Navier-Stokes Seminar: Caffarelli-Kohn-Nirenberg Theory

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Preface

These are lecture notes geberated by the seminar course on the Caffarelli-Kohn-Nirenberg Theory for the Navier-Stokes equations at the Universität Ulm in the summer term of 2019. We mainly follow the [**CKN82**] in a modern fashion. This work is aimed at enthusiastic Masters and PhD students.

I would like to thank everyone taking the seminar for typing parts of these notes.

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CHAPTER 1

Talk 7: The Blow-up estimate part 1

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The aim of this talk is to provide a partial proof of the following Proposition 2. This proposition gives a criterion for the regularity of certain points of suitable weak solutions by means of control of the parabolic mean of the gradient of u in cylinders shrinking to that point. Let us recall some notation first. Given any point (t,x) and a radius $r>0$ we introduce the cylinders

$$Q_r(t,x) = \left\{(s,y)\in\mathbb{R}^4 \mid t-r^2<s<t, |x-y|<r\right\}$$
$$Q_r^*(t,x) = \left\{(s,y)\in\mathbb{R}^4 \;\middle|\; t-\frac{7}{8}r^2<s<t+\frac{1}{8}r^2, |x-y|<r\right\}.$$

The cylinders $Q_r^*(t,x)$ are useful in the sense that $(t,x)\in Q^*_{\frac{r}{2}}(t,x)$, while $(t,x)\notin Q_r(t,x)$. Therefore we may apply Corollary 1 to the cylinders $Q_r^*(t,x)$ to show that the point $(t,x)\in Q^*_{\frac{r}{2}}(t,x)$ is regular.

PROPOSITION 2. *There is an aabsolute constant* $\varepsilon_3>0$ *such that for all suitable weak solutions* (u,p) *of the Navier-Stokes in a neighborhood of a given point (t,x) satisfying*

$$\limsup_{r\to 0}\frac{1}{r}\int_{Q_r^*(t,x)}|\nabla u|^2\,\mathrm{d}(t,x)\le\frac{1}{2}\varepsilon_3$$

are regular in (t,x).

This theorem is going to be used to show Theorem B in [**CKN82**], namely that the singular set S satisfies $\mathscr{P}^1(S)=0$.

The proof of Proposition 2 is based on a rather technical decay estimate for a quantity $M_*(r)$ in terms of M_*,δ_* and F_*. These quantities are analogues to the quantities introduced in section 3. However they are defined on the translated cylinders $Q_r^*(t,x)$ instead of on the cylinder $Q_r(t,x)$. The estimate and its proof are going to be subject of the next talk. We are going to use it to prove Proposition 2 for now. To provide a shorthand way of writing it down we introduce several dimension-less quantities depending on u,p and f. Without loss of generality we may restrict to the case $(t,x)=(0,0)$ by translation in space and time. Writing $Q_r^*=Q_r^*(0,0)$, we define

$$G_*(r) = r^{-2}\int_{Q_r^*}|u|^3\,\mathrm{d}(t,x) \qquad K_*(r) = r^{-\frac{13}{4}}\int_{-\frac{7}{8}r^2}^{\frac{1}{8}r^2}\left(\int_{B_r(0)}|p|\,\mathrm{d}x\right)^{\frac{5}{4}}\mathrm{d}t$$
$$J_*(r) = r^{-2}\int_{Q_r^*}|u|\,|p|\,\mathrm{d}(t,x) \qquad H_*(r) = r^{-2}\int_{Q_r^*}|u|\left||u|^2-\overline{|u|_r^2}\right|\mathrm{d}(t,x)$$
$$\delta_*(r) = r^{-1}\int_{Q_r^*}|\nabla u|^2\,\mathrm{d}(t,x) \qquad F_*(r) = r^{-\frac{1}{2}}\int_{Q_r^*}|f|^{\frac{3}{2}}\,\mathrm{d}(t,x),$$

where

$$\overline{|u|_r^2} = \fint_{B_r(0)}|u|^2\,\mathrm{d}x.$$

Let us compare these quantities to their analogues from section 3. Clearly δ, G, K are exactly the same integral, with the only difference that they are now defined on the translated cylinder $Q_r^*(0,0)$. The quantity $F_*(r)$ corresponds to the quantity $F(r)$ with $q = \frac{3}{2}$ fixed and again $Q_r(0,0)$ swapped by $Q_r^*(0,0)$. The function $\delta_*(r)$ is used to provide a shorthand way of writing down the regularity condition in Proposition 2, i.e. $\limsup_{r\to 0} \delta_*(r) \le \frac{1}{2}\varepsilon_3$. We define the function

$$M_*(r) = G_*^{\frac{2}{3}}(r) + H_*(r) + J_*(r) + K_*^{\frac{8}{5}}(r),$$

which satisfies the following decay estimate.

Proposition 3. *Let $\rho > 0$ and let (u,p) be a suitable weak solution of the Navier-Stokes System with force f on the cylinder $Q_\rho^*(0,0)$. If it holds $\delta_*(\rho) \le 1$ and $F_*(\rho) \le 1$, then the following decay estimate holds*

$$M_*(r) \le C\left[\left(\frac{r}{\rho}\right)^{\frac{1}{5}} M_*(\rho) + \left(\frac{\rho}{r}\right)^2 \left(M_*^{\frac{1}{2}}(\rho)\delta_*^{\frac{1}{2}}(\rho) + M_*(\rho)\delta_*(\rho) + F_*(\rho) + \delta_*(\rho)\right)\right]$$

for some constant $C > 0$ and all $0 < r \le \frac{1}{4}\rho$. Moreover $M_*(r)$ is finite for all $r \le \frac{1}{4}\rho$.

Corollary 1. *There exists absolute constants $\varepsilon_1, \varepsilon_2 > 0$ such that the following holds. We consider a cylinder $Q_r(t,x)$ and any suitable weak solution of the Navier Stokes system in the given cylinders with a force term $f \in L^q$ for $q > \frac{5}{2}$. Suppose that*

$$r^{-2}\int_{Q_r(t,x)} |u|^3 + |u|\,|p| \,\mathrm{d}(s,y) + r^{-\frac{13}{4}}\int_{t-r^2}^{t}\left(\int_{B_r(x)} |p| \,\mathrm{d}y\right)^{\frac{5}{4}} \mathrm{d}s \le \varepsilon_1$$

and

$$F_q(r) = r^{3q-5}\int_{Q_r(t,x)} |f|^q \,\mathrm{d}(s,y) \le \varepsilon_2,$$

then it must hold $|u| \le Cr^{-1}$ Lebesgue almost everywhere in the smaller cylinder $Q_{\frac{r}{2}}(t,x)$. In particular u is regular on $Q_{\frac{r}{2}}(t,x)$.

Proof of Proposition 2. By translation of (u,p) we may assume that $(t,x) = (0,0)$. Let (u,p) be a suitable weak solution of the Navier Stokes System in a neighborhood D of $(0,0)$. We want to apply Corollary 1 and verify its assumptions to prove that $(0,0)$ is a regular point. It holds $Q_r^* = Q_r(\frac{1}{8}r^2, 0)$ which suggest that we can use Corollary 1 applied to the point $(\frac{1}{8}r^2, 0)$. Let $r \le 1$ such that $Q_r^* \subset D$, then it holds

$$F_q(r) = r^{3q-5}\int_{Q_r} |f|^q \,\mathrm{d}(t,x) \le r^{\frac{5}{2}}\int_D |f|^q \,\mathrm{d}(t,x),$$

whence $\lim_{r\to 0} F_q(r) = 0$ due to the fact that $f \in L^1(D)$. This shows that, by Corollary 1, the point $(0,0) \in Q_{\frac{r}{2}}(\frac{1}{8}r^2, 0)$ is regular if for example it holds

$$\liminf_{r\to 0} r^{-2}\int_{Q_r(0,\frac{1}{8}r^2)} |u|^3 + |u|\,|p| \,\mathrm{d}(t,x) + r^{-\frac{13}{4}}\int_{-\frac{7}{8}r^2}^{\frac{1}{8}r^2}\left(\int_{B_r(0)} |p| \,\mathrm{d}y\right)^{\frac{5}{4}} \mathrm{d}s \le \varepsilon_1$$

which can be written as

$$\liminf_{r\to 0} G_*(r) + J_*(r) + K_*(r) \le \varepsilon_1.$$

Due to the nonnegativity of the involved terms the latter condition is clearly verified if it holds

$$\liminf_{r\to 0} M_*(r) \le \tilde{\varepsilon}_1 := \min\left\{\frac{\varepsilon_1}{3}, \left(\frac{\varepsilon_1}{3}\right)^{\frac{2}{3}}, \left(\frac{\varepsilon_1}{3}\right)^{\frac{8}{5}}\right\}.$$

We claim that there are constants $\varepsilon_3 \in (0,1]$ and $\gamma \in (0, \frac{1}{4}]$ such that whenever it holds

$$M_*(\rho) > \tilde{\varepsilon}_1, F_*(\rho) \le \varepsilon_3 \text{ and } \delta_*(\rho) \le \varepsilon_3$$

for some $\rho > 0$ with $Q^*_\rho \subset D$ it follows that $M_*(\gamma\rho) \le \frac{1}{2}M_*(\rho)$. To show the existence of such constants we choose

$$\gamma < \min\left\{\frac{1}{(C6)^5}, \frac{1}{4}\right\}$$

and then $\varepsilon_3 > 0$ such that

$$\varepsilon_3 < \min\left\{\frac{1}{12C}\gamma^2\tilde{\varepsilon}_1, 1\right\} \text{ and } \varepsilon_3 + \left(\frac{\varepsilon_3}{\tilde{\varepsilon}_1}\right)^{\frac{1}{2}} \le \frac{\gamma^2}{6C}.$$

Let us suppose that $M_*(\rho) > \tilde{\varepsilon}_1$, that $F_*(\rho) \le \varepsilon_3$ and that $\delta_*(\rho) \le \varepsilon_3$. In this case it holds

$$M_*^{\frac{1}{2}}(\rho) < \tilde{\varepsilon}_1^{-\frac{1}{2}} M_*(\rho).$$

Using the decay estimate from Proposition 3 we deduce

$$M_*(r) \le C\left\{\left(\frac{r}{\rho}\right)^{\frac{1}{5}} M_*(\rho) + \left(\frac{\rho}{r}\right)^2 \left[\varepsilon_3 + \left(\frac{\varepsilon_3}{\varepsilon_1}\right)^{\frac{1}{2}}\right] M_*(\rho) + 2\left(\frac{\rho}{r}\right)^2 \varepsilon_3\right\}$$

for all $r \le \frac{1}{4}\rho$. Choosing $r = \gamma\rho \le \frac{1}{4}\rho$ and using the assumptions on γ and ε_3 we deduce

$$\begin{aligned} M_*(\gamma\rho) &\le C\left\{\gamma^{\frac{1}{5}} M_*(\rho) + \left(\frac{1}{\gamma}\right)^2 \left[\varepsilon_3 + \left(\frac{\varepsilon_3}{\varepsilon_1}\right)^{\frac{1}{2}}\right] M_*(\rho) + 2\left(\frac{1}{\gamma}\right)^2 \varepsilon_3\right\} \\ &\le \frac{1}{6}M_*(\rho) + \frac{1}{6}M_*(\rho) + \frac{1}{6}\tilde{\varepsilon}_1 \le \frac{1}{2}M_*(\rho). \end{aligned}$$

Now let us show that

$$\liminf_{r\to 0} M_*(r) \le \tilde{\varepsilon}_1.$$

We first note that due to $q > \frac{5}{4}$ it holds

$$F_*(r) = r^{-\frac{1}{2}} \int_{Q^*_r} |f|^{\frac{3}{2}} \,\mathrm{d}(t,x) \le C\left(\int_{Q^*_r} |f|^q \,\mathrm{d}(t,x)\right)^{\frac{3}{2q}} r^{\frac{9}{2}-\frac{15}{2}q} \le Cr^{\frac{3}{2}} \left(\int_D |f|^q \,\mathrm{d}(t,x)\right)^{\frac{3}{2q}}$$

for all $r \le 1$ such that $Q^*_r \subset D$ by Hölder's inequality. This shows $\lim_{r\to 0} F_*(r) = 0$, which together with the assumption yields a radius $r_0 > 0$ such that $F_*(r) \le \varepsilon_3$ and $\delta_*(r) \le \varepsilon_3$ for all $r < r_0$. This is due to the assumption that $\limsup_{r\to 0} \delta_*(r) \le \frac{1}{2}\varepsilon_3 < \varepsilon_3$. Let us now suppose that $\liminf_{r\to 0} M_*(r) > \tilde{\varepsilon}_1$. We claim that there is $N \in \mathbb{N}$ such that $M_*(\gamma^N r_0) \le \tilde{\varepsilon}_1$. Assuming the opposite would be true it must hold that $M_*(\gamma^n r_0) > \tilde{\varepsilon}_1$ for all $n \in \mathbb{N}$. Consequently as we have proven before it follows that

$$M_*(\gamma^n r_0) \le \left(\frac{1}{2}\right)^n M_*(r_0)$$

for all $n \in \mathbb{N}$, which is a contradiction to $\liminf_{r\to 0} M_*(r) > \tilde{\varepsilon}_1$. This is only due to the fact that $M_*(r_0)$ is finite. Hence, we may assume that $M_*(\gamma^N r_0) \le \tilde{\varepsilon}_1$ for some $N \in \mathbb{N}$. Now if it were true that $M_*(\gamma^{N+1} r_0) > \tilde{\varepsilon}_1$ we could conclude that $\tilde{\varepsilon}_1 < M_*(\gamma^{N+1} r_0) \le \frac{1}{2}M_*(\gamma^N r_0) \le \frac{1}{2}\tilde{\varepsilon}_1$ which is a contradiction. By induction it follows that $M_*(\gamma^{N+k} r_0) \le \tilde{\varepsilon}_1$ for all $k \in \mathbb{N}$, whence $\liminf_{r\to 0} M_*(r) \le \tilde{\varepsilon}_1$. This shows that $(0,0)$ is a regular point. □

In preparation of the proof of the decay estimate we are going to start with an bound of H_* in terms of $G_*(r)$, $\delta_*(r)$ and in terms of $A_*(r)$, which is given by

$$A_*(r) = \sup_{-\frac{7}{8}r^2<t<\frac{1}{8}r^2} r^{-1}\int_{\{t\}\times B_r(0)} |u|^2(t,\cdot)\mathrm{d}x.$$

Let us fix a suitable weak solution (u,p) of the Navier Stokes system in a neighborhood D of $(0,0)$. Let $r>0$ such that $Q_r^* \subset D$. Clearly it holds that $A_*(r) \le r^{-1}E_0 < \infty$.

LEMMA 5.1. *For any r such that $Q_r^* \subset D$ it holds*

$$H_*(r) \le C(G_*^{\frac{2}{3}}(r) + A_*(r)\delta_*(r))$$

for some constant $C>0$.

PROOF. At almost every time t it holds

$$\begin{aligned}
&\int_{B_r(0)} |u(t,x)|\left||u|^2(t,x) - \overline{|u|_r^2}(t)\right|\mathrm{d}x\\
&\le \left(\int_{B_r(0)}|u|^3(t)\mathrm{d}x\right)^{\frac{1}{3}}\left(\int_{B_r(0)}\left||u|^2(t) - \overline{|u|_r^2}(t)\right|^{\frac{3}{2}}\mathrm{d}x\right)^{\frac{2}{3}}\\
&\le C\left(\int_{B_r(0)}|u|^3(t)\mathrm{d}x\right)^{\frac{1}{3}}\int_{B_r(0)}\left|\nabla|u|^2\right|(t)\mathrm{d}x\\
&\le C\left(\int_{B_r(0)}|u|^3(t)\mathrm{d}x\right)^{\frac{1}{3}}\int_{B_r(0)}|\nabla u|(t)\,|u|(t)\mathrm{d}x\\
&\le C\left(\int_{B_r(0)}|u|^3(t)\mathrm{d}x\right)^{\frac{1}{3}}\left(\int_{B_r(0)}|\nabla u|^2(t)\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{B_r(0)}|u|^2(t)\mathrm{d}x\right)^{\frac{1}{2}}\\
&\le C\left(\int_{B_r(0)}|u|^3(t)\mathrm{d}x\right)^{\frac{1}{3}}(rA_*(r))^{\frac{1}{2}}\left(\int_{B_r(0)}|u|^2(t)\mathrm{d}x\right)^{\frac{1}{2}},
\end{aligned}$$

where we have used Hölder's inequality, the Poincaré inequality on the ball $B_r(0)$, the Cauchy-Schwarz inequality and the definition of $A_*(r)$. Integration in time from $-\frac{7}{8}r^2$ to $\frac{1}{8}r^2$ yields

$$\begin{aligned}
r^2H_*(r) &\le C\,(rA_*(r))^{\frac{1}{2}}\int_{-\frac{7}{8}r^2}^{\frac{1}{8}r^2}\left(\int_{B_r(0)}|u|(t)^3\mathrm{d}x\right)^{\frac{1}{3}}\left(\int_{B_r(0)}|u|^2(t)\mathrm{d}x\right)^{\frac{1}{2}}\mathrm{d}t\\
&\le (rA_*(r))^{\frac{1}{2}}\left(\int_{Q_r^*}|u|^3\mathrm{d}(t,x)\right)^{\frac{1}{3}}\left(\int_{Q_r^*}|\nabla u|^2\mathrm{d}(t,x)\right)^{\frac{1}{2}} r^{\frac{1}{3}}\\
&= r^2A_*^{\frac{1}{2}}(r)G_*^{\frac{1}{3}}(r)\delta_*(r)^{\frac{1}{2}}
\end{aligned}$$

by Hölders inequality. Applying Young's inequality we conclude

$$H_*(r) \le Cr^{-\frac{1}{6}}A_*^{\frac{1}{2}}(r)G_*^{\frac{1}{3}}(r)\delta_*(r)^{\frac{1}{2}} \le C(G_*^{\frac{2}{3}}(r) + A_*(r)\delta_*(r)).$$

□

REMARK 1. Let $r>0$ such that $Q_r^* \subset D$, then $A_*(r) \le r^{-1}E_0$ and $\delta_*(r) \le r^{-1}E_1$. It can be shown similar to Lemma 3.1 that $G_*(r)$ can be bounded by $A_*(r)$ and $\delta_*(r)$ and thus must be finite. This is going to be proven in the next talk given by Marius. By Lemma 5.1 we deduce that $H_*(r)$ is finite as well. Furthermore, due to the fact that $p \in L^{\frac{5}{4}}(D)$ it holds that $K_*(r)$ is finite. Finally $J_*(r)$ can be bounded by $A_*(r), \delta_*(r), G_*(r)$ and

$K_*(r)$, whence $M_*(r)$ must be finite. The latter claim is going to be shown in the talk given by Marius, too.

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