, "A"] \arrow[r, hook, "\iota"] & \mathbf{D} = D_1 \oplus D_2 = D(B_{F_1}) \oplus D(B_{F_2}) \arrow[u, hook, "\iota"'] \arrow[ur, "B_F"] & \\
D(A_D) = D(A_{D_1}) \oplus D(A_{D_2}) \arrow[u, hook, "\iota"] \arrow[ur, "A_D"] & &
\end{tikzcd}
$$

Here, ι denotes a suitable injection in each case. Of course, this diagram can be extended on both sides. Compare this also with the Sobolev semigroups described by Nagel in [91, A-1:3.5] (see also [53, II.5]).

Summarising the above results, we obtain the following characterisation theorem of bisectorial operators.

Theorem 1.6.7 *Let A be a densely defined, bisectorial operator on a Banach space E. Then there exists an operator B on a the direct sum of two Banach spaces $X = X_1 \oplus X_2$ such that B is of the form $B(x_1, x_2) = (B_1 x_1, -B_2 x_2)$ where B_1 and B_2 are generators of holomorphic semigroups. Moreover, E becomes an intermediate space between $D(B)$ and X, i.e.*

$$D(B) \hookrightarrow E \hookrightarrow X,$$

and $A = B_E$ becomes the part of B in E.

Remark, that if the corresponding spectral projections of the bisectorial operator A are already bounded, then $X = E = E_1 \oplus E_2$ and $B = A$ (see Theorem 1.5.2).

Chapter 2

Bounded uniformly continuous solutions of first-order differential equations

In this chapter, we consider the first-order differential equation on the line

$$(\mathrm{I}) \qquad u'(t) = Au(t) + f(t) \quad (t \in \mathbb{R}),$$

where A is a closed, linear operator on a Banach space E and $f \in BUC(\mathbb{R}, E)$, the space of all bounded uniformly continuous functions on $\mathbb{R}$ with values in the Banach space E. We write $(\mathrm{I})_f$ if we want to specify the inhomogeneity f.

2.1 Uniqueness of mild solutions

In this section, we give a sufficient condition on the operator A for uniqueness of mild solutions of the above Equation (I). This will be done by relating the spectrum of the operator A to the spectrum of the corresponding mild solution of the homogeneous equation, i.e. the inhomogeneity $f = 0$.

First, we give the definition of solutions of (I).

Definition 2.1.1 Let $f \in BUC(\mathbb{R}, E)$. We say that $u \in BUC(\mathbb{R}, E)$ is a *mild solution* of $(\mathrm{I})_f$ if $\int_0^t u(s)ds \in D(A)$ and

$$u(t) - u(0) = A\int_0^t u(s)ds + \int_0^t f(s)ds \tag{2.1}$$

for all $t \in \mathbb{R}$.

We call u a *classical solution* of $(\mathrm{I})_f$ if $u \in C^1(\mathbb{R}, E) \cap C(\mathbb{R}, D(A))$ and $u'(t) = Au(t) + f(t)$ for all $t \in \mathbb{R}$.

Lemma 2.1.2 *Let $u \in BUC(\mathbb{R}, E) \cap C^1(\mathbb{R}, E)$ be a mild solution of $(I)_f$ and assume that $\rho(A) \neq \emptyset$. Then u is a classical solution.*

Proof. Since u is a mild solution of (I), we have by Definition (2.1.1)

$$u(t) - u(0) - \int_0^t f(s)ds = A \int_0^t u(s)ds$$

for all $t \in \mathbb{R}$. We see that the left hand side of the equation is once continuously differentiable, and so is the right hand side. Hence we have $\{t \mapsto (\lambda - A) \int_0^t u(s)ds\} \in C^1(\mathbb{R}, E)$. If we now take $\lambda \in \rho(A)$, it follows that

$$R(\lambda, A)\left(\frac{d}{dt}(\lambda - A)\int_0^t u(s)ds\right) = \frac{d}{dt}\int_0^t u(s)ds = u(t).$$

We conclude that $u(t) \in D(A)$ and $u'(t) = Au(t) + f(t)$ for all $t \in \mathbb{R}$. Since $u \in C^1(\mathbb{R}, E)$, it follows that $u \in C(\mathbb{R}, D(A))$. $\square$

In the following, we want to consider the spectrum of bounded uniformly continuous functions. Recall that the spectrum of such a function can be defined in several ways (see for example [8]).

Denote by $\mathcal{F}$ the *Fourier transform* of a function $f \in L^1(\mathbb{R})$ which is given by

$$(\mathcal{F}f)(s) := \int_{-\infty}^{+\infty} e^{-ist} f(t)dt \tag{2.2}$$

for all $s \in \mathbb{R}$. The *Beurling spectrum* of $u \in BUC(\mathbb{R}, E)$ is defined by

$$\begin{aligned} sp_B(u) \;:=\; & \{\xi \in \mathbb{R} : \forall \epsilon > 0\, \exists f \in L^1(\mathbb{R}) \text{ such that} \\ & supp(\mathcal{F}f) \subseteq (\xi - \epsilon, \xi + \epsilon) \text{ and } f * u \neq 0\}. \end{aligned} \tag{2.3}$$

The *Carleman transform* of a function $u \in BUC(\mathbb{R}, E)$ is given by

$$\hat{u}(\lambda) \;:=\; \begin{cases} \int_0^\infty e^{-\lambda t} u(t)dt, & \mathcal{R}e(\lambda) > 0 \\ -\int_{-\infty}^0 e^{-\lambda t} u(t)dt, & \mathcal{R}e(\lambda) < 0. \end{cases} \tag{2.4}$$

Clearly, $\hat{u}$ is a holomorphic function on $\mathbb{C} \setminus i\mathbb{R}$. A point $\eta \in \mathbb{R}$ is called a *regular point* if the Carleman transform has a holomorphic extension in a neighbourhood of $i\eta$. Now, the *Carleman spectrum* of $u \in BUC(\mathbb{R}, E)$ is defined by

$$sp_C(u) := \{\xi \in \mathbb{R} \, : \, \xi \text{ is not regular}\} \tag{2.5}$$

which coincides with the Beurling spectrum $sp_B(u)$ ([99, Proposition 0.5]). Hence, we can denote the spectrum of u simply by $sp(u) := sp_B(u) = sp_C(u)$.

In the following, we show how the spectrum of mild solutions of Equation (I) is related to the spectrum of the operator A and the inhomogeneity f (see also [8, Theorem 4.3]).

Proposition 2.1.3 *Let A be a closed linear operator on a Banach space E, $f \in BUC(\mathbb{R}, E)$ and u be a mild solution of (I)$_f$. Then*

$$sp(u) \subseteq \{\eta \in \mathbb{R} : i\eta \in \sigma(A)\} \cup sp(f).$$

Proof. By taking Carleman transforms on both sides of Equation (2.1), we obtain

$$(\lambda - A)\hat{u}(\lambda) = u(0) + \hat{f}(\lambda)$$

for all $\lambda \notin i\mathbb{R}$. Thus, for $\lambda \in \rho(A)$ it follows

$$\hat{u}(\lambda) = R(\lambda, A)u(0) + R(\lambda, A)\hat{f}(\lambda).$$

From this, we see that if $\eta \in \mathbb{R}$ is a regular point of f and $i\eta \in \rho(A)$, then $\hat{u}$ has a holomorphic extension in a neighbourhood of $i\eta$, i.e. η is a regular point of u. □

Corollary 2.1.4 *If A is a closed linear operator on a Banach space E and u is a mild solution of the homogeneous first-order equation (I)$_0$, $u' = Au$, then*

$$i\, sp(u) \subseteq \sigma(A) \cap i\mathbb{R}.$$

Theorem 2.1.5 *Let A be a closed linear operator on a Banach space E with $\sigma(A) \cap i\mathbb{R} = \emptyset$. Then the mild solutions of (I) are unique.*

Proof. Let $f \in BUC(\mathbb{R}, E)$ and suppose that u, v are two mild solutions of (I)$_f$. It follows that $u - v$ is a solution of the homogeneous equation (I)$_0$. By Corollary 2.1.4 we obtain that $sp(u - v) = \emptyset$. Hence, $u = v$ (see [99, Proposition 0.5]). □

2.2 Well-posedness of first-order differential Equations

In this section, we first give a necessary condition on the spectrum of an operator A for well-posedness of Equation (I). Second, we recall results by Vũ Quôc Phóng and Schüler (see [107], [120] and [123]) who show the relation between Equation (I) and a suitable operator equation. In Section 5.2, we examine similar facts for the second-order differential equation. For the results of this section, see also [110].

We are interested in the following property of well-posedness.

Definition 2.2.1 We say that Equation (I) is *well-posed* if for all $f \in BUC(\mathbb{R}, E)$ there exists a unique mild solution $u \in BUC(\mathbb{R}, E)$ of Equation $(\mathrm{I})_f$.

Further, we consider the solution operator M for Equation (I), defined by

$$\begin{aligned} D(M) &:= \{f \in BUC(\mathbb{R}, E) : \exists!\, u_f \in BUC(\mathbb{R}, E) \text{ such that} \\ & \qquad u_f \text{ is a mild solution of } (\mathrm{I})_f\} \\ Mf &:= u_f. \end{aligned} \tag{2.6}$$

Remark that if one mild solution of Equation (I) is not unique, then no mild solution of (I) is unique and $D(M) = \emptyset$. Moreover, it is easy to see that M is a closed operator. By a standard application of the closed graph theorem, it follows that if Equation (I) is well-posed then the solution operator M is bounded.

The following theorem gives a necessary condition for the well-posedness of Equation (I).

Theorem 2.2.2 *Let A be a closed operator on a Banach space E and assume that Equation (I) is well-posed. Then $i\mathbb{R} \subseteq \rho(A)$ and there exists a constant $C \geq 0$ such that*

$$\|R(i\xi, A)\| \leq C \quad \text{for all } \xi \in \mathbb{R}.$$

Proof. Take $\xi \in \mathbb{R}$ and $y \in E$. Now define $f_s(t) := e^{i\xi(s+t)}y = f_0(s+t) = e^{i\xi s} f_0(t)$ for all $t, s \in \mathbb{R}$, where $f_0(t) := e^{i\xi t}y$. It follows that there exists a unique function $u_s := u_{f_s} \in BUC(\mathbb{R}, E)$ which is a mild solution of $(\mathrm{I})_{f_s}$. We claim that $u_s(t) = e^{i\xi s}u_0(t) = u_0(s+t)$.

Since u_s solves $(\mathrm{I})_{f_s}$ we obtain by Definition 2.1.1

$$e^{-i\xi s}u_s(t) - e^{-i\xi s}u_s(0) \;=\; A\int_0^t e^{-i\xi s}u_s(r)dr + \int_0^t f_0(r)dr$$

for all $t \in \mathbb{R}$. Hence $e^{-i\xi s}u_s$ is a mild solution of $(\mathrm{I})_{f_0}$. From the uniqueness of solutions, it follows $u_s = e^{i\xi s}u_0$.

For the second equality, let $\tilde{u}(t) := u_0(s+t)$. Since u_0 is a mild solution of $(\mathrm{I})f_0$ we get, again by Definition 2.1.1,

$$\begin{aligned}\tilde{u}(t) - \tilde{u}(0) &= A\int_0^{s+t} u_0(r)dr + \int_0^{s+t} f_0(r)dr - (A\int_0^s u_0(r)dr + \int_0^s f_0(r)dr) \\ &= A\int_0^t \tilde{u}(r)dr + \int_0^t f_s(r)dr\end{aligned}$$

for all $t \in \mathbb{R}$. Again from the uniqueness of solutions it follows that $\tilde{u} = u_s$, i.e. $u_0(s+t) = u_s(t)$ for all $s, t \in \mathbb{R}$.

Now define $z := u_0(0)$. Then $u_0(t) = e^{i\xi t}z \in BUC(\mathbb{R}, E) \cap C^1(\mathbb{R}, E)$ is a mild solution of $(\mathrm{I})_{f_0}$ and therefore a classical solution, i.e. $u_0(t) \in D(A)$ and $u_0'(t) = Au_0(t) + f_0(t)$ for all $t \in \mathbb{R}$. Hence, $u_0(0) = z \in D(A)$ and

$$i\xi z = u_0'(0) = Au_0(0) + f_0(0) = Az + y.$$

We obtain that $(i\xi - A)z = y$ and, since $y \in \mathbb{R}$ was arbitrary, that $i\xi - A$ is surjective for all $\xi \in \mathbb{R}$.

Assume that $i\xi - A$ is not injective, then there exists $z \neq 0$ such that $Az = i\xi z$. Define $u(t) := e^{i\xi t}z$ and we obtain

$$u'(t) = i\xi e^{i\xi t}z = e^{i\xi t}Az = Au(t),$$

i.e. u is a non-trivial solution of the homogeneous equation. It follows that the solutions of (I) are not unique which is a contradiction. Hence, $i\xi - A$ is injective.

Since the solution operator M is bounded, we get

$$\|z\|_E = \|u_0\|_\infty = \|Mf_0\|_\infty \leq \|M\|\|f_0\|_\infty = \|M\|\|y\|_E.$$

It follows that $i\xi \in \rho(A)$ and $\|R(i\xi, A)\| \leq \|M\| =: C$ for all $\xi \in \mathbb{R}$. □

The proof above is inspired by a result of Mielke ([88]) who considers strong solutions on $L^p(\mathbb{R}, E)$ and establishes maximal L^p-regularity (see also Chapter 3). The theorem (and the proof) is also related to a result of Datko ([45], see also [92, Sections 3.3 and 3.4]) that if A generates a C_0-semigroup and the solutions of the inhomogeneous Cauchy problem on $\mathbb{R}^+$ are in L^p (respectively bounded) whenever $f \in L^p$ (respectively bounded), then the semigroup tends to 0 in operator norm as t tends to infinity.

Moreover, we see that the situation on the line is different from initial value problems of first-order:

$$\text{(CP)} \qquad \begin{cases} u'(t) &= Au(t) + f(t) \qquad (t \geq 0) \\ u(0) &= x_0 \end{cases}$$

Here, well-pesedness of (CP) is equivalent to A being a generator of a C_0-semigroup on E (see for example [56], [53], [95] and many others).

The following lemma shows that in the first-order equation, the role of A can be replaced by $-A$. This will be useful in Chapter 5.

Lemma 2.2.3 *Let A be a closed operator on a Banach space E. Then the well-posedness of Equation (I) is equivalent to*

$\forall f \in BUC(\mathbb{R}, E)\ \exists!$ *mild solution* $u \in BUC(\mathbb{R}, E)$ *of* $u'(t) = -Au(t) + f(t) \quad (t \in \mathbb{R})$,

i.e. Equation (I) with A replaced by $-A$ is also well-posed.

Proof. Define $P : BUC(\mathbb{R}, E) \longrightarrow BUC(\mathbb{R}, E)$ by $(Pu)(t) := u(-t)$. Then P is an isomorphism on $BUC(\mathbb{R}, E)$ and the following equivalences hold

$$\begin{aligned} & u \in BUC(\mathbb{R}, E) \text{ is a mild solution of } (\mathrm{I})_f \\ \Leftrightarrow \quad & u(t) - u(0) = A \int_0^t u(r)dr + \int_0^t f(r)dr, \quad t \in \mathbb{R} \\ \Leftrightarrow \quad & (Pu)(t) - u(0) = -A \int_0^t (Pu)(r)dr - \int_0^t (Pf)(r)dr, \quad t \in \mathbb{R} \\ \Leftrightarrow \quad & Pu \text{ is a mild solution of } u'(t) = -Au(t) - Pf(t), \quad t \in \mathbb{R}, \end{aligned}$$

and $\int_0^t u(r)dr \in D(A)$ iff $\int_0^t (Pu)(r)dr \in D(A)$. Since P is a bijection, we obtain the result. □

Next we recall results on operator equations. Let A and B be closed operators on Banach spaces E and F, respectively, and let C be a bounded linear operator from F to E.

Definition 2.2.4 A bounded linear operator $X : F \longrightarrow E$ is called a *solution of the operator equation*

$$AX - XB = C$$

if for each $f \in D(B)$, $Xf \in D(A)$ and $AXf - XBf = Cf$.

This operator equation is naturally related to the operator $\tau_{A,B}$ on $\mathcal{L}(F,E)$ defined by

$$\begin{aligned} D(\tau_{A,B}) &:= \{X \in \mathcal{L}(F,E) : XD(B) \subseteq D(A) \ \&\ \exists Y \in \mathcal{L}(F,E) \\ &\qquad \text{such that } AXf - XBf = Yf \ \ \forall f \in D(B)\} \\ \tau_{A,B}(X) &:= Y. \end{aligned} \tag{2.7}$$

It is clear that existence and uniqueness of solutions of the operator equation (Definition 2.2.4) is equivalent to saying that $\tau_{A,B}$ is invertible.

Further, denote by $(S(t))_{t\in\mathbb{R}}$ the shift group on $BUC(\mathbb{R},E)$ which is defined by $(S(t)f)(s) := f(s+t)$ for all $s,t \in \mathbb{R}$ and all $f \in BUC(\mathbb{R},E)$, and denote by D its generator.

The first-order problem is related to certain operator equations. The proofs of the following theorem can be found in [14],[107] and [123]. Remark that the proof of [107, Theorem 3.1] works also if A is not a generator of a C_0-semigroup.

Theorem 2.2.5 *Let A be a closed operator on a Banach space E. Then the following are equivalent:*

(i) Equation (I) is well-posed.

(ii) The operator equation

$$AX - XD = -\delta_0, \tag{2.8}$$

where $\delta_0 \in \mathcal{L}(BUC(\mathbb{R},E),E)$ is given by $\delta_0(f) = f(0)$, has a unique bounded solution.

(iii) For every bounded linear operator $C : BUC(\mathbb{R},E) \longrightarrow E$ the operator equation $AX - XD = C$ has a unique bounded solution.

(iv) The operator $\tau_{A,-D}$ is invertible.

2.3 Examples of well-posed operators of first-order differential equations

Here, we want to give examples of operators such that Equation (I) will be well-posed. First we consider generators of C_0-semigroups (For more theory about C_0-semigroups see for example the monographs of Engel and Nagel ([53] and [91]), van Neerven [92] and Pazy [95]). Second, we show that if the operator A is a bisectorial with $0 \in \rho(A)$, then Equation (I) is well-posed. For that, some results of Chapter 1 are needed.

First, let A be a generator of a C_0-semigroup $(T(t))_{t\geq 0}$. One can characterise well-posedness of Equation (I) as follows. Note that the proofs of the following theorems can be found in [98] and [53, V.c].

Theorem 2.3.1 *Let A be a generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space E. Then the following are equivalent*

(i) Equation (I) is well-posed.

(ii) The C_0-semigroup $(T(t))_{t\geq 0}$ is hyperbolic.

(iii) $\sigma(T(t)) \cap \{\lambda \in \mathbb{C} : |\lambda| = 1\} = \emptyset$ for some (and hence for all) $t > 0$.

Next, we consider a special class of operators - the multiplication operators.

Definition 2.3.2

(i) Let Ω be a locally compact space and $m : \Omega \to \mathbb{C}$ be a continuous function. Then the *multiplication operator* M_m on $C_0(\Omega)$ induced by m is given by

$$\begin{aligned} D(M_m) &:= \{f \in C_0(\Omega) : m \cdot f \in C_0(\Omega)\} \\ M_m f &:= m \cdot f \qquad \forall f \in D(M_m) \end{aligned}$$

(ii) For a σ-finite measure space (Ω, Σ, μ) and a measurable function $m : \Omega \to \mathbb{C}$ the *multiplication operator* M_m on $L^p(\Omega)$ induced by m is defined by

$$\begin{aligned} D(M_m) &:= \{f \in L^p(\Omega) : m \cdot f \in L^p(\Omega)\} \\ M_m f &:= m \cdot f \qquad \forall f \in D(M_m) \end{aligned}$$

The above defined multiplication operators are closed, densely defined, linear operators where the spectrum of M_m is given by the range, respectively the essential range of the function m, i.e. $\sigma(M) = \mathrm{rg}(m)$, respectively $\sigma(M_m) = \text{ess–rg}(m)$ (see [53, I.4]). If $\sup_{\omega\in\Omega} \mathcal{R}e(m(\omega)) < \infty$, respectively $\text{ess–sup}_{\omega\in\Omega} \mathcal{R}e(m(\omega)) < \infty$, then M_m is the generator of a so-called *multiplication semigroup* $(T_m(t))_{t\geq 0}$ on $C_0(\Omega)$, respectively $L^p(\Omega)$. Since the Weak Spectral Mapping Theorem holds for such multiplication semigroups (see [53, Proposition IV.3.13]) we obtain from the above Theorem 2.3.1

Corollary 2.3.3 *Let M_m be the generator of a multiplication semigroup $(T_m(t))_{t\geq 0}$ on $C_0(\Omega)$ (or $L^p(\Omega)$) induced by an appropriate function $m : \Omega \to \mathbb{C}$. Then Equation (I) is well-posed if, and only if, $\{z \in \mathbb{C} : |\mathcal{R}e(z)| < c\} \cap \mathrm{rg}(m) = \emptyset$ (or $\{z \in \mathbb{C} : |\mathcal{R}e(z)| < c\} \cap \mathrm{ess{-}rg}(m) = \emptyset$) for a constant $c > 0$.*

In the case when the underlying Banach space is a Hilbert space, we have a similar situation (see [98]). This is a consequence of Gearhart's Theorem (see [53, V.1.11]).

Theorem 2.3.4 *Let A be a generator of a C_0-semigroup on a Hilbert space H. Then Equation (I) is well-posed if, and only if, $i\mathbb{R} \subseteq \rho(A)$ and there exists a constant $M \geq 0$ such that $\|R(i\xi, A)\| \leq M$ for all $\xi \in \mathbb{R}$.*

Second, let A be a bisectorial operator on a Banach space E with $0 \in \rho(A)$, then we get existence and uniqueness of mild solutions of the first-order differential equation on the line.

Theorem 2.3.5 *Let A be a densely defined, bisectorial operator on a Banach space E with $0 \in \rho(A)$, then Equation (I) is well-posed and the unique mild solutions are given by*

$$u(t) := \int_{\mathbb{R}} K(t-s)f(s)ds,$$

where the kernel $K(s)$ is given by : $K(s) := \begin{cases} T^-(\ s\ ), & s > 0 \\ -T^+(-s), & s < 0. \end{cases}$

Proof. Note that by Lemma 1.2.3, the kernel $K(\cdot)$ is integrable and, hence, the above integral is well-defined. Let $f \in BUC(\mathbb{R}, D(A))$ and define u as above. Then we can also write

$$u(t) \;=\; \int_{-\infty}^{t} T^-(t-s)f(s)ds - \int_{t}^{\infty} T^+(s-t)f(s)ds.$$

By Lemma 1.2.4, we see that u is continuously differentiable. By the properties of the spectral projections (see Proposition 1.3.6), we obtain

$$\begin{aligned} u'(t) &= Q^- f(t) + \int_{-\infty}^{t} AT^-(t-s)f(s)ds + Q^+ f(t) + \int_{t}^{\infty} (-A)T^+(s-t)f(s)ds \\ &= Au(t) + f(t) \end{aligned}$$

for all $t \in \mathbb{R}$, where Q^- and Q^+ are the initial projections corresponding to A (see Definition 1.2.6) and equal the spectral projections P^- and P^+ (see Corollaries 1.5.3 and 1.6.3). Hence, u is a classical solution of (I). Since $BUC(\mathbb{R}, D(A))$ is dense in $BUC(\mathbb{R}, E)$, there exists for arbitrary $f \in BUC(\mathbb{R}, E)$ a sequence $(f_n)_{n\in\mathbb{N}} \subseteq BUC(\mathbb{R}, D(A))$, such that $\lim_n f_n = f$. Then $\lim_n u_n = u$ and since the solution operator is closed, it follows that u is a mild solution of $(\mathrm{I})_f$, which proves existence of solutions.

The uniqueness of solutions follows immediately from Theorem 2.1.5. □

2.4 Bounded uniformly continuous functions with discrete spectrum

We will show that a bounded uniformly continuous function with discrete spectrum is almost periodic without any further conditions on the Banach space E except that $E \neq \{0\}$ to exclude trivial cases. Recall that a subset of $\mathbb{R}$ is called *discrete* if it contains only isolated points.

Note that the results of this section for vector-valued functions on $\mathbb{R}$ can be generalised to vector-valued functions on locally compact Abelian groups. This is done in [15, Section 5]. For more information about representations of locally compact Abelian groups it can be referred to [65], [81] [109] and the references therein.

But first, we have to recall some facts about spectral theory of bounded C_0-groups. Therefore, assume that A generates a bounded C_0-group $\mathcal{U} = (U(t))_{t\in\mathbb{R}}$. Recall the following definitions (see [8] or [16]):

Definition 2.4.1 Let A be the generator of a bounded C_0-group $\mathcal{U}$.

(i) The *Arveson spectrum* of $\mathcal{U}$ is given by

$$\begin{aligned} Sp(\mathcal{U}) \;:=\; & \{\xi \in \mathbb{R} : \forall \epsilon > 0\, \exists f \in L^1(\mathbb{R}) \text{ such that} \\ & supp(\bar{\mathcal{F}} f) \subseteq (\xi - \epsilon, \xi + \epsilon) \text{ and } f(\mathcal{U}) \neq 0\} \end{aligned}$$

where the operator $f(\mathcal{U}) \in \mathcal{L}(E)$ is given by $f(\mathcal{U})x := \int_{-\infty}^{+\infty} f(t)U(t)x dt$ for all $x \in E$ and let $\bar{\mathcal{F}} f(s) := \int_{-\infty}^{+\infty} e^{ist} f(t)dt$ for all $s \in \mathbb{R}$.

(ii) For an element $x \in E$ the *Arveson spectrum of x with respect to $\mathcal{U}$* is defined by

$$\begin{aligned} sp^{\mathcal{U}}(x) \;:=\; & \{\xi \in \mathbb{R} : \forall \epsilon > 0\, \exists f \in L^1(\mathbb{R}) \text{ such that} \\ & supp(\bar{\mathcal{F}} f) \subseteq (\xi - \epsilon, \xi + \epsilon) \text{ and } f(\mathcal{U})x \neq 0\}. \end{aligned}$$

It is known that (see [46, Theorem 8.19]) $iSp(\mathcal{U}) = \sigma(A)$. From the definitions, it follows that the Arveson spectrum of x with respect to $\mathcal{U}$ coincides with the Arveson spectrum of the group $\mathcal{U}_x$ given by $U_x(t) := U(t)|_{E_x}$, where $E_x = \overline{span}\{U(t)x : t \in \mathbb{R}\}$. Hence, we have

$$isp^{\mathcal{U}}(x) = iSp(\mathcal{U}_x) = \sigma(A_x) \tag{2.9}$$

where A_x is the generator of $\mathcal{U}_x$.

Now consider a special C_0-group, the shift group $\mathcal{S} = (S(t))_{t\in\mathbb{R}}$ on $BUC(\mathbb{R}, E)$ defined by $(S(t)u)(s) := u_t(s) = u(s + t)$. Denote by D its generator; then the domain of D consists of all $u \in BUC(\mathbb{R}, E) \cap C^1(\mathbb{R}, E)$ such that $u' \in BUC(\mathbb{R}, E)$ and $Du = u'$.

Let $BUC_u := \overline{span}\{S(t)u : t \in \mathbb{R}\}$ and denote by $\mathcal{S}_u = (S_u(t))_{t\in\mathbb{R}}$ the shift group on BUC_u with generator D_u. Then D_u is the part of D in BUC_u and by [8, (2.4)] and (2.9) we have

$$isp(u) = \sigma(D_u) = iSp(\mathcal{S}_u). \tag{2.10}$$

In the following, let $AP(\mathbb{R}, E)$ be the space of all *almost periodic* functions on $\mathbb{R}$ with values in the Banach space E. For the definition and various characterisations we refer to [75]. In particular, it is known that

$$AP(\mathbb{R}, E) = \overline{span}\{e_\eta \otimes x \, : \, \eta \in \mathbb{R}, x \in E\}, \tag{2.11}$$

where $(e_\eta \otimes x)(s) = e^{i\eta s}x$, $s \in \mathbb{R}$.

For $u \in BUC(\mathbb{R}, E)$ define the *reduced spectrum of u with respect to $AP(\mathbb{R}, E)$* by (see [8])

$$\begin{aligned} sp_{AP}(u) \; := \; & \{\xi \in \mathbb{R} \, : \, \forall \epsilon > 0 \, \exists f \in L^1(\mathbb{R}) \text{ such that} \\ & supp(\mathcal{F}f) \subseteq (\xi - \epsilon, \xi + \epsilon) \text{ and } f * u \notin AP(\mathbb{R}, E)\}. \end{aligned} \tag{2.12}$$

As before, we consider the shift group $\mathcal{S} = (S(t))_{t\in\mathbb{R}}$ on $BUC(\mathbb{R}, E)$. Since $\mathcal{S}$ leaves $AP(\mathbb{R}, E)$ invariant, we can define the quotient group $\bar{\mathcal{S}} = (\bar{S}(t))_{t\in\mathbb{R}}$ on $Y := BUC(\mathbb{R}, E)/AP(\mathbb{R}, E)$ by

$$\bar{S}(t)\bar{u} = \overline{(S(t)u)}$$

for all $t \in \mathbb{R}$ and $u \in BUC(\mathbb{R}, E)$, where $^- : BUC(\mathbb{R}, E) \to Y$ denotes the quotient mapping. The generator of $\bar{\mathcal{S}}$ is denoted by $\bar{D}$.

Again we let $Y_{\bar{u}}$ be the closed linear span of the orbit $\{\bar{S}(t)\bar{u} : t \in \mathbb{R}\}$ in Y and we denote by $\bar{\mathcal{S}}_{\bar{u}} = (\bar{S}_u(t))_{t\in\mathbb{R}}$ the restricted group on $Y_{\bar{u}}$ with generator $\bar{D}_{\bar{u}}$.

Proposition 2.4.2 *For $u \in BUC(\mathbb{R}, E)$ we have with the above notations*

$$isp_{AP}(u) = \sigma(\bar{D}_{\bar{u}}).$$

Proof. Let $f \in L^1(\mathbb{R})$ and $u \in BUC(\mathbb{R}, E)$. Then

$$\begin{aligned}(f * u)(t) &= \int_{-\infty}^{+\infty} f(s)u(t-s)ds \\ &= \int_{-\infty}^{+\infty} f(-s)(S(s)u)(t)ds \;=\; (f_-(\mathcal{S})u)(t)\end{aligned}$$

for all $t \in \mathbb{R}$, where $f_-(s) := f(-s)$. Hence $f * u \in AP(\mathbb{R}, E)$ if, and only if, $f_-(\mathcal{S})u \in AP(\mathbb{R}, E)$ which is equivalent to $f_-(\bar{\mathcal{S}}_{\bar{u}}) = 0$ in $\mathcal{L}(Y_{\bar{u}})$. Since $\bar{\mathcal{F}} f_- = \mathcal{F} f$, we have $sp_{AP}(u) = Sp(\bar{\mathcal{S}}_{\bar{u}})$ and the claim follows from (2.9). □

Since the spectrum of the generator of a bounded group on a Banach space different from 0 is never empty (see [91, A-III 7.6.] or [92, Lemma 2.4.3]), we obtain as a consequence (see also [23, Prop. 2.5]) the following.

Corollary 2.4.3 *Let $u \in BUC(\mathbb{R}, E)$. Then $u \in AP(\mathbb{R}, E)$ if, and only if, $sp_{AP}(u) = \emptyset$.*

Lemma 2.4.4 *Let $u \in BUC(\mathbb{R}, E)$ and assume that $\eta \in \mathbb{R}$ is an isolated point of $sp(u)$. Then $u = u_0 + u_1$, where $u_0 = e_\eta \otimes x$ for some $x \in E$ and $\eta \notin sp(u_1)$.*

Proof. Since η is an isolated point of $sp(u)$ there exists a function $\psi \in L^1(\mathbb{R})$ with $\mathcal{F}\psi = 1$ in a neighbourhood of $\{\eta\}$ and $\mathcal{F}\psi = 0$ in a neighbourhood of $sp(u) \setminus \{\eta\}$. Then we can write

$$u = (u * \psi) + (u - u * \psi) =: u_0 + u_1.$$

It follows from the definition of the Beurling spectrum (see (2.3) and also [22, 4.1.4]) that $sp(u_0) \subseteq sp(u) \cap supp(\mathcal{F}\psi) = \{\eta\}$ and $sp(u_1) \subseteq sp(u) \cap supp(1 - \mathcal{F}\psi) = sp(u) \setminus \{\eta\}$. Hence $u_0 = e_\eta \otimes x$ for some $x \in E$ and $\eta \notin sp(u_1)$. □

Theorem 2.4.5 *Let $u \in BUC(\mathbb{R}, E)$ and assume that $\eta \in sp_{AP}(u)$. Then η is an accumulation point of $sp(u)$.*

Proof. Assume that η is an isolated point of $sp(u)$. Then by Lemma 2.4.4 we have

$$u = u_0 + u_1,$$

where $u_0 = e_\eta \otimes x \in AP(\mathbb{R}, E)$ and $\eta \notin sp(u_1)$. Hence the closed linear spans $Y_{\bar{u}}$ and $Y_{\bar{u}_1}$ of the orbit of the elements $\overline{u}$, respectively $\overline{u_1}$, in the quotient space $Y = BUC(\mathbb{R}, E)/AP(\mathbb{R}, E)$ coincide. It follows that

$$\sigma(\bar{D}_{\bar{u}}) = \sigma(\bar{D}_{\bar{u}_1}).$$

Since $\sigma(\bar{D}_{\bar{u}_1}) \subseteq \sigma(D_{u_1})$ and $isp(u_1) = \sigma(D_{u_1})$, we obtain from Lemma 2.4.4 that $i\eta \notin \sigma(\bar{D}_{\bar{u}}) = isp_{AP}(u)$ (by Proposition 2.4.2), which is a contradiction. □

From this we deduce immediately the following result (see also [24, Theorem 8, p. 78]).

Theorem 2.4.6 *Let $u \in BUC(\mathbb{R}, E)$ and assume that $sp(u)$ is discrete. Then $u \in AP(\mathbb{R}, E)$.*

Proof. Since $sp(u)$ does not have accumulation points, it follows from Theorem 2.4.5 that $sp_{AP}(u) = \emptyset$. Hence $u \in AP(\mathbb{R}, E)$ by Corollary 2.4.3. □

Remark 2.4.7 Remark that a more direct argument is possible: Let $u \in BUC(\mathbb{R}, E)$. Take an approximate unit $\rho_n \in L^1(\mathbb{R})$ $(n \in \mathbb{N})$ such that $\mathcal{F}\rho_n$ has compact support and $\rho_n * u \to u$ $(n \to \infty)$ in $BUC(\mathbb{R}, E)$. Then $sp(\rho_n * u) \subseteq supp(\mathcal{F}\rho_n) \cap sp(u)$. Thus if $sp(u)$ is discrete, $\rho_n * u$ has finite spectrum. This implies that $\rho_n * u$ is a trigonometric polynomial, and hence $u \in AP(\mathbb{R}, E)$.

The condition about the discreteness of the spectrum is the best possible if we compare it with Loomis' Theorem ([78]) where discrete is replaced by countable, but then an additional condition on the geometry of the Banach space is needed. This important result in the spectral theory of almost periodic functions proved Loomis in the scalar case ([78]), whereas the vector-valued version (see for example [75, p. 92] and [8, Theorem 3.2]) is a consequence of Kadets' theorem ([21], [66] or [75, p. 86]).

Theorem 2.4.8 (Loomis) *Let $u \in BUC(\mathbb{R}, E)$ and assume that $sp(u)$ is countable. Then $u \in AP(\mathbb{R}, E)$ provided that $c_0 \not\subseteq E$.*

The following example (compare with [75, p. 81]) shows that Theorem 2.4.8 fails on c, the space of all convergent sequences. Note that c is isomorphic to c_0.

Counterexample 2.4.9 *Let $u \in BUC(\mathbb{R}, c)$ given by*

$$u(t) := (e^{\frac{i}{k}t})_{k \in \mathbb{N}}.$$

Then $u \notin AP(\mathbb{R}, c)$, but $sp(u) = \{\frac{1}{k} | k \in \mathbb{N}\} \cup \{0\}$ is countable.

2.5 Asymptotic behaviour of mild solutions of first-order differential equations

In this section, we give conditions on the operator A and the inhomogeneity f such that the mild solutions of Equation (I) satisfy certain asymptotic behaviour. Asymptotic behaviour of a bounded uniformly continuous function u characterises the behaviour of the function for large $t \in \mathbb{R}$. This is equivalent to say that u belongs to an associated translation-invariant subspace of $BUC(\mathbb{R}, E)$ (see [36, Section 1.2]).

The most interesting classes of closed, translation-invariant subspaces of $BUC(\mathbb{R}, E)$ are the following:

(i) the space $AP(\mathbb{R}, E)$;

(ii) the space $W(\mathbb{R}, E)$ of all *weakly almost periodic functions in the sense of Eberlein*

$$\begin{aligned} W(\mathbb{R}, E) \quad := \quad & \{u \in BUC(\mathbb{R}, E) \,:\, \{S(t)u : t \in \mathbb{R}\} \text{ is relatively weakly} \\ & \text{compact in } BUC(\mathbb{R}, E)\}; \end{aligned}$$

(iii) the space $WAP(\mathbb{R}, E)$ of all *weakly almost periodic functions*

$$WAP(\mathbb{R}, E) \quad := \quad \{u \in BUC(\mathbb{R}, E) \,:\, x' \circ u \in AP(\mathbb{R}) \text{ for all } x' \in E'\};$$

(iv) the space $E(\mathbb{R}, E)$ of all *uniformly ergodic functions*

$$E(\mathbb{R}, E) := \{u \in BUC(\mathbb{R}, E) \,:\, \lim_{\alpha \downarrow 0} \alpha \int_0^\infty e^{-\alpha t} S(t)u \, dt \text{ exists in } BUC(\mathbb{R}, E)\};$$

(v) the space $TE(\mathbb{R}, E)$ of all *totally uniformly ergodic functions*

$$TE(\mathbb{R}, E) := \{u \in BUC(\mathbb{R}, E) \,:\, e^{i\eta\cdot} u \in E(\mathbb{R}, E) \;\forall \eta \in \mathbb{R}\};$$

(vi) the space $AAP(\mathbb{R}, E)$ of all *asymptotically almost periodic functions*

$$AAP(\mathbb{R}, E) := C_0(\mathbb{R}, E) \oplus AP(\mathbb{R}, E).$$

Remark that the space $AP(\mathbb{R}, E)$ is included in all these spaces.

A slight generalisation of Proposition 2.4.2 and Corollary 2.4.3 is the following. Let $\mathcal{G}$ be a closed, translation-invariant subspace of $BUC(\mathbb{R}, E)$ and define the *reduced spectrum of $u \in BUC(\mathbb{R}, E)$ with respect to $\mathcal{G}$* by

$$\begin{aligned} sp_{\mathcal{G}}(u) \quad := \quad & \{\xi \in \mathbb{R} \,:\, \forall \epsilon > 0 \, \exists f \in L^1(\mathbb{R}) \text{ such that} \\ & supp(\mathcal{F}f) \subseteq (\xi - \epsilon, \xi + \epsilon) \text{ and } f * u \notin \mathcal{G}\}. \end{aligned} \tag{2.13}$$

Note that $sp_{\mathcal{G}}(u) \subseteq sp_{AP}(u)$, if $AP(\mathbb{R}, E) \subseteq \mathcal{G}$. Since $\mathcal{G}$ is closed and translation-invariant we can consider the quotient group $\tilde{\mathcal{S}}$ on $\tilde{Y} := BUC(\mathbb{R}, E)/\mathcal{G}$ defined by $\tilde{S}(t)\tilde{u} := (S(t)u)\tilde{}$ where $\tilde{}: BUC(\mathbb{R}, E) \to \tilde{Y}$ denotes the quotient mapping.

Then we obtain by the same proof as in Proposition 2.4.2

$$isp_{\mathcal{G}}(u) = \sigma(\tilde{D}_{\tilde{u}}), \tag{2.14}$$

where $\tilde{D}_{\tilde{u}}$ is the generator of the group $\tilde{\mathcal{S}}_{\tilde{u}} = (\tilde{S}_{\tilde{u}}(t))_{t\in\mathbb{R}}$ on $\tilde{Y}_{\tilde{u}} = \overline{span}\{\tilde{S}(t)\tilde{u} | t \in \mathbb{R}\}$. And as in Corollary 2.4.3, we have for functions $u \in BUC(\mathbb{R}, E)$ that $u \in \mathcal{G}$ if, and only if, $sp_{\mathcal{G}}(u) = \emptyset$.

In Section 2.4, we have seen that we are interested in the discrete and non-discrete points of the spectrum of bounded uniformly continuous functions. Thus, for an arbitrary set $M \subseteq \mathbb{C}$, we denote by M' *the set of accumulation points* of M. If $f \in BUC(\mathbb{R}, E)$ and $M = sp(f)$, we obtain

$$sp'(f) := \{\eta \in sp(f) \,:\, \exists (\eta_n)_{n\in\mathbb{N}} \subseteq sp(f) \setminus \{\eta\} \text{ such that } \lim_n \eta_n = \eta\}. \tag{2.15}$$

For the proof of the next theorem we first use the Laplace transform argument of [8, Theorem 4.3]. But here an additional argument is needed since we do not assume that $c_0 \not\subseteq E$.

Theorem 2.5.1 *Let $u \in BUC(\mathbb{R}, E)$ be a mild solution of Equation $(I)_f$ and assume that $\sigma(A) \cap i\mathbb{R}$ is discrete as a subset of $i\mathbb{R}$. Let $f \in AP(\mathbb{R}, E)$ with $isp'(f) \cap \sigma(A) = \emptyset$. Then $u \in AP(\mathbb{R}, E)$.*

Proof. By replacing u by u_s and by taking Carleman transform on both sides of Equation (I), we obtain (see also the proof of Proposition 2.1.3)

$$\hat{u}_s(\lambda) = R(\lambda, A)u(s) + R(\lambda, A)\hat{f}_s(\lambda).$$

Furthermore $\hat{u}_s(\lambda) = \int_{-\infty}^{+\infty} e^{-\lambda t} u(s+t)dt = (R(\lambda, D)u)(s)$, where D is the generator of the shift group $\mathcal{S}$ on $BUC(\mathbb{R}, E)$. Similarly, $\hat{f}_s(\lambda) = (R(\lambda, D)f)(s)$. So we finally obtain

$$(R(\lambda, D)u)(s) = R(\lambda, A)u(s) + R(\lambda, A)(R(\lambda, D)f)(s)$$

for all $s \in \mathbb{R}$. We now consider, as before, the quotient space $BUC(\mathbb{R}, E)/AP(\mathbb{R}, E)$ with the induced shift group $\bar{\mathcal{S}}$ and its generator $\bar{D}$. Since $f \in AP(\mathbb{R}, E)$, we obtain

$$R(\lambda, \bar{D})\bar{u} = \overline{(R(\lambda, A) \circ u)}.$$

It follows from Proposition 2.4.2 (see also [8, Theorem 4.3]) that

$$isp_{AP}(u) = \sigma(\bar{B}_{\bar{u}}) \subseteq \sigma(A) \cap i\mathbb{R}.$$

By Theorem 2.4.5 we deduce from this

$$isp_{AP}(u) \subseteq (\sigma(A) \cap i\mathbb{R}) \cap isp'(u). \tag{2.16}$$

On the other hand, since by hypothesis $\sigma(A) \cap i\mathbb{R}$ is discrete, Proposition 2.1.3 implies that

$$sp'(u) \subseteq sp'(f).$$

Hence by (2.16), $isp_{AP}(u) \subseteq \sigma(A) \cap isp'(f) = \emptyset$. This implies by Corollary 2.4.3 that $u \in AP(\mathbb{R}, E)$. □

Note that in Theorem 2.5.1 we merely need that $\sigma(A) \cap i\mathbb{R}$ is discrete in $i\mathbb{R}$, but $\sigma(A) \cap i\mathbb{R}$ is allowed to contain limit points of $\sigma(A)$.

From [8, Proposition 4.2] (see also [20, 2. Theorem]) we obtain the following corollary.

Corollary 2.5.2 *Let A be the generator of a bounded C_0-group on a Banach space E and assume that $\sigma(A)$ is discrete. Then*

$$E = E_{AP} := \overline{span}\{x \in D(A) \,:\, \exists \eta \in \mathbb{R} \text{ such that } Ax = i\eta x\}$$

and the C_0-group is almost periodic.

Theorem 2.5.1 is in some sense the best possible result. This is shown by the following example where $A = 0$.

Example 2.5.3 *Define $f \in AP(\mathbb{R}, c)$ by $f(t) := (\frac{i}{k} e^{\frac{i}{k}t})_{k \in \mathbb{N}}$. Remark that $sp(f) = \{\frac{1}{k} | k \in \mathbb{N}\} \cup \{0\}$; thus $sp'(f) = \{0\}$ has non-empty intersection with the spectrum of the operator $A = 0$. The function $u \in BUC(\mathbb{R}, c)$ given by $u(t) = (e^{\frac{i}{k}t})_{k \in \mathbb{N}}$ is a solution of*

$$u'(t) = Au(t) + f(t) \qquad (t \in \mathbb{R}),$$

(see also Counterexample 2.4.9). But $u \notin AP(\mathbb{R}, c)$.

Let us now consider closed, translation-invariant subspaces $\mathcal{G}$ of $BUC(\mathbb{R}, E)$ where $AP(\mathbb{R}, E)$ is included in $\mathcal{G}$. For examples of closed, translation-invariant subspaces of $BUC(\mathbb{R}, E)$ see the beginning of this section.

By a slight modification of the proof we obtain the following generalisation of Theorem 2.5.1.

Theorem 2.5.4 *Let $\mathcal{G} \subseteq BUC(\mathbb{R}, E)$ be a closed, translation-invariant subspace of $BUC(\mathbb{R}, E)$ containing $AP(\mathbb{R}, E)$, and suppose $f \in \mathcal{G}$. Assume that $\sigma(A) \cap i\mathbb{R}$ is discrete and that $isp'(f) \cap \sigma(A) = \emptyset$. Let $u \in BUC(\mathbb{R}, E)$ be a mild solution of Equation $(I)_f$. Then $u \in \mathcal{G}$.*

Proof. It follows from Proposition 2.1.3 that $isp(u) \subseteq (\sigma(A) \cap i\mathbb{R}) \cup isp(f)$. And hence, since $(\sigma(A) \cap i\mathbb{R})$ is discrete, we obtain

$$sp'(u) \subseteq sp'(f).$$

Now, we consider the quotient space $\tilde{Y} := BUC(\mathbb{R}, E)/\mathcal{G}$ with the induced shift group $\tilde{\mathcal{S}}$ and generator $\tilde{D}$. Then we obtain as in the proof of Theorem 2.5.1 (see also in the proof of [8, Theorem 4.3]) that $R(\lambda, \tilde{D})\tilde{u} = (R(\lambda, A) \circ u)\tilde{}$. It follows (by (2.14)) that

$$isp_{\mathcal{G}}(u) = \sigma(\tilde{D}_{\tilde{u}}) \subseteq \sigma(A) \cap i\mathbb{R}.$$

Since $AP(\mathbb{R}, E) \subseteq \mathcal{G}$ we have that $sp_{\mathcal{G}}(u) \subseteq sp_{AP}(u)$ and $sp_{AP}(u) \subseteq sp'(u)$ by Theorem 2.4.5. So we conclude with the above equations that $sp_{\mathcal{G}}(u) \subseteq (\sigma(A) \cap i\mathbb{R}) \cap sp'(f) = \emptyset$. It follows that $u \in \mathcal{G}$. □

It is interesting to compare Theorem 2.5.1 and Theorem 2.5.4 with the following stronger result [8, Theorem 4.3] which holds if $c_0 \not\subseteq E$:

Theorem 2.5.5 *Let A be a closed operator on a Banach space E such that $c_0 \not\subseteq E$. Assume that $\sigma(A) \cap i\mathbb{R}$ is countable. Let $u \in BUC(\mathbb{R}, E)$ be a solution of Equation (I). If $f \in AP(\mathbb{R}, E)$, then $u \in AP(\mathbb{R}, E)$.*

In contrast to Theorem 2.5.1 which has an extension to more general spaces (Theorem 2.5.4), this result does not extend to other spaces than $AP(\mathbb{R}, E)$. We give an example in the scalar case.

Example 2.5.6 *Let $\mathcal{G} = TE(\mathbb{R})$. Then there exists $f \in \mathcal{G}$ such that $u(t) = \int_0^t f(s)ds$ is bounded, but $u \notin \mathcal{G}$. Thus, if we choose $A = 0$, then $u \in BUC(\mathbb{R})$ is a solution of Equation (I), but $u \notin \mathcal{G}$. Such f can be defined by*

$$f(t) \quad := \quad \begin{cases} \frac{1}{2\sqrt{t}} \cos\sqrt{t}, & \text{if } t \geq \frac{\pi^2}{4} \\ 0, & \text{if } t < \frac{\pi^2}{4}. \end{cases}$$

Then

$$u(t) \quad := \quad \begin{cases} \sin\sqrt{t} - \sin\frac{\pi}{2}, & \text{if } t \geq \frac{\pi^2}{4} \\ 0, & \text{if } t < \frac{\pi^2}{4}. \end{cases}$$

It has been shown in [26, Example 4.2] that this function is not totally uniformly ergodic.

A similar result for slowly oscillating functions with countable spectrum is given by Arendt and Batty in [10].

Chapter 3

Solutions of first-order differential equations in $\mathbf{L^p(\mathbb{R}, E)}$

In this chapter, we consider again the following Equation

$$\text{(I)} \qquad u'(t) = Au(t) + f(t) \quad (t \in \mathbb{R}),$$

where A is a closed, linear operator on a Banach space E, but now $f \in L^p(\mathbb{R}, E)$ for $1 < p < \infty$.

3.1 Mild solutions in $\mathbf{L^p(\mathbb{R}, E)}$

We give a necessary condition on the operator A for existence and uniqueness of mild solutions in $L^p(\mathbb{R}, E)$.

Definition 3.1.1 For $f \in L^p(\mathbb{R}, E)$, we call $u \in L^p(\mathbb{R}, E)$ a *mild solution* of Equation (I) if $\int_0^t u(s)ds \in D(A)$ and there exists $x \in E$ such that

$$u(t) = x + A\int_0^t u(s)ds + \int_0^t f(s)ds \tag{3.1}$$

for almost all $t \in \mathbb{R}$.

We call u a *strong solution* of Equation (I), if $u \in W^{1,p}(\mathbb{R}, E) \cap L^p(\mathbb{R}, D(A))$ and u satisfies Equation (I) for almost all $t \in \mathbb{R}$.

Note that a strong solution of Equation (I) is of course also a mild solution of Equation (I).

For functions $u \in L^p(\mathbb{R}, E)$, we can define the spectrum as in the case of bounded uniformly continuous functions by using the Carleman transform (compare with Section 2.1 (2.4) and (2.5)).

Denote by $\hat{u}$ the *Carleman transform* of the function $u \in L^p(\mathbb{R}, E)$, i.e.

$$\hat{u}(\lambda) \quad := \quad \begin{cases} \int_0^\infty e^{-\lambda t} u(t)dt & (\mathcal{R}e(\lambda) > 0) \\ -\int_{-\infty}^0 e^{-\lambda t} u(t)dt & (\mathcal{R}e(\lambda) < 0). \end{cases} \tag{3.2}$$

Obviously, $\hat{u}$ is a holomorphic function on $\mathbb{C} \setminus i\mathbb{R}$. A point $\eta \in \mathbb{R}$ is called a *regular point* if the Carleman transform has a holomorphic extension to a neighbourhood of $i\eta$ and the *Carleman spectrum* of u is given by $sp(u) := \{\xi \in \mathbb{R} : \xi \text{ is not regular}\}$ (see for example [99, Section 0]).

With the fact that a function $u = 0$ if $sp(u) = \emptyset$ (see [99, Propositon 0.5]), we can prove uniqueness of mild solutions.

Proposition 3.1.2 *Let A be a closed linear operator on a Banach space E such that $\sigma(A) \cap i\mathbb{R} = \emptyset$. If $u \in L^p(\mathbb{R}, E)$ is a mild solution of the homogeneous equation (I)$_0$, then $u = 0$.*

Proof. Taking Carleman-transform on both sides of Equation (3.1) with $f = 0$, it follows for $\mathcal{R}e(\lambda) \neq 0$ that

$$\hat{u}(\lambda) = \frac{1}{\lambda} x + \frac{1}{\lambda} A\hat{u}(\lambda).$$

Since $\lambda \in \rho(A)$ for all $\lambda \in i\mathbb{R}$, we obtain that $\hat{u}(\lambda) = R(\lambda, A)x$ has a holomorphic extension in a neighbourhood of λ for all $\lambda \in i\mathbb{R}$. Hence, $sp(u) = \emptyset$ and $u = 0$. □

Next, we consider the solution operator for Equation (I) in $L^p(\mathbb{R}, E)$ which is defined by

$$\begin{aligned} D(M_p) \quad &:= \quad \{f \in L^p(\mathbb{R}, E) : \exists!\, u_f \in L^p(\mathbb{R}, E) \text{ such that} \\ &\qquad u_f \text{ is a mild solution of Equation } (\mathrm{I})_f\} \\ M_p f \quad &:= \quad u_f. \end{aligned} \tag{3.3}$$

Remark that if one mild solution of Equation (I) is not unique, then no mild solution of (I) is unique and $D(M_p) = \emptyset$. Moreover, since the operator A is closed, it follows that $\tilde{M}_p f := (M_p f, x_f)$, where x_f is given by Equation (3.1), is also a closed operator.

Thus $M_p = p_1 \circ \tilde{M}_p$ is bounded if $D(M_p) = L^p(\mathbb{R}, E)$, since the projection on the first coordinate p_1 is bounded and $\tilde{M}_p$ is bounded by the closed graph theorem.

For $\alpha > 0$ we define the following weighted L^p-space.

$$L^p_\alpha(\mathbb{R}, E) := \{f : \mathbb{R} \to E \text{ measurable } : \|f\|_{p,\alpha} < \infty\} \tag{3.4}$$

where the norm is given by $\|f\|_{p,\alpha} := (\int_{\mathbb{R}} \|e^{-\alpha|t|} f(t)\|^p dt)^{\frac{1}{p}}$.

As in Definition 3.1.1, we call for $f \in L^p_\alpha(\mathbb{R}, E)$ the function $u \in L^p_\alpha(\mathbb{R}, E)$ a *mild solution* of Equation (I), if u satisfies the integrated equation (3.1) for almost all $t \in \mathbb{R}$. Similarly, we can define *strong solutions* in $L^p_\alpha(\mathbb{R}, E)$.

Further, we define the following mapping

$$\begin{array}{rcl} \tilde{} : L^p_\alpha(\mathbb{R}, E) & \to & L^p(\mathbb{R}, E) \\ u & \mapsto & \tilde{u}, \text{ where } \tilde{u}(t) := e^{-\alpha|t|} u(t). \end{array} \tag{3.5}$$

It is easy to see that $\tilde{}$ is an isomorphism between $L^p_\alpha(\mathbb{R}, E)$ and $L^p(\mathbb{R}, E)$.

We obtain the following connection between solutions in $L^p(\mathbb{R}, E)$ and solutions in $L^p_\alpha(\mathbb{R}, E)$.

Lemma 3.1.3 *Let $p \in (1, \infty)$, $\alpha > 0$ and $f \in L^p_\alpha(\mathbb{R}, E)$. Then $u \in L^p_\alpha(\mathbb{R}, E)$ is a mild solution of $(I)_f$ if, and only if, $\tilde{u} \in L^p(\mathbb{R}, E)$ is a mild solution of*

$$(\tilde{\mathrm{I}})_{\tilde{f}} \qquad \tilde{u}'(t) = A\tilde{u}(t) + \tilde{f}(t) - \alpha \operatorname{sgn}(t)\tilde{u}(t) \qquad (t \in \mathbb{R}).$$

Proof. Let $u \in L^p_\alpha(\mathbb{R}, E)$ be a mild solution of $(\mathrm{I})_f$, i.e. there exist $x \in E$ such that the integrated equation (3.1) holds for almost all $t \in \mathbb{R}$. With integration by parts, it follows

$$\begin{aligned}
& A\int_0^t \tilde{u}(s)ds + \int_0^t \tilde{f}(s)ds \\
&= A\left(e^{-\alpha|t|}\int_0^t u(s)ds + \alpha\int_0^t (\operatorname{sgn}(s)e^{-\alpha|s|}\int_0^s u(r)dr)ds\right) \\
&\quad +e^{-\alpha|t|}\int_0^t f(s)ds + \alpha\int_0^t (\operatorname{sgn}(s)e^{-\alpha|s|}\int_0^s f(r)dr)ds \\
&= e^{-\alpha|t|}\left(A\int_0^t \tilde{u}(s)ds + \int_0^t f(s)ds\right) \\
&\quad +\alpha \operatorname{sgn}(t)\int_0^t e^{-\alpha|s|}(A\int_0^s u(r)dr + \int_0^s f(r)dr)ds
\end{aligned}$$

$$\begin{aligned} &= e^{-\alpha|t|}(u(t)-x) + \alpha\,\mathrm{sgn}(t)\int_0^t e^{-\alpha|s|}(u(s)-x)ds \\ &= \tilde{u}(t) + \alpha\,\mathrm{sgn}(t)\int_0^t \tilde{u}(s)ds + x \end{aligned}$$

for almost all $t \in \mathbb{R}$. Thus, $\tilde{u}$ is a mild solution of $(\mathrm{I})_{\tilde{f}}$. The reverse can be proved similar. □

Lemma 3.1.4 *Let $p \in (1,\infty)$, $\alpha > 0$, the mappings M_p and $\tilde{}$ defined as above and $f \in L^p_\alpha(\mathbb{R},E)$. Let $\tilde{u} \in L^p(\mathbb{R},E)$ be a mild solution of $(\tilde{I})_{\tilde{f}}$. Then*

$$M_p\tilde{f} = \tilde{u} + \alpha M_p(\,\mathrm{sgn}\ \tilde{u}).$$

Proof. By hypothesis, there exist $x_1 \in E$ such that

$$\tilde{u}(t) = x_1 + A\int_0^t \tilde{u}(s)ds + \int_0^t \tilde{f}(s)ds - \alpha\,\mathrm{sgn}(t)\int_0^t \tilde{u}(s)ds.$$

Let $v = M_p(\,\mathrm{sgn}\ \tilde{u})$, i.e. there exists $x_2 \in E$ such that

$$v(t) = x_2 + A\int_0^t v(s)ds + \ \mathrm{sgn}(t)\int_0^t \tilde{u}(s)ds.$$

It follows that

$$\begin{aligned} &\tilde{u}(t) + \alpha v(t) \\ &\quad= (x_1+\alpha x_2) + A\int_0^t (\tilde{u}(s)+\alpha v(s))ds + \int_0^t \tilde{f}(s)ds \\ &\quad= x' + A\int_0^t (\tilde{u}(s)+\alpha v(s))ds + \int_0^t \tilde{f}(s)ds \end{aligned}$$

with $x' = x_1 + \alpha x_2$. Hence, $\tilde{u}+\alpha v$ is a mild solution of $(\mathrm{I})_{\tilde{f}}$, i.e. $M_p\tilde{f} = \tilde{u}+\alpha M_p(\,\mathrm{sgn}\ \tilde{u})$.
□

Theorem 3.1.5 *Let $1 < p < \infty$ and let A be a closed linear operator on a Banach space E such that for each $f \in L^p(\mathbb{R},E)$ there exists a unique mild solution $u \in L^p(\mathbb{R},E)$ of Equation $(I)_f$. Then $i\mathbb{R} \in \rho(A)$ and there exists a constant $c \geq 0$ such that $\|R(i\xi,A)\| \leq c$ for all $\xi \in \mathbb{R}$.*

Proof. By hypothesis, $D(M_p) = L^p(\mathbb{R}, E)$, thus the solution operator M_p of Equation (I) is a bounded operator on $L^p(\mathbb{R}, E)$. Moreover, if α is small enough, i.e. $\alpha < \frac{1}{\|M_p\|}$, the mapping $\tilde{u} \mapsto \tilde{u} + \alpha M_p(\operatorname{sgn}(.)\tilde{u})$ is invertible. Hence, the mapping

$$\begin{array}{rcl} M_{p,\alpha} : L^p_\alpha(R, E) & \longrightarrow & L^p_\alpha(R, E) \\ M_{p,\alpha} f & := & (\tilde{\ })^{-1}(I + \alpha M_p(\operatorname{sgn}\ .\ ))^{-1} M_p \tilde{f} \end{array}$$

maps each $f \in L^p_\alpha(R, E)$ to the unique mild solution $M_{p,\alpha} f = u \in L^p_\alpha(R, E)$ of Equation $(\mathrm{I})_f$. Again with the closed graph theorem, we have that $M_{p,\alpha}$ is bounded.

Now, let $y \in E$, $\xi \in \mathbb{R}$ and define

$$f_s(t) := e^{i\xi(s+t)} y = f_0(s+t) = e^{i\xi s} f_0(t) \tag{3.6}$$

for all $s, t \in \mathbb{R}$, where $f_0(t) = e^{i\xi t} y$. Remark that $f_s \notin L^p(\mathbb{R}, E)$, but $f_s \in L^p_\alpha(\mathbb{R}, E)$ for all $\alpha > 0$ with

$$\|f_s\|_{p,\alpha} = \left(\frac{2}{p\alpha}\right)^{\frac{1}{p}} \|y\|.$$

By the considerations above, there exists for each f_s a unique mild solution $u_s \in L^p_\alpha(\mathbb{R}, E)$ of Equation $(\mathrm{I})_{f_s}$. Similar as in the proof of Theorem 2.2.2 one can show that for all $s \in \mathbb{R}$ it holds that $u_s(t) = u_0(s+t) = e^{i\xi s} u_0(t)$ for almost all $t \in \mathbb{R}$. Thus, u is an exponential function, i.e. there exists $z \in E$ such that $u_s(t) = e^{i\xi(s+t)} z$. Since $\int_0^t u_s(r)dr \in D(A)$, it follows that $z \in D(A)$. Thus u_0 is differentiable and

$$i\xi e^{i\xi t} z = u_0'(t) = A u_0(t) + f_0(t) = e^{i\xi t} A z + e^{i\xi t} y.$$

Hence, we obtain $(i\xi - A)z = y$ and $(i\xi - A)$ is surjective since y was chosen arbitrary.

To prove injectivity, let $Az = i\xi z$ and define $u(t) := e^{i\xi t} z \in L^p_\alpha(\mathbb{R}, E)$. Then it is easy to see that u is a solution of the homogeneous Equation $(\mathrm{I})_0$, thus by uniqueness of solutions $z = 0$.

Further, since $M_{p,\alpha}$ is bounded, we obtain

$$\left(\frac{2}{p\alpha}\right)^{\frac{1}{p}} \|z\| = \|u_0\|_{p\alpha} \le \|M_{p,\alpha}\| \|f_0\|_{p\alpha} = \left(\frac{2}{p\alpha}\right)^{\frac{1}{p}} \|y\|.$$

Thus $i\xi \in \rho(A)$ for all $\xi \in \mathbb{R}$ and with $c := \|M_{p,\alpha}\|$ it follows that $\|R(i\xi, A)\| \le c$. □

Remark 3.1.6 If we compare Theorem 3.1.5 with Theorem 2.2.2, we establish the same condition on the spectrum of the operator A from existence and uniqueness of solutions. But, since the functions f_s (see (3.6)) are in $BUC(\mathbb{R}, E)$, but not in $L^p(\mathbb{R}, E)$, the proof of Theorem 3.1.5 is much more complicated (we had to introduce the spaces $L^p_\alpha(\mathbb{R}, E)$).

Finally, we consider again bisectorial operators and establish existence and uniqueness of mild solutions of Equation (I).

Theorem 3.1.7 *Let A be a densely defined, bisectorial operator on E with $0 \in \rho(A)$. Then for all $f \in L^p(\mathbb{R}, E)$ there exists a unique mild solution $u_f \in L^p(\mathbb{R}, E)$ of Equation (I)$_f$ which is given by*

$$u_f(t) = \int_{\mathbb{R}} K(t-s)f(s)ds,$$

where $K(s) := \begin{cases} T^{-}(\ s\ ), & s > 0 \\ -T^{+}(-s), & s < 0. \end{cases}$

Proof. From Young's Inequality, it follows that $\|u_f\|_p \leq \|K\|_1 \|f\|_p$.

Now assume first that $f \in C_c^\infty(\mathbb{R}, D(A))$, i. e. f is infinitely differentiable with compact support and values in $D(A)$. Then it follow by Theorem 2.3.5 that

$$u_f'(t) = Au_f(t) + f(t)$$

for all $t \in \mathbb{R}$. Thus, u_f is a strong solution of (I)$_f$.

For arbitrary $\in L^p(\mathbb{R}, E)$, note that we can approximate f with functions $f_n \in C_c^\infty(\mathbb{R}, D(A))$. By the closedness of the operator A, the limit of the functions u_{f_n} becomes a mild solution of (I)$_f$.

Uniqueness of solutions follows immediately from Proposition 3.1.2. □

3.2 Maximal regularity of first-order differential equations

Maximal L^p-regularity of first-order differential equations has been intensively studied in the last few years. See for example in [48], [74], [80], [88] and a lot more. In this section, we recall important results about necessary conditions and p-independence of maximal L^p-regularity and about first-order equations on Hilbert spaces. We will see that the occurring operators are bisectorial with 0 included in the resolvent set.

Definition 3.2.1 The operator A satisfies *maximal L^p-regularity* for Equation (I) if for all $f \in L^p(\mathbb{R}, X)$ there exists a unique solution $u_f \in W^{1,p}(\mathbb{R}, X) \cap L^p(\mathbb{R}, D(A))$ of Equation (I)$_f$.

From the closed graph theorem it follows that in this case the solution operator $M_p : L^p(\mathbb{R}, E) \to W^{1,p}(\mathbb{R}, E) : f \mapsto u_f$ is bounded.

A. Mielke has proved in [88] a necessary condition for maximal L^p-regularity of an operator A for the first-order equation. He got the following result:

Theorem 3.2.2 *Let A be a closed, densely defined, linear operator on a Banach space E and let Equation (I) be maximally L^p-regular for one $p \in (1, \infty)$. Then $i\mathbb{R} \subseteq \rho(A)$ and there exists a constant $c \geq 0$ such that $\|R(i\xi, A)\| \leq \frac{c}{1+|\xi|}$ for all $\xi \in \mathbb{R}$.*

We want to look more carefully at the spectrum of this kind of operators.

Proposition 3.2.3 *Let A be a closed, densely defined linear operator on a Banach space E such that $i\mathbb{R} \subseteq \rho(A)$ and there exists a constant $c \geq 0$ such that $\|R(i\xi, A)\| \leq \frac{c}{1+|\xi|}$ for all $\xi \in \mathbb{R}$. Then there exists a constant $b \geq 0$ such that*

$$V_b := \{z \in \mathbb{C} \,:\, |Re(z)| \leq \frac{1}{b}(i + |\Im m(z)|)\} \subseteq \rho(A)$$

and $\|R(z, A)\| \leq \frac{b}{1+|z|}$ for all $z \in V_b$.

Proof. For $i\xi \in i\mathbb{R} \subseteq \rho(A)$, we get the following equation:

$$\begin{aligned} R(z, A) &= (z - A)^{-1} R(i\xi, A)^{-1} R(i\xi, A) \\ &= (((z - i\xi) + (i\xi - A))R(i\xi, A))^{-1} R(i\xi, A) \\ &= (I + (z - i\xi)R(i\xi, A))^{-1} R(i\xi, A). \end{aligned}$$

Now let $z = \eta + i\xi \in \mathbb{C}$ where $\eta = \mathcal{R}e(z)$ and $\xi = \Im m(z)$. Then by hypothesis $\|(z - i\xi)R(i\xi, A)\| \leq \frac{|\eta|c}{1+|\xi|}$. With the above equation, we obtain that $z \in \rho(A)$ if $\frac{|\eta|c}{1+|\xi|} < 1$. Hence $V_b \in \rho(A)$ for all $b > c$. Finally, let $b = \frac{c+1}{2} + \sqrt{\frac{(c+1)^2}{4} + 1}$ which is a positive solution of the equation $b = \frac{1}{1-\frac{c}{b}}(\frac{1}{b} + 1)$. With this choice of b we obtain the following resolvent estimate:

$$\begin{aligned} \|R(z, A)\| &\leq \|(I + (z - i\xi)R(i\xi, A))^{-1}\| \|R(i\xi, A)\| \\ &\leq \frac{1}{1 - \frac{|\eta|c}{1+|\xi|}} \cdot \frac{1}{1 + |\xi|} \\ &\leq \frac{1}{1 - \frac{c}{b}} \cdot \frac{1 + |\xi| + |\eta|}{1 + |\xi|} \cdot \frac{1}{1 + |z|} \\ &\leq \frac{1}{1 - \frac{c}{b}} \cdot (\frac{1}{b} + 1) \cdot \frac{1}{1 + |z|} \\ &= \frac{b}{1 + |z|} \end{aligned}$$

for all $z \in V_b$. □

Lemma 1.1.2, Theorem 3.2.2 and Lemma 3.2.3 give the following.

Theorem 3.2.4 *Let the operator A satisfy maximal L^p-regularity for some $p \in (1, \infty)$. Then A is bisectorial with $0 \in \rho(A)$.*

Remark 3.2.5 Comparing this result about first-order equations on the line with initial value problems

$$\text{(CP)} \qquad \begin{cases} u'(t) &= Au(t) + f(t) \qquad (t \geq 0) \\ u(0) &= x_0 \end{cases}$$

where A is again closed, linear, $f \in L^p(\mathbb{R}, E)$ and the initial value $x_0 \in E$, we get a different result. Namely, a necessary condition on the operator A to satisfy maximal regularity for (CP) is that A is the generator of an analytic C_0-semigroup (see [48] and the references therein).

The next result - p-independence of maximal L^p-regularity - holds for equations on the line as well as for initial value problems. It was first proved for Hilbert spaces by De Simon ([47]), and then later for general Banach spaces by Cannarsa and Vespri (see [35]) and Coulhon and Lamberton (see [41]) for the initial value problem (CP) and by Mielke ([88]) for the equation on the line (I).

Theorem 3.2.6 *Let A be a closed, densely defined, linear operator on a Banach space E that satisfies maximal L^p-regularity for Equation (I) (respectively for (CP)) for one $p \in (1, \infty)$. Then A satisfies maximal L^q-regularity for all $q \in (1, \infty)$.*

Thus, we can say the operator A satisfies *maximal regularity* for Equation (I) if A satisfies maximal L^p-regularity for some, and hence for all $p \in (1, \infty)$.

In the remaining section, we want to give a necessary condition for maximal regularity on UMD spaces, i.e. E possesses the property of **u***nconditionality of* **m***artingale* **d***ifferences* (see for example [3, III.4.4], [31] and [34]).

Now, let $\mathcal{F}$ be the *Fourier transform* on $L^1(\mathbb{R}, E)$ defined by

$$\mathcal{F}u(\lambda) = \int_{\mathbb{R}} e^{-i\lambda t} u(t)dt, \qquad \lambda \in \mathbb{R} \tag{3.7}$$

for all $u \in L^1(\mathbb{R}, E)$ (compare this with the scalar Fourier transform (2.2)).

We denote by $\mathcal{D}(\mathbb{R}, E)$ the space of E-valued C^∞-functions with compact support and let $\mathcal{D}'(\mathbb{R}, E) = \mathcal{L}(\mathcal{D}(\mathbb{R}, E), E)$ be the space of E-valued distributions. Similarly, let $\mathcal{S}(\mathbb{R}, E)$ the *Schwartz* space of smooth rapidly decreasing E-valued functions on $\mathbb{R}$ and $\mathcal{S}'(\mathbb{R}, E) = \mathcal{L}(\mathcal{S}(\mathbb{R}, E), E)$.

Then, given a function $M \in L^1_{loc}(\mathbb{R}, \mathcal{L}(E))$, we may define the *pseudo-differential operator* $M(D) : \mathcal{F}^{-1}\mathcal{D}(\mathbb{R}, E) \to \mathcal{S}'(\mathbb{R}, E)$ by

$$M(D)\phi := \mathcal{F}^{-1}M\mathcal{F}\phi \tag{3.8}$$

for all $\mathcal{F}\phi \in \mathcal{D}(\mathbb{R}, E)$. Since $\mathcal{F}^{-1}\mathcal{D}(\mathbb{R}, E)$ is dense in $L^p(\mathbb{R}, E)$, $M(D)$ is defined on a dense subset of $L^p(\mathbb{R}, E)$.

Now, one can ask what conditions have to be imposed on M so that $M(D)$ extends to a bounded linear operator on $L^p(\mathbb{R}, E)$. In the scalar case, the famous Mikhlin multiplier theorem gives a satisfactory answer ([89], see also [64]). This theorem was then extended by Bourgain [32], McConnell [85] and Zimmermann [127] to a vector-valued version on UMD spaces. Pisier proved in [97] that the operator-valued case of Mikhlin's theorem is only valid in Hilbert spaces. But recently a new operator-valued version was found by Weis ([124]and [125], see also [39] and [114]) in which boundedness of the multiplier function is replaced by $\mathcal{R}$-boundedness. This version is also useful to study the first-order equation on the line, but first, we will give the definition of $\mathcal{R}$-boundedness (see [114]).

Definition 3.2.7 A family $\mathcal{M} \subseteq \mathcal{L}(E)$ of bounded, linear operators on a Banach space E is called *$\mathcal{R}$-bounded* if there is a constant $C \geq 0$ such that for all $n \in \mathbb{N}$, all elements $x_j \in E$, selections $T_j \in \mathcal{M}$ $(1 \leq j \leq n)$ and n independent symmetric $\{-1, 1\}$-valued random variables ϵ_j on a probability space (Ω, Σ, μ) the following inequality holds:

$$\|\sum_{j=1}^{n} \epsilon_j T_j x_j\|_{L^1(\Omega,E)} \leq C\|\sum_{j=1}^{n} \epsilon_j x_j\|_{L^1(\Omega,E)}. \tag{3.9}$$

The *$\mathcal{R}$-bound* of the family $\mathcal{M}$ is given by

$$\mathcal{R}(\mathcal{M}) := \inf\{C \geq 0 \,:\, (3.9) \text{ holds}\}. \tag{3.10}$$

We turn now to the operator-valued version of Mikhlin's theorem due to Weis ([124],[125], see also [39]).

Theorem 3.2.8 *Let E be a UMD space and $1 < p < \infty$. Suppose $M \in C^1(\mathbb{R} \setminus \{0\}, \mathcal{L}(E))$ such that the following conditions are satisfied:*

(i) $\mathcal{R}(\{M(s) : s \in \mathbb{R} \setminus \{0\}\}) < \infty$

(ii) $\mathcal{R}(\{sM'(s) : s \in \mathbb{R} \setminus \{0\}\}) < \infty$

Then, the pseudo-differential operator $M(D)$ defined by (3.8) can be extended to a bounded, linear operator on $L^p(\mathbb{R}, E)$.

If we apply the above Theorem 3.2.8 to Equation (I), we obtain the following criterion for maximal regularity on UMD spaces.

Theorem 3.2.9 *Let E be a UMD space and A be a bisectorial operator such that $\{isR(is, A) : s \in \mathbb{R} \setminus \{0\}\}$ is $\mathcal{R}$-bounded. Then A satisfies maximal regularity for Equation (I).*

Proof. Let $f \in \mathcal{F}^{-1}\mathcal{D}(\mathbb{R}, E)$. If we take Fourier transform of Equation (I), we obtain

$$is\mathcal{F}u(s) = A\mathcal{F}u(s) + \mathcal{F}f(s), \quad \text{or} \quad \mathcal{F}u(s) = R(is, A)\mathcal{F}f(s).$$

Hence, $M(s) := R(is, A) \in C^1(\mathbb{R}, \mathcal{L}(E))$ is the multiplier function of the pseudo-differential operator $f \mapsto u := M(D)f$. If we now consider the multiplier function $\tilde{M}(s) := AR(is, A) \in C^1(\mathbb{R}, \mathcal{L}(E))$, we see that $\tilde{M}$ satisfies the conditions (i) and (ii) of Theorem 3.2.8. It follows that $Au \in L^p(\mathbb{R}, E)$, or $u \in L^p(\mathbb{R}, D(A))$. Similarly, one shows that $u' \in L^p(\mathbb{R}, E)$, i.e. $u \in W^{1,p}(\mathbb{R}, E)$. Hence, $u = M(D)f$ is a strong solution of Equation (I) (which is also unique by Proposition 3.1.2). From Theorem 3.2.8 it follows that $M(D)$ extends to a bounded operator $L^p(\mathbb{R}, E) \to L^p(\mathbb{R}, D(A)) \cap W^{1,p}(\mathbb{R}, E)$ which maps each $f \in L^p(\mathbb{R}, E)$ to the unique strong solution $u = M(D)f$. Hence, A satisfies maximal regularity for Equation (I). □

Since bounded sets correspond to $\mathcal{R}$-bounded sets if, and only if, E is a Hilbert space, we get as a simple consequence from the previous theorem that the necessary condition of Theorem 3.2.4 is also sufficient if the underlying Banach space is a Hilbert space (see also [88]).

Corollary 3.2.10 *Let A be a bisectorial operator on a Hilbert space H with $0 \in \rho(A)$. Then A satisfies maximal regularity for Equation (I).*

Remark 3.2.11 For the initial value problem, we obtain that if A is the generator of an analytic semigroup with negative exponential type on a Hilbert space H, then A satisfies maximal L^p-regularity for (CP) for all $p \in (1, \infty)$ (see [47]).

Note, that if A or $-A$ is a generator of an analytic semigroup and $\sigma(A) \cap i\mathbb{R} = \emptyset$ then A is a special kind of a bisectorial operator where $\sigma^+(A)$ or $\sigma^-(A)$ is bounded. In this case, we have seen in Section 1.4 that the corresponding spectral projections are bounded. Moreover, it follows that if E is a Hilbert space, then, by Theorem 3.2.10 maximal L^p-regularity holds for the operators occurring in Example 1.4.1 and 1.4.2. But maximal L^p-regularity does not hold in general Banach spaces (see [67] or [74]).

And the other way round, there exists an example of a bisectorial operator on a Hilbert space H such that corresponding spectral projections are unbounded (see Example 1.4.3). But in this case, we have maximal regularity by Corollary 3.2.10.

Summarising, we can see that there is no connection between boundedness of spectral projections, which induce a spectral decomposition of the underlying Banach space, and maximal regularity of Equation (I). Thus,

Proposition 3.2.12

(i) There exists a Banach space E and a bisectorial operator A with bounded spectral projections, but A does not satisfy maximal regularity for Equation (I).

(ii) There exists a bisectorial operator on a Hilbert space H, i.e. A satisfies maximal regularity of Equation (I), such that the spectral projections are unbounded.

Part II

Second-Order Differential Equations on the Line

In the second part of this monograph, we examine the second-order differential equation on the line, i.e.

$$\text{(II)} \qquad u''(t) = Au(t) + f(t), \qquad t \in \mathbb{R},$$

where A is again a closed, linear operator on a Banach space E and $f \in BUC(\mathbb{R}, E)$ or $f \in L^p(\mathbb{R}, E)$. If we want to specify the inhomogeneity f we will write $(\text{II})_f$ instead of (II).

As in the first part, we will denote by $D(A)$, $\sigma(A)$ and $\rho(A)$ the domain of A, the spectrum of A and the resolvent set of A. For $\lambda \in \rho(A)$ let $R(\lambda, A) = (\lambda - A)^{-1}$. Moreover, let $\mathbb{R}^- = (-\infty, 0]$.

This part is, like the first part, divided in three Chapters:

- Chapter 4: Sectorial operators
- Chapter 5: Bounded uniformly continuous solutions
- Chapter 6: Solutions in $\mathrm{L}^p(\mathbb{R},\mathrm{E})$.

Sectorial operators play an important role in discussing the second-order equation as bisectorial operators do in the first-order case. In Chapter 4, we will examine sectorial operators. Since these operators are much more studied than bisectorial operators, we just give the definition and recall some basic properties, like the functional calculus and fractional powers (see Section 4.1).

In Chapter 5, we study bounded uniformly continuous solutions of Equation (II) with inhomogeneity $f \in BUC(\mathbb{R}, E)$. First, we give a condition on the spectrum of A for uniqueness of mild solutions (see Section 5.1).

In Section 5.2, we study the relation between the second-order differential equation and a suitable operator equation. We give a necessary condition on the operator A for existence and uniqueness of mild solutions of Equation (II) and show that well-posedness of Equation (II) is equivalent to existence and uniqueness of a unique bounded solution of the operator equation.

In the Section 5.3, we give a variety of examples for which existence and uniqueness of mild solutions is satisfied.

As for cosine families (see [57], [115], [118]), it is possible to reduce the second-order problem to a first-order system. We show in the fourth paragraph that Equation (II) is solvable for all $f \in BUC(\mathbb{R}, E)$ if, and only if, the reduced system is solvable for all $f \in BUC^1(\mathbb{R}, E) = \{f \in BUC(\mathbb{R}, E) \cap C^1(\mathbb{R}, E) \mid f' \in BUC(\mathbb{R}, E)\}$. Moreover, this is equivalent to the existence of a unique bounded solution of another operator equation.

In Section 5.4, we consider the case, where $A = B^2$ and the first-order equation for B is well-posed. In this case, we get well-posedness of the second-order equation for A and more "regularity" for the mild solutions of (II). More precisely, we show that every solution of Equation (II) is in $BUC(\mathbb{R}, D(B)) \cap BUC^1(\mathbb{R}, E)$. Furthermore, we obtain the solvability for the corresponding first-order system for all $f \in BUC(\mathbb{R}, E)$ (and not only for $f \in BUC^1(\mathbb{R}, E)$). Moreover, we obtain a unique bounded solution for a further operator equation.

In the last Section of this chapter, we concentrate again on the asymptotic behaviour of bounded uniformly continuous solutions of Equation (II). We show that a mild solution of the first-order differential equation on the line with an inhomogeneity f that satisfies a certain asymptotic behaviour, has the same asymptotic behaviour provided that $\sigma(A) \cap R^-$ is discrete and the set $\{\eta \in \mathbb{R} : -\eta^2 \in \sigma(A)\}$ contains no accumulation points of the spectrum of f.

In the last chapter, Chapter 6, the case where the inhomogeneity $\in L^p(\mathbb{R}, E)$ is studied. First, in Section 6.1, we examine mild solutions, i.e. solutions of Equation (II) which belong also to $L^p(\mathbb{R}, E)$. We give a uniqueness result and some further properties. Moreover, we show, that for sectorial operators there exist unique mild solutions of Equation (II).

Finally, in Section 6.2, strong solutions of Equation (II) are considered, i.e. $u \in W^{2,p}(\mathbb{R}, E) \cap L^p(\mathbb{R}, D(A))$. We will see that a necessary condition for the operator A to satisfy maximal $\mathrm{L^p}$-regularity is, that A is sectorial and $0 \in \rho(A)$. P-independence of maximal regularity of the first-order differential equation was first proved for Hilbert spaces by De Simon ([47]), and then later for general Banach spaces by Cannarsa and Vespri (see [35]) and Coulhon and Lamberton (see [41]). We prove that maximal regularity of the second-order differential equation on the line is also independent of $p \in (1, \infty)$. With an operator valued Mikhlin multiplier theorem (see [124]) we give a sufficient condition for maximal regularity of sectorial operators on UMD spaces. Furthermore, it is shown that the necessary condition for maximal $\mathrm{L^p}$-regularity, i.e. A is sectorial and $0 \in \rho(A)$, is also sufficient if the underlying Banach space is a Hilbert space. However, for a large class of UMD spaces which are not Hilbert spaces, there exist counterexamples to maximal $\mathrm{L^p}$-regularity. This is a consequence of recent results from Clément and Guerre-Delabrière [38] and from Kalton and Lancien [67].

Chapter 4

Sectorial operators

Sectorial operators are much more discussed in recent time than bisectorial operators. So that we just recall some basic facts about these kind of operators which we will need later. We will see that sectorial operators for the second-order equation on the line are as important as bisectorial operators for the first-order equation on the line.

4.1 Definition of a sectorial operator

We first give 'our' definition of sectorial operators. Remark that the definition varies in the different publications concerning sectorial operators depending on the fact if $\mathbb{R}^-$ or $\mathbb{R}^+$ is contained in the resolvent set. Often, such operators are also called *operators of type θ* instead of *sectorial* (see [87] or [86]). Remark further, that there exists an earlier definition of "sectorial" operators on Hilbert spaces connected with elliptic forms from Kato (see [68, page 280]).

Definition 4.1.1 A closed, densely defined, linear operator A is called *sectorial*, if

(i) there exists $\theta \in [0, \pi)$ such that $\sigma(A) \subseteq \Sigma_\theta := \{z \in \mathbb{C} : |\arg(z)| \leq \theta\} \cup \{0\}$ and

(ii) $\forall \mu > \theta \;\; \exists M_\mu \geq 0$ such that $\|\lambda R(\lambda, A)\| \leq M_\mu \quad \forall \lambda \in \mathbb{C} \setminus \Sigma_\mu$.

The *spectral angle* ω_A of a sectorial operator A is given by

$$\omega_A := \inf\{\, \theta \in [0, \pi) \,:\, (i) \;\&\; (ii) \text{ hold}\,\}.$$

We have the following simple connection between sectorial and bisectorial operators.

Remark 4.1.2 If A is sectorial of spectral angle $\omega_A < \frac{\pi}{2}$, then A is bisectorial with spectral angle $\varpi_A = \omega_A$.

Moreover, there is also a connection between sectorial operators and analytic semigroups. It is well-known that if the operator A is sectorial with spectral angle ω_A smaller than $\frac{\pi}{2}$ then $-A$ is the generator of an analytic C_0-semigroup on E (see for example [95, Section 2.5]).

4.2 A functional calculus for sectorial operators

It is well-known that there exists a functional calculus for densely defined, sectorial operators. This is an extension of the classical Dunford-functional calculus for bounded operators (see [51]). Here, we will just give the definitions for the functional calculus. For the more interested reader we want to refer to the huge amount of literature about this topic (see [1], [42], [73], [86], [87], [117] and the references therein).

From now on, let A be a densely defined, sectorial operator with spectral angle ω_A which is one-one with dense range. This is for example the case when $0 \in \rho(A)$.

For $\omega_A < \theta < \mu < \pi$, let $\Sigma_{\mu,0} := \{z \in \mathbb{C} \setminus \{0\} \, : \, |\arg(z)| < \mu\}$ and define the following spaces of holomorphic functions on $\Sigma_{\mu,0}$.

$$\begin{aligned}
H(\Sigma_{\mu,0}) &:= \{f : \Sigma_{\mu,0} \to \mathbb{C} : \, f \text{ holomorphic}\}, \\
H^\infty(\Sigma_{\mu,0}) &:= \{f \in H(\Sigma_{\mu,0}) \, : \, \|f\|_\infty < \infty\}, \\
H_0^\infty(\Sigma_{\mu,0}) &:= \{f \in H(\Sigma_{\mu,0}) \, : \, \exists s > 0 : \, f\psi^{-s} \in H^\infty(\Sigma_{\mu,0})\} \text{ and} \\
\mathcal{F}(\Sigma_{\mu,0}) &:= \{f \in H(\Sigma_{\mu,0}) \, : \, \exists s > 0 : \, f\psi^{s} \in H^\infty(\Sigma_{\mu,0})\}
\end{aligned}$$

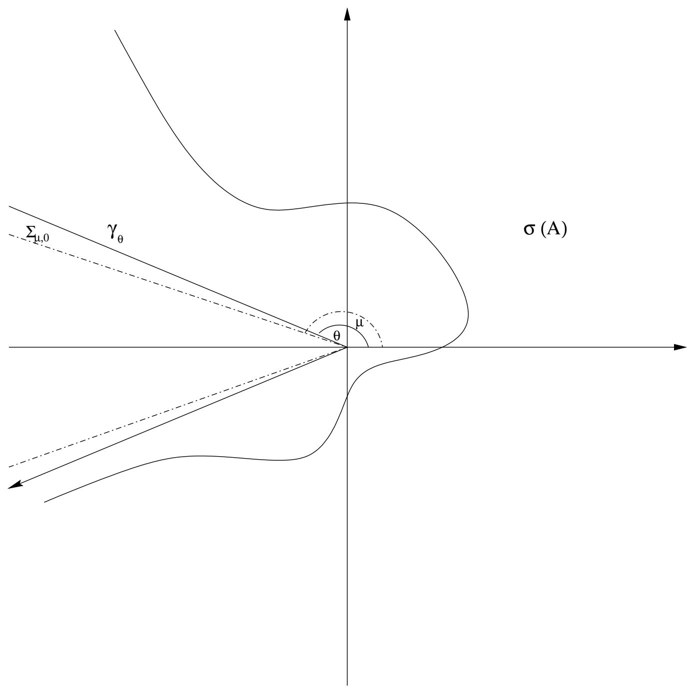
where the function ψ is given by $\psi(\xi) := \frac{\xi}{1+\xi^2}$. It is easy to see that

$$H_0^\infty(\Sigma_{\mu,0}) \subseteq H^\infty(\Sigma_{\mu,0}) \subseteq \mathcal{F}(\Sigma_{\mu,0}) \subseteq H(\Sigma_{\mu,0}).$$

Moreover, let γ_θ be the contour defined by

$$\gamma_\theta := \begin{cases} -te^{\,i\theta}, & t \leq 0 \\ te^{-i\theta}\,, & t \geq 0, \end{cases} \tag{4.1}$$

oriented via increasing parameter. This setting is described in the following picture.



Definition 4.2.1 Let A be a densely defined, sectorial operator, $\omega_A < \theta < \mu < \pi$ and γ_θ as above. Then define the linear operator $f(A)$ by

(i) For $f \in H_0^\infty(\Sigma_{\mu,0})$ let

$$f(A) := \frac{1}{2\pi i} \int_{\gamma_\theta} f(\lambda) R(\lambda, A) d\lambda \in \mathcal{L}(E).$$

(ii) For $f \in \mathcal{F}(\Sigma_{\mu,0})$, choose $k \in \mathbb{N}$ such that $f\psi^k \in H_0^\infty(\Sigma_{\mu,0})$ and let

$$f(A) := \psi^{-k}(A)(f\psi^k)(A),$$

where $D(f(A)) := \{x \in E \,:\, (f\psi^k)(A)x \in D(\psi^{-k}(A))\}$.

The integral in (i) is absolutely norm-convergent since A is sectorial. The definition in (ii) is independent from the choice of $k \in \mathbb{N}$. It can be shown that the definitions coincide with the usual Dunford calculus if A is bounded. Moreover, the definition of $f(A)$ is independent of the concrete choice of $\theta \in (\omega_A, \mu)$, thus, $f(A)$ are well-defined, closed linear operators on E.

Remark that there exists also a joint functional calculus for a pair of sectorial operators with commuting resolvents (see [72] and [2]).

4.3 Fractional powers of sectorial operators

Via the functional calculus described in the last section, we can now define *fractional powers* (or *complex powers*) of sectorial operators, i.e. $A^z = f(A)$ where $f(\xi) = \xi^z$ for $z \in \mathbb{C}$ (see [117, Section 2.2.4]).

But, fractional powers of sectorial operators were already studied before the functional calculus was developed. Consequently fractional powers of sectorial operators were defined in a different way. In fact, fractional powers of bounded operators were first studied by Hille [62]. Afterwards, fractional powers for the negatives of generators of bounded C_0-semigroups have been discussed by Bochner [28], Phillips [96] and Yosida [126]. Balakrishnan [19] extended this theory to sectorial operators which was then further developed by many other authors (see for example [70], [83], [84], [93], [113] and in the case of (purely) imaginary powers see [49] and [101]).

We will give the definition of *fractional powers* of sectorial operators which was given by Balakrishnan in [19] (see also [84]).

Definition 4.3.1 Let $\mathcal{R}e(\alpha) > 0$ and define the linear operator J^α by the following.

For $0 < \mathcal{R}e(\alpha) < 1$ and $x \in D(A)$ let

$$J^\alpha x \;:=\; \frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{\alpha-1}(\lambda + A)^{-1} Ax \, d\lambda.$$

For $0 < \mathcal{R}e(\alpha) < 2$ and $x \in D(A^2)$ let

$$J^\alpha x \;:=\; \frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{\alpha-1}\Big((\lambda + A)^{-1} - \frac{\lambda}{1+\lambda^2}\Big) Ax \, d\lambda + \sin(\frac{\pi\alpha}{2})Ax.$$

For $n < \mathcal{R}e(\alpha) < n+1$ and $x \in D(A^{n+1})$ let

$$J^\alpha x \;:=\; J^{\alpha-n}A^n x.$$

For $n < \mathcal{R}e(\alpha) < n+2$ and $x \in D(A^{n+2})$ let

$$J^\alpha x \;:=\; J^{\alpha-n}A^n x.$$

These definitions coincide for overlapping ranges of α. Moreover, these operators can be extended to be closed linear [19, Lemma 2.1]. Now define the *fractional power* of the sectorial operator A by

$$A^\alpha := \overline{J^\alpha}. \tag{4.2}$$

This definition coincide with the one given by the functional calculus if $D(A)$ is dense in E (see [117, Theorem 2.2.30]). Moreover, we have the following important properties of A^α (see [19] and [84]).

Proposition 4.3.2 *Let A be a sectorial operator on E. Then*

(i) *For $\mathcal{R}e(\alpha) > 0$ and $\mathcal{R}e(\beta) > 0$:*

$$A^{\alpha+\beta} = A^\alpha A^\beta.$$

(ii) *For $0 < \alpha < 1$ and $\mathcal{R}e(\beta) > 0$:*

$$(A^\alpha)^\beta = A^{\alpha\beta}.$$

(iii) *For $\mathcal{R}e(\alpha) > 0$:*

$$\sigma(A^\alpha) = \sigma(A)^\alpha.$$

(iv) *If $\overline{D(A)} = E$ and $0 < \alpha \leq \frac{1}{2}$, then $-A^\alpha$ is the generator of an analytic C_0-semigroup on E.*

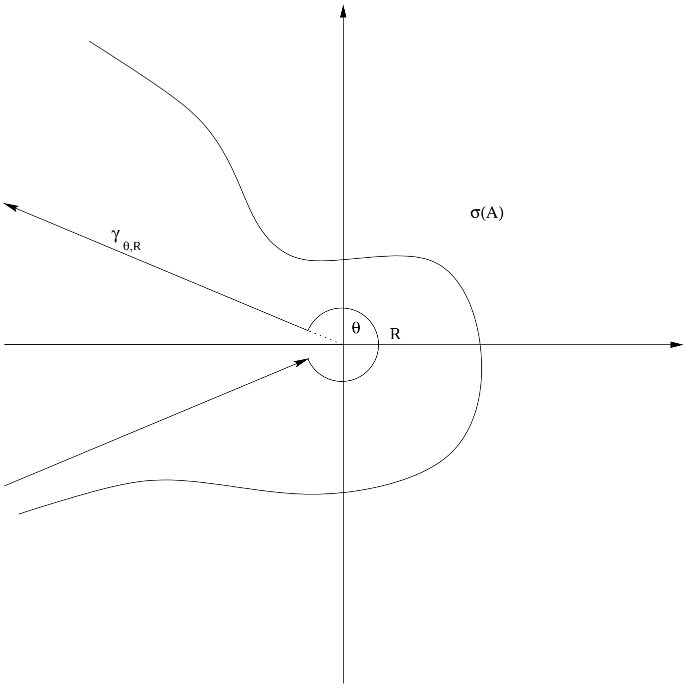
For the further applications of sectorial operators in second-order differential equations on the line, we will see, as in the case of bisectorial operators in the first part, that we are mainly interested in sectorial operators where 0 is included in the resolvent set (see 5.1.6, 5.3.2, 6.1.5 or 6.2.5).

Hence, let in the following A be a sectorial operator with spectral angle ω_A and assume that $0 \in \rho(A)$.

In this case, there exists $R > 0$ such that the disk with centre 0 and radius R is contained in $\rho(A)$. Now let $\pi > \theta > \max\{\frac{\pi}{2}, \omega_A\}$ and denote by $\gamma_{\theta,R}$ the unbounded contour which is the boundary of the domain $B := \{\lambda \in \mathbb{C} : |\arg(\lambda)| > \theta\} \cup \{\lambda \in \mathbb{C} : |\lambda| < R\}$, oriented in such a way that the domain B remains on the left side of $\gamma_{\theta,R}$. Thus,

$$\gamma_{\theta,R} := \begin{cases} -tRe^{-i\theta}, & t \leq -1 \\ Re^{\,i\theta t}, & -1 < t < 1. \\ tRe^{\,i\theta}, & t \geq 1 \end{cases} \tag{4.3}$$

We will need this curve with the described orientation in Example 5.3.2. The situation is described in the following picture.



In this situation, it follows that A possesses fractional powers, especially, we can define

$$B := -A^{\frac{1}{2}}$$

and from Proposition 4.3.2, we obtain the following.

Proposition 4.3.3 *Let A be a densely defined, sectorial operator with $0 \in \rho(A)$. Then $B := -A^{\frac{1}{2}}$ is the generator of an exponentially stable, analytic C_0-semigroup $(T(t))_{t\geq 0}$ with $0 \in \rho(B)$ and $B^2 = A$*

Chapter 5

Bounded uniformely continuous solutions of second-order differential equations

Let A be a closed, linear operator on a Banach space E. In this chapter, we examine the second-order differential equation on the real line, i.e.

$$(\mathrm{II}) \qquad u''(t) = Au(t) + f(t) \qquad (t \in \mathbb{R}),$$

where $f \in BUC(\mathbb{R}, E)$ and consider bounded, uniformely continuous, i.e. mild solutions of the above equation.

5.1 Uniqueness of mild solutions

In this section, we introduce mild solutions of the second-order differential equation on the line (Equation (II)) and show that these solutions are unique if $\sigma(A) \cap \mathbb{R}^- = \emptyset$. To do this, we use the characterisation of the spectrum of bounded, uniformly continuous functions via the Carleman-transform (see (2.4) and (2.5)).

Definition 5.1.1 Let $f \in BUC(\mathbb{R}, E)$. We call a function $u \in BUC(\mathbb{R}, E)$ a *mild solution* of (II) if $\int_0^t (t-s)u(s)ds \in D(A)$ for all $t \in \mathbb{R}$ and there exists a $y \in E$ such that

$$u(t) - u(0) = ty + A\int_0^t (t-s)u(s)ds + \int_0^t (t-s)f(s)ds \tag{5.1}$$

for all $t \in \mathbb{R}$.

By partial integration, this definition is equivalent to requiring that $\int_0^t \int_0^s u(r)drds \in D(A)$ and

$$u(t) - u(0) = ty + A\int_0^t \int_0^s u(r)drds + \int_0^t \int_0^s f(r)drds \tag{5.2}$$

for all $t \in \mathbb{R}$. Remark that the y occurring in this definition is unique. To see this, assume that $u \in BUC(\mathbb{R}, E)$ is a mild solution of $(\mathrm{II})_f$ such that there exist $y_1, y_2 \in E$ satisfying Equation (5.1). By subtracting, we obtain $0 = t(y_1 - y_2)$ for all $t \in \mathbb{R}$. Thus, $y_1 = y_2$.

Definition 5.1.2 Let $f \in BUC(\mathbb{R}, E)$. We say that a function $u \in BUC(\mathbb{R}, E)$ is a *classical solution* of $(\mathrm{II})_f$ if u is twice continuously differentiable, $u \in C(\mathbb{R}, D(A))$ and $u''(t) = Au(t) + f(t)$ for all $t \in \mathbb{R}$.

As in the first-order case, we have the following (compare with Lemma 2.1.2).

Lemma 5.1.3 *Let A be a closed, linear operator on a Banach space E with non-empty resolvent set, let $f \in BUC(\mathbb{R}, E)$ and $u \in BUC(\mathbb{R}, E)$ be a mild solution of (II) and assume that $u \in C^2(\mathbb{R}, E)$. Then u is a classical solution.*

Proof. Since u is a mild solution we have by definition (see (5.2))

$$A\int_0^t \int_0^s u(r)drds = u(t) - u(0) - ty - \int_0^t \int_0^s f(r)drds$$

for all $t \in \mathbb{R}$ and some $y \in E$. Since $u \in C^2(\mathbb{R}, E)$, the right hand side of the above equation is twice continuously differentiable, and so is the left hand side. Now, for $\lambda \in \rho(A)$, it follows that $R(\lambda, A)\frac{d^2}{dt^2}(\lambda - A)\int_0^t \int_0^s u(r)drds = u(t)$ for all $t \in \mathbb{R}$. Hence, $u(t) \in D(A)$ and $u''(t) = Au(t) + f(t)$ for all $t \in \mathbb{R}$. Since $u \in C^2(\mathbb{R}, E)$, it follows that $u \in C(\mathbb{R}, D(A))$. $\square$

Here, we can also relate the spectrum of mild solutions to the spectra of the operator A and the inhomogeneity f. Compare this result also with the first-order case (see Proposition 2.1.3). Recall also the definition of the spectrum of uniformly continuous functions via the Carleman transform (see (2.4) and (2.5)).

Proposition 5.1.4 *Let A be a closed, linear operator on a Banach space E, $f \in BUC(\mathbb{R}, E)$ and let $u \in BUC(\mathbb{R}, E)$ be a mild solution of $(II)_f$. Then*

$$sp(u) \subseteq \{\eta \in \mathbb{R} : -\eta^2 \in \sigma(A)\} \cup sp(f).$$

Proof. Taking Carleman transforms on both sides of Equation (5.1.1) we obtain

$$\hat{u}(\lambda) - \frac{1}{\lambda}u(0) = \frac{1}{\lambda^2}\left(y + A\hat{u}(\lambda) + \hat{f}(\lambda)\right)$$

for all $Re\lambda \neq 0$. Hence, it follows for $\lambda \in i\mathbb{R}$ with $\lambda^2 \in \rho(A)$ that

$$\hat{u}(\lambda) = \lambda R(\lambda^2, A)u(0) + R(\lambda^2, A)y + R(\lambda^2, A)\hat{f}(\lambda).$$

If now $\eta \in \mathbb{R}$ is a regular point of f and $-\eta^2 = (i\eta)^2 \in \rho(A)$, then $\hat{u}$ has a holomorphic extension in a neighbourhood of $i\eta$, and we obtain the result. $\square$

As a consequence this gives:

Corollary 5.1.5 *If A is a closed, linear operator on E and u is a mild solution of the homogeneous second-order equation (II)$_0$; $u'' = Au$. Then*

$$sp(u) \subseteq \{\eta \in \mathbb{R} : -\eta^2 \in \sigma(A)\}.$$

Theorem 5.1.6 *Let A be a linear operator on a Banach space E such that $\sigma(A) \cap (-\infty, 0] = \emptyset$. Then the mild solutions of (II) are unique.*

Proof. Let $f \in BUC(\mathbb{R}, E)$ and suppose that $u, v \in BUC(\mathbb{R}, E)$ are two mild solutions of (II)$_f$. Then $u - v$ is a solution of the homogeneous second-order equation (II)$_0$ and, by hypothesis and Corollary 5.1.5, it follows that $sp(u - v) = \emptyset$. Thus, $u = v$ [99, Proposition 0.5]. $\square$

5.2 Well-posedness of second-order differential equations

Well-posedness of the second-order differential equation on the line corresponds to existence and uniqueness of solutions. If Equation (II) is well-posed, then we have for the mild solutions that not only $\int_0^t \int_0^s u(r)dr \in D(A)$, but also $\int_0^t u(r)dr \in D(A)$ for all $t \in \mathbb{R}$ (see Proposition 5.2.5 and Corollary 5.2.6). With this, we can give a necessary condition on the operator A such that Equation (II) will be well-posed (see Theorem 5.2.7). Finally, we show how the well-posedness of Equation (II) is connected to the solvability of a suitable operator equation (see Theorem 5.2.12). For the results of this section see also [110].

Definition 5.2.1 Equation (II) is called *well-posed* if for each $f \in BUC(\mathbb{R}, E)$ exists a unique mild solution $u \in BUC(\mathbb{R}, E)$ of Equation $(\mathrm{II})_f$.

Next, we consider the solution operator for the second-order problem. Define the linear operator $(N, D(N))$ in $BUC(\mathbb{R}, E)$ by

$$\begin{aligned} D(N) &:= \{f \in BUC(\mathbb{R}, E) : \exists!\, u_f \in BUC(\mathbb{R}, E) \text{ such that} \\ &\qquad u_f \text{ is a mild solution of } (\mathrm{II})_f\} \\ Nf &:= u_f. \end{aligned} \tag{5.3}$$

Again, as in the first-order case, $D(N) = \emptyset$ if for some $f \in BUC(\mathbb{R}, E)$ there exist two different mild solutions of $(\mathrm{II})_f$.

Define the operator $\tilde{N} : D(\tilde{N}) = D(N) \longrightarrow BUC(\mathbb{R}, E) \oplus E$ by $\tilde{N}f := (u_f, y_f)$, where $u_f(t) - u_f(0) = ty_f + A\int_0^t (t-s)u(s)ds + \int_0^t (t-s)f(s)ds$ $(t \in \mathbb{R})$. Note, that $\tilde{N}$ is well-defined since the y_f occurring in the definition of mild solutions are unique (see Definition 5.1.1 and the succeeding remark).

Lemma 5.2.2 *The operator $(\tilde{N}, D(N))$ is a closed, linear operator.*

Proof. It is clear that $\tilde{N}$ is linear. To prove that N is closed, let $(f_n)_{n\in\mathbb{N}} \subseteq D(G)$, $\lim_n f_n = f$, $Gf_n = (u_n, y_n)$ and $\lim_n (u_n, y_n) = (u, y)$. It follows by (5.1.1) that

$$A\int_0^t (t-s)u_n(s)ds = u_n(t) - u_n(0) - ty_n - \int_0^t (t-s)f(s)ds$$

for all $t \in \mathbb{R}$. Since A is closed, we obtain upon letting $n \to \infty$ that $\int_0^t (t-s)u(s)ds \in D(A)$ and

$$A\int_0^t (t-s)u(s)ds = u(t) - u(0) - ty - \int_0^t (t-s)f(s)ds,$$

for all $t \in \mathbb{R}$. It follows that u is a mild solution of $(\mathrm{II})_f$ which is unique by the above remark. Hence, $u \in D(N)$ and $\tilde{N}f = (u, f)$. Thus $\tilde{N}$ is closed. $\square$

A simple application of the closed graph theorem gives the following corollary.

Corollary 5.2.3 *Equation (II) is well-posed if, and only if, N is bounded with $D(N) = BUC(\mathbb{R}, E)$.*

Proof. Let $p_1 : BUC(\mathbb{R}, E) \oplus E \longrightarrow BUC(\mathbb{R}, E) : (u, y) \mapsto u$ be the projection on the first coordinate, which is bounded. Then $N = p_1 \circ \tilde{N}$ is bounded if, and only if, $D(N) = BUC(\mathbb{R}, E)$, which is the case when (II) is well-posed. □

In the following, by $R(\lambda, A)$ we will denote also the operator on $BUC(\mathbb{R}, E)$ defined by $(R(\lambda, A)f)(t) := R(\lambda, A)(f(t))$ for all $f \in BUC(\mathbb{R}, E)$ and all $t \in \mathbb{R}$.

Lemma 5.2.4 *Let A and N as above and assume that $D(N) \neq \emptyset$. Then if $f \in D(N)$, we have $R(\lambda, A)f \in D(N)$ and $R(\lambda, A)(Nf) = N(R(\lambda, A)f)$ for all $\lambda \in \rho(A)$.*

Proof. Let $f \in D(N)$ and u_f be the unique mild solution of (II). Then it follows that

$$\begin{aligned} & R(\lambda, A)u_f(t) - R(\lambda, A)u_f(0) \\ = \ & tR(\lambda, A)y + A\int_0^t \int_0^s R(\lambda, A)u_f(r)drds + \int_0^t \int_0^s R(\lambda, A)f(r)drds \end{aligned}$$

for all $t \in \mathbb{R}$. Hence, $R(\lambda, A)u_f$ is a mild solution of $(\text{II})_{R(\lambda,A)f}$ which is unique since u_f is unique. It follows that $R(\lambda, A)f \in D(N)$ and $N(R(\lambda, A)f) = u_{R(\lambda,A)f} = R(\lambda, A)u_f = R(\lambda, A)(Nf)$. □

Recall that $(S(t))_{t\in\mathbb{R}}$ denotes the translation group on $BUC(\mathbb{R}, E)$ with generator D.

Proposition 5.2.5 *Let A be a closed, linear operator with non-empty resolvent set and let N be the solution operator for the second-order problem (II). Assume that $D(N) \neq \emptyset$. Then for $u, f \in BUC(\mathbb{R}, E)$ the following are equivalent:*

(1) $S(h)f \in D(N)$ for all $h \in \mathbb{R}$ and $Nf = u$.

(2) $S(h)f \in D(N)$ and $NS(h)f = S(h)u$ for all $h \in \mathbb{R}$.

(3) For all $s \in \mathbb{R}$, $\int_s^t (t-r)u(r)dr \in D(A)$ for all $t \in \mathbb{R}$ and there exist $y_s \in E$ such that

$$u(t) - u(s) = (t-s)y_s + A\int_s^t (t-r)u(r)dr + \int_s^t (t-r)f(r)dr \qquad (t \in \mathbb{R}).$$

(4) $f \in D(N)$, $Nf = u$ and $\int_0^t u(r)dr \in D(A)$ for all $t \in \mathbb{R}$.

Proof. To prove that (1) $\Rightarrow$ (2), let $\lambda \in \rho(A)$ and $Nf = u$. It follows that there exists $y \in E$ such that

$$\begin{aligned}
&S(h)R(\lambda,A)u_f(t) - S(h)R(\lambda,A)u_f(0)\\
&= (t+h)y + A\int_0^{t+h}\int_0^s R(\lambda,A)u_f(r)drds + \int_0^{t+h}\int_0^s R(\lambda,A)f(r)drds\\
&\quad - \left(hy + A\int_0^h\int_0^s R(\lambda,A)u_f(r)drds + \int_0^h\int_0^s R(\lambda,A)f(r)drds\right)\\
&= tx_h + A\int_0^t\int_0^s S(h)R(\lambda,A)u_f(r)drds + \int_0^t\int_0^s S(h)R(\lambda,A)f(r)drds,
\end{aligned}$$

where $x_h := y + A\int_0^h R(\lambda,A)u_f(r)dr + \int_0^h R(\lambda,A)f(r)dr$. Since $R(\lambda,A)$ and $S(h)$ commute, we obtain from the above equation and Lemma 5.2.4 that

$$R(\lambda,A)S(h)Nf = S(h)R(\lambda,A)u_f = NR(\lambda,A)S(h)f = R(\lambda,A)NS(h)f.$$

Since $R(\lambda,A)$ is injective, it follows that $S(h)u = S(h)Nf = NS(h)f$ for all $f \in D(N)$.

Let $s \in \mathbb{R}$. Then $S(s)f \in D(N)$ and $NS(s)f = S(s)u$. It follows that $\int_s^t (t-r)u(r)dr = \int_0^{t-s}((t-s)-r)(S(s)u)(r)dr \in D(A)$ for all $t \in \mathbb{R}$. Moreover, there exists $y_s \in E$, where y_s is given by $\tilde{N}f = (S(s)u, y_s)$, such that

$$\begin{aligned}
&u(t) - u(s)\\
&= (S(s)u)(t-s) - (S(s)u)(0)\\
&= (t-s)y_s + A\int_0^{t-s}((t-s)-r)(S(s)u)(r)dr + \int_0^{t-s}((t-s)-r)(S(s)f)(r)dr\\
&= (t-s)y_s + A\int_s^t (t-r)u(r)dr + \int_s^t f(r)dr
\end{aligned}$$

for all $t \in \mathbb{R}$, which proves (2) $\Rightarrow$ (3).

By setting $s = 0$ in (3), we see that $Nf = u$. Furthermore,

$$\int_0^t u(r)dr = \int_0^{t+1}((t+1)-r)u(r)dr - \int_0^t (t-r)u(r)dr - \int_t^{t+1}((t+1)-r)u(r)dr$$

which proves (4).

It remains to prove (4) $\Rightarrow$ (1). Let $h \in \mathbb{R}$. Since $Nf = u$, it follows that there exists $y \in E$ such that

$$(S(h)u)(t) - (S(h)u)(0)$$

$$
\begin{aligned}
&= u(t+h) - u(0) - (u(h) - u(0)) \\
&= ty + A\left(\int_0^{t+h} (t+h-r)u(r)dr - \int_0^h (h-r)u(r)dr\right) \\
&\quad + \int_0^{t+h} (t+h-r)f(r)dr - \int_0^h (h-r)f(r)dr \\
&= t\left(y + A\int_0^h u(r)dr + \int_0^h f(r)dr\right) + A\int_0^t (t-r)(S(h)u)(r)dr \\
&\quad + \int_0^t (t-r)(S(h)f)(r)dr
\end{aligned}
$$

for all $t \in \mathbb{R}$. Hence, $S(h)u$ is a solution of $(\mathrm{II})_{S(h)f}$ which is unique by the remark given after the definition of the operator N (see (5.3)). It follows that $S(h)f \in D(N)$ for all $h \in \mathbb{R}$. □

Remark, that in the case when Equation (II) is well-posed, then (1) from Proposition 5.2.5 is satisfied. Hence, we obtain the following corollary from (4) in Proposition 5.2.5.

Corollary 5.2.6 *Let A be a linear operator such that $\rho(A) \neq \emptyset$ and let Equation (II) be well-posed. If $f \in BUC(\mathbb{R}, E)$ and $u \in BUC(\mathbb{R}, E)$ is the unique mild solution of $(II)_f$, it follows that*

$$\int_0^t u(r)dr \in D(A) \qquad (t \in \mathbb{R}).$$

Now, we can give a necessary condition for existence and uniqueness of mild solutions of the second-order Equation (II).

Theorem 5.2.7 *Let A be a linear operator on a Banach space E with non-empty resolvent set. Assume that Equation (II) is well-posed. Then $\mathbb{R}^- \subseteq \rho(A)$ and there exists a constant $C \geq 0$ such that $\|R(-\lambda, A)\| \leq C$ for all $\lambda \geq 0$.*

Proof. Choose arbitrary $\lambda \in \mathbb{R}$ and $y \in E$. We define $f_s(t) := e^{i\lambda(s+t)}y = e^{i\lambda s}f_0(t) = f_0(s+t)$ for all $s, t \in \mathbb{R}$, where $f_0(t) := e^{i\lambda t}y$. Then, since (II) is well-posed, there exists a unique function $u_s \in BUC(\mathbb{R}, E)$ such that u_s is a mild solution of $(\mathrm{II})_{f_s}$. We claim that

$$u_s(t) = e^{i\lambda s}u_0(t) = u_0(s+t)$$

for all $s, t \in \mathbb{R}$.

Since u_s is a mild solution of $(\mathrm{II})_{f_s}$, there exists $y_s \in E$ such that

$$u_s(t) - u_s(0) = ty_s + A\int_0^t (t-r)u_s(r)dr + \int_0^t (t-r)e^{i\lambda s}f_0(r)dr \qquad (*)$$

for all $s, t \in \mathbb{R}$. If we multiply both sides by $e^{-i\lambda s}$, we obtain

$$e^{-i\lambda s}u_s(t) - e^{-i\lambda s}u_s(0) = te^{-i\lambda s}y_s + A\int_0^t (t-r)e^{-i\lambda s}u_s(r)dr + \int_0^t (t-r)f_0(r)dr.$$

Hence, $e^{-i\lambda s}u_s$ is a mild solution of $(\mathrm{II})_{f_0}$. From uniqueness of solutions it follows that $e^{-i\lambda s}u_s(t) = u_0(t)$ for all $s, t \in \mathbb{R}$ and we get the first equality.

Further, since u_0 is a mild solution of $(\mathrm{II})_{f_0}$, there exists $y_0 \in E$ such that

$$\begin{aligned} u_0(s+t) - u_0(s) &= ty_0 + A\left(\int_0^{s+t}(s+t-r)u_0(r)dr - \int_0^s (s-r)u_0(r)dr\right) \quad (**) \\ &\quad + \int_0^{s+t}(s+t-r)f_0(r)dr - \int_0^s (s-r)f_0(r)dr. \end{aligned}$$

If we subtract $(**)$ from $(*)$ and use that $\int_0^t u(r)dr \in D(A)$ (see Corollary 5.2.6), we obtain

$$\begin{aligned} &(u_s(t) - u_0(s+t)) - (u_s(0) - u_0(s)) \\ &= t(y_s - y_0) + A\left(\int_0^t (t-r)(u_s(r) - u_0(s+r))dr - t\int_0^s u_0(r)dr\right) \\ &\quad -t\int_0^s f_0(r)dr \\ &= t\left(y_s - y_0 - A\int_0^s u_0(r)dr - \int_0^s f_0(r)dr\right) \\ &\quad +A\left(\int_0^t (t-r)(u_s(r) - u_0(s+r))dr\right) \end{aligned}$$

Hence $u_s(.) - u_0(s+.)$ is a mild solution of the homogeneous second-order equation $u'' = Au$. By uniqueness of solutions, it follows that $u_s(t) = u_0(s+t)$ for all $s, t \in \mathbb{R}$.

If we now set $z := u_0(0) \in E$, then $u_0(t) = e^{i\lambda t}z \in C^2(\mathbb{R}, E)$. Thus by Lemma 5.1.3, u_0 is a classical solution of $(\mathrm{II})_{f_0}$, i.e. $u_0(t) \in D(A)$ and $u_0''(t) = Au_0(t) + f_0(t)$ for all $t \in \mathbb{R}$. For $t = 0$ we obtain $z \in D(A)$ and

$$-\lambda^2 z = u_0''(0) = Au_0(0) + f_0(0) = Az + y.$$

It follows that $(-\lambda^2 - A)z = y$ and $(-\lambda^2 - A)$ is surjective since $y \in E$ was arbitrary.

Next, let $(-\lambda^2 - A)z = 0$ for some $z \in A$ and define $u(t) := e^{i\lambda t}z$. Then $u''(t) = -\lambda^2 e^{i\lambda t}z = Au(t)$. It follows by uniqueness of solutions that $z = 0$, and hence, $-\lambda^2 - A$ is injective.

Since the solution operator N is bounded, we get

$$\|z\|_E = \|u_0\|_\infty = \|Nf_0\|_\infty \leq \|N\|\|f_0\|_\infty = \|N\|\|y\|_E.$$

Hence, $\mathbb{R}^- \in \rho(A)$ and $\|R(-\lambda^2, A)\| \leq C$ for $C := \|N\|$. □

Since we have used for the proof of Theorem 5.2.7 only the functions $f_s(t) = e^{i\lambda(s+t)}y$, with $s, t \in \mathbb{R}$ and $y \in E$, we get the same estimate for $R(-\lambda, A)$, $\lambda \geq 0$, if we only consider almost periodic functions. We denote again by $AP(\mathbb{R}, E)$ the set of all almost periodic functions on $\mathbb{R}$ with values in the Banach space E (see (2.11)).

Corollary 5.2.8 *Let A be a linear operator on a Banach space E with non-empty resolvent set. Assume that for all $f \in AP(\mathbb{R}, E)$ there exists a unique mild solution $u_f \in BUC(\mathbb{R}, E)$ of $(II)_f$. Then $\mathbb{R}^- \subseteq \rho(A)$ and there exists a constant $C \geq 0$ such that $\|R(-\lambda, A)\| \leq C$ for all $\lambda \geq 0$. Moreover, $u_f \in AP(\mathbb{R}, E)$.*

Proof. As in the proof of Theorem 5.2.7 we get that $\mathbb{R}^- \in \rho(A)$ and that there exists a constant $0 \leq C := \|N|_{AP(\mathbb{R},E)}\| < \infty$ such that $\|R(-\lambda, A)\| \leq C$ for all $\lambda \geq 0$. It follows that $\{\eta \in \mathbb{R} : -\eta^2 \in \sigma(A)\} = \emptyset$. And, by [15, Theorem 4.6] or Lemma 5.1.4, we obtain $u_f \in AP(\mathbb{R}, E)$. □

For generators of bounded cosine families we obtain from Theorem 5.2.7 the following corollary. Recall that the spectrum of such a generator is contained in $\mathbb{R}^-$ and not empty if the underlying Banach space is non-trivial (see [118, 3.3.6]). For more information about operators that generate cosine functions, we want to refer to [57], [69], [115] and [118].

Corollary 5.2.9 *Let A be the generator of a bounded cosine function on a Banach space $E \neq \{0\}$. Then Equation (II) is not well-posed.*

Definition 5.2.10 For a Banach space E, define the following spaces of differentiable functions:

$$\begin{aligned} BUC^1(\mathbb{R}, E) &= (D(D), \|.\|_D) \\ &= \{f \in BUC(\mathbb{R}, E) \cap C^1(\mathbb{R}, E) : f' \in BUC(\mathbb{R}, E)\} \end{aligned} \tag{5.4}$$

the Banach space $D(D)$ with the graph norm $\|f\|_D = \|f\| + \|f'\|$, where D is the generator of the shift group on $BUC(\mathbb{R}, E)$. Similarly, we define

$$\begin{aligned} BUC^2(\mathbb{R}, E) &= (D(D^2), \|.\|_{D^2}) \\ &= \{f \in BUC^1(\mathbb{R}, E) \cap C^2(\mathbb{R}, E) : f' \in BUC^1(\mathbb{R}, E)\}. \end{aligned} \tag{5.5}$$

Proposition 5.2.11 *Let A be a linear operator such that $\rho(A) \neq \emptyset$ and let Equation (II) be well-posed. Assume that $f \in BUC^1(\mathbb{R}, E)$, respectively $f \in BUC^2(\mathbb{R}, E)$. Then $Nf = u_f \in BUC^1(\mathbb{R}, E)$ and $(Nf)' = N(f')$, respectively $Nf = u_f \in BUC^2(\mathbb{R}, E)$ and $(Nf)'' = N(f'')$.*

Proof. Since N is bounded and N commutes with $S(h)$ (Proposition 5.2.5), we obtain for $f \in BUC^1(\mathbb{R}, E)$ that

$$\begin{aligned} \lim_{h\to 0} \frac{Nf(t+h) - Nf(t)}{h} &= \lim_{h\to 0} \frac{(S(h)Nf - Nf)(t)}{h} \\ &= \left(N \lim_{h\to 0} \frac{S(h)f - f}{h} \right)(t) = Nf'(t) \end{aligned}$$

for all $t \in \mathbb{R}$. Hence, $Nf \in BUC^1(\mathbb{R}, E)$ and $(Nf)' = Nf'$.

For $f \in BUC^2(\mathbb{R}, E)$, we get similarly that $\lim_{h\to 0} \frac{Nf'(t+h) - Nf'(t)}{h} = Nf''(t)$. Thus $Nf \in BUC^2(\mathbb{R}, E)$ and $(Nf)'' = (Nf')' = Nf''$. □

The second-order equation is, like the first-order equation (see Theorem 2.2.5), related to an operator equation.

Theorem 5.2.12 *Let A be a linear operator on a Banach space E with non-empty resolvent set. Then the following conditions are equivalent:*

(i) Equation (II) is well-posed.

(ii) The operator equation

$$AX - XD^2 = -\delta_0, \tag{5.6}$$

where $\delta_0 f = f(0)$, has a unique bounded solution $X : BUC(\mathbb{R}, E) \to E$.

(iii) For every bounded linear operator $C : BUC(\mathbb{R}, E) \longrightarrow E$ the operator equation $AX - XD^2 = C$ has a unique bounded solution $X : BUC(\mathbb{R}, E) \to E$.

(iv) The operator τ_{A,D^2} is invertible.

Proof. It is clear that (iii)⇔(iv).

Since (iii)⇒(ii) is trivial, we prove next (ii)⇒(i): Let X be the unique solution of the operator equation (5.6) and $f \in BUC(\mathbb{R}, E)$. Define a bounded uniformly continuous function by

$$u(t) := XS(t)f.$$

Then, for $f \in D(D^2)$, u is a classical solution of (II) since $u \in C^2(\mathbb{R}, E)$ and

$$u''(t) = XD^2S(t)f = AXS(t)f + \delta_0 S(t)f = Au(t) + f(t)$$

for all $t \in \mathbb{R}$. Now let $f \in BUC(\mathbb{R}, E)$ and $(f_n)_{n\in\mathbb{N}} \subseteq D(D^2)$ with $\lim_n f_n = f$. Then $\lim_n u_n := \lim_n XS(.)f_n = XS(.)f = u$. By the closedness of the operator A, it follows that $u = XS(.)f$ is a mild solution of $(\mathrm{II})_f$ for arbitrary $f \in BUC(\mathbb{R}, E)$. This proves existence.

Remark that from (iv), and hence also from (iii), it follows that $\sigma(A) \cap \mathbb{R}^- = \emptyset$ (see [14, Theorem 2.1]). To prove now uniqueness, assume that $v \in BUC(\mathbb{R}, E)$ is a mild solution of the homogeneous equation $v''(t) = Av(t)$, $(t \in \mathbb{R})$. Then, by Proposition 5.1.5, $sp(v) \subseteq \{\eta \in \mathbb{R} : -\eta^2 \subseteq \sigma(A)\} = \emptyset$. Hence $v = 0$.

(i)$\Rightarrow$(ii): Define the operator $X : BUC(\mathbb{R}, E) \to E$ by

$$Xf := (Nf)(0) = (\delta_0 \circ N)(f),$$

where N is the solution operator of the second-order equation (see (5.3)). Remark that X is bounded since δ_0 and N are bounded. Let $f \in D(D^2)$. Then, by Proposition 5.2.11, $Nf \in D(D^2)$ and hence, Nf is a classical solution of (II). It follows that $Xf = Nf(0) \in D(A)$ and

$$XD^2f = N(f'')(0) = (Nf)''(0) = A(Nf)(0) + f(0) = AXf + \delta_0 f.$$

Hence, X is a solution of the operator equation $AX - XD^2 = -\delta_0$.

For the proof of uniqueness assume that X is a non-trivial bounded solution of the operator equation $AX - XD^2 = 0$. Then, since $D(D^2)$ is dense, it follows that there exists $f \in D(D^2)$ such that $u(.) := XS(.)f \neq 0$. But, by the same proof as (ii)$\Rightarrow$(i), u is a mild solution of the homogeneous equation $u''(t) = Au(t)$, $(t \in \mathbb{R})$. This is a contradiction to uniqueness.

(ii)$\Rightarrow$(iii) Define the bounded operator $Y : BUC(\mathbb{R}, E) \to E$ by $Yf := X\tilde{f}$, where $\tilde{f}(t) := -CS(t)f$ and X is the bounded solution of the operator equation (5.6). Let $f \in D(D^2)$. Then $(D^2f)^{\sim}(t) = -CS(t)D^2f = -CD^2S(t)f = D^2\tilde{f}(t)$ and

$$AYf = AX\tilde{f} = XD^2\tilde{f} - \delta_0\tilde{f} = X(D^2f)^{\sim} + Cf = YD^2f + Cf,$$

i.e. Y is a bounded solution of (iii).
The uniqueness of the solutions of the operator equation $AX - XD^2 = C$ follows immediately from the uniqueness of solutions of the operator equation $AX - XD^2 = -\delta_0$. $\square$

5.3 Examples of well-posed operators of second-order equations

In this section we give some examples of closed, linear operators A such that Equation (II) is well-posed, i.e. operators A for which we have existence and uniqueness of mild solutions of the second-order problem

$$\text{(II)} \qquad u''(t) = Au(t) + f(t), \quad (t \in \mathbb{R}).$$

Denote again by D the generator of the shift group $(S(t))_{t\in\mathbb{R}}$ on $BUC(\mathbb{R}, E)$. Recall that D^2 is the generator of the *Gaussian semigroup* $(G(t))_{t\geq 0}$ on $BUC(\mathbb{R}, E)$ which is given by $G(0)f = f$ and

$$(G(t)f)(s) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(s-r)^2}{4t}} f(r)dr \tag{5.7}$$

for all $f \in BUC(\mathbb{R}, E)$, $s \in \mathbb{R}$ and $t > 0$.

Example 5.3.1 *Let $-A$ be the generator of a uniformly exponentially stable C_0-semigroup $(T(t))_{t\geq 0}$. Then Equation (II) is well-posed and for $f \in BUC(\mathbb{R}, E)$ the unique mild solution of Equation (II)$_f$ is given by*

$$u(t) = -\int_0^\infty T(s)(G(s)f)(t)ds. \tag{5.8}$$

Proof. Define the bounded operator $X : BUC(\mathbb{R}, E) \to E$ by

$$Xf := -\int_0^\infty T(s)\delta_0 G(s)fds.$$

The integral exists since $(T(t))_{t\geq 0}$ is exponentially stable. Further, we obtain for $f \in D(D^2)$

$$XD^2f = -\int_0^\infty T(s)\delta_0 G(s)D^2fds = -\int_0^\infty T(s)\left(D^2G(s)f\right)(0)ds = f(0) + AXf.$$

Hence, X is a bounded solution of the operator equation (5.6).

Now assume that this solution is not unique, then there exists a non-trivial bounded solution Y of the operator equation $AY - YD^2 = 0$. Hence, there exists also a function $f \in D(D^2)$ such that $u(.) := YS(.)f \neq 0$ (see in the proof of Theorem 5.2.12 (ii)$\Rightarrow$(i)). It follows that u is a solution of the homogeneous equation $u'' = Au$. Thus, by Proposition 5.1.5, $\emptyset \neq sp(u) \subseteq \{\eta \in \mathbb{R} : -\eta^2 \in \sigma(A)\}$. Hence, $\sigma(A) \cap \mathbb{R}^- \neq \emptyset$ which is a contradiction, since $-A$ is the generator of a uniformly exponentially stable C_0-semigroup.

It follows from Theorem 5.2.12 that Equation (II) is well-posed and that the solution is given by

$$u(t) = XS(t)f = -\int_0^\infty T(s)\Big(G(s)g\Big)(t)ds.$$

□

Now, let A be a sectorial operator with spectral angle $\omega_A < \pi$ and $0 \in \rho(A)$. Then there exists $R > 0$ such that the disk with centre 0 and radius R is contained in $\rho(A)$ and let $\pi > \theta > \max\{\frac{\pi}{2}, \omega_A\}$. Define $\gamma_{\theta,R}$ as in (4.3).

For the next example see also [121, Theorem 15].

Example 5.3.2 *Let A be a sectorial operator with spectral angle $\omega_A < \pi$ and $0 \in \rho(A)$. Then Equation (II) is well-posed and for $f \in BUC(\mathbb{R}, E)$ the solution of (II)$_f$ is given by*

$$u(t) = -\frac{1}{2\pi i}\int_{\gamma_{\theta,R}} R(\lambda, A)\Big(R(\lambda, D^2)f\Big)(t)d\lambda. \tag{5.9}$$

Proof. As in Example 5.3.1, we want to show the existence and uniqueness of bounded solutions of the operator equation $AX - XD^2 = C$, where $C : BUC(\mathbb{R}, E) \to E$ is an arbitrary bounded linear operator.

Recall that the operator $-D^2$ is sectorial with spectral angle $\omega_{D^2} = 0$ and, since $(G(t))_{t\geq 0}$ is analytic, we have (see [95, Theorem 1.7.7])

$$G(t)f = \frac{1}{2\pi i}\int_{\gamma_{\theta,R}} e^{\lambda t}R(\lambda, D^2)fd\lambda$$

for all $f \in BUC(\mathbb{R}, E)$. Now define the bounded linear operator $X : BUC(\mathbb{R}, E) \to E$ by

$$Xf := \frac{1}{2\pi i}\int_{\gamma_{\theta,R}} R(\lambda, A)CR(\lambda, D^2)fd\lambda.$$

The integral exists and is uniformly bounded since

$$\begin{aligned}
\int_{\gamma_{\theta,R}} \|R(\lambda, A)CR(\lambda, D^2)f\|d\lambda &\leq \int_{\gamma_{\theta,R}} \frac{M_1}{|\lambda|}\,\|C\|\,\frac{M_2}{|\lambda|}\,\|f\|\,d\lambda \\
&= M_1\,M_2\,\|C\|\int_{\gamma_{\theta,R}} \frac{1}{|\lambda|^2}d\lambda\,\|f\| \\
&\leq \mathit{const}\,\|f\| < \infty,
\end{aligned}$$

where the constants M_1 and M_2 exist since A and $-D^2$ are sectorial. Now let $t > 0$. Then the integral

$$X_t f := \frac{1}{2\pi i} \int_{\gamma_{\theta,R}} e^{\lambda t} R(\lambda, A) C R(\lambda, D^2) f d\lambda$$

converges also for all $f \in BUC(\mathbb{R}, E)$ and defines a bounded linear operator from $BUC(\mathbb{R}, E)$ to E. Thus for $f \in D(D^2)$ we have $X_t f \in D(A)$, since $\|A(e^{\lambda t} R(\lambda, A) C R(\lambda, D^2) f)\| \leq const \frac{|e^{\lambda t}|}{|\lambda|}$ which is integrable on $\gamma_{\theta,R}$.

Further, by a standard application of Cauchy's Theorem, we obtain that $\int_{\gamma_{\theta,R}} e^{\lambda t} R(\lambda, A) C f d\lambda = 0$. Thus

$$\begin{aligned}
&AX_t f - X_t D^2 f \\
&= \frac{1}{2\pi i} \int_{\gamma_{\theta,R}} e^{\lambda t} A R(\lambda, A) C R(\lambda, D^2) f d\lambda - \frac{1}{2\pi i} \int_{\gamma_{\theta,R}} e^{\lambda t} R(\lambda, A) C R(\lambda, D^2) D^2 f d\lambda \\
&= \frac{1}{2\pi i} \int_{\gamma_{\theta,R}} e^{\lambda t} C R(\lambda, D^2) f d\lambda - \frac{\lambda}{2\pi i} \int_{\gamma_{\theta,R}} e^{\lambda t} R(\lambda, A) C f d\lambda \\
&= C \left(\frac{1}{2\pi i} \int_{\gamma_{\theta,R}} e^{\lambda t} R(\lambda, D^2) f d\lambda \right) \\
&= CG(t) f.
\end{aligned}$$

Therefore, letting $t \to 0$, we obtain $AXf - XD^2 f = Cf$ for all $f \in D(D^2)$.

To prove uniqueness, let Y be a bounded solution of the operator equation $AX - XD^2 = 0$. Then, for each $\lambda \in \rho(A) \cap \rho(D^2)$, we have

$$\begin{aligned}
YR(\lambda, D^2) - R(\lambda, A) Y &= R(\lambda, A) \big((\lambda - A) Y - Y(\lambda - D^2) \big) R(\lambda, D^2) \\
&= R(\lambda, A) \big(-AY + YD^2 \big) R(\lambda, D^2) \;=\; 0.
\end{aligned}$$

It follows that

$$YG(t) f = \frac{1}{2\pi i} \int_{\gamma_{\theta,R}} e^{\lambda t} Y R(\lambda, D^2) f d\lambda = \frac{1}{2\pi i} \int_{\gamma_{\theta,R}} e^{\lambda t} R(\lambda, A) Y f d\lambda = 0$$

for all $f \in BUC(\mathbb{R}, E)$. Letting $t \to 0$, we obtain $Y = 0$.

Finally, by Theorem 5.2.12, we obtain that well-posedness is satisfied and, if we set $C = -\delta_0$, the unique solution of $(\mathrm{II})_f$ is given by

$$u(t) = XS(t) f = \frac{1}{2\pi i} \int_{\gamma_{\theta,R}} R(\lambda, A) \Big(R(\lambda, D^2) f \Big)(t) d\lambda.$$

□

The idea of the next example is used later to establish existence and uniqueness of mild solutions of the second order differential equation, provided that $A = B^2$ and Equation (I) with the operator A replaced by B is well-posed (see Theorem 5.5.1). The solution is here given by a generalised Green's function.

Example 5.3.3 *Let A be a closed, linear operator on E and assume that $A = B^2$, where B is the generator of a uniformly exponentially stable C_0-semigroup $(T(t))_{t\geq 0}$. Then Equation (II) with the operator A is well-posed and the unique mild solutions of (II) are given by*

$$u(t) = -\frac{1}{2}R(0,B)\int_{\mathbb{R}} T(|t-s|)f(s)ds. \tag{5.10}$$

Proof. Let $f \in BUC(\mathbb{R}, D(B))$ and define u by Equation (5.10). Then we can also write

$$u(t) = -\frac{1}{2}R(0,B)\int_{-\infty}^{t} T(t-s)f(s)ds - \frac{1}{2}R(0,B)\int_{t}^{\infty} T(s-t)f(s)ds.$$

One sees that u is continuously differentiable and

$$u'(t) \;=\; \frac{1}{2}\int_{-\infty}^{t} T(t-s)f(s)ds - \frac{1}{2}\int_{t}^{\infty} T(s-t)f(s)ds \tag{5.11}$$

for all $t \in \mathbb{R}$. From this we see that u is twice continuously differentiable and, noting that $u(t) \in D(A)$, we obtain

$$\begin{aligned} u''(t) &= \frac{1}{2}f(t) + \frac{1}{2}\int_{-\infty}^{t} BT(t-s)f(s)ds + \frac{1}{2}f(t) + \frac{1}{2}\int_{t}^{\infty} BT(s-t)f(s)ds \\ &= f(t) - B^2R(0,B)\int_{-\infty}^{\infty} T(|s-t|)f(s)ds \\ &= Au(t) + f(t) \end{aligned}$$

for all $t \in \mathbb{R}$. Hence, u is a classical solution of $(\mathrm{II})_f$. Since $BUC(\mathbb{R}, D(B))$ is dense in $BUC(\mathbb{R}, E)$, there exists for arbitrary $f \in BUC(\mathbb{R}, E)$ a sequence $(f_n)_{n\in\mathbb{N}} \subseteq BUC(\mathbb{R}, D(B))$, such that $\lim_n f_n = f$. Then $\lim_n Nf_n = \lim_n u_n = u$ by Equation (5.10) and $\lim u_n'(0) = u'(0) =: y$ by Equation (5.11). Hence $\lim_n \tilde{N}f_n = (u, y)$, and since $\tilde{N}$ is closed, it follows that u is a mild solution of $(\mathrm{II})_f$.

We notice that for $\lambda \in \mathbb{R}$

$$\lambda^2 + A = \lambda^2 + B^2 = -(i\lambda - B)(i\lambda + B)$$

and the right hand side is invertible. Hence $\lambda^2 + A$ is invertible and it follows that $\mathbb{R}^- \subseteq \rho(A)$. Thus, uniqueness of solutions follows immediately from Theorem 5.1.6. $\square$

For completeness, we mention the scalar case (see also [59, p. 23]). This is a special case of the previous three examples.

Example 5.3.4 *Let $a \in \mathbb{C}^+ = \{\lambda \in \mathbb{C} : Re\lambda > 0\}$. Then for each $f \in BUC(\mathbb{R})$ exists a unique mild solution $u \in BUC(\mathbb{R})$ of the equation*

$$u''(t) = a^2 u(t) + f(t), \qquad (t \in \mathbb{R}). \tag{5.12}$$

Proof. Existence and uniqueness of the mild solutions of equation (5.12) follows from Example 5.3.2 and Example 5.3.3, and the solutions are given by

$$\begin{aligned} u(t) &= -\frac{1}{2\pi i}\int_{\gamma_{\theta,R}} \frac{1}{\lambda - a^2}(R(\lambda, D^2)f)(t)d\lambda \\ &= -\frac{1}{2a}\int_{\mathbb{R}} e^{-a|t-s|}f(s)ds, \end{aligned}$$

for all $t \in \mathbb{R}$ and arbitrary $f \in BUC(\mathbb{R})$. A direct proof of the second equality is sketched in [59, p.23]. If $a^2 \in \mathbb{C}^+$ then the solution is also given by (see Example 5.3.1)

$$u(t) = -\int_0^\infty e^{-a^2 s}(G(s)f)(t)ds.$$

$\square$

Last, we consider multiplication operators on $C_0(\Omega)$, respectively $L^p(\Omega)$ (see Definition 2.3.2).

A sufficient condition for well-posedness of Equation (II) is that a parabola is included in the resolvent set of the multiplication operator.

Theorem 5.3.5 *Let M_m be a multiplication operator on $C_0(\Omega)$, respectively $L^p(\Omega)$ $(p \geq 1)$, induced by an appropriate function $m : \Omega \to \mathbb{C}$. Let $a > 0$ and*

$$\mathrm{rg}(m) \subseteq \mathbb{C} \setminus P_a \qquad (\textit{respectively } \mathrm{ess\text{–}rg}(m) \subseteq \mathbb{C} \setminus P_a),$$

where $P_a := \{z \in \mathbb{C} : \mathcal{R}e(z) \leq a^2 - \frac{1}{4a^2}(\Im m(z))^2\}$. Then Equation (II) is well-posed.

Proof. First we claim that $0 \leq \mathcal{R}e(z) \leq a$ if, and only if $z^2 \in P_a$. For this, let $z = x + iy \in \mathbb{C}^+ = \{z \in \mathbb{C} : \mathcal{R}e(z) \geq 0\}$, thus $z^2 = (x^2 - y^2) + 2ixy$. If $\mathcal{R}e(z) \leq a$, then

$$\mathcal{R}e(z^2) = x^2 - y^2 = x^2 - \frac{\Im m(z^2)^2}{4x^2} \leq a^2 - \frac{\Im m(z^2)^2}{4a^2}.$$

Hence, $z^2 \in P_a$. If $z^2 \in P_a$, then

$$x^2 - \frac{\Im m(z^2)^2}{4x^2} \leq a^2 - \frac{\Im m(z^2)^2}{4a^2}.$$

From this, one obtains

$$4a^2(x^2)^2 + (\Im m(z^2)^2 - 4a^4)x^2 - a^2 \Im m(z^2)^2 \leq 0.$$

Solving this equation for x^2 by using that $x^2 \geq 0$, it follows that $x^2 \leq a^2$. Thus, $\mathcal{R}e(z) = x \leq a$.

From the above considerations, it follows that the function $-m^{\frac{1}{2}} : \omega \to \{z \in \mathbb{C} : \mathcal{R}e(z) \leq -a\}$ induces a multiplication operator $M_{-m^{1/2}}$ which is the generator of an exponentially stable multiplication semigroup and $(M_{-m^{1/2}})^2 = M_m$. Hence, by Example 5.3.1, we obtain that Equation (II) is well-posed. $\square$

Remark 5.3.6 If any parabola P symmetric to the real-axis and with $\mathbb{R}^- \subseteq P^\circ$ is included in the resolvent set of M_m, then there exists $a > 0$ such that $P_a \subseteq P$. Thus Equation (II) is also well-posed.

In contrast to Equation (I) (see Corollary 2.3.3), the necessary condition of Theorem 5.2.7 is not a sufficient condition for well-posedness of Equation (II). To show this, we will give a counterexample on $C_0(\mathbb{R})$.

Counterexample 5.3.7 *Let M_m be the multiplication operator on $E = C_0(\mathbb{R})$ induced by the continuous function $m : \mathbb{R} \to \mathbb{C}$ given by*

$$m(r) := \begin{cases} \frac{1}{r^2} - r^2 - 2i\,, & r \leq -1 \\ 1 - r^2 + 2ir, & |r| < 1 \\ \frac{1}{r^2} - r^2 + 2i\,, & r \geq 1. \end{cases}$$

Thus, $\mathbb{R}^- \in \rho(M_m)$ and $\|R(\lambda, M_m)\| \leq 1$ for all $\lambda < 0$. But Equation (II) is not well-posed.

Proof. We assume that Equation (II) is well-posed. For $f \in BUC(\mathbb{R}, C_0(\mathbb{R}))$, let $u_f \in BUC(\mathbb{R}, C_0(\mathbb{R}))$ be the unique mild solution of Equation (II). Then for each $r \in \mathbb{R}$ the function $u(\cdot, r) \in BUC(\mathbb{R})$ is the unique mild solution of

$$u''(t, r) = m(r)u(t, r) + f(t, r) \qquad (t \in \mathbb{R}).$$

It follows from Example 5.3.4 that u is of the form

$$u(t,r) = -\frac{1}{2\,m(r)^{1/2}} \int_{\mathbb{R}} e^{-m(r)^{1/2}|t-s|} f(s,r)dr \tag{5.13}$$

for all $t, r \in \mathbb{R}$. Here, the function $m^{1/2}$ is given by

$$m^{1/2}(r) := \begin{cases} \frac{1}{|r|} + ir, & |r| \geq 1 \\ 1 + ir, & |r| < 1. \end{cases}$$

Thus, the integral in (5.13) is well-defined and u is bounded, since $\frac{1}{m^{1/2}\,\mathcal{R}e(m^{1/2})}$ is bounded.

Now, let $n \in \mathbb{N}$ and define the function $f \in BUC(\mathbb{R}, C_0(\mathbb{R}))$ by

$$f(s,r) := \begin{cases} e^{ir|s|} & , & |r| \leq s \\ e^{ir|s|}\,(nr + sn + 1) & , & -s - \frac{1}{n} < r < -s \\ e^{ir|s|}\,(-nr + sn + 1) & , & s < r < s + \frac{1}{n} \\ 0 & , & |r| \geq s + \frac{1}{n}. \end{cases}$$

Let u_f be the mild solution of Equation $(\mathrm{II})_f$. Then for $r > 1$, we obtain by (5.13)

$$\begin{aligned} |u_f(0,r)| &= \left| \frac{1}{2(\frac{1}{r} + ir)} \int_{\mathbb{R}} e^{-(\frac{1}{r}+ir)|s|} f(s,r)ds \right| \\ &= \frac{1}{2|\frac{1}{r} + ir|} \left| \int_{-\infty}^{-r} e^{\frac{s}{r}} e^{irs} e^{-irs} ds + \int_{-r}^{-r+1/n} e^{\frac{s}{r}} e^{irs} e^{-irs} (nr + sn + 1) ds \right. \\ &\quad \left. + \int_{r-1/n}^{r} e^{-\frac{s}{r}} e^{-irs} e^{irs} (-nr + sn + 1) ds + \int_{r}^{\infty} e^{-\frac{s}{r}} e^{-irs} e^{irs} ds \right| \\ &\geq \frac{2}{r+1} \left(\int_r^{\infty} e^{-\frac{s}{r}} ds - \int_{r-1/n}^{r} e^{-\frac{s}{r}} ds \right) \\ &\geq \frac{2}{r+1} \left(re^{-1} - \frac{1}{n} e^{-\frac{r-1/n}{r}} \right) \end{aligned}$$

which is bigger than 0 if $n \in \mathbb{N}$ is small enough. It follows that $|u_f(0,r)|$ does not tend to 0 for $r \to \infty$, and hence, $u_f(0) \notin C_0(\mathbb{R})$. Thus, Equation (II) cannot be well-posed. $\square$

5.4 Reduction of the second-order equation to a first-order system

As before, we denote by $(S(t))_{t\in\mathbb{R}}$ the shift group on $BUC(\mathbb{R}, E)$ with generator D. Moreover, denote by $\mathcal{D} = \frac{d}{dt} : D(\mathcal{D}) \to BUC^1(\mathbb{R}, E)$ the first derivative, realised as a densely defined closed operator on $BUC^1(\mathbb{R}, E)$ (see (5.4)) with domain $D(\mathcal{D}) = D(D^2)$.

For a closed, linear operator A on a Banach space E define

$$\mathcal{A} := \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}.$$

Then $\mathcal{A}$ is a closed, linear operator on $E \times E$ with domain $D(\mathcal{A}) = D(A) \times E$. In the following theorem we give another equivalent condition to the well-posedness of Equation (II).

Theorem 5.4.1 *Let A be a linear operator on E with non-empty resolvent set. Then the following are equivalent:*

(i) Equation (II) is well-posed.

(ii) The operator equation

$$\mathcal{A}\mathcal{X} - \mathcal{X}\mathcal{D} = -\tilde{\delta}_0, \tag{5.14}$$

where $\tilde{\delta}_0 f = \binom{0}{\delta_0 f}$, has a unique bounded solution $\mathcal{X} : BUC^1(\mathbb{R}, E) \to E \times E$.

Proof. Recall that in Theorem 5.2.12 we have proved that (i) is equivalent to existence of a unique bounded solution $X \in \mathcal{L}(BUC(\mathbb{R}, E), E)$ of the operator equation $AX - XD^2 = -\delta_0$, (5.6) where $\delta_0 f = f(0)$.

First, let X be the unique bounded solution of the operator equation (5.6) and define the operator $\mathcal{X} : BUC^1(\mathbb{R}, E) \to E \times E$ by

$$\mathcal{X}f := \begin{pmatrix} Xf \\ XDf \end{pmatrix}.$$

Then $\mathcal{X}$ is bounded. Now, let $f \in D(\mathcal{D}) = D(D^2)$. Then, since X is a solution of (5.6), $Xf \in D(A)$ and hence, $\mathcal{X}f \in D(\mathcal{A})$. Thus we obtain

$$\mathcal{A}\mathcal{X}f - \mathcal{X}\mathcal{D}f = \begin{pmatrix} XDf - XDf \\ AXf - XD^2 f \end{pmatrix} = \begin{pmatrix} 0 \\ -\delta_0 f \end{pmatrix} = -\tilde{\delta}_0 f.$$

It follows that $\mathcal{X}$ is a bounded solution of the operator equation (5.14).

Second, let $\mathcal{X} = (X_1, X_2)$ be the unique solution of equation (5.14). It follows that

$$X_2 f = X_1 D f \qquad \text{and} \qquad A X_1 f - X_1 D^2 f = -\delta_0 f$$

for all $f \in D(\mathcal{D}) = D(D^2)$. And, define the bounded operator $X : BUC(\mathbb{R}, E) \to E$ by

$$Xg = X((1+D)f) := X_1 f + X_2 f = (X_1 + X_2)(I + D)^{-1} g$$

for all $g \in BUC(\mathbb{R}, E)$, where $f = R(1, -D)g \in BUC^1(\mathbb{R}, E)$. For $g \in D(D^2)$, we have $Df = DR(1, -D)g \in D(D^2)$ and

$$\begin{aligned} AXg - XD^2 g &= AX_1 f + AX_2 f - X_1 D^2 f - X_2 D^2 f \\ &= AX_1 f - X_1 D^2 f + AX_1(Df) - X_2 D^2(Df) \;=\; -\delta_0 f. \end{aligned}$$

Hence, X is a bounded solution of the operator equation (5.6).

Finally, one sees that the two mappings, $X \mapsto (X, XD)$ and $\mathcal{X} = (X_1, X_2) \mapsto (X_1 + X_2)(I + D)^{-1}$, are mutually inverse. This sets up a bijection between the solutions of (5.6) and those of (5.14), establishing both existence and uniqueness simultaneously. □

We next show that the second-order differential equation is related to a first-order equation on $E \times E$.

To be more precise, we consider the following equation on $BUC(\mathbb{R}, E) \times BUC(\mathbb{R}, E)$:

$$\begin{pmatrix} u \\ v \end{pmatrix}'(t) = \mathcal{A} \begin{pmatrix} u \\ v \end{pmatrix}(t) + \begin{pmatrix} 0 \\ f \end{pmatrix}(t) \qquad (t \in \mathbb{R}), \tag{5.15}$$

where $\mathcal{A}$ is defined as above and $f \in BUC(\mathbb{R}, E)$.

A pair of functions $(u, v) \in BUC(\mathbb{R}, E) \times BUC(\mathbb{R}, E)$ is called a *mild solution* of (5.15) if $\int_0^t u(r)dr \in D(A)$, $u(t) - u(0) = \int_0^t v(r)dr$ and

$$v(t) - v(0) \;=\; A \int_0^t u(r)dr + \int_0^t f(r)dr \tag{5.16}$$

for all $t \in \mathbb{R}$.

As a simple consequence of this definition we get that if $(u, v) \in BUC(\mathbb{R}, E) \times BUC(\mathbb{R}, E)$ is a mild solution of (5.15) then $u \in BUC^1(\mathbb{R}, E)$, $u' = v$ and u is a mild solution of (II).

Remark that if $(u,v) \in C^1(\mathbb{R},E) \times C^1(\mathbb{R},E)$ is a mild solution of (5.15) and $\rho(A) \neq \emptyset$ then (u,v) is a *classical solution* of (5.15), i.e. $u \in C(\mathbb{R},D(A))$, $u' = v$ and $v' = Au+f$. In this case, $u \in C^2(\mathbb{R},E)$ and u is a classical solution of $(\mathrm{II})_f$.

We will see that in the context of equation (5.15) the space $BUC^1(\mathbb{R},E)$ is the correct space to consider for the functions f. Because of that, we will define the solution operator G for equation (5.15) on $BUC^1(\mathbb{R},E)$. To be more precise, we define

$$\begin{aligned} D(G) &:= \{f \in BUC^1(\mathbb{R},E) : \exists!\,(u_f,v_f) \in BUC(\mathbb{R},E) \times BUC(\mathbb{R},E) \\ &\qquad \text{such that } (u_f,v_f) \text{ is a mild solution of (5.15)}\} \qquad (5.17) \\ Gf &:= \begin{pmatrix} u_f \\ v_f \end{pmatrix}. \end{aligned}$$

Remark that, as for the solution operators for the first- and second-order equations, either all mild solutions of Equation (5.15) are unique or no mild solution is unique. Moreover, the operator G is closed.

If we assume existence and uniqueness of mild solutions for all $f \in BUC^1(\mathbb{R},E)$, we obtain by the closed graph theorem that G is bounded.

One can show that the solution operator G commutes with translations.

Lemma 5.4.2 *Let G be the solution operator of Equation (5.15). Then for all $f \in D(G)$ we have $S(h)f \in D(G)$ and*

$$\mathcal{S}(h)Gf = GS(h)f$$

for all $h \in \mathbb{R}$, where $\mathcal{S}(h)\binom{u}{v} = \binom{S(h)u}{S(h)v}$.

Proof. Let $f \in D(G)$ and (u,v) be the unique mild solution of (5.15). Then by the definition of a mild solution

$$S(h)u(t) - S(h)u(0) = \int_h^{t+h} v(r)dr = \int_0^t S(h)v(r)dr,$$

and

$$S(h)v(t) - S(h)v(0) = A\int_0^t S(h)u(r)dr + \int_0^t S(h)f(r)dr$$

for all $t,h \in \mathbb{R}$. Hence $(S(h)u, S(h)v)$ is a mild solution of (5.15) with f replaced by $S(h)f$. This solution is unique, since the solution (u,v) is unique. Hence, we have shown that $S(h)f \in D(G)$ and $\mathcal{S}(h)Gf = GS(h)f$. $\square$

As in in Proposition 5.2.11 we obtain from Lemma 5.4.2 the following regularity result.

Proposition 5.4.3 *Assume that for each $f \in BUC^1(\mathbb{R}, E)$ exists a unique mild solution $(u, v) \in BUC(\mathbb{R}, E) \times BUC(\mathbb{R}, E)$ of Equation (5.15). Then for $f \in BUC^2(\mathbb{R}, E)$, the mild solution $(u, v) \in BUC^1(\mathbb{R}, E) \times BUC^1(\mathbb{R}, E)$ and*

$$(Gf)' = G(f').$$

Remark that in this case, (u, v) is a classical solution of equation (5.15).

Next we show that existence and uniqueness of mild solutions of (5.15) for functions $f \in BUC^1(\mathbb{R}, E)$ is equivalent to the solvability of the operator equation (5.14).

Theorem 5.4.4 *Let A be a linear operator on E with $\rho(A) \neq \emptyset$ and define $\mathcal{A}$, $\mathcal{D}$ and $\tilde{\delta_0}$ as above. Then the following are equivalent:*

(i) The operator equation (5.14) has a unique bounded solution $\mathcal{X} : BUC^1(\mathbb{R}, E) \to E \times E$.

(ii) For all $f \in BUC^1(\mathbb{R}, E)$ there exists a unique mild solution $(u, v) \in BUC(\mathbb{R}, E) \times BUC(\mathbb{R}, E)$ of Equation (5.15).

Proof. (i)⇒(ii): For $f \in BUC^1(\mathbb{R}, E)$, define $\binom{u}{v}(t) := \mathcal{X}S(t)f$, where $\mathcal{X}$ is the unique solution of $\mathcal{A}\mathcal{X} - \mathcal{X}\mathcal{D} = -\tilde{\delta_0}$. Now let $f \in D(\mathcal{D}) = D(D^2)$ then $u, v \in BUC^1(\mathbb{R}, E)$ and

$$\begin{pmatrix} u \\ v \end{pmatrix}'(t) = \mathcal{X}DS(t)f = \mathcal{A}\mathcal{X}S(t)f + \tilde{\delta_0}S(t)f = \mathcal{A}\begin{pmatrix} u \\ v \end{pmatrix}(t) + \begin{pmatrix} 0 \\ f \end{pmatrix}(t)$$

for all $t \in \mathbb{R}$. Hence $\binom{u}{v} = \mathcal{X}S(.)f$ is a solution of (5.15). Next let $f \in BUC^1(\mathbb{R}, E)$ be arbitrary. Then there exists a sequence $(f_n)_{n\in\mathbb{N}} \subseteq D(D^2)$ such that $f_n \xrightarrow{\|\cdot\|_D} f$ as $n \to \infty$. It follows that $\binom{u_n}{v_n} := \lim_n Gf_n = \mathcal{X}S(.)f = \binom{u}{v}$ in $BUC(\mathbb{R}, E) \times BUC(\mathbb{R}, E)$. Since G is closed, we get $f \in D(G)$ and $Gf = \binom{u}{v}$, i.e. (u, v) is a mild solution of (5.15).

To prove uniqueness, assume that (u, v) is a mild solution of $\binom{u}{v}' = \mathcal{A}\binom{u}{v}$. It follows that u is a mild solution of $u'' = Au$. Since, by Theorem 5.4.1 and Theorem 5.2.12, (i) is equivalent to existence and uniqueness of mild solutions of the second-order equation (II), we obtain $u = 0$. From the definition, it follows that $v = 0$.

(ii)⇒(i): Define $\mathcal{X} : BUC^1(\mathbb{R}, E) \to E \times E$ by

$$\mathcal{X}f := (Gf)(0).$$

Note that $\mathcal{X}$ is bounded since G is bounded. Let $f \in D(D^2)$. Then by Proposition 5.4.3

$$\mathcal{X}\mathcal{D}f \;=\; (Gf')(0) \;=\; (Gf)'(0) \;=\; \mathcal{A}(Gf)(0) + \begin{pmatrix} 0 \\ f(0) \end{pmatrix} \;=\; \mathcal{A}\mathcal{X}f + \tilde{\delta}_0 f.$$

Hence, $\mathcal{X}$ is a solution of the operator equation (5.14).

Finally, let $\mathcal{X}$ be a non-trivial solution of the operator equation $\mathcal{A}\mathcal{X} - \mathcal{X}\mathcal{D} = 0$. Since $D(D^2)$ is dense, there exists a function $f \in D(D^2)$ such that $\binom{u}{v} := \mathcal{X}S(.)f \neq 0$. But

$$\begin{pmatrix} u \\ v \end{pmatrix}'(t) = \mathcal{X}DS(t)f = \mathcal{A}\mathcal{X}S(t)f = \mathcal{A}\begin{pmatrix} u \\ v \end{pmatrix}(t).$$

It follows from (ii) that $\binom{u}{v} = 0$, which is a contradiction. □

From the theorem, it follows that if $f \in BUC^1(\mathbb{R}, E)$ and $u \in BUC(\mathbb{R}, E)$ is a mild solution of $(\mathrm{II})_f$, then $u \in BUC^1(\mathbb{R}, E)$ and $\binom{u}{u'}$ is a mild solution of Equation (5.15). Moreover, if u is a classical solution of $(\mathrm{II})_f$, then $\binom{u}{u'}$ becomes a classical solution of (5.15).

5.5 The case: $\mathbf{A} = \mathbf{B}^2$

In this section we assume that $A = B^2$, where B is a closed, linear operator such that the first-order equation on the line with the operator B is well-posed, i.e. Equation (I) for B is uniquely solvable. Under this condition we show the existence and uniqueness of mild solutions of the second-order equation on the line, i.e. Equation (II) with the operator A is well-posed. The idea of the following theorem comes from Example 5.3.3.

Theorem 5.5.1 *Let $A = B^2$, where B is a closed, linear operator on E with non-empty resolvent set. Then, the following are equivalent:*

(i) For all $f \in BUC(\mathbb{R}, E)$ there exists a unique mild solution $u \in BUC(\mathbb{R}, E)$ of $u'(t) = Bu(t) + f(t)$, i.e. Equation (I) is well-posed.

(ii) For all $f \in BUC(\mathbb{R}, E)$ there exists a unique mild solution $w \in BUC(\mathbb{R}, D(B)) \cap BUC^1(\mathbb{R}, E)$ of $w''(t) = Aw(t) + f(t)$, especially Equation (II) is well-posed.

Proof. To prove $(i) \Rightarrow (ii)$, let $f \in BUC(\mathbb{R}, E)$ and $u \in BUC(\mathbb{R}, E)$ be the mild solution of $u' = Bu + f$. Thus, we have $\int_0^t u(r)dr \in D(B)$ and (see (2.1.1))

$$u(t) - u(0) = B \int_0^t u(r)dr + \int_0^t f(r)dr.$$

By Lemma 2.2.3, there exists also a function $v \in BUC(\mathbb{R}, E)$ which is a mild solution of $v' = -Bv - f$. Thus $\int_0^t v(r)dr \in D(B)$ and

$$v(t) - v(0) = -B \int_0^t v(r)dr - \int_0^t f(r)dr.$$

From Theorem 2.2.2, it follows that $i\mathbb{R} \subseteq \rho(B)$, in particular $0 \in \rho(B)$. Now define $w \in BUC(\mathbb{R}, E)$ by

$$w(t) := -\frac{1}{2}R(0, B)(u(t) + v(t)). \tag{5.18}$$

We show that w is a mild solution of $(\mathrm{II})_f$. Note that $w \in BUC(\mathbb{R}, D(B))$ by definition. First observe, that $\int_0^t \int_0^s w(r)dr\, ds = -\frac{1}{2}R(0, B) \int_0^t \left(\int_0^s u(r)dr + \int_0^s v(r)dr\right) ds \in D(B^2) = D(A)$. From the above equations, we get

$$\begin{aligned}
B^2 \int_0^t \int_0^s w(r)dr\, ds \;&=\; \frac{1}{2} \int_0^t \left(B \int_0^s u(r)dr - (-B) \int_0^s v(r)dr\right) ds \\
&= \; t\left(\frac{1}{2}(v(0) - u(0))\right) + \frac{1}{2}\left(\int_0^t u(s)ds - \int_0^t v(s)ds\right) - \int_0^t \int_0^s f(r)drds \\
&= \; -tz + \frac{1}{2}\left(\int_0^t u(s)ds - \int_0^t v(s)ds\right) - \int_0^t \int_0^s f(r)drds,
\end{aligned}$$

if we set $z := \frac{1}{2}(u(0) - v(0))$. From this, we obtain

$$\begin{aligned}
&w(t) - w(0) \\
&\quad= \; -\frac{1}{2}R(0, B)\left(B \int_0^t u(r)dr + \int_0^t f(r)dr - B \int_0^t v(r)dr - \int_0^t f(r)dr\right) \\
&\quad= \; \frac{1}{2} \int_0^t u(r)dr - \frac{1}{2} \int_0^t v(r)dr \\
&\quad= \; tz + B^2 \int_0^t \int_0^s w(r)dr + \int_0^t \int_0^s f(r)drds.
\end{aligned}$$

Hence w is a mild solution of $u'' = B^2 u + f = Au + f$. Remark also that

$$w' = \frac{1}{2}(u - v)$$

and therefore, $w \in BUC^1(\mathbb{R}, E)$.

Since $i\mathbb{R} \subseteq \rho(B)$ (again by Theorem 2.2.2) and $-\xi^2 - A = -(i\xi - B)(-i\xi - B)$ for all $\xi \in \mathbb{R}$, it follows that $\mathbb{R}^- \subseteq \rho(A)$. Hence by Theorem 5.1.6, it follows that the mild solutions of Equation (II) are unique.

For the direction $(ii) \Rightarrow (i)$, let $f \in BUC(\mathbb{R}, E)$ and $w \in BUC(\mathbb{R}, D(B)) \cap BUC^1(\mathbb{R}, E)$ be the unique mild solution of $w''(t) = Aw(t) + f(t)$. Now define

$$u(t) := Bw(t) + w'(t).$$

It follows that for all $t \in \mathbb{R}$

$$\begin{aligned} u(t) - u(0) &= Bw(t) - Bw(0) + w'(t) - w'(0) \\ &= B \int_0^t w'(r)dr + B^2 \int_0^t w(r)dr + \int_0^t f(r)dr \\ &= B \int_0^t u(r)dr + \int_0^t f(r)dr. \end{aligned}$$

Hence, u is a mild solution of Equation (I) with A replaced by B.

To prove uniqueness, let $u \in BUC(\mathbb{R}, E)$ be a mild solution of the homogeneous equation $u'(t) = Bu(t)$. Then, by Lemma 2.2.3, $v = Pu$ is a mild solution of $v'(t) = -Bv(t)$. It follows as in the first part of the proof that $w = -\frac{1}{2}R(0, B)(u + v)$ is a mild solution of $w''(t) = Aw(t)$. Thus, $w = 0$ and $w' = \frac{1}{2}(u - v) = 0$. Hence, $u = v = 0$. □

Corollary 5.5.2 *Let $A = B^2$, where B is a closed, linear operator such that the equation $u'(t) = Bu(t) + f(t)$ is well-posed. Assume that $f \in BUC^1(\mathbb{R}, E)$. Then the unique mild solution of (II)$_f$ is a classical solution.*

Proof. Since $f \in BUC^1(\mathbb{R}, E)$, it follows that the unique mild solutions of $u' = Bu + f$ and $v' = -Bv + f$ are classical solutions. By Theorem 5.5.1, the unique mild solution w of (II)$_f$ satisfies $w'' = \frac{1}{2}(u' - v')$ and therefore $w \in C^2(\mathbb{R}, E)$. Hence w is a classical solution. □

From Theorem 5.5.1 (and Theorem 2.3.1) we obtain another class of operators such that Equation (II) is well-posed. This extends Example 5.3.3.

Example 5.5.3 *Let $A = B^2$, where B is the generator of a hyperbolic C_0-semigroup. Then Equation (II) is well-posed.*

In this case, where $A = B^2$ and the first-order equation $u'(t) = Bu(t) + f(t)$ is well-posed, we obtain that the corresponding first-order system is solvable in $BUC(\mathbb{R}, E)$ instead of only in $BUC^1(\mathbb{R}, E)$ (compare with 5.4.4).

Theorem 5.5.4 *Let B be a closed operator on E, such that the first-order equation $u'(t) = Bu(t) + f(t)$ is well-posed. Assume that $A = B^2$. Then*

(i) For all $f \in BUC(\mathbb{R}, E)$ there exists a unique mild solution $(u, v) \in BUC(\mathbb{R}, E) \times BUC(\mathbb{R}, E)$ of Equation (5.15).

(ii) The operator equation

$$\mathcal{A}\mathcal{X} - \mathcal{X}D = -\tilde{\delta}_0$$

has a unique bounded solution $\mathcal{X} : BUC(\mathbb{R}, E) \to E \times E$.

Proof. (i): From Theorem 5.5.1, it follows that $(\mathrm{II})_f$ has a unique mild solution $w \in BUC^1(\mathbb{R}, E)$. Define $u := w$ and $v := w'$. Then (u, v) is a mild solution of Equation (5.15) and this solution is unique since the solution w of $(\mathrm{II})_f$ is unique.

(ii): Now let $\mathcal{G} : BUC(\mathbb{R}, E) \longrightarrow BUC^1(\mathbb{R}, E) \times BUC(\mathbb{R}, E)$, $\mathcal{G}(f) := (u_f, v_f)$ the solution operator for the first-order system. Remark that $\mathcal{G}$ is bounded by (i) and a simple application of the closed graph theorem. Define $\mathcal{X} \in \mathcal{L}(BUC(\mathbb{R}, E), E \times E)$ by

$$\mathcal{X} := (\mathcal{G}f)(0).$$

Let $f \in D(D) = BUC^1(\mathbb{R}, E)$. Then, it follows by Corollary 5.5.2 and (i) that $\mathcal{G}f \in BUC^2(\mathbb{R}, E) \times BUC^1(\mathbb{R}, E)$ and (see Proposition 5.4.3)

$$\mathcal{X}\mathcal{D}f = (Gf')(0) = (Gf)'(0) = \mathcal{A}(Gf)(0) + \begin{pmatrix} 0 \\ f(0) \end{pmatrix} = \mathcal{A}\mathcal{X}f + \tilde{\delta}_0 f.$$

Hence, $\mathcal{X}$ is a solution of the operator equation $\mathcal{A}\mathcal{X} - \mathcal{X}D = -\tilde{\delta}_0$.

The uniqueness of the operator $\mathcal{X}$ follows easily from the uniqueness of the mild solutions of Equation (5.15) (compare with the proof of Theorem 5.4.4). □

For a closed, linear operator B on a Banach space E define the closed operator $\tilde{\mathcal{B}}$ on $E \times E$ with domain $D(\tilde{\mathcal{B}}) = D(B) \times D(B)$ by

$$\tilde{\mathcal{B}} := \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix}.$$

Note, that if we have $A = B^2$, then $\mathcal{A}^2 = \tilde{\mathcal{B}}^2 = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$. Compare this also with results from Fattorini ([54] and [55]) in the case when A is the generator of cosine family and $B = -(-A + b^2 I)^{1/2}$ for a suitable $b \in \mathbb{R}$.

Theorem 5.5.5 *Let $A = B^2$, where B is a closed, linear operator on E. Assume that Equation (I) with the operator B is well-posed. Then, it follows that the operator equation*

$$\tilde{\mathcal{B}}\mathcal{Y} - \mathcal{Y}D = -\tilde{\delta}_0 \tag{5.19}$$

has a unique bounded solution $\mathcal{Y} : BUC(\mathbb{R}, E) \to E \times E$.

Proof. For $f \in BUC(\mathbb{R}, E)$, let $Nf = w_f \in BUC(\mathbb{R}, D(B)) \cap BUC^1(\mathbb{R}, E)$ the unique mild solution of (II) (Theorem 5.5.1). Define the closed operator $\mathcal{Y}$ on $BUC^1(\mathbb{R}, E)$ with values in the Banach space $E \times E$ by

$$\mathcal{Y}f := \begin{pmatrix} Bw_f(0) \\ w_f'(0) \end{pmatrix}.$$

Let $f \in D(D) = BUC^1(\mathbb{R}, E)$. Then, by Corollary 5.5.2, w_f is a classical solution of (II). Thus, since the solution operator N commutes with differentiation (see Proposition 5.2.11), we obtain

$$\tilde{\mathcal{B}}Yf - \mathcal{Y}\mathcal{D}f = \begin{pmatrix} Bw_f'(0) \\ B^2 w_f(0) \end{pmatrix} - \begin{pmatrix} Bw_{f'}(0) \\ w_{f'}'(0) \end{pmatrix} = \begin{pmatrix} 0 \\ -f(0) \end{pmatrix} = -\tilde{\delta}_0 f.$$

We obtain that $\mathcal{Y}$ is a bounded solution of the operator equation (5.19).

Now suppose that $\mathcal{Y}$ is a bounded solution of the operator equation $\tilde{\mathcal{B}}Y - YD = 0$ and define the bounded operator $\mathcal{X} : BUC(\mathbb{R}, E) \to E \times E$ by

$$\mathcal{X} := \begin{pmatrix} B^{-1} & 0 \\ 0 & I \end{pmatrix} \mathcal{Y}.$$

Note that $0 \in \rho(B)$ by Theorem 2.2.2. Then, we obtain $\mathcal{X}f \in D(\mathcal{A})$ for all $f \in D(D)$ and

$$\begin{aligned} \mathcal{A}\mathcal{X}f - \mathcal{X}Df &= \begin{pmatrix} 0 & I \\ B^2 & 0 \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & I \end{pmatrix} \mathcal{Y}f - \begin{pmatrix} B^{-1} & 0 \\ 0 & I \end{pmatrix} \mathcal{Y}Df \\ &= \begin{pmatrix} B^{-1} & 0 \\ 0 & I \end{pmatrix} \left[\tilde{\mathcal{B}}Y - YD\right] f \;=\; 0. \end{aligned}$$

It follows from Theorem 5.5.4 that $\mathcal{X} = 0$. Hence, if $\mathcal{Y} = (Y_1, Y_2)$, $B^{-1}Y_1 = 0$ and $Y_2 = 0$. Since the resolvent is injective, it follows that $\mathcal{Y} = 0$, this proves uniqueness. $\square$

If $u'(t) = Bu(t) + f(t)$ is well-posed, then $BX - XD = -\delta_0$ is uniquely solvable. Also, by Theorem 5.5.1, Equation (II) with $A = B^2$ is well-posed, and therefore $AY - YD^2 = -\delta_0$ is uniquely solvable. The following results relate these solutions X and Y to each other. We use the operator $\tau_{B,D}$ on $\mathcal{L}(BUC(\mathbb{R}, E), E)$ defined in (2.7).

Theorem 5.5.6 *Let B be a closed operator on E such that Equation (I) is well-posed. Then the operator equation $B^2Y - YD^2 = -\delta_0$ has a unique bounded solution and the solution is given by*

$$Y \;=\; \tau_{B,D}^{-1}\, X,$$

where X is the unique solution of the operator equation $BX - XD = -\delta_0$.

Proof. The existence and uniqueness of the solutions of the operator equations $BX - XD = -\delta_0$ and $B^2Y - YD^2 = -\delta_0$ follow from Theorem 2.2.5 and Theorem 5.2.12, respectively. It follows from Lemma 2.2.3 and Theorem 2.2.5 that $\tau_{B,D}$ is invertible. Since $X = BY + YD$, we obtain

$$-\delta_0 = BX - XD = B(BY + YD) - (BY + YD)D = B^2Y - YD^2.$$

Thus $Y = \tau_{B,D}X$ is the unique solution of $B^2Y - YD^2 = -\delta_0$. □

5.6 Asymptotic behaviour of mild solutions of second-order differential equations

In the last section of this chapter, we consider again the asymptotic behaviour of mild solutions (compare with Section 2.5). Here, we use once more the properties of bounded, uniformly continuous functions with discrete spectrum (see Section 2.4).

Theorem 5.6.1 *Let A be a closed, linear operator such that $\sigma(A)\cap(-\infty, 0]$ is discrete. Let $u \in BUC(\mathbb{R}, E)$ be a mild solution of Equation $(II)_f$. Assume that $f \in AP(\mathbb{R}, E)$ and that $\{\eta \in \mathbb{R} : -\eta^2 \in \sigma(A)\} \cap sp'(f) = \emptyset$. Then $u \in AP(\mathbb{R}, E)$.*

Proof. Taking Laplace transform we obtain from (II) that $\hat{u}(\lambda) = \lambda R(\lambda^2, A)u(0) + R(\lambda^2, A)y + R(\lambda^2, A)\hat{f}(\lambda)$, for $Re\,\lambda \neq 0$ and $\lambda^2 \in \rho(A)$. Hence (see Proposition 5.1.4)

$$sp(u) \subseteq \{\eta \in \mathbb{R} \,:\, -\eta^2 \in \sigma(A)\} \cup sp(f).$$

If we replace u by u_s we obtain

$$\hat{u}_s(\lambda) = \lambda R(\lambda^2, A)u(s) + R(\lambda^2, A)y + R(\lambda^2, A)\hat{f}_s(\lambda) + \varphi_s(\lambda)$$

where $\varphi_s(\lambda) := AR(\lambda^2, A)\int_0^s u(r)dr + R(\lambda^2, A)\int_0^s f(r)dr$. Again by considering the shift group $\mathcal{S}$ on $BUC(\mathbb{R}, E)$ with generator D we obtain

$$R(\lambda, D)u = \lambda R(\lambda^2, A) \circ u + R(\lambda^2, A)y + R(\lambda^2, A) \circ R(\lambda, D)f + g(\lambda),$$

where $g(\lambda)(s) := \varphi_s(\lambda)$. Now let $\eta \in \mathbb{R}$ and $r > 0$ such that $\lambda^2 \in \rho(A)$ whenever $|\lambda - i\eta| \leq 2r$ or $|\lambda + i\eta| \leq 2r$. In [8, Theorem 4.5] it is shown that the function $g : B(i\eta, r) \to BUC(\mathbb{R}, E)$ given by $\lambda \mapsto g(\lambda)$ is holomorphic. If we now consider the quotient space $BUC(\mathbb{R}, E)/AP(\mathbb{R}, E)$ with the induced shift group $\bar{\mathcal{S}}$ and its generator $\bar{D}$, the above equation becomes

$$R(\lambda, \bar{D})\bar{u} = \overline{(\lambda R(\lambda^2, A) \circ u + R(\lambda^2, A)y + g(\lambda))}$$

since $f \in AP(\mathbb{R}, E)$. Here, $^-$denotes the quotient mapping as in Section 2.4. So we see that

$$sp_{AP}(u) \subseteq \{\mu \in \mathbb{R} : -\mu^2 \in \sigma(A)\}. \tag{5.20}$$

Now suppose that $\eta \in sp_{AP}(u)$. Then by (5.20) $\eta \in \{\mu \in \mathbb{R} : -\mu^2 \in \sigma(A)\}$ and by Theorem 2.4.5 η is an accumulation point of $sp(u)$. Since $\sigma(A) \cap (-\infty, 0]$ is discrete it follows from Proposition 5.1.4 that $\eta \in sp'(f)$. Hence $\eta \in \{\mu \in \mathbb{R} : -\mu^2 \in \sigma(A)\} \cap sp'(f)$ which is a contradiction. We conclude that $sp_{AP}(u) = \emptyset$ and so $u \in AP(\mathbb{R}, E)$ by Corollary 2.4.3. □

With help of [8, Proposition 4.8] we obtain the following corollary from Theorem 5.6.1. For more theory about cosine functions see [57], [69], [115] and [118].

Corollary 5.6.2 *Let A be the generator of a bounded cosine function on a Banach space E. Assume that $\sigma(A)$ is discrete. Then the eigenvectors of A are total in E, i.e.,*

$$E = \overline{span}\{x \in D(A) : \exists \eta \in \mathbb{R} \text{ such that } Ax = -\eta^2 x\}$$

and the cosine function is almost periodic.

To characterise asymptotic behaviour different from almost periodicity, we use closed, translation-invariant subspaces of $BUC(\mathbb{R}, E)$ (compare with Section 2.5). With that, we obtain the following generalisation of Theorem 5.6.1.

Theorem 5.6.3 *Let $\mathcal{G} \subseteq BUC(\mathbb{R}, E)$ be a closed, translation-invariant subspace of $BUC(\mathbb{R}, E)$ containing $AP(\mathbb{R}, E)$, and suppose $f \in \mathcal{G}$. Assume that $\sigma(A) \cap (-\infty, 0]$ is discrete and that $\{\eta \in \mathbb{R} : -\eta^2 \in \sigma(A)\} \cap sp'(f) = \emptyset$. Let $u \in BUC(\mathbb{R}, E)$ be a mild solution of $(II)_f$. Then $u \in \mathcal{G}$.*

Proof. If we use in the proof of Theorem 5.6.1 the quotient mapping $\sim$ on the quotient space $BUC(\mathbb{R}, E)/\mathcal{G}$ instead of the quotient mapping $-$, we obtain

$$R(\lambda, \tilde{D})\tilde{u} = (\lambda R(\lambda^2, A) \circ u + R(\lambda^2, A)y + g(\lambda))^{\sim},$$

since $f \in \mathcal{G}$. Here, $\tilde{D}$ denotes the generator of the induced shift group on $BUC(\mathbb{R}, E)/\mathcal{G}$ and g is the holomorphic function defined as in the proof of Theorem 5.6.1. Hence, we obtain

$$sp_{\mathcal{G}}(u) \subseteq \{\eta \in \mathbb{R} : -\eta^2 \in \sigma(A)\}.$$

Moreover, we obtain from Proposition 5.1.4 and the fact that $AP(\mathbb{R}, E) \subseteq \mathcal{G}$ that

$$sp_{\mathcal{G}}(u) \subseteq sp_{AP}(u) \subseteq sp'(u) \subseteq sp'(f).$$

Finally, from the last two equations it follows by hypothesis that $sp_{\mathcal{G}}(u) = \emptyset$. Thus, $u \in \mathcal{G}$. $\square$

Chapter 6

Solutions of second-order differential equations in $\mathbf{L^p(\mathbb{R}, E)}$

In this chapter, we examine the second-order differential equation on the real line

$$\text{(II)} \qquad u''(t) = Au(t) + f(t) \qquad (t \in \mathbb{R}),$$

where A is a closed, linear operator on a Banach space E and $f \in L^p(\mathbb{R}, E)$. Throughout the whole chapter, let $p \in (1, \infty)$.

6.1 Mild solutions of second-order differential equations in $\mathbf{L^p(\mathbb{R}, E)}$

In this section, we prove some properties of mild solutions of Equation (II), among other things, we prove a sufficient condition for uniqueness of solutions. Moreover, we examine the case when the operator A is sectorial. First of all, we give the definition.

Definition 6.1.1 For $f \in L^p(\mathbb{R}, E)$, we call $u \in L^p(\mathbb{R}, E)$ a *mild solution* of Equation $(\mathrm{II})_f$ if $\int_0^t (t-s)u(s)ds \in D(A)$ for all $t \in \mathbb{R}$ and there exist $x, y \in E$ such that

$$u(t) = x + ty + A\int_0^t (t-s)u(s)ds + \int_0^t (t-s)f(s)ds \tag{6.1}$$

for almost all $t \in \mathbb{R}$.

We call u a *strong solution* of Equation $(\mathrm{II})_f$, if $u \in W^{2,p}(\mathbb{R}, E) \cap L^p(\mathbb{R}, D(A))$ and u satisfies Equation $(\mathrm{II})_f$ for almost all $t \in \mathbb{R}$.

Note that by integration by parts the integral equation (6.1) is equivalent to requiring $\int_0^t \int_0^s u(r)drds \in D(A)$ and

$$u(t) = x + ty + A\int_0^t \int_0^s u(r)drds + \int_0^t \int_0^s f(r)drds \tag{6.2}$$

for almost all $t \in \mathbb{R}$.

Remark that a strong solution of Equation (II) is also a mild solution of (II).

Recall that the spectrum of a function $u \in L^p(\mathbb{R}, E)$ can be defined via the Carleman transform (see Section 3.1 (3.2)).

Proposition 6.1.2 *Let A be a closed, linear operator on E with $(-\infty, 0] \in \rho(A)$ and $u \in L^p(\mathbb{R}, E)$ be a mild solution of the homogeneous Equation (II)$_0$. Then $u = 0$.*

Proof. By taking Carleman transform of Equation (6.1) with $f = 0$, we obtain

$$\hat{u}(\lambda) = \frac{1}{\lambda}x + \frac{1}{\lambda^2}(y + A\hat{u}(\lambda)) \qquad (\mathcal{R}e(\lambda) \neq 0).$$

Since $\lambda^2 \in \rho(A)$ for all $\lambda \in i\mathbb{R}$, it follows

$$\hat{u}(\lambda) = \lambda R(\lambda^2, A)x + R(\lambda^2, A)y.$$

Hence, $\hat{u}$ has a holomorphic extension in a neighbourhood of λ for all $\lambda \in i\mathbb{R}$. Thus, $sp(u) = \emptyset$. It follows that $u = 0$ (see [99, Propositon 0.5]). $\square$

Thus, if A is a closed, linear operator with $(-\infty, 0] \in \rho(A)$ then the mild solutions of Equation (II) are unique (if there are any).

Now, let us consider the solution operator of Equation (II). Let $1 < p < \infty$ and define

$$\begin{aligned} D(N_p) \;&:=\; \{f \in L^p(\mathbb{R}, E) : \exists! u_f \in L^p(\mathbb{R}, E) \text{ such that} \\ &\qquad u_f \text{ is a mild solution of Equation } (\text{II})_f\} \\ N_p f \;&:=\; u_f. \end{aligned} \tag{6.3}$$

Note that either all solutions of Equation (II) are unique or $D(N_p) = \emptyset$. Moreover, one can show, since the operator A is closed, that the operator $\tilde{N}_p : D(N_p) \to L^p(\mathbb{R}, E) \oplus E^2$ defined by $\tilde{N}_p f := (N_p f, x_f, y_f)$ where $N_p f(t) = u_f(t) = x_f + ty_f + A\int_0^t (t-s)u_f(s)ds + \int_0^t (t-s)f(s)ds$ for almost all $t \in \mathbb{R}$, is closed. Since the projection on the first coordinate p_1 is bounded, it follows from the closed graph theorem that N_p is bounded if, and only if, $D(N_p) = D(\tilde{N}_p) = L^p(\mathbb{R}, E)$.

Lemma 6.1.3 *Let $f \in D(N_p)$ and $\lambda \in \rho(A)$. Then $R(\lambda, A)f \in D(N_p)$ and $R(\lambda, A)N_pf = N_pR(\lambda, A)f$.*

Proof. Let $u \in L^p(\mathbb{R}, E)$ be the unique mild solution of Equation $(\mathrm{II})_f$. Then by Definition 6.1.1 there exist $x, y \in E$ such that

$$R(\lambda, A)u(t) = R(\lambda, A)x+tR(\lambda, A)y+A\int_0^t (t-s)R(\lambda, A)u(s)ds+\int_0^t (t-s)R(\lambda, A)f(s)ds$$

for almost all $t \in \mathbb{R}$, i.e. $R(\lambda, A)u$ is the mild solution of Equation $(\mathrm{II})_{R(\lambda,A)f}$. Thus, $R(\lambda, A)f \in D(N_p)$ and $N_pR(\lambda, A)f = R(\lambda, A)u = R(\lambda, A)N_pf$. □

Since there is no confusion possible, we denote also by $(S(t))_{t\in\mathbb{R}}$ the shift on $L^p(\mathbb{R}, E)$.

Proposition 6.1.4 *Let $f \in L^p(\mathbb{R}, E)$ and $S(h)f \in D(N_p)$ for all $h \in \mathbb{R}$. Then $N_pS(h)f = S(h)N_pf$ and $\int_0^h u(r)dr \in D(A)$ for all $h \in \mathbb{R}$.*

Proof. Let $N_pf =: u$, $N_pS(h)f =: u_h$ and $\lambda \in \rho(A)$. By the above Lemma 6.1.3 there exist x, y, x_h & $y_h \in E$ such that for almost all $t \in \mathbb{R}$

$$\begin{aligned}
&R(\lambda, A)u(t + h) - R(\lambda, A)u_h(t)\\
&= R(\lambda, A)x + (t + h)R(\lambda, A)y + A\int_0^{t+h}\int_0^s R(\lambda, A)u(r)drds\\
&\quad+\int_0^{t+h}\int_0^s R(\lambda, A)f(r)drds - \Big(R(\lambda, A)x_h + tR(\lambda, A)y_h\\
&\quad+A\int_0^t\int_0^s R(\lambda, A)u_h(r)drds + \int_0^t\int_0^s R(\lambda, A)(S(h)f)(r)drds\Big)\\
&= \tilde{x} + t\tilde{y} + A\int_0^t\int_0^s (S(h)R(\lambda, A)u - R(\lambda, A)u_h)(r)drds,
\end{aligned}$$

where $\tilde{x} = R(\lambda, A)(x-x_h+hy+A\int_0^h\int_0^s u(r)drds+\int_0^h\int_0^s f(r)drds)$ and $\tilde{y} = R(\lambda, A)(y-y_h)+A\int_0^h R(\lambda, A)u(r)dr+R(\lambda, A)\int_0^h f(r)dr$. It follows that $S(h)R(\lambda, A)u-R(\lambda, A)u_h$ is the mild solution of the homogeneous Equation $(\mathrm{II})_0$. Hence, since $R(\lambda, A)$ and $S(h)$ commute and by uniqueness of solutions for the functions $S(h)f \in D(N_p)$ it follows $R(\lambda, A)S(h)N_pf = R(\lambda, A)N_pS(h)f$. Since $R(\lambda, A)$ is injective, we obtain

$$S(h)u = S(h)N_pf = N_pS(h)f.$$

Moreover, $\int_0^t\int_0^s(S(h)u)(r)drds \in D(A)$ for all $h \in \mathbb{R}$ and from

$$\int_0^t\int_0^s (S(h)u)(r)drds = \int_0^{t+h}\int_0^s u(r)drds - \int_0^h\int_0^s u(r)drds - t\int_0^h u(r)dr,$$

it follows that $\int_0^h u(r)dr \in D(A)$ for all $h \in \mathbb{R}$. □

Finally, we consider the case of sectorial operators. Recall that if A is a sectorial operator, we can define the fractional power $A^{\frac{1}{2}}$ (see 4.3.3).

Theorem 6.1.5 *Let A be a densely defined, sectorial operator with $0 \in \rho(A)$ and let $B := -A^{\frac{1}{2}}$. Then for all $f \in L^p(\mathbb{R}, E)$ there exists a unique mild solution $u \in L^p(\mathbb{R}, E)$ of Equation (II)$_f$ which is given by*

$$u(t) = -\frac{1}{2}R(0, B)\int_{\mathbb{R}} T(|t-s|)f(s)ds \tag{6.4}$$

where $(T(t))_{t\geq 0}$ is the semigroup generated by B.

Proof. Let $f \in L^p(\mathbb{R}, E)$ and define u_f as in (6.4). It follows from Young's Inequality that $u_f \in L^p(\mathbb{R}, E)$ and $\|u_f\|_p \leq \|R(0, B)\|\|T(\cdot)\|_1\|f\|_p$. It is sufficient to show that u_f is a mild solution of (II)$_f$ since uniqueness follows from the previous Proposition 6.1.2.

First, assume that $f \in C_c^\infty(\mathbb{R}, D(A))$, i.e. f is infinitely differentiable with compact support and values in $D(A)$. Then it follows that

$$u_f'(t) = \frac{1}{2}\int_{-\infty}^t T(t-s)f(s)ds - \frac{1}{2}\int_t^\infty T(s-t)f(s)ds \tag{6.5}$$

and

$$u_f''(t) = Au_f(t) + f(t) \tag{6.6}$$

for all $t \in \mathbb{R}$. Thus, $u_f \in W^{2,p}(\mathbb{R}, E) \cap L^p(\mathbb{R}, D(A))$ is a strong solution of (II)$_f$ and hence, also a mild solution.

Now, let $f \in L^p(\mathbb{R}, E)$. Since $C_c^\infty(\mathbb{R}, D(A))$ is dense in $L^p(\mathbb{R}, E)$ there exist $f_n \in C_c^\infty(\mathbb{R}, D(A))$ converging to f. It follows by Equations (6.4) and (6.5) that the functions u_{f_n} and u_{f_n}' converge to u_f, respectively u_f', uniformly on compact intervals. By the closedness of the operator $\tilde{N}_p$ this is a solution of the integrated Equation (6.1), i.e. a mild solution of Equation (II)$_f$. □

6.2 Maximal regularity of second-order differential equations

In this section, we examine maximal Lp-regularity for the second-order differential equation on the line. First, we show that a necessary condition for the operator A to satisfy maximal L^p-regularity is that A is sectorial. Second, we show that maximal L^p-regularity is independent of $p \in (1, \infty)$. Third, we consider UMD spaces and give a sufficient condition for maximal regularity. Forth, we consider Hilbert spaces and show that there each sectorial operator satisfies maximal L^p-regularity. Last, we give examples that this is not the case in general Banach spaces.

Definition 6.2.1 We say that the operator A satisfies *maximal L^p-regularity* for Equation (II), if for all $f \in L^p(\mathbb{R}, E)$ there exists a unique strong solution u of $(\mathrm{II})_f$.

From the closed graph theorem, it follows that in this case the solution operator $N_p : L^p(\mathbb{R}, E) \longrightarrow W^{2,p}(\mathbb{R}, E) : f \mapsto u_f$ is bounded.

For this we define the weighted L^p- and Sobolev spaces on $\mathbb{R}$ with values in the Banach space E. Let $\alpha \geq 0$ and $\beta(t) := \frac{t^2}{\sqrt{1+t^2}}$ be a weight on $L^p(\mathbb{R}, E)$, then define

$$\begin{aligned} L^p_{\alpha,\beta}(\mathbb{R}, E) &:= \{f : \mathbb{R} \longrightarrow E \text{ measurable} \mid |f|_{p,\alpha,\beta} < \infty\} \\ W^{2,p}_{\alpha,\beta}(\mathbb{R}, E) &:= \{f, f', f'' \in L^p_{\alpha,\beta}\}, \end{aligned}$$

where $|f|_{p,\alpha,\beta} := \left(\int_{\mathbb{R}} \|e^{-\alpha\beta(t)} f(t)\|^p dt\right)^{\frac{1}{p}}$ is the norm in $L^p_{\alpha,\beta}(\mathbb{R}, E)$ and $|f|_{W^{2,p}_{\alpha,\beta}(\mathbb{R},E)} := |f|_{p,\alpha,\beta} + |f'|_{p,\alpha,\beta} + |f''|_{p,\alpha,\beta}$ is the norm in $W^{2,p}_{\alpha,\beta}(\mathbb{R}, E)$.

Lemma 6.2.2 *The mapping $f \mapsto \bar{f}(t) := e^{-\alpha\beta(t)} f(t)$ is an homeomorphism from $L^p_{\alpha,\beta}(\mathbb{R}, E) \longrightarrow L^p(\mathbb{R}, E)$ and from $W^{2,p}_{\alpha,\beta}(\mathbb{R}, E) \longrightarrow W^{2,p}(\mathbb{R}, E)$.*

Proof. From $\beta(t) := \frac{t^2}{\sqrt{1+t^2}}$ we can calculate $\beta'(t) = \frac{2t}{\sqrt{1+t^2}} - \frac{t^3}{(1+t^2)^{\frac{3}{2}}}$ and $\beta''(t) = \frac{2\sqrt{1+t^2}}{1+t^2} - \frac{2t^2}{(1+t^2)^{\frac{3}{2}}} - \frac{3t^2}{(1+t^2)^{\frac{3}{2}}} + \frac{3t^4\sqrt{1+t^2}}{(1+t^2)^3}$. We see, that β' and β'' are bounded on $\mathbb{R}$. From the definition

$$\bar{f}(t) = e^{-\alpha\beta(t)} f(t),$$

it follows that $\bar{f} \in L^p(\mathbb{R}, E)$ whenever $f \in L^p_{\alpha,\beta}(\mathbb{R}, E)$. For the first and second derivative of $\bar{f}$, we obtain the following

$$(\bar{f})'(t) = -\alpha\beta'(t)\bar{f}(t) + e^{-\alpha\beta(t)} f'(t) \tag{6.7}$$

and

$$(\bar{f})''(t) = (-\alpha\beta''(t) - \alpha^2\beta'(t)^2)\bar{f}(t) - 2\alpha\beta'(t)(\bar{f})'(t) + e^{-\alpha\beta(t)} f''(t). \tag{6.8}$$

Hence, $\bar{f} \in W^{2,p}(\mathbb{R}, E)$, whenever $f \in W^{2,p}_{\alpha,\beta}(\mathbb{R}, E)$. It follows that $\bar{\ }$ is well-defined. Since $e^{-\alpha\beta(t)} \neq 0$ for all $t \in \mathbb{R}$, $\bar{\ }$ is invertible. And finally, it is easy to see that $\bar{\ }$ and its inverse are continuous. $\square$

Lemma 6.2.3 *Let A be a closed linear operator that satisfies maximal L^p-regularity for Equation (II). Then there exists $\alpha > 0$ such that for all $f \in L^p_{\alpha,\beta}(\mathbb{R}, E)$ there exists a unique solution $u_f \in W^{2,p}_{\alpha,\beta}(\mathbb{R}, E)$ of Equation (II) and the mapping $N_{p,\alpha} : L^p_{\alpha,\beta}(\mathbb{R}, E) \longrightarrow W^{2,p}_{\alpha,\beta}(\mathbb{R}, E) : f \mapsto u_f$ is continuous.*

Proof. Let $f \in L^p_{\alpha,\beta}(\mathbb{R}, E)$. From (6.8), it follows, that u is a solution of Equation (II), i.e. $u'' = Au + f$, if, and only if $(\bar{u})'' = A\bar{u} + \bar{f} + (-\alpha\beta'' - \alpha^2\beta'^2)\bar{u} - 2\alpha\beta'(\bar{u})'$. Hence, we define the mapping $\bar{N}_{p,\alpha} : W^{2,p}(\mathbb{R}, E) \longrightarrow W^{2,p}(\mathbb{R}, E)$ by

$$\bar{N}_{p,\alpha}x := N_p((\beta'' + \alpha\beta'^2)x + 2\beta'x'),$$

for all $x \in W^{2,p}(\mathbb{R}, E)$, where N_p is the solution mapping of Equation (II) (see Definition (6.3)). Since β', β'' and N_p are bounded, $\bar{N}_{p,\alpha}$ is also bounded, and $\|\bar{N}_{p,\alpha}\| \leq \|N_p\|(\|\beta'' + \alpha\beta'^2\|_\infty + 2\|\beta'\|_\infty)$. Moreover, with (6.7)

$$\begin{aligned}
((1 + \alpha\bar{N}_{p,\alpha})\bar{u})'' &= (\bar{u})'' + \alpha(N_p((\beta'' + \alpha\beta'^2)\bar{u} + 2\beta'(\bar{u})'))'' \\
&= A\bar{u} + \bar{f} - \alpha((\beta'' + \alpha\beta'^2)\bar{u} + 2\beta'(\bar{u})') \\
&\quad + \alpha(AN_p((\beta'' + \alpha\beta'^2)\bar{u} + 2\beta'(\bar{u})') + (\beta'' + \alpha\beta'^2)\bar{u} + 2\beta'(\bar{u})') \\
&= A(\bar{u} + \alpha\bar{N}_{p,\alpha}\bar{u}) + \bar{f},
\end{aligned}$$

i.e. $N_p\bar{f} = (1 + \alpha\bar{N}_{p,\alpha})\bar{u}$. If now, α is small enough, we obtain that $(1 + \alpha\bar{N}_{p,\alpha})$ is invertible, and finally, it follows from the above considerations that

$$N_{p,\alpha}f = (\bar{\ })^{-1}(1 + \alpha\bar{N}_{p,\alpha})^{-1}N_p\bar{f}.$$

So, it is also easily seen that $N_{p,\alpha}$ is continuous. □

For $\xi \in \mathbb{R}$, and $y \in E$ define

$$f_s(t) := e^{i\xi(s+t)}y = f_0(t + s) = e^{i\xi s}f_0(t), \tag{6.9}$$

for all $s, t \in \mathbb{R}$. Remark, that $f_s \in L^p_{\alpha,\beta}(\mathbb{R}, E)$ for all $s \in \mathbb{R}$, since

$$|f_s|_{p,\alpha,\beta} = (\int_{\mathbb{R}} (e^{-\alpha\beta(t)})^p dt)^{\frac{1}{p}}\|y\| =: c_{p,\alpha,\beta}\|y\|. \tag{6.10}$$

Lemma 6.2.4 *Let A be a closed, linear operator that satisfies maximal L^p-regularity for Equation (II) and let α be small enough, as in Lemma 6.2.3. For $\xi \in \mathbb{R}$, and $y \in E$ define $f_s(t)$ as above. Then it follows for the unique solution $u_s = N_{p,\alpha}f_s \in W^{2,p}_{\alpha,\beta}(\mathbb{R}, E)$ of Equation (II)$_{f_s}$ that*

$$u_s(t) = u_0(s + t) = e^{i\xi s}u_0(t),$$

for all $s, t \in \mathbb{R}$.

Proof. Since u_s and u_0 are solutions of $(\mathrm{II})_{f_s}$ and $(\mathrm{II})_{f_0}$, respectively, it follows that

$$\begin{aligned} u_s''(t) - u_0''(s+t) &= Au_s(t) + f_s(t) - (Au_0(s+t) + f_0(s+t)) \\ &= A(u_s(t) - u_0(s+t)) \end{aligned}$$

for $s, t \in \mathbb{R}$. Hence, $u_s(.) - u_0(s + .)$ is a solution of the homogeneous equation. From uniqueness of solutions, it follows that $u_s(t) = u_0(s+t)$ for all $s, t \in \mathbb{R}$. Moreover,

$$u_s''(t) = Au_s(t) + f_s(t) = Au_s(t) + e^{i\xi s} f_0(t)$$

for $s, t \in \mathbb{R}$. We obtain

$$e^{-i\xi s} u_s''(t) = A(e^{-i\xi s} u_s(t)) + f_0(t).$$

Hence, $e^{-i\xi s} u_s(t)$ is a solution of $(\mathrm{II})_{f_0}$. Again from uniqueness of solutions, it follows that $u_s(t) = e^{i\xi s} u_0(t)$. □

Now, we are able to prove the first theorem of this section.

Theorem 6.2.5 *Let A be a closed linear operator on a Banach space E that satisfies maximal L^p-regularity for Equation (II) for some $p \in (1, \infty)$. Then A is sectorial with $0 \in \rho(A)$.*

Proof. Let $\xi \in \mathbb{R}$ and $y \in E$ be arbitrary and let α be small enough as in Lemma 6.2.3. Then there exists for $f_s \in L^p_{\alpha,\beta}(\mathbb{R}, E)$ (defined by (6.9)) a unique solution $u_s \in W^{2,p}_{\alpha,\beta}(\mathbb{R}, E)$ of $(\mathrm{II})_{f_s}$. By Lemma 6.2.4, it follows that $u_0(t) = e^{i\xi t} z$ with $z := u_0(0) \in E$. We obtain that

$$-\xi^2 e^{i\xi t} z = A e^{i\xi t} z + e^{i\xi t} y.$$

Hence, $(-\xi^2 - A)z = y$ and, since $y \in E$ was arbitrary, it follows that $(-\xi^2 - A)$ is surjective for all $\xi \in \mathbb{R}$.

To prove injectivity, assume that $Az = -\xi^2 z$. Define $u(t) := e^{i\xi t} z$ for all $t \in \mathbb{R}$. Then $u \in W^{2,p}_{\alpha,\beta}(\mathbb{R}, E)$ and , since

$$u''(t) = -\xi^2 e^{i\xi t} z = Au(t),$$

u is a solution of the homogeneous equation. From uniqueness of solutions (Lemma 6.2.3), it follows that $z = 0$, and hence, $(-\xi^2 - A)$ is injective.

Moreover, we obtain

$$\begin{aligned} |u_s|_{p,\alpha,\beta} &= c_{p,\alpha,\beta}\, \|z\|, \\ |u_s'|_{p,\alpha,\beta} &= |\xi|\, c_{p,\alpha,\beta}\, \|z\| \quad \text{and} \\ |u_s''|_{p,\alpha,\beta} &= \xi^2\, c_{p,\alpha,\beta}\, \|z\|, \end{aligned}$$

where the constant $c_{p,\alpha,\beta}$ is given in (6.10). It follows that

$$(1+|\xi|+\xi^2)\, c_{p,\alpha,\beta}\, \|z\| \;=\; \|u_0\|_{W^{2,p}_{\alpha,\beta}} \;\leq\; \|N_{p,\alpha}\|\; |f_0|_{p,\alpha,\beta} \;=\; \|N_{p,\alpha}\|\; c_{p,\alpha,\beta}\, \|y\|.$$

Hence, $-\xi^2 \in \rho(A)$ for all $\xi \in \mathbb{R}$, especially $0 \in \rho(A)$, and there exists a constant C, such that

$$\|R(-\xi^2, A)\| \leq \frac{C}{1+\xi^2}.$$

From this it follows easily, that A is sectorial with a spectral angle smaller than π. $\square$

In the following, we suppose that A is a densely defined, sectorial operator with $0 \in \rho(A)$. Moreover, we let $p \in (1, \infty)$. Recall that in this case, we can define the operator $B = A^{\frac{1}{2}}$ such that $B^2 = A$ and B generates an analytic C_0-semigroup $(T(t))_{t\geq 0}$ on E.

Remark 6.2.6 According to Theorem 6.1.5, a densely defined, sectorial operator A satisfies maximal L^p-regularity if each mild solution (which exists by Theorem 6.1.5) is already in $W^{2,p}(\mathbb{R}, E)$.

Moreover, we obtain from (6.5) in the proof of Theorem 6.1.5 that $u \in W^{1,p}(\mathbb{R}, E)$ for each mild solution u. If $f \in L^p(\mathbb{R}, D(A))$ then it follows from (6.5) and (6.6) that u is a strong solution and that $Au \in L^p(\mathbb{R}, E)$. Thus, we can define the operator $K : L^p(\mathbb{R}, D(A)) \longrightarrow L^p(\mathbb{R}, E)$ by

$$Kf(t) := Au(t) = -\frac{1}{2}\int_{\mathbb{R}} T(|t-s|)Bf(s)ds, \tag{6.11}$$

where $B = A^{\frac{1}{2}}$ (see Proposition 4.3.3). Remark that, since A is a densely defined, linear operator, $L^p(\mathbb{R}, D(A))$ is dense in $L^p(\mathbb{R}, E)$.

Lemma 6.2.7 *Let A and K be as above, then the following are equivalent:*

(i) The operator K has a continuous extension $K : L^p(\mathbb{R}, E) \longrightarrow L^p(\mathbb{R}, E)$.

(ii) The operator A satisfies maximal L^p-regularity for Equation (II).

Proof. $(i) \Rightarrow (ii)$: Assume that $K : L^p(\mathbb{R}, E) \longrightarrow L^p(\mathbb{R}, E)$ is bounded, then

$$u''(t) = Au(t) + f(t) = Kf(t) + f(t)$$

for almost all $t \in \mathbb{R}$. It follows that $\|u''\|_p \leq (\|K\| + 1)\|f\|_p$ and that $u'' \in L^p(\mathbb{R}, E)$. Further, from (6.5), we obtain that $\|u'\|_p \leq \|T(\cdot)\|_1 \|f\|_p$ and that $u' \in L^p(\mathbb{R}, E)$. Thus, A satisfies maximal regularity for Equation (II).

$(ii) \Rightarrow (i)$: Let $f \in L^p(\mathbb{R}, E)$, then since the operator A satisfies maximal $\mathrm{L^p}$-regularity for Equation (II), it follows that

$$\|Kf\|_p = \|Au\|_p = \|u'' - f\|_p \leq (\|N_p\| + 1)\|f\|_p,$$

where N_p is the solution operator of Equation (II) (see Definition 6.2.1). Hence, $K : L^p(\mathbb{R}, E) \longrightarrow L^p(\mathbb{R}, E)$ is bounded. □

In the next theorem, we show that maximal regularity for Equation (II) is independent of $p \in (1, \infty)$. For this, we will use a result of Rubio de Francia, Ruiz and Torrea [105] which is based on an article of Benedek, Calderon and Panzone [27].

Theorem 6.2.8 *Let A be a densely defined, linear operator on a Banach space E. Assume that A satisfies maximal L^p-regularity for Equation (II) for one $p \in (1, \infty)$. Then A satisfies maximal L^q-regularity for Equation (II) for all $q \in (1, \infty)$.*

Proof. Let A satisfy maximal $\mathrm{L^p}$-regularity for Equation (II) for one $p \in (1, \infty)$, then, by Theorem 6.2.5, A is sectorial with $0 \in \rho(A)$ and we can define the mapping K as above. From (6.11) it follows that K is a convolution operator with kernel

$$k(t) := \frac{1}{2} BT(|t|). \tag{6.12}$$

Since the operator B is the generator of the analytic semigroup $(T(t))_{t \geq 0}$ we obtain the following estimate (see for example [95, p. 70])

$$\|\frac{d}{dt} BT(|t|)\| \quad \leq \quad \frac{c}{t^2}$$

for all $t \in \mathbb{R}$. Thus, k satisfies the Hörmander Condition, i.e.

$$\begin{aligned}
&\int_{|t|>2|s|} \|k(t-s) - k(t)\| dt \\
&= \frac{1}{2}\left(\int_{t>2|s|} \|BT(t-s) - BT(t)\| dt + \int_{t<-2|s|} \|BT(s-t) - BT(-t)\| dt\right) \\
&= \frac{1}{2}\left(\int_{t>2|s|} \|\int_t^{t-s} \frac{d}{du} BT(u) du\| dt + \int_{t<-2|s|} \|\int_{-t}^{s-t} \frac{d}{du} BT(u) du\| dt\right) \\
&\leq \frac{1}{2}\left(\int_{t>2|s|} |s| \frac{c}{(t-|s|)^2} dt + \int_{t<-2|s|} |s| \frac{c}{(|s|-t)^2} dt\right) \\
&\leq c,
\end{aligned}$$

for a constant c not depending on s. Finally, the mapping $K : L^p(\mathbb{R}, E) \longrightarrow L^p(\mathbb{R}, E)$ is bounded by Lemma 6.2.7. Hence, K is a *singular integral operator* in the sense of [105, Definition 1.2] and it follows that K can be extended to an operator on $L^q(\mathbb{R}, E)$ for all $q \in (1, \infty)$ and

$$\|Kf\|_q \leq c_q \|f\|_q$$

for a constant c_q not depending on f (see [105, Theorem 1.3]). Thus $K : L^q(\mathbb{R}, E) \longrightarrow L^q(\mathbb{R}, E)$ is bounded and hence, A satisfies maximal L^q-regularity for Equation (II) for all $q \in (1, \infty)$. □

Thus, we can say the operator A satisfies *maximal regularity* for Equation (II) if A satisfies maximal L^p-regularity for one, and hence for all $p \in (1, \infty)$.

Remark 6.2.9 In a recent article from Clément and Guerre-Delabrière it is shown that on a UMD space a sectorial operator B with spectral angle $\omega_B < \frac{\pi}{2}$ satisfies maximal regularity for the first-order differential equation if and only if $A := B^2$ satisfies maximal regularity for the second-order differential equation (see [38, Theorem 2.1]). From this, one can deduce the p-independence for the second-order equation on UMD spaces. Note that in the above Theorem 6.2.8, we have not used any geometric condition on the Banach space E.

In the remaining part, we concentrate again on UMD spaces (compare with Section 3.2). We give a sufficient condition for a sectorial operator defined on a UMD space to satisfy maximal regularity for Equation (II). Therefore, recall the definition of $\mathcal{R}$-bounded families of bounded linear operators (see Definition 3.2.7) and the operator-valued version of Mikhlin's theorem due to Weis (Theorem 3.2.8).

Theorem 6.2.10 *Let E be a UMD space and A be a sectorial operator such that $\{sR(s, A) : s < 0\}$ is $\mathcal{R}$-bounded. Then A satisfies maximal regularity for Equation (II).*

Proof. Let $f \in \mathcal{F}^{-1}\mathcal{D}(\mathbb{R}, E)$ and take Fourier transforms of Equation (II). Thus,

$$-s^2 \mathcal{F}u(s) = A\mathcal{F}u(s) + \mathcal{F}f(s),$$

or equivalently

$$\mathcal{F}u(s) = R(-s^2, A)\mathcal{F}f(s).$$

It follows that $M(s) := R(-s^2, A) \in C^1(\mathbb{R}, \mathcal{L}(E))$ is the multiplier function of the pseudo-differential operator $f \mapsto u := M(D)f$. Since the solution operator is closed, it follows that $u := M(D)f \in L^p(\mathbb{R}, E)$ is for each $f \in L^p(\mathbb{R}, E)$ the unique (by Proposition 6.1.2) mild solution of Equation (II). Since by hypothesis, the multiplier functions $s \mapsto AR(-s^2, A)$ and $s \mapsto -s^2R(-s^2, A)$ satisfy the conditions of Theorem 3.2.8, it follows that $u = M(D)f \in L^p(\mathbb{R}, D(A)) \cap W^{2,p}(\mathbb{R}, E)$, thus u is a strong solution of Equation (II). This is equal to saying that the operator A satisfies maximal regularity for Equation (II). □

The following Corollary follows from the above Theorem 6.2.10 since bounded and $\mathcal{R}$-bounded sets correspond on Hilbert spaces. But this result can also proved directly by using the Fourier-Plancherel transformation on $L^2(\mathbb{R}, H)$ for a Hilbert space H. For completion, we will also give this proof.

Corollary 6.2.11 *Let A be a sectorial operator on a Hilbert space H and let $0 \in \rho(A)$. Then A satisfies maximal regularity for Equation (II).*

Proof. The Fourier-Plancherel transformation $\mathcal{F} : L^2(\mathbb{R}, H) \longrightarrow L^2(\mathbb{R}, H)$ is an isomorphism that maps $W^{2,2}(\mathbb{R}, H)$ onto the space $\{g \in L^2(\mathbb{R}, H) \mid \lambda \mapsto \lambda^2 g(\lambda) \in L^2(\mathbb{R}, H)\}$ and $\mathcal{F}u''(\lambda) = -\lambda^2\mathcal{F}u(\lambda)$ $(\lambda \in \mathbb{R})$ for all $u \in W^{2,2}(\mathbb{R}, H)$.

Now, let $f \in L^2(\mathbb{R}, H)$, and define

$$\mathcal{F}u(\lambda) := R(-\lambda^2, A)\mathcal{F}f(\lambda) \qquad (\lambda \in \mathbb{R}).$$

Since A is sectorial with $0 \in \rho(A)$, $\mathcal{F}u$ is well-defined and $u \in W^{2,2}(\mathbb{R}, H)$. Moreover,

$$-\lambda^2\mathcal{F}u(\lambda) = A\mathcal{F}u(\lambda) + \mathcal{F}f(\lambda),$$

and hence, $u''(t) = Au(t) + f(t)$ $(t \in \mathbb{R})$. It follows that u is the unique strong solution of Equation (II). Thus, A satisfies maximal L^2-regularity.

Now it follows by the p-independence of maximal regularity (see Theorem 6.2.8) that A satisfies maximal L^p-regularity for all $p \in (1, \infty)$. □

It follows from recent results from Clément and Guerre-Delabrière [38] and Kalton and Lancien [67] that, in general, for sectorial operators maximal regularity of Equation (II) is not satisfied, even not on the spaces $L^p(0, 1)$ with $p \in (1, \infty)$ and $p \neq 2$. Recall, that $L^p(0, 1)$ are UMD spaces for $p \in (1, \infty)$ and that the Haar system is an unconditional basis (see [77, II:2.c] and [94]).

Theorem 6.2.12 *Let E be a UMD Banach space with an unconditional basis which is not isomorphic to a Hilbert space. Then there exists a sectorial operator A on E with $0 \in \rho(A)$ such that A does not satisfy maximal regularity for Equation (II).*

Proof. Since E has an unconditional bases, it follows from [67, Theorem 3.4 and Final Remarks] that there exists a generator B of an analytic semigroup on E with $0 \in \rho(B)$ such that B does not satisfy maximal L^{p}-regularity for the first-order differential equation

$$u'(t) = Bu(t) + f(t) \qquad (t \in \mathbb{R}). \tag{6.13}$$

Now, set $A := B^2$. It is easy to see, that the operator A is sectorial with $0 \in \rho(A)$. But, since E is a UMD space, it follows from [38, Theorem 2.1] that A does not satisfy maximal L^{p}-regularity for the second-order Equation (II). □

Using [67, Theorem 3.7] in the above proof instead of [67, Theorem 3.4], we obtain the following theorem.

Theorem 6.2.13 *Let X be a UMD space which is not isomorphic to a Hilbert space. Then there exists a sectorial operator A on $E := L^2((0,1),X)$ with $0 \in \rho(A)$ such that A does not satisfy maximal regularity for Equation (II).*

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Zusammenfassung

Gegenstand dieser Dissertation sind Differentialgleichungen erster Ordnung (Teil I) und zweiter Ordnung (Teil II), bei denen keine Randwert- oder Anfangswertbedingungen gegeben sind. Untersucht werden dabei die Lösungen der Gleichungen

$$\text{(I)} \qquad u'(t) = Au(t) + f(t), \quad t \in \mathbb{R},$$

bzw.

$$\text{(II)} \qquad u''(t) = Au(t) + f(t), \quad t \in \mathbb{R}.$$

Dabei ist A ein abgeschlossener, linearer Operator auf einem Banachraum E und $u, f : \mathbb{R} \longrightarrow E$ sind Banachraum-wertige Funktionen.

Bei einer genaueren Untersuchung der Gleichungen stellt sich heraus, dass im Falle der Gleichung (I) bisektorielle und im Falle der Gleichung (II) sektorielle Operatoren eine besondere Rolle spielen. Zu Beginn eines jeden Teilabschnittes, d.h. in den Kapiteln 1 und 4, untersuchen wir diese Klassen von Operatoren. Wir zeigen, wie bisektorielle Operatoren auf natürliche Weise mit analytischen Halbgruppen verbunden sind. Mit Hilfe des Funktionalkalküls definieren wir Spektralprojektionen, welche im allgemeinen nicht beschränkt sind. Desweiteren werden im beschränkten wie auch im unbeschränkten Fall die entsprechenden Spektralzerlegungen durchleuchtet. Bei den sektoriellen Operatoren werden im Rahmen dieser Arbeit lediglich die für die weiteren Betrachtungen wichtigen Ergebnisse hinsichtlich des Funktionalkalküls und gebrochener Potenzen aufgeführt.

Danach untersuchen wir in Kapitel 2 für die Gleichung erster Ordnung, bzw. in Kapitel 5 für die Gleichung zweiter Ordnung den Fall, bei dem die Inhomogenität f sowie die Lösung u gleichmäßig stetige, beschränkte Funktionen auf $\mathbb{R}$ mit Werten in E sind. Wir charakterisieren die Wohlgestelltheit der jeweiligen Gleichung mit Hilfe von verschiedenen Operatorgleichungen und wenden die Ergebnisse auf verschiedenste Beispiele, insbesondere auch auf bisektorielle, bzw. sektorielle Operatoren, an. Außerdem werden wir sehen, dass die Diskretheit des Spektrums bei der Betrachtung des asymptotischen Verhaltens der Lösungen eine wesentliche Rolle spielt.

In den Kapiteln 3 und 6 betrachten wir die Gleichungen (I), bzw. (II), wobei nun f und u Funktionen aus $L^p(\mathbb{R}, E)$ sind. Hier existieren für bisektorielle, bzw. sektorielle Operatoren eindeutige milde Lösungen. Für die maximale Regularität der jeweiligen Gleichung auf $L^p(\mathbb{R}, E)$ geben wir eine notwendige Bedingung an und zeigen p-Unabhängigkeit auf beliebigen Banachräumen. Außerdem wird eine hinreichende Bedingung für maximale Regularität auf UMD-Räumen angegeben.

Erklärung:

Hiermit erkläre ich, dass ich die Arbeit selbständig und nur mit den angegebenen Hilfsmitteln angefertigt habe. Alle Stellen, die anderen Werken entnommen sind, wurden durch Angabe der Quellen kenntlich gemacht.

Dettingen, den 5. Juli 2000

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