Navier-Stokes Seminar: Caffarelli-Kohn-Nirenberg Theory

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Preface

These are lecture notes geberated by the seminar course on the Caffarelli-Kohn-Nirenberg Theory for the Navier-Stokes equations at the Universität Ulm in the summer term of 2019. We mainly follow the [**CKN82**] in a modern fashion. This work is aimed at enthusiastic Masters and PhD students.

I would like to thank everyone taking the seminar for typing parts of these notes.

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Contents

CHAPTER 1

Talk 9: Estimating the Singular Set and Estimates for u and p in Weighted Norms

(Dennis Gallenmüller)

This talk splits into two independent sections. On the one hand, we prove the main theorem subject to this seminar, namely Theorem B in [**CKN82**], which corresponds to section 6 in the paper. On the other hand, this talk will prepare the proof of Theorems C and D in [**CKN82**], which corresponds to section 7 of the paper. For this we provide two lemmas concerning estimates of the velocity field u and the pressure p of a suitable weak solution in some specific weighted norms.

1.1. Estimating the Singular Set

1.1.1. Completing the Proof of the Main Theorem. For convenience we recall the main theorem and Proposition 2 from [**CKN82**].

THEOREM 1.1 (Caffarelli, Kohn, Nirenberg (Theorem B)). *For any suitable weak solution of the Navier-Stokes system on an open set in space-time, the associated singular set satisfies* $\mathcal{P}^1(S) = 0$.

PROPOSITION 1.2 (Caffarelli, Kohn, Nirenberg (Proposition 2)). *There is an absolute constant* $\varepsilon_3 > 0$ *with the following property. If* (u,p) *is a suitable weak solution of the Navier-Stokes system near* (x,t) *and if* $\limsup\limits_{r\to 0} \frac{1}{r}\int_{Q_r^*(x,t)} |\nabla u|^2 \le \varepsilon_3$, *then* (x,t) *is a regular point.*

The idea of the proof of Theorem 1.1 is to use Proposition 1.2 (cf. Proposition 1.6 in talk 1) and a variant of Vitali's covering lemma for parabolic cylinders (see Lemma 1.3) to estimate the one-dimensional parabolic Hausdorff measure of S.
First, let us state and prove this variant of Vitali's covering lemma for parabolic cylinders. The classical Vitali lemma considers balls, but cylinders are more convenient for our discussion due to the structure of the Navier-Stokes equations.

LEMMA 1.3. *Let* $\mathcal{C} = \left(Q^*_{r_i}(x_i,t_i)\right)_{i\in I}$ *be any collection of parabolic cylinders contained in a bounded subset of* $\mathbb{R}^3\times\mathbb{R}$. *Then there exists a finite or countable subcollection* $\mathcal{C}' = \left(Q^*_{r_{i_j}}(x_{i_j},t_{i_j})\right)_{j\in I'}$, *i.e.* $I' \subset I$, *which is disjoint and has the property that for all* $Q^* \in \mathcal{C}$ *there is a* $j \in I'$ *such that* $Q^* \subset Q^*_{5r_{i_j}}(x_{i_j},t_{i_j})$.

REMARK 1.4. As in the other talks we use the notation

$$Q_r^*(x,t) \coloneqq \left\{(y,\tau)\ :\ |y-x|<r,\ t-\frac{7}{8}r^2<\tau<t+\frac{1}{8}r^2\right\}.$$

PROOF. Set $\mathcal{C}_0 \coloneqq \mathcal{C}$. Moreover, since $\left(Q^*_{r_i}\right)$ is contained in a bounded subset, we have $\sup\limits_{i\in I} r_i < \infty$. Hence, we can choose $Q_1^* \subset \mathcal{C}_0$ such that $\frac{3}{2}r_{Q_1^*} \ge \sup\limits_{i\in I} r_i$. Note that this is possible, since by the definition of the supremum and $\frac{1}{3}\sup\limits_{i\in I} r_i > 0$ we find some r such that

$r \geq \sup_{i\in I} r_i - \frac{1}{3}\sup_{i\in I} r_i = \frac{2}{3}\sup_{i\in I} r_i$.
Let us choose a countable subcollection of $\mathcal{C}$ inductively as follows:
Assume for $n \in \mathbb{N}$ we have already chosen $\left(Q_k^*\right)_{k=1}^n$. Then set

$$\mathcal{C}_n := \{Q^* \in \mathcal{C} \ : \ Q^* \cap Q_k^* = \varnothing, \ k = 1, ..., n\}.$$

Moreover, as long as $\mathcal{C}_n \neq \varnothing$ choose $Q_{n+1}^* \in \mathcal{C}_n$ such that $\sup_{Q^*\in\mathcal{C}_n} r_{Q^*} \leq \frac{3}{2} r_{Q_{n+1}^*}$. This is again possible by the definition of the supremum as above.
Thus, the subcollection $\mathcal{C}' := \left(Q_k^*\right)_k$ is disjoint and countable or finite by construction. The latter case is considered if $\mathcal{C}_n = \varnothing$ for some $n \in \mathbb{N}$.
Now, we claim that given a $Q^* \in \mathcal{C}\backslash\mathcal{C}'$ there exists a $n \in \mathbb{N}_0$ such that $Q^* \in \mathcal{C}_n$ but $Q^* \notin \mathcal{C}_{n+1}$. In the case that $\mathcal{C}'$ is finite this is obvious, since $\mathcal{C}_{n+1} = \varnothing$ for some $n \in \mathbb{N}_0$. In the case that $\mathcal{C}'$ is countably infinite, the pairwise distjointness of $\mathcal{C}'$ and the fact that $\mathcal{C}$ is contained in a bounded set imply that $r_{Q_n^*}$ tend to zero as $n \to \infty$. Now, given a $Q^* \in \mathcal{C}\backslash\mathcal{C}'$ by the same reasoning as just mentioned there are only finitely many pairwise disjoint cylinders $\tilde{Q}^* \in \mathcal{C}$ such that $\frac{3}{2} r_{\tilde{Q}^*} \geq r_{Q^*}$. Assume now that Q^* would not be deleted by intersecting one of these $\tilde{Q}^*$. Then eventually after finitely many, say $n \in \mathbb{N}$ many, selection processes holds

$$r_{Q^*} > \frac{3}{2} r_{Q'^*}$$

for all $Q'^* \in \mathcal{C}_n$. As Q^* has not yet been deleted, we have $Q^* \in \mathcal{C}_n$. Therefore, we have to make the selection $Q^* = Q_{n+1}^*$ contradicting the fact that $Q^* \notin \mathcal{C}'$. Thus, Q^* has to be deleted after finitely many steps yielding the claim.
The claim implies by definition of the selection process, that for every $Q^* \in \mathcal{C}\backslash\mathcal{C}'$ there is a $n \in \mathbb{N}_0$ such that $Q^* \cap Q_{n+1}^* \neq \varnothing$ and $r_{Q^*} \leq \frac{3}{2} r_{Q_{n+1}^*}$.
Let us write $r_{n+1} := r_{Q_{n+1}^*}$. Therefore, the diameter of Q^* in space direction is at most $3r_{n+1}$ and in time direction at most $\left(\frac{3}{2} r_{n+1}\right)^2$. Hence, the maximal distance of a point $(x,t) \in Q^*$ to the parabolic center of Q_{n+1}^* in space is $4r_{n+1}$. In time direction the maximal distance of (x,t) to the parabolic center of Q_{n+1}^* has to be considered for forewards and backwards direction seperately, since the definition of the Q^* involves different scaling forewards and backwards in time. To be precise, the maximal distance backwards in time is

$$\frac{7}{8} r_{n+1}^2 + \frac{9}{4} r_{n+1}^2 \overset{!}{\leq} \frac{7}{8} (a r_{n+1})^2. \tag{1.1}$$

Here, we introduced some $a \in \mathbb{R}$ to be chosen such that the cylinder $Q_{ar_{n+1}}^*$ contains Q^*. From (1.1) it follows that $a \geq \sqrt{\frac{25}{7}}$, where the latter is less than 2.
For the forewards time direction we have to ensure that

$$\frac{1}{8} r_{n+1}^2 + \frac{9}{4} r_{n+1}^2 \leq \frac{1}{8} (a r_{n+1})^2.$$

Thus, $a \geq \sqrt{19}$, which is less than 5. All in all, we showed that $Q^* \subset Q_{5r_{n+1}}^*$ for all $Q^* \in \mathcal{C}$ as the latter is obviously true for $Q^* \in \mathcal{C}'$, as $5 > 1$. □

We have collected all tools needed to prove Theorem 1.1.

Proof. Let (u,p) be a suitable weak solution of the Navier-Stokes system. It suffices to assume that (u,p) is only defined on an open bounded subset of $\mathbb{R}^3 \times \mathbb{R}$. Indeed, let $(D_i)_{i=0}^\infty$ be a countable open covering of the potentially unbounded domain of definition of (u,p). Then (u,p) is also a suitable weak solution on $\mathcal{S} \cap D_i$ for all i by restricting the set of testfunctions to those with support in D_i. Assume we have already shown the theorem for bounded domains, then $\mathcal{P}^1(\mathcal{S} \cap D_i) = 0$ for all i. Let $\delta > 0$ and $\varepsilon > 0$. Now, for all i choose a countable collection of parabolic cylinders $\left(Q_{r_j^i}^*\left(x_j^i, t_j^i\right)\right)$ covering $\mathcal{S} \cap D_i$ with

$r^i_j < \delta$ such that $\sum_j r^i_j < \mathcal{P}^1_\delta(\mathcal{S}\cap D_i) + \frac{\varepsilon}{2^i}$. Then $\bigcup_i\bigcup_j Q^*_{r^i_j}\left(x^i_j, t^i_j\right) = \bigcup_{i,j} Q^*_{r^i_j}\left(x^i_j, t^i_j\right)$ is a countable collection covering $\mathcal{S}$. Thus,

$$\mathcal{P}^1_\delta(\mathcal{S}) \le \sum_{i,j} r^i_j = \sum_i\sum_j r^i_j \le \sum_i \left(\mathcal{P}^1_\delta(\mathcal{S}\cap D_i) + \frac{\varepsilon}{2^i}\right) = \sum_i \mathcal{P}^1_\delta(\mathcal{S}\cap D_i) + \varepsilon.$$

We can let $\varepsilon \to 0$ to infer the countable subadditivity of $\mathcal{P}^1_\delta$ for all $\delta > 0$. By definition of $\mathcal{P}^1_\delta$ and the infimum, we know that $\mathcal{P}^1_{\delta'}(\mathcal{S}\cap D^i) \le \mathcal{P}^1_\delta(\mathcal{S}\cap D^i)$ for $\delta \le \delta'$. Thus, by monotone convergence

$$\mathcal{P}^1(\mathcal{S}) = \lim_{\delta\to 0}\mathcal{P}^1_\delta(\mathcal{S}) \le \lim_{\delta\to 0}\sum_i \mathcal{P}^1_\delta(\mathcal{S}\cap D_i) = \sum_i \lim_{\delta\to 0}\mathcal{P}^1_\delta(\mathcal{S}\cap D_i) = \sum_i \mathcal{P}^1(\mathcal{S}\cap D_i) = \sum_i 0 = 0.$$

Therefore, we assume (u,p) to be defined on the open bounded set $D \subset \mathbb{R}^3\times\mathbb{R}$.
By Proposition 1.2 there is a constant $\varepsilon_3 > 0$ such that for all $(x,t)\in\mathcal{S}$ holds

$$\limsup_{r\to 0}\frac{1}{r}\int_{Q^*_r(x,t)} |\nabla u|^2 > \varepsilon_3. \tag{1.2}$$

Now, let $\delta > 0$ and $V \subset D$ be a neighborhood of $\mathcal{S}$. By the strictness of the inequality in (1.2) for every $(x,t)\in\mathcal{S}$ we can choose a parabolic cylinder $Q^*_r(x,t)$ with $0 < r < \delta$ such that

$$\frac{1}{r}\int_{Q^*_r(x,t)} |\nabla u|^2 > \varepsilon_3 \tag{1.3}$$

and $Q^*_r(x,t)\subset V$.
Now the covering Lemma for parabolic cylinders (Lemma 1.3) yields a disjoint countable subcollection $\left(Q^*_{r_i}(x_i,t_i)\right)_i$ such that

$$\mathcal{S} \subset \bigcup_{(x,t)\in\mathcal{S}} Q^*_r(x,t) \subset \bigcup_i Q^*_{5r_i}(x_i,t_i).$$

Moreover, since $Q^*_{r_i}\subset V$ are disjoint and by (1.3) we obtain

$$\sum_i r_i \le \sum_i \frac{1}{\varepsilon_3}\int_{Q^*_{r_i}(x_i,t_i)} |\nabla u|^2 \le \frac{1}{\varepsilon_3}\int_V |\nabla u|^2, \tag{1.4}$$

where the right hand side is independend of the choice of r_i and hence independent of δ.
Therefore, we estimate the Lebesgue measure of the singular set by using $r < \delta$

$$|\mathcal{S}| \le \left|\bigcup_i Q^*_{5r_i}(x_i,t_i)\right| \le \sum_i \left|Q^*_{5r_i}(x_i,t_i)\right| = \sum_i C r_i^5 \le C\delta^4\sum_i r_i \le C\frac{1}{\varepsilon_3}\int_V |\nabla u|^2\cdot\delta^4,$$

where in the last inequality we used (1.4). Since $\delta > 0$ was arbitrary, we conclude that $|\mathcal{S}| = 0$.
Also (1.4) and $r_i < \delta$ imply that

$$\mathcal{P}^1_\delta(\mathcal{S}) \le \sum_i 5r_i \le \frac{5}{\varepsilon_3}\int_V |\nabla u|^2$$

for all $\delta > 0$, hence also the limit for $\delta\to 0$, i.e. the parabolic Hausdorff measure $\mathcal{P}^1(\mathcal{S})$, is less or equal than $\frac{5}{\varepsilon_3}\int_V |\nabla u|^2$.
Still the neighborhood V is arbitrary. Thus, we can choose a sequence V_n for example by

$$V_n := D \cap \left\{(y,s) : \operatorname{dist}(\mathcal{S},(y,s)) < \frac{1}{n}\right\}.$$

We need to show that the indicator function of V_n tends pointwise almost everywhere in D to zero as $n\to\infty$. Indeed, $\mathcal{S}$ is closed, since its complement $\mathcal{R}$, defined as

$$\mathcal{R} := \{(x,t) : \exists \text{ neighborhood } U \text{ of } (x,t) \text{ such that } u \in L^\infty(U)\},$$

is obviously an open set, because every point in $\mathcal{R}$ has an open neighborhood that contains again only points in $\mathcal{R}$ by definition. Now, let $(x,t) \in \mathcal{R}$. Then, there is some ball $B_\varepsilon(x,t)$ around this point lying in $\mathcal{R}$. Hence, for all $n \in \mathbb{N}$ large enough such that $\frac{1}{n} < \frac{\varepsilon}{2}$ we achieve $(x,t) \notin V_n$, because else $B_\varepsilon(x,t) \cap \mathcal{S} \neq \emptyset$ would be a contradiction. Since, $\mathcal{R}^c = \mathcal{S}$ is a Lebesgue null set, we infer that the indicator of V_n tends to zero as $n \to \infty$ almost everywhere.
We also have $\int_{V_n} |\nabla u|^2 \leq \int_D |\nabla u|^2 < \infty$. So, dominated convergence implies that

$$\mathcal{P}^1(\mathcal{S}) \leq \frac{5}{\varepsilon_3} \int_{V_n} |\nabla u|^2 \overset{n\to\infty}{\to} 0,$$

finishing the proof. □

1.1.2. Some Corollaries. In general, Theorem 1.1 is not strong enough to imply the uniqueness or strong time-continuity for suitable weak solutions, since still $\mathcal{S}$ could be non-empty. On the other hand, there are interesting direct consequences, some of them listed in the following.

COROLLARY 1.5. *On $\mathbb{T}^3$ holds $\mathcal{H}^{\frac{1}{2}}(\mathcal{T}) = 0$, where $\mathcal{T}$ denotes the set of positive singular times.*

PROOF. From Lemma 16.3 in **[RRS16]** we know that on $\mathbb{T}^3$ holds $\mathcal{T} = \mathrm{pr}_t(\mathcal{S})$, where the latter denotes the projection of $\mathcal{S}$ onto the time coordinate. Thus, it is sufficient to prove the inequality

$$\mathcal{H}^{\frac{1}{2}}(\mathrm{pr}_t X) \leq C\mathcal{P}^1(X) \tag{1.5}$$

for all $X \subset \mathbb{R}^3 \times \mathbb{R}$.
Indeed, for every covering by parabolic cylinders $(Q_{r_i}(x_i,t_i))_{i=0}^\infty$ of X holds in particular that $\mathrm{pr}_t X \subset \bigcup_i \mathrm{pr}_t Q_{r_i}$. Thus, for all $\delta > 0$ holds

$$\begin{aligned}
\mathcal{P}^1_\delta(X) &= \inf\left\{\sum_i r_i \,:\, X \subset \bigcup_i Q_{r_i},\ r_i < \delta\right\} \\
&\geq \inf\left\{\sum_i r_i \,:\, \mathrm{pr}_t X \subset \bigcup_i \mathrm{pr}_t Q_{r_i},\ r_i < \delta\right\} \\
&= \inf\left\{\sum_i r_i \,:\, \mathrm{pr}_t X \subset \bigcup_i (t_i - r_i^2, t_i),\ r_i < \delta\right\} \\
&= \inf\left\{\sum_i \sqrt{s_i} \,:\, \mathrm{pr}_t X \subset \bigcup_i (t_i - s_i, t_i),\ s_i < \delta^2\right\} \\
&\geq \inf\left\{\sum_i \sqrt{\mathrm{diam}([t_i - s_i, t_i])} \,:\, \mathrm{pr}_t X \subset \bigcup_i [t_i - s_i, t_i],\ s_i < \delta^2\right\} \\
&\geq C \cdot \mathcal{H}^{\frac{1}{2}}_{\delta^2}(\mathrm{pr}_t X).
\end{aligned}$$

Passing to the limit $\delta \to 0$ on both sides yields the desired inequality (1.5), completing the proof. □

COROLLARY 1.6. *Let $\int \left(\int |\nabla u|^2 dx\right)^2 dt < \infty$ for a suitable weak solution (u,p) defined on $D \subset \mathbb{R}^3 \times \mathbb{R}$. Then (u,p) is regular on D.*

PROOF. Let $(x,t) \in D$ be any point. We estimate using Hölder in the time-integration

$$\begin{aligned}
\limsup_{r\to 0} \frac{1}{r} \int_{Q_r^*(x,t)} |\nabla u|^2 &= \limsup_{r\to 0} \frac{1}{r} \int_{t-\frac{7}{8}r^2}^{t+\frac{1}{8}r^2} 1 \cdot \int_{B_r(x)} |\nabla u|^2 dyds \\
&\le \limsup_{r\to 0} \frac{1}{r} \left(\int_{t-\frac{7}{8}r^2}^{t+\frac{1}{8}r^2} \left(\int_{B_r(x)} |\nabla u|^2 dy \right)^2 ds \right)^{\frac{1}{2}} \cdot \sqrt{r^2} \\
&\le \limsup_{r\to 0} \left(\int_{t-\frac{7}{8}r^2}^{t+\frac{1}{8}r^2} \left(\int_{D\cap\{s=t\}} |\nabla u|^2 dy \right)^2 ds \right)^{\frac{1}{2}} \\
&= 0,
\end{aligned}$$

where in the last step we used dominated convergence as clearly the indicator of the time interval $\chi_{[t-\frac{7}{8}r^2, t+\frac{1}{8}r^2]}(s) \to 0$ for all $s \ne t$ and $\left(\int_{t-\frac{7}{8}r^2}^{t+\frac{1}{8}r^2} \left(\int_{D\cap\{s=t\}} |\nabla u|^2 dy \right)^2 ds \right) \le \left(\int_{\mathrm{pr}_t D} \left(\int_{D\cap\{s=t\}} |\nabla u|^2 dy \right)^2 ds \right)$ which is bounded by assumption.
Thus, Proposition 1.2 implies that (x,t) is regular. □

A similar result follows by Proposition 1 in the paper.

COROLLARY 1.7. *Let* $\int \left(\int |u|^s + |p|^{\frac{s}{2}} dx \right)^{\frac{s'}{s}} dt < \infty$ *for a suitable weak solution* (u,p) *defined on* $D \subset \mathbb{R}^3 \times \mathbb{R}$ *and* $3 < s \le s'$ *satisfying* $\frac{3}{s} + \frac{2}{s'} = 1$. *Then* (u,p) *is regular on* D.

PROOF. Let $(x,t) \in D$ be any point. We estimate using Hölder with exponents $\frac{s'}{3}$ and $\frac{1}{1-\frac{3}{s'}}$ in the time-integration, but first we Hölder in space in both summands with Hölder exponents $\frac{s}{3}$ and $\frac{1}{1-\frac{3}{s}} = \frac{s'}{2}$.

$$\begin{aligned}
&\limsup_{r\to 0} \frac{1}{r^2} \int_{Q_r^*(x,t)} |u|^3 + |p|^{\frac{3}{2}} \\
=&\limsup_{r\to 0} \frac{1}{r^2} \int_{t-\frac{7}{8}r^2}^{t+\frac{1}{8}r^2} \int_{B_r(x)} 1 \cdot |u|^3 + 1 \cdot |p|^{\frac{3}{2}} dyd\tau \\
\le&\limsup_{r\to 0} \frac{1}{r^2} \int_{t-\frac{7}{8}r^2}^{t+\frac{1}{8}r^2} \left(\left(\int_{B_r(x)} |u|^s dy \right)^{\frac{3}{s}} + \left(\int_{B_r(x)} |p|^{\frac{s}{2}} dy \right)^{\frac{3}{s}} \right) \cdot |B_r(x)|^{\frac{2}{s'}} d\tau \\
\le&\limsup_{r\to 0} \frac{1}{r^2} 2 \int_{t-\frac{7}{8}r^2}^{t+\frac{1}{8}r^2} \left(\int_{B_r(x)} |u|^s + |p|^{\frac{s}{2}} dy \right)^{\frac{3}{s}} \cdot 1 \, d\tau \cdot |B_r(x)|^{\frac{2}{s'}} \\
\le&\limsup_{r\to 0} \frac{2}{r^2} \left(\int_{t-\frac{7}{8}r^2}^{t+\frac{1}{8}r^2} \left(|u|^s + |p|^{\frac{s}{2}} dy \right)^{\frac{s'}{s}} d\tau \right)^{\frac{3}{s'}} (r^2)^{1-\frac{3}{s'}} |B_r(x)|^{\frac{2}{s'}}.
\end{aligned}$$

Now the second factor involving the time integral over the measure of the ball $B_r(x)$ is proportional to $r^{(2-\frac{6}{s'})+\frac{6}{s'}} = r^2$. Hence, this factors cancelles the prefactor of $\frac{1}{r^2}$ and infer that the limit $r \to 0$ tends to zero on the right hand side by dominated convergence similarly as in the previous proof, which is valid due to $\int \left(\int |u|^s + |p|^{\frac{s}{2}} dx \right)^{\frac{s'}{s}} dt$ being bounded by assumption. □

COROLLARY 1.8. *Let* (u,p) *be a suitable weak solution of the Navier-Stokes system which has cylindrical symmetry about some axis. Then singularities can only occur on the symmetry axis.*

PROOF. Assume there would be an off-axis singularity. Then, due to symmetry, this would give rise to a whole circle on which the solution would be singular. But this contradicts the fact that $\mathcal{H}^1(\mathcal{S}) \le C\mathcal{P}^1(\mathcal{S}) = 0$. So, possible singularities can only lie on the axis of symmetry. □

1.2. Estimates for u and p in Weighted Norms

In the following we will prove two lemmas that play an important role in the proof of Theorems C and D, which are subject to the subsequent talk.
The first lemma provides a weighted interpolation estimate.

LEMMA 1.9. *Let $\alpha, \beta, \gamma, r, s$ be such that:*
(i) $r \ge 2$, $\gamma + \frac{3}{r} > 0$, $\alpha + \frac{3}{2} > 0$, $\beta + \frac{3}{2} > 0$, *and* $s \in \left[\frac{1}{2}, 1\right]$,
(ii) $\gamma + \frac{3}{r} = s\left(\alpha + \frac{1}{2}\right) + (1-s)\left(\beta + \frac{3}{2}\right)$,
(iii) $s(\alpha - 1) + (1-s)\beta \le \gamma \le s\alpha + (1-s)\beta$.
Then there exists a constant $C = C(\alpha, \beta, \gamma, r, s)$ such that for all $\varepsilon \ge 0$ holds the inequality

$$\|(\varepsilon + |x|^2)^{\frac{\gamma}{2}} u\|_{L^r(\mathbb{R}^3)} \le C \|(\varepsilon + |x|^2)^{\frac{\alpha}{2}} |\nabla u|\|_{L^2(\mathbb{R}^3)}^s \|(\varepsilon + |x|^2)^{\frac{\beta}{2}} u\|_{L^2(\mathbb{R}^3)}^{1-s} \tag{1.6}$$

for all $u \in H^1(\mathbb{R}^3)$ with $\left\|(\varepsilon + |x|^2)^{\frac{\alpha-1}{2}} u\right\|_{L^2(\mathbb{R}^3)} < \infty$.

REMARK 1.10. Note that for functions $u \in H^1(\mathbb{R}^3)$ with compact support of course we do not need to assume the weighted L^2-norm of u with exponent $\alpha - 1$ to be finite. Also in the original paper, this assumption is not stated even for non compactly supported functions u. Nevertheless, it is not clear to us how to relax this condition or even skip it. Hence, we kept this assumption for completeness of the present notes.

PROOF. Suppose we have already proven the lemma for $\varepsilon = 1$, then for $\varepsilon > 0$ we have by rescaling

$$\begin{aligned}
&\left\|(\varepsilon + |x|^2)^{\frac{\gamma}{2}} u\right\|_{L^r} \\
=&\left\|\left(1 + \frac{|x|^2}{\varepsilon}\right)^{\frac{\gamma}{2}} u(x)\right\|_{L^r} \varepsilon^{\frac{\gamma}{2}} \\
=&\left\|(1 + |y|^2)^{\frac{\gamma}{2}} u(\sqrt{\varepsilon} y)\right\|_{L^r} \varepsilon^{\frac{\gamma}{2} + \frac{3}{2r}} \\
\le& C \left\|(1 + |y|^2)^{\frac{\alpha}{2}} |\nabla_y (u(\sqrt{\varepsilon} y))|^2\right\|_{L^2}^s \left\|(1 + |y|^2)^{\frac{\beta}{2}} u(\sqrt{\varepsilon} y)\right\|_{L^2}^{1-s} \varepsilon^{\frac{\gamma + \frac{3}{r}}{2}} \\
=& C \left\|(1 + |y|^2)^{\frac{\alpha}{2}} \left|\nabla_{\sqrt{\varepsilon} y} (u(\sqrt{\varepsilon} y)) \sqrt{\varepsilon}\right|^2\right\|_{L^2}^s \left\|(1 + |y|^2)^{\frac{\beta}{2}} u(\sqrt{\varepsilon} y)\right\|_{L^2}^{1-s} \varepsilon^{\frac{1}{2}(s(\alpha + \frac{1}{2}) + (1-s)(\beta + \frac{3}{2}))} \\
=& C \left\|(\varepsilon + |\sqrt{\varepsilon} y|^2)^{\frac{\alpha}{2}} \left|\nabla_{\sqrt{\varepsilon} y} (u(\sqrt{\varepsilon} y))\right|^2\right\|_{L^2}^s \left\|(\varepsilon + |\sqrt{\varepsilon} y|^2)^{\frac{\beta}{2}} u(\sqrt{\varepsilon} y)\right\|_{L^2}^{1-s} \varepsilon^{\frac{1}{2}(s\frac{3}{2} + (1-s)\frac{3}{2})} \\
=& C \left\|(\varepsilon + |x|^2)^{\frac{\alpha}{2}} |\nabla u|^2\right\|_{L^2}^s \left\|(\varepsilon + |x|^2)^{\frac{\beta}{2}} u\right\|_{L^2}^{1-s},
\end{aligned}$$

where we used assumption (ii) from the lemma. So, the case $\varepsilon > 0$ follows from $\varepsilon = 1$. For $\varepsilon = 0$ we let $\varepsilon \to 0$ in the inequality for $\varepsilon > 0$. To do so, we use dominated convergence, which is valid as the pointwise almost everywhere convergence $(\varepsilon + |x|^2)^{\frac{\gamma}{2} r} |u|^r \to |x|^{r\gamma} |u|^r$ is clear and for ε small enough and $\gamma \ge 0$ holds $(\varepsilon + |x|^2)^{\frac{\gamma}{2} r} |u|^r \le (1 + |x|^2)^{\frac{\gamma}{2} r} |u|^r$. This last function is integrable because

$$\begin{aligned}
\left\|(1 + |x|^2)^{\frac{\gamma}{2}} u\right\|_{L^r} &\le C \left\|(1 + |x|^2)^{\frac{\alpha}{2}} |\nabla u|^2\right\|_{L^2}^s \left\|(1 + |x|^2)^{\frac{\beta}{2}} u\right\|_{L^2}^{1-s} \\
&\le M + C \left\||x|^{\alpha} |\nabla u|^2\right\|_{L^2}^s \left\||x|^{\beta} u\right\|_{L^2}^{1-s} < \infty
\end{aligned}$$

for some number $M < \infty$, and we assume the right hand side of (1.6) to be finite for $\varepsilon = 0$, since else the statement of the lemma is trivially true. For $\gamma < 0$, we simply estimate $(\varepsilon + |x|^2)^{\frac{\gamma}{2}r}|u|^r \leq |u|^r \in L^1$ and respecting (1.9). The convergence $\varepsilon \to 0$ on the right hand side of (1.6) is treated similarly.

We now want to simplify the proof in terms of which functions u need to be considered. Again, without loss of generality, we assume that both weighted norms of u on the right hand side of (1.6) are finite, since else the inequality is trivially satisfied.

Assume further that we have shown the inequality (1.6) already for all functions in H^1 with compact support. Then for a general $u \in H^1(\mathbb{R}^3)$ we define $\varphi \in C_0^\infty(\mathbb{R}^3)$ to be a radially-symmetric function with $0 \leq \varphi \leq 1$, $\varphi = 1$ on $B_1(0)$ and $\varphi = 0$ outside $B_2(0)$, and $|\nabla\varphi| \leq 2$. Then, the sequence of smooth functions $(\varphi_n)_n := \left(\varphi\left(\frac{\cdot}{n}\right)\right)_n$ tends to 1 a.e. on $\mathbb{R}^3$ and $(\nabla\varphi_n)_n$ tends to zero almost everywhere. So, $\varphi_n u \in H^1(\mathbb{R}^3)$ has compact support and we can estimate

$$\begin{aligned}\left\|(1+|x|^2)^{\frac{\gamma}{2}}\varphi_n u\right\|_{L^r(\mathbb{R}^3)} \leq & C\left\|(1+|x|^2)^{\frac{\alpha}{2}}|\nabla(\varphi_n u)|\right\|_{L^2(\mathbb{R}^3)}^s \left\|(1+|x|^2)^{\frac{\beta}{2}}(\varphi_n u)\right\|_{L^2(\mathbb{R}^3)}^{1-s} \\ \leq & C\left\|(1+|x|^2)^{\frac{\alpha}{2}}|\nabla\varphi_n|u|\right\|_{L^2(\mathbb{R}^3)}^s \left\|(1+|x|^2)^{\frac{\beta}{2}}\varphi_n u\right\|_{L^2(\mathbb{R}^3)}^{1-s} \\ & + C\left\|(1+|x|^2)^{\frac{\alpha}{2}}\varphi_n|\nabla u|\right\|_{L^2(\mathbb{R}^3)}^s \left\|(1+|x|^2)^{\frac{\beta}{2}}\varphi_n u\right\|_{L^2(\mathbb{R}^3)}^{1-s} \qquad (1.7)\end{aligned}$$

All norms in (1.7) involving φ_n tend to the desired norm by monotone convergence, e.g.

$$\left\|(1+|x|^2)^{\frac{\gamma}{2}}\varphi_n u\right\|_{L^r(\mathbb{R}^3)} \overset{n\to\infty}{\to} \left\|(1+|x|^2)^{\frac{\gamma}{2}}u\right\|_{L^r(\mathbb{R}^3)}.$$

The only problematic term is $\left\|(1+|x|^2)^{\frac{\alpha}{2}}|\nabla\varphi_n|u|\right\|_{L^2(\mathbb{R}^3)}$ as $(\nabla\varphi_n)_n$ is not a monotonically increasing sequence of functions. But here (and in fact only here) the assumption discussed in Remark 1.10 comes into play and we have

$$\begin{aligned}\left|(1+|x|^2)^{\frac{\alpha}{2}}|\nabla\varphi_n|u\right|^2 &\leq \left|(1+|x|^2)^{\frac{\alpha}{2}}\frac{2}{n}u\cdot\chi_{B_{2n}(0)\setminus B_n(0)}\right|^2 \\ &= \left|(1+|x|^2)^{\frac{\alpha}{2}}\frac{4\sqrt{2}}{\sqrt{2}2n}u\cdot\chi_{B_{2n}(0)\setminus B_n(0)}\right|^2 \\ &\leq \left|\frac{4\sqrt{2}(1+|x|^2)^{\frac{\alpha}{2}}}{\sqrt{1+|x|^2}}u\cdot\chi_{B_{2n}(0)\setminus B_n(0)}\right|^2 \\ &\leq \left|4\sqrt{2}(1+|x|^2)^{\frac{\alpha-1}{2}}u\right|^2 \in L^1(\mathbb{R}^3),\end{aligned}$$

since $\sqrt{|x|^2+1} \leq \sqrt{2|x|^2} \leq \sqrt{2}2n$ for all $n \in \mathbb{N}$. By dominated convergence we infer that all terms in (1.7) converge to the desired norms.

Now it suffices to show the inequality (1.6) only for smooth compactly supported functions. Indeed, assume u is supported in $B_M(0)$, i.e. $u \in H^1(B_M(0))$. Clearly, for all $1 \leq q \leq \infty$ and all measurable functions f holds

$$\|f\|_{L^q(B_M(0))} \leq \left\|(1+|x|^2)^{\frac{\delta}{2}}f\right\|_{L^q(\mathbb{R}^3)} \leq (1+M^2)^{\frac{\delta}{2}}\|f\|_{L^q(B_M(0))}.$$

Thus, the weighted norm and the unweighted norm on $L^q(B_M(0))$ are equivalent. Note that by (1.9) (see below) and assumption (i) we have $r \in [2,6]$. So by the continuous Sobolev embedding $H^1(B_M(0)) \hookrightarrow L^6(B_M(0))$ and the fact that $u \in H_0^1(B_M(0))$ we can choose a sequence $u_n \in C_0^\infty(B_M(0))$ with $u_n \to u$ in H^1 as $n \to \infty$. Note that the convergence in H^1 implies the convergence in L^6 and hence in L^r and L^2 by Hölder. Thus,

$$\begin{aligned}\left\|(1+|x|^2)^{\frac{\gamma}{2}}u\right\|_{L^r(B_M(0))} &\leftarrow \left\|(1+|x|^2)^{\frac{\gamma}{2}}u_n\right\|_{L^r(B_M(0))}\\ &\le C\left\|(1+|x|^2)^{\frac{\alpha}{2}}|\nabla u_n|\right\|^s_{L^2(B_M(0))}\left\|(1+|x|^2)^{\frac{\beta}{2}}u_n\right\|^{1-s}_{L^2(B_M(0))}\\ &\to C\left\|(1+|x|^2)^{\frac{\alpha}{2}}|\nabla u|\right\|^s_{L^2(B_M(0))}\left\|(1+|x|^2)^{\frac{\beta}{2}}u\right\|^{1-s}_{L^2(B_M(0))},\end{aligned}$$

hence the lemma follows for u compactly supported.
To sum up, it suffices to prove the lemma for $\varepsilon = 1$ and $u \in C_0^\infty(\mathbb{R}^3)$.
We introduce the notation $\tau := (1+|x|^2)^{\frac{1}{2}}$, $A := \|\tau^\alpha|\nabla u|\|_{L^2}$, and $B := \|\tau^\beta u\|_{L^2}$.
We first consider the case $r = 2$:
Assumption (i) implies $\gamma > -\frac{3}{2}$ and assumption (ii) implies

$$\gamma = s\left(\alpha + \frac{1}{2}\right) + (1-s)\left(\beta + \frac{3}{2}\right) - \frac{3}{2} = s(\alpha - 1) + (1-s)\beta.$$

We now introduce spherical coordinates (ρ, θ) on $\mathbb{R}^3$ to obtain

$$\begin{aligned}\int_{\mathbb{R}^3}\tau^{2\gamma}|u|^2dx &= \int_{S^2}\int_0^\infty \tau^{2\gamma}|u|^2\rho^2 d\rho d\theta\\ &= \int_{S^2}\int_0^\infty \tau^{2\gamma}|u|^2\tau\rho d\rho d\theta + \int_{S^2}\int_0^\infty \tau^{2\gamma}|u|^2(\rho^2 - \tau\rho)d\rho d\theta.\end{aligned}$$

By partial integration in ρ, while recalling $u \in C_0^\infty$ and the triviality $\rho\big|_{\rho=0} = 0$, we get

$$\begin{aligned}\int_{S^2}\int_0^\infty \tau^{2\gamma+1}\rho|u|^2 d\rho d\theta &= -\int_{S^2}\int_0^\infty \frac{\rho^2}{2}\left(u\cdot\partial_r u\tau^{2\gamma+1} + |u|^2(2\gamma+1)\tau^{2\gamma}\frac{\rho}{\tau}\right)d\rho d\theta\\ &\le -\int_{S^2}\int_0^\infty \rho^3\tau^{2\gamma-1}|u|^2\left(\gamma+\frac{1}{2}\right) + \rho^2\tau^{2\gamma+1}|u||\nabla u|d\rho d\theta.\end{aligned}$$

Thus,

$$\int_{\mathbb{R}^3}\tau^{2\gamma}|u|^2dx \le \int_{S^2}\int_0^\infty \tau^{2\gamma+1}\rho^2|u||\nabla u| + |u|^2\tau^{2\gamma}\rho^2\left(1 - \frac{\tau}{\rho} - \left(\gamma + \frac{1}{2}\frac{\rho}{\tau}\right)\right)d\rho d\theta.$$

Now notice that by $\gamma > -\frac{3}{2}$ we have $\gamma \ge -\frac{3}{2} + \bar{C}$ for some constant $1 > \bar{C} > 0$. Hence,

$$\begin{aligned}\frac{\tau}{\rho} + \left(\gamma + \frac{1}{2}\right)\frac{\rho}{\tau} &\ge (1-\bar{C})\frac{\sqrt{1+|x|^2}}{|x|} - (1-\bar{C})\frac{|x|}{\sqrt{1+|x|^2}} + \bar{C}\frac{\sqrt{1+|x|^2}}{|x|}\\ &\ge \bar{C}\frac{\sqrt{1+|x|^2}}{|x|} \ge \bar{C}.\end{aligned}$$

Using the last estimate and rearranging yields

$$\begin{aligned}\int_{\mathbb{R}^3}\tau^{2\gamma}|u|^2 &\le \frac{1}{\bar{C}}\int_{S^2}\int_0^\infty \tau^{2\gamma+1}\rho^2|u||\nabla u|d\rho d\theta\\ &= \frac{1}{\bar{C}}\int_{\mathbb{R}^3}\tau^{2\gamma+1}|u||\nabla u|dx\\ &= \frac{1}{\bar{C}}\int_{\mathbb{R}^3}\tau^{1+\frac{\gamma}{2}}(\tau^\gamma|u|)^{2-\frac{1}{s}}|\nabla u||u|^{\frac{1}{s}-1}dx\\ &= \frac{1}{\bar{C}}\int_{\mathbb{R}^3}(\tau^\gamma|u|)^{2-\frac{1}{s}}(\tau^\alpha|\nabla u|)(\tau^\beta|u|)^{\frac{1}{s}-1}dx,\end{aligned}$$

where we used (ii), i.e. $1 + \frac{\gamma}{s} = \alpha + \beta\left(\frac{1}{s} - 1\right)$.
Note that $\frac{1}{s} \in [1,2]$ and $\frac{1}{2} + \frac{1}{2}\left(\frac{1}{s} - 1\right) + \frac{1}{2}\left(2 - \frac{1}{s}\right) = 1$. Thus, we can apply the Hölder inequality for three factors with exponents 2, $\frac{2}{\frac{1}{s}-1}$, and $\frac{2}{2-\frac{1}{s}}$ to obtain

$$\begin{aligned}\int_{\mathbb{R}^3} \tau^{2\gamma}|u|^2 &\le \frac{1}{\bar{C}}\left(\int_{\mathbb{R}^3} (\tau^\gamma |u|)^2 dx\right)^{1-\frac{1}{2s}} \left(\int_{\mathbb{R}^3} (\tau^\beta |u|)^2 dx\right)^{\frac{1}{2s}-\frac{1}{2}} \left(\int_{\mathbb{R}^3} \tau^\alpha |\nabla u|^2 dx\right)^{\frac{1}{2}} \\ &= \frac{1}{\bar{C}}\left(\int_{\mathbb{R}^3} (\tau^\gamma |u|)^2 dx\right)^{1-\frac{1}{2s}} B^{\frac{1}{s}-1} A.\end{aligned}$$

Rearranging and taking the whole inequality to the power s yields

$$\left(\int_{\mathbb{R}^3} \tau^{2\gamma}|u|^2 dx\right)^{\frac{1}{2}} \le C B^{1-s} A^s,$$

finishing the proof in the case $r = 2$.

Now consider $r > 2$:
Define $R_k := \{2^{k-1} < |x| \le 2^k\}$. We note that $q := \frac{1}{\frac{1}{2} - \frac{s}{3}} = \frac{6}{3-2s} \in [3,6]$ as $s \in \left[\frac{1}{2}, 1\right]$. Therefore, by standard Lebesgue space interpolation on R_k with $\frac{1}{q} = \frac{1-s}{2} + \frac{s}{6}$ and Poincare's inequality on balls (cf. section 4.5.2 in [**EG92**]) holds

$$\begin{aligned}\|u\|_{L^q(R_k)} &\le \|u - \bar{u}\|_{L^q(R_k)} + \|\bar{u}\|_{L^q(R_k)} \\ &\le \|u - \bar{u}\|_{L^2(R_k)}^{1-s} \|u - \bar{u}\|_{L^6(R_k)}^{s} + \|\bar{u}\|_{L^q(R_k)} \\ &\le C\|u - \bar{u}\|_{L^2(R_k)}^{1-s} \|\nabla u\|_{L^2(R_k)}^{s} + C d_k^{-3+\frac{3}{q}} \int_{R_k} |u| dx \\ &\le C\|u\|_{L^2(R_k)}^{1-s} \|\nabla u\|_{L^2(R_k)}^{s} + C d_k^{-3+\frac{3}{q}} \sqrt{|R_k|} \cdot \|u\|_{L^2(R_k)} \\ &\le C\|\nabla u\|_{L^2(R_k)}^{s} \|u\|_{L^2(R_k)}^{1-s} + \frac{C}{d_k^s}\|u\|_{L^2(R_k)}, \end{aligned} \tag{1.8}$$

where we introduced the shorthand notation $d_k := \mathrm{diam}(R_k)$ and $\bar{u} := \frac{1}{|R_k|}\int_{R_k} u\,dx$. We also used the estimate

$$\begin{aligned}\|u - \bar{u}\|_{L^2(R_k)} &= \left(\int_{R_k} \left|u - \frac{1}{|R_k|}\int_{R_k} u\,dy\right|^2 dx\right)^{\frac{1}{2}} \\ &\le C\|u\|_{L^2(R_k)} + C\left(\int_{R_k} \left|\frac{1}{|R_k|}\int_{R_k} u\,dy\right|^2\right)^{\frac{1}{2}} \\ &\le C\|u\|_{L^2(R_k)} + C\left(\int_{R_k} dx \frac{1}{|R_k|}\int_{R_k} |u|^2 dy\right)^{\frac{1}{2}} \\ &= C\|u\|_{L^2(R_k)},\end{aligned}$$

which follows from Jensen's inequality. Note that C does not depend on d_k. Another remark on (1.8) is concerning the use of the Poincare inequality for balls: By Theorem 1 in section 5.4 in [**Eva10**] there is a continuous linear extension operator $E\colon H^1(R_k) \to H^1(B_{22^k}(0))$. This means we estimate as follows

$$\begin{aligned}\|u - \bar{u}\|_{L^6(R_k)} &= \|Eu - \overline{Eu}\|_{L^6(R_k)} \le \|Eu - \overline{Eu}\|_{L^6(B_{22^k}(0))} \\ &\le C\|\nabla Eu\|_{L^2(B_{22^k}(0))} \le C\|\nabla u\|_{L^2(R_k)}.\end{aligned}$$

Moreover, assumptions (ii) and (iii) yield $s\left(\alpha+\frac{1}{2}\right)+(1-s)\left(\beta+\frac{3}{2}\right)=\gamma+\frac{3}{r}\leq s\alpha+(1-s)\beta+\frac{3}{r}$. Hence, $\frac{s}{2}+\frac{3}{2}-\frac{3s}{2}\leq\frac{3}{r}$, which rearranged gives

$$\frac{1}{r}\geq\frac{1}{2}-\frac{s}{3}. \tag{1.9}$$

Or in other words, $r\leq q$. So, Hölder with exponents $\frac{q}{r}$ and $\frac{q}{q-r}$leads to

$$\|\tau^\gamma u\|_{L^r(R_k)}\leq C|R_k|^{\frac{q-r}{qr}}\|\tau^\gamma u\|_{L^q(R_k)}\leq Cd_k^{3\left(\frac{1}{r}-\frac{1}{q}\right)+\gamma}\|u\|_{L^q(R_k)}, \tag{1.10}$$

where we used that d_k is comparable to τ on R_k. Indeed, $d_k=2\cdot2^k$, hence

$$\tau=\sqrt{1+|x|^2}\geq|x|\geq2^{k-1}=\frac{1}{4}d_k.$$

On the other hand,

$$\tau=\sqrt{1+|x|^2}\leq\sqrt{1+2^{2k}}\leq\sqrt{2\cdot2^{2k}}=\sqrt{2}2^k=\sqrt{2}d_k.$$

Thus, we can combine the non-weighted interpolation estimate (1.8) with the Hölder estimate (1.10) to obtain

$$\begin{aligned}\|\tau^\gamma u\|_{L^r(R_k)}&\leq Cd_k^{3\left(\frac{1}{r}-\frac{1}{q}\right)+\gamma}\left(\|\nabla u\|^s_{L^2(R_k)}\|u\|^{1-s}_{L^2(R_k)}+\frac{1}{d_k^s}\|u\|_{L^2(R_k)}\right)\\&\leq C\|\tau^\alpha|\nabla u|\|^s_{L^2(R_k)}\|\tau^\beta u\|^{1-s}_{L^2(R_k)}+C\|\tau^\delta u\|_{L^2(R_k)},\end{aligned} \tag{1.11}$$

where we used the definition $\delta:=\gamma+\frac{3}{r}-\frac{3}{2}=\gamma+\frac{3}{r}+\frac{3}{q}-s$ and the identity

$$\begin{aligned}\frac{3}{r}-\frac{3}{q}+\gamma&=s\left(\alpha+\frac{1}{2}\right)+(1-s)\left(\beta+\frac{3}{2}\right)-\frac{3}{q}=s\alpha+(1-s)\beta+\frac{s}{2}+(1-s)\frac{3}{2}-\frac{3}{q}\\&=s\alpha+(1-s)\beta,\end{aligned}$$

which follows from (ii).

We now take the sum of the inequalities (1.11) over k to obtain the inequality on the whole $\mathbb{R}^3$. Note that then the left hand side is estimated by $a^{\frac{1}{r}}+b^{\frac{1}{r}}\leq2^{1-\frac{1}{r}}(a+b)^{\frac{1}{r}}$ for $a,b\geq0$ by concavity, and by using that $R_k\cap R_j=\varnothing$ for $k\neq j$ the sum over the integrals is simply the integral over $\mathbb{R}^3$. On the right hand side we use Minkowski's inequality on the summands with $\tau^\delta u$ and for the others we estimate using concavity of the square root and Hölder with exponents $\frac{1}{s}$ and $\frac{1}{1-s}$ in the sum over k to get

$$\begin{aligned}&\sum_{k=0}^\infty\left(\int_{R_k}\tau^{2\alpha}|\nabla u|^2dx\right)^{\frac{s}{2}}\left(\int_{R_k}\tau^{2\beta}|u|^2dx\right)^{\frac{1-s}{2}}\\&\leq\sqrt{2}\left(\sum_{k=0}^\infty\left(\int_{R_k}\tau^{2\alpha}|\nabla u|^2dx\right)^s\cdot\left(\int_{R_k}\tau^{2\beta}|u|^2dx\right)^{1-s}\right)^{\frac{1}{2}}\\&\leq\sqrt{2}\left(\left(\sum_{k=0}^\infty\int_{R_k}\tau^{2\alpha}|\nabla u|^2dx\right)^s\cdot\left(\sum_{k=0}^\infty\int_{R_k}\tau^{2\beta}|u|^2dx\right)^{1-s}\right)^{\frac{1}{2}}\\&=\sqrt{2}A^sB^{1-s}.\end{aligned}$$

We conclude the proof by applying the case $r=2$ to the term $C\|\tau^\delta u\|_{L^2(\mathbb{R}^3)}$ on the right hand side with δ playing the role of γ, which is valid since $\delta=\gamma+\frac{3}{r}-\frac{3}{2}>-\frac{3}{2}$ by (i). This means, we can estimate

$$\|\tau^\delta u\|_{L^2(\mathbb{R}^3)}\leq CA^sB^{1-s},$$

which finishes the proof also for the case $r>2$. □

Let us now prove a lemma concerning weighted-norm bounds of the singular integral operator $(-\Delta)^{-1}\operatorname{div}\operatorname{div}$, i.e. for later use we want to relate weighted norms of the pressure and the velocity field of a suitable weak solution.

LEMMA 1.11. *If $p \in L^3(\mathbb{R}^3)$ is the solution of the differential equation*

$$-\Delta p = \operatorname{div}\operatorname{div}(u \otimes u) \tag{1.12}$$

on $\mathbb{R}^3$ for a function $u \in H^1(\mathbb{R}^3)$. Then for r, γ satisfying $1 < r < \infty$ and $-\frac{3}{r} < \gamma < 3 - \frac{3}{r}$ there exists a constant C such that for all $\varepsilon \geq 0$ holds

$$\left\|(\varepsilon + |x|^2)^{\frac{\gamma}{2}} p\right\|_{L^r(\mathbb{R}^3)} \leq C \left\|(\varepsilon + |x|^2)^{\frac{\gamma}{4}} u\right\|^2_{L^{2r}(\mathbb{R}^3)}. \tag{1.13}$$

PROOF. Without loss of generality assume the right hand side of (1.13) is finite, else the statement is trivial.
Define the operator $T_{ij} f := c\partial_i\partial_j\left(\frac{1}{x}\right) \star f$ for $f \in C_0^\infty(\mathbb{R}^3)$. By Theorem B.6 in [**RRS16**] this extends to a linear bounded operator from $L^3(\mathbb{R}^3)$ to itself. Note that $u_i u_j \leq |u|^2 \in L^3(\mathbb{R}^3)$ for all i, j, since $u \in H^1$. Thus, for all i, j choose a sequence of testfunctions $\varphi_n^{ij} \overset{L^3}{\to} u_i u_j$. Moreover, define

$$p := -|u|^2 + \sum_{i \neq j} T_{ij}(u_i u_j).$$

By the above we know $p \in L^3(\mathbb{R}^3)$. This is also the unique solution to (1.12) in L^3. Indeed, let $\tilde{p} \in L^3$ be another distributional solution. We check that p is a distributional solution. For that let $\psi \in C_c^\infty(\mathbb{R}^3)$ and observe

$$\begin{aligned}
\langle -\Delta p, \psi\rangle &= \sum_i \langle -u_i u_i, -\partial_i\partial_i\psi\rangle + \sum_{i\neq j}\langle T_{ij}(u_i u_j), -\Delta\psi\rangle \\
&\leftarrow \sum_i \langle u_i u_i, \partial_i\partial_i\psi\rangle + \sum_{i\neq j}\langle T_{ij}(\varphi_n^{ij}), -\Delta\psi\rangle \\
&= \sum_i \langle \partial_i\partial_i u_i u_i, \psi\rangle + \sum_{i\neq j}\langle \varphi_n^{ij}, \partial_i\partial_j(-\Delta)^{-1}(-\Delta)\psi\rangle \\
&\to \sum_i \langle \partial_i\partial_i u_i u_i, \psi\rangle + \sum_{i\neq j}\langle u_i u_j, \partial_i\partial_j\psi\rangle \\
&= \sum_i \langle \partial_i\partial_i u_i u_i, \psi\rangle + \sum_{i\neq j}\langle \partial_i\partial_j(u_i u_j), \psi\rangle \\
&= \langle \operatorname{div}\operatorname{div}(u\otimes u), \psi\rangle.
\end{aligned}$$

Therefore, $\Delta(p - \tilde{p}) = 0$ and $p - \tilde{p} \in L^3$, hence in particular $p - \tilde{p} \in L^1_{\text{loc}}(\mathbb{R}^3)$. By Weyl's Lemma (cf. Theorem C.3 in [**RRS16**]) we obtain that $p - \tilde{p}$ is smooth. But since $p - \tilde{p}$ also lies in L^3 we get by the mean value property for all $x \in \mathbb{R}^3$ using Hölder

$$|(p - \tilde{p})(x)| \leq \frac{C}{r^3}\int_{B_r(x)} |p - \tilde{p}| dx \leq \frac{C}{r^{3-\frac{3}{2}}}\|p - \tilde{p}\|_{L^3(\mathbb{R}^3)} \overset{r\to\infty}{\to} 0.$$

Thus, p defined above is the unique solution in L^3.
We show that for $i \neq j$ holds

$$\|(1 + |x|^2)^{\frac{\gamma}{2}} T_{ij}(f)\|_{L^r(\mathbb{R}^3)} \leq C\|(1 + |x|^2)^{\frac{\gamma}{2}} f\|_{L^r(\mathbb{R}^3)} \tag{1.14}$$

for all f such that the right hand side is finite. This then proves the lemma, since by scaling we infer for $\varepsilon > 0$

$$\begin{aligned}
\left(\int_{\mathbb{R}^3} (\varepsilon + |x|^2)^{\frac{r\gamma}{2}} (T_{ij}f)^r(x)dx\right)^{\frac{1}{r}} &= \varepsilon^{\frac{\gamma}{2}} \left(\int_{\mathbb{R}^3} \left(1 + \left(\frac{|x|}{\sqrt{\varepsilon}}\right)^2\right)^{\frac{r\gamma}{2}} (T_{ij}f)^r(x)dx\right)^{\frac{1}{r}} \\
&= \varepsilon^{\frac{\gamma}{2}+\frac{3}{2r}} \left(\int_{\mathbb{R}^3} (1+|y|^2)^{\frac{r\gamma}{2}} (T_{ij}f)^r(\sqrt{\varepsilon}y)dy\right)^{\frac{1}{r}} \\
&= \varepsilon^{\frac{\gamma}{2}+\frac{3}{2r}} \left(\int_{\mathbb{R}^3} (1+|y|^2)^{\frac{r\gamma}{2}} (T_{ij}(f(\sqrt{\varepsilon}\cdot))(y))^r dy\right)^{\frac{1}{r}} \\
&\le C\varepsilon^{\frac{\gamma}{2}+\frac{3}{2r}} \left(\int_{\mathbb{R}^3} (1+|y|^2)^{\frac{r\gamma}{2}} f^r(\sqrt{\varepsilon}y)dy\right)^{\frac{1}{r}} \\
&= C\varepsilon^{\frac{\gamma}{2}} \left(\int_{\mathbb{R}^3} \left(1 + \left(\frac{|x|}{\sqrt{\varepsilon}}\right)^2\right)^{\frac{r\gamma}{2}} f^r(x)dx\right)^{\frac{1}{r}} \\
&= C \left(\int_{\mathbb{R}^3} (\varepsilon + |x|^2)^{\frac{r\gamma}{2}} f^r(x)dx\right)^{\frac{1}{r}}.
\end{aligned}$$

In the above we used

$$\begin{aligned}
(T_{ij}f)(\sqrt{\varepsilon}y) &= \mathrm{p.v.} \int_{\mathbb{R}^3} C\left(\partial_i\partial_j \frac{1}{|\cdot|}\right)(x-\sqrt{\varepsilon}y)f(x)dx \\
&= \mathrm{p.v.} \int_{\mathbb{R}^3} C\frac{(\sqrt{\varepsilon}y_i - x_i)(\sqrt{\varepsilon}y_j - x_j)}{|\sqrt{\varepsilon}y - x|^5} f(x)dx \\
&= \mathrm{p.v.}\, C\varepsilon^{-\frac{3}{2}} \int_{\mathbb{R}^3} \frac{\left(y_i - \frac{x_i}{\sqrt{\varepsilon}}\right)\left(y_j - \frac{x_j}{\sqrt{\varepsilon}}\right)}{\left|y - \frac{x}{\sqrt{\varepsilon}}\right|^5} f(x)dx \\
&= \mathrm{p.v.}\, C\varepsilon^{-\frac{3}{2}+\frac{3}{2}} \int_{\mathbb{R}^3} \frac{(y_i - z_i)(y_j - z_j)}{|y-z|^5} f(\sqrt{\varepsilon}z)dz \\
&= T(f(\sqrt{\varepsilon}\cdot))(y)
\end{aligned}$$

Note that the case $\varepsilon = 0$, i.e.

$$\left\| |x|^\gamma T_{ij}(f) \right\|_{L^r(\mathbb{R}^3)} \le C \left\| |x|^\gamma f \right\|_{L^r(\mathbb{R}^3)},$$

is proven in [**Ste57**], where the bounds on γ stated in the lemma are exactly chosen to fit into Stein's theorem. Indeed, using $\partial_i\partial_j\left(\frac{1}{|x|}\right) = 3\frac{x_i x_j}{|x|^5}$ the function $H(x, x-y)$ appearing in Stein's theorem, which is defined by

$$\left(\partial_i\partial_j \frac{1}{|\cdot|}\right)(x-y) =: \frac{1}{|x-y|^3} \cdot H(x, x-y),$$

satisfies the bound

$$|H(x, x-y)| = 3\frac{|x_i - y_i||x_j - y_j|}{|x-y|^2} \le 3.$$

Thus, all requirements for Stein's theorem are satisfied.
So, assume (1.14) holds, then for $\varepsilon \geq 0$ we get

$$\begin{aligned}\left\|(\varepsilon+|x|^2)^{\frac{\gamma}{2}}p\right\|_{L^r(\mathbb{R}^3)} &= \left\|(\varepsilon+|x|^2)^{\frac{\gamma}{2}}\left(-|u|^2+\sum_{i\neq j}T_{ij}(u_iu_j)\right)\right\|_{L^r(\mathbb{R}^3)}\\ &\leq \left\|(\varepsilon+|x|^2)^{\frac{\gamma}{2}}u\right\|^2_{L^{2r}(\mathbb{R}^3)}+\sum_{i\neq j}C\left\|(\varepsilon+|x|^2)^{\frac{\gamma}{2}}u_iu_j\right\|_{L^r}\\ &\leq \left\|(\varepsilon+|x|^2)^{\frac{\gamma}{2}}u\right\|^2_{L^{2r}(\mathbb{R}^3)}+\sum_{i\neq j}C\left\|(\varepsilon+|x|^2)^{\frac{\gamma}{2}}u\right\|^2_{L^{2r}}\\ &\leq \left\|(\varepsilon+|x|^2)^{\frac{\gamma}{2}}u\right\|^2_{L^{2r}(\mathbb{R}^3)}.\end{aligned}$$

Hence, it is left to prove (1.14). The case $\gamma = 0$ corresponds to the classical Calderon-Zygmund estimate

$$\|T_{ij}(f)\|_{L^r} \leq C\|f\|_{L^r}.$$

Now decompose f into $f = f_1 + f_2$, where $f_1 = \chi_{|x|\leq 1}$ and $f_2 = \chi_{|x|>1}$. Note that $(1+|x|^2)^{\frac{\gamma}{2}} \leq 1 \leq 2^{\frac{\gamma}{2}}(1+|x|^{\gamma})$ for $\gamma \leq 0$ and else for $|x| \leq 1$ holds

$$(1+|x|^2)^{\frac{\gamma}{2}} \leq 2^{\frac{\gamma}{2}} \leq 2^{\frac{\gamma}{2}}(1+|x|^{\gamma}),$$

whereas for $|x| > 1$ holds

$$(1+|x|^2)^{\frac{\gamma}{2}} \leq 2^{\frac{\gamma}{2}}|x|^{\gamma} \leq 2^{\frac{\gamma}{2}}(1+|x|^{\gamma}).$$

Now using the estimates for $\gamma = 0$ and $\varepsilon = 0$ proven before we obtain for positive γ

$$\begin{aligned}\left\|(1+|x|^2)^{\frac{\gamma}{2}}T_{ij}f\right\|^r_{L^r} &\leq \left\|(1+|x|^2)^{\frac{\gamma}{2}}T_{ij}f_1\right\|^r_{L^r}+\left\|(1+|x|^2)^{\frac{\gamma}{2}}T_{ij}f_2\right\|^r_{L^r}\\ &\leq C\left\|T_{ij}f_1\right\|^r_{L^r}+C\left\||x|^{\gamma}T_{ij}f_1\right\|^r_{L^r}+C\left\|T_{ij}f_2\right\|^r_{L^r}+C\left\||x|^{\gamma}T_{ij}f_2\right\|^r_{L^r}\\ &\leq C\left\|f_1\right\|^r_{L^r}+C\left\||x|^{\gamma}f_1\right\|^r_{L^r}+C\left\|f_2\right\|^r_{L^r}+C\left\||x|^{\gamma}f_2\right\|^r_{L^r}\\ &\leq C\left\|(1+|x|^2)^{\frac{\gamma}{2}}f_1\right\|^r_{L^r}+C\left\|(1+|x|^2)^{\frac{\gamma}{2}}f_1\right\|^r_{L^r}\\ &\quad +C\left\|(1+|x|^2)^{\frac{\gamma}{2}}f_2\right\|^r_{L^r}+C\left\|(1+|x|^2)^{\frac{\gamma}{2}}f_2\right\|^r_{L^r}\\ &= C\left\|(1+|x|^2)^{\frac{\gamma}{2}}f\right\|^r_{L^r}.\end{aligned}$$

For $\gamma \leq 0$ we estimate

$$\begin{aligned}\left\|(1+|x|^2)^{\frac{\gamma}{2}}T_{ij}f\right\|^r_{L^r} &\leq \left\|(1+|x|^2)^{\frac{\gamma}{2}}T_{ij}f_1\right\|^r_{L^r}+\left\|(1+|x|^2)^{\frac{\gamma}{2}}T_{ij}f_2\right\|^r_{L^r}\\ &\leq \left\|T_{ij}f_1\right\|^r_{L^r}+\left\||x|^{\gamma}T_{ij}f_2\right\|^r_{L^r}\\ &\leq C\left\|\frac{2}{2}\cdot f_1\right\|^r_{L^r}+C\left\|\frac{2}{2|x|^{-\gamma}}f_2\right\|^r_{L^r}\\ &\leq C\left\|2(1+|x|^2)^{\frac{\gamma}{2}}f_1\right\|^r_{L^r}+C\left\|2(1+|x|^2)^{\frac{\gamma}{2}}f_2\right\|^r_{L^r}\\ &= C\left\|(1+|x|^2)^{\frac{\gamma}{2}}f\right\|^r_{L^r}.\end{aligned}$$

This completes the proof of the lemma. $\square$

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