Navier-Stokes Seminar: Caffarelli-Kohn-Nirenberg Theory

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Universität Ulm, Summer 2019

Preface

These are lecture notes geberated by the seminar course on the Caffarelli-Kohn-Nirenberg Theory for the Navier-Stokes equations at the Universität Ulm in the summer term of 2019. We mainly follow the [**1**] in a modern fashion. This work is aimed at enthusiastic Masters and PhD students.

I would like to thank everyone taking the seminar for typing parts of these notes.

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CHAPTER 1

Talk 1: Introduction

In this seminar course, we consider the three-dimensional Navier-Stokes equations given by

$$\begin{aligned} \partial_t u(x,t) + (u \cdot \nabla) u(x,t) + \nabla p(x,t) - \Delta u(x,t) &= f(x,t) \\ \operatorname{div} u(x,t) &= 0. \end{aligned} \tag{1.1}$$

Here, $(x,t) \in \Omega \times [0,T]$, where $\Omega \subset \mathbb{R}^3$ some domain, and we have the unknown velocity field

$$u \colon \Omega \times [0,T] \to \mathbb{R}^3;$$

the unknown pressure field

$$p \colon \Omega \times [0,T] \to \mathbb{R};$$

and the given force $f \colon \Omega \times [0,T] \to \mathbb{R}^3$ with $\operatorname{div} f = 0$ in $\Omega \times [0,T]$. Together with initial data and boundary data, (1.1) turns into an initial boundary value problem

$$\begin{aligned} u(x,0) &= u_0(x), && x \in \Omega, \\ u(x,t) &= 0, && x \in \partial\Omega, 0 < t < T. \end{aligned} \tag{1.2}$$

CHAPTER 2

Talk 4: Background and Definitions

2.1. On the initial boundary value problem

First, note that the condition $\operatorname{div} f = 0$ is not a restriction at all. Indeed, suppose we want to solve (1.1) for a general force $f \in L^q(\Omega)$ with $1 < q < \infty$. We may apply a L^q-Helmholtz decomposition to write $f = \nabla\Phi + f_1$ with $\operatorname{div} f_1 = 0$ and $\|f_1\|_{L^q(\Omega\times[0,T])} \le C(q,\Omega)\,\|f\|_{L^q(\Omega\times[0,T])}$. If (u,p) is a solution of (1.1) with the force term f_1, it is easy to see that $(u, p+\Phi)$ is a solution to (1.1) with the right hand side $\nabla\Phi + f_1 = f$ as desired.

To obtain an existence theory for arbitrary time intervals, we study weak solutions of (1.1) for which the energy

$$\operatorname{ess\,sup}_{0<t<T} \int_\Omega |u|^2 \,\mathrm{d}x + \int_0^T \int_\Omega |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}t < \infty, \tag{2.1}$$

is finite, where $|\nabla u|^2 := \sum_{i,j} \left|\partial_i u^j\right|^2$. This choice is motivated by multiplying (1.1) by u, integration and using integration by parts. (2.1) justifies why requiring a solution u to have space derivatives of first order is a somewhat physical assumption.

If one instead multiplies (1.1) by $2u\phi$ for some $\phi \in \mathcal{C}^\infty(\Omega\times[0,T])$ and integrates one obtains

$$\int_0^t \int_\Omega 2\partial_t u\cdot u\phi + 2\left((u\cdot\nabla)u\right)\cdot u\phi - 2\Delta u\cdot u\phi + 2\nabla p\cdot u\phi \,\mathrm{d}x = \int_0^t \int_\Omega 2f\cdot u\phi \,\mathrm{d}x. \tag{2.2}$$

Since $u|_{\partial\Omega} = 0$ by (1.2), we may use integration by parts without creating any boundary terms. For the first term, we use $\partial_t |u|^2 = 2\partial_t u\cdot u$, so

$$\begin{aligned}\int_0^t \int_\Omega 2\partial_t u\cdot u\phi \,\mathrm{d}x\,\mathrm{d}t &= \int_0^t \partial_t \int_\Omega |u|^2 \phi \,\mathrm{d}x\,\mathrm{d}t - \int_\Omega |u|^2 \partial_t\phi \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_\Omega |u(t)|^2 \phi \,\mathrm{d}x - \int_\Omega |u(0)|^2 \phi \,\mathrm{d}x - \int_\Omega |u|^2 \partial_t\phi \,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{2.3}$$

For the second part, integration by parts yields, using summation convention,

$$\begin{aligned}\int_0^t \int_\Omega 2u^i \partial_i u^j u^j \phi \,\mathrm{d}x\,\mathrm{d}t &= -\int_0^t \int_\Omega |u|^2 \partial_i u^i \phi \,\mathrm{d}x\,\mathrm{d}t - \int_\Omega |u|^2 u^i \partial_i\phi \,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_0^t \int_\Omega |u|^2 u\cdot\nabla\phi \,\mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{2.4}$$

since $\partial_i |u|^2 = 2\partial_i u^j u^j$ and $\operatorname{div} u = 0$ by (1.1). For the third term, we get using $\partial_i |u|^2 = 2\partial_i u^j u^j$ again

$$\begin{aligned}-2\int_0^t \int_\Omega \partial_i\partial_i u^j u^j \phi \,\mathrm{d}x &= 2\int_0^t \int_\Omega |\nabla u|^2 \phi \,\mathrm{d}x\,\mathrm{d}t + 2\int_0^t \int_\Omega \partial_i u^j u^j \partial_i\phi \,\mathrm{d}x\,\mathrm{d}t \\ &= 2\int_0^t \int_\Omega |\nabla u|^2 \phi \,\mathrm{d}x\,\mathrm{d}t - \int_0^t \int_\Omega |u|^2 \partial_i\partial_i\phi \,\mathrm{d}x\,\mathrm{d}t \\ &= 2\int_0^t \int_\Omega |\nabla u|^2 \phi \,\mathrm{d}x\,\mathrm{d}t - \int_0^t \int_\Omega |u|^2 \Delta\phi \,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{2.5}$$

Finally, for the last term, using $\operatorname{div} u = 0$, we have

$$2\int_0^t\int_\Omega \partial_i p u^i \phi \,\mathrm{d}x\,\mathrm{d}t = -2\int_0^t\int_\Omega p\partial_i u^i\phi\,\mathrm{d}x\,\mathrm{d}t - 2\int_0^t\int_\Omega pu^i\partial_i\phi\,\mathrm{d}x\,\mathrm{d}t \tag{2.6}$$
$$= -2\int_0^t\int_\Omega pu\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t.$$

Combining, (2.2),(2.3),(2.4),(2.5) and (2.6), we get

$$\int_\Omega |u(t)|^2\phi\,\mathrm{d}x + 2\int_0^t\int_\Omega |\nabla u|^2\phi\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega |u_0|^2\phi\,\mathrm{d}x + \int_0^t\int_\Omega |u|^2(\partial_t\phi+\Delta\phi)\,\mathrm{d}x\,\mathrm{d}t \tag{2.7}$$
$$+\int_0^t\int_\Omega (|u|^2+2p)u\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t + 2\int_0^t\int_\Omega f\cdot u\phi\,\mathrm{d}x\,\mathrm{d}t.$$

Pluggin in $\phi\equiv 1$ in (2.7) we obtain

$$\int_\Omega |u(t)|^2\,\mathrm{d}x + 2\int_0^t\int_\Omega |\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega |u_0|^2 + 2\int_0^t\int_\Omega f\cdot u\,\mathrm{d}x. \tag{2.8}$$

Note that for $f\equiv 0$ in (2.8), we may formally conclude (2.1) with an explicit bound depending on the initial date $u_0\in L^2(\Omega)$. The key point in proving existence of weak *Leray-Hopf solutions* is the *energy inequality*, an inequality form of (2.8).

$$\int_\Omega |u(t)|^2\,\mathrm{d}x + 2\int_0^t\int_\Omega |\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t \le \int_\Omega |u_0|^2 + 2\int_0^t\int_\Omega f\cdot u\,\mathrm{d}x, \tag{2.9}$$

for almost every t.

For the main result, the localized version of (2.9) is crucial. Taking any $\phi\ge 0$ with compact support in $\Omega\times(0,T)$ in (2.7), one may conclude the following *generalized energy inequality* by estimating the first term by zero

$$2\int_0^T\int_\Omega |\nabla u|^2\phi\,\mathrm{d}x\,\mathrm{d}t \le \int_0^T\int_\Omega \left[|u|^2(\partial_t\phi+\Delta\phi) + (|u|^2+2p)u\cdot\nabla\phi + 2u\cdot f\phi\right]\,\mathrm{d}x\,\mathrm{d}t. \tag{2.10}$$

By definition, any *suitable weak solution* satisfies (2.10). Last week, we saw that such a suitable weak solution in fact exists (cf. David's talk Lemma 2.2, Theorem 2.5, Farid's talk Lemma 1.3).

DEFINITION 2.1. We call a pair (u,p) a *suitable weak solution* to the Navier-Stokes equation with force f on $\Omega\times(0,T)$ if the following conditions are satisfied.

(1) u,p,f are measureable on $\Omega\times(0,T)$ and
   (a) $f\in L^q(\Omega\times(0,T))$ for $q>\frac{5}{2}$ and $\operatorname{div} f = 0$,
   (b) $p\in L^{5/4}(\Omega\times(0,T))$
   (c) for some $E_0, E_1<\infty$ we have

$$\int_\Omega |u|^2\,\mathrm{d}x \le E_0 \text{ for almost every } t\in(0,T), \text{ and} \tag{2.11}$$
$$\int_0^T\int_\Omega |\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t \le E_1. \tag{2.12}$$

(2) u,p and f satisfy (1.1) in the sense of distributions on $\Omega\times(0,T)$.
(3) For each $\phi\in\mathcal{C}_0^\infty(\Omega\times(0,T))$ with $\phi\ge 0$, inequality (2.10) holds.

Even for a suitable weak solution, it is not immediately clear that the right hand side of (2.10) is well, defined, i.e. it is not obvious that the integrals

$$\int_0^T \int_\Omega |u|^2 u \cdot \nabla\phi \, \mathrm{d}x \, \mathrm{d}t \quad \text{and} \quad \int_0^T \int_\Omega pu \cdot \nabla\phi \, \mathrm{d}x \, \mathrm{d}t$$

do exist. We will prove that this is the case.

2.2. Higher Regularity

Recall that a point (x,t) in space-time is *regular* if $u \in L^\infty_{loc}(U)$ for an open neighborhood U of (x,t). This is justified by the following result due to Serrin [**2**]. If u is a weak solution of (1.1) on a cylinder $\Omega \times (a,b)$ satisfying

$$\int_a^b \left(\int_\Omega |u|^q \, \mathrm{d}x\right)^{s/q} \mathrm{d}t < \infty \text{ with } \frac{3}{q} + \frac{2}{s} < 1, \tag{2.13}$$

then u us necessarily $\mathcal{C}^{m+2,\beta}$ in space on compact subsets of Ω, provided f is $\mathcal{C}^{m,\beta}$ in space with $m \geq 0$ and $0 < \beta < 1$. In particular if f is $\mathcal{C}^\infty$ in space and (2.13) is satisfied, then u is $\mathcal{C}^\infty$ in space. Regularity in time is more difficult. If $u \in L^\infty(0,T;L^3(U))$, then u is Hölder continuous in time. From this, if $u \in L^\infty_{loc}(U)$ in a neighborhood U of (x,t), then (2.13) clearly holds, so u is smooth in space, provided f is smooth in space.

2.3. Recurrent Themes

The following three observations will be used frequently.

2.3.1. Interpolation inequalities for u and p. If $B_r \subset \mathbb{R}^3$ be a ball of radius $r > 0$ and let $u \in H^1(B_r)$. Then, the *Gagliardo-Nirenberg-Sobolev inequality* yields

$$\int_{B_r} |u|^q \, \mathrm{d}x \leq C\left(\int_{B_r} |\nabla u|^2 \, \mathrm{d}x\right)^a \left(\int_{B_r} |u|^2 \, \mathrm{d}x\right)^{q/2-a} + \frac{C}{r^{2a}}\left(\int_{B_r} |u|^2 \, \mathrm{d}x\right)^{q/2}, \tag{2.14}$$

where $C > 0$, $2 \leq q \leq 6$ and $a = \frac{3}{4}(q-2)$. If B_r is replaced by $\mathbb{R}^3$ the second term on the right in (2.14) can be omitted. Inequality (2.14) follows from the classical Gagliardo-Nirenberg-Sobolev inequality [**3**] by applying an extension operator to $u \in H^1(B_r)$. The term $\frac{1}{r^{2a}}$ makes (2.14) scaling invariant with respect to $r > 0$.

We will now use (2.14) to interpolate between (2.11) and (2.12). Take $q = \frac{10}{3}$ so $a = 1$ in (2.14) and integrate in time. Then

$$\int_0^T \int_{B_r} |u|^{10/3} \, \mathrm{d}x \, \mathrm{d}t \leq C\left(E_0^{2/3} E_1 + r^{-2} E_0^{5/3} T\right). \tag{2.15}$$

A particular consequence is that $u \in L^3(\Omega \times (0,T))$, hence

$$\left|\int_0^T \int_\Omega |u|^2 u \cdot \nabla\phi \, \mathrm{d}x \, \mathrm{d}t\right| \leq \|\nabla\phi\|_{L^\infty(\Omega\times(0,T))} \|u\|_{L^3(\Omega\times(0,T))} < \infty,$$

so the corresponding term in (2.10) is in fact finite if u is a suitable weak solution and $\phi \in \mathcal{C}^\infty(\Omega \times (0,T))$. Moreover, if $q = \frac{5}{2}$, so $a = \frac{3}{8}$ we get

$$\int_0^T \left(\int_{B_r} |u|^{5/2} \, \mathrm{d}x\right)^{8/3} \mathrm{d}t \leq C(E_0^{7/3} E_1 + r^{-2} E_0^{10/3} T). \tag{2.16}$$

If we take the (distributional) divergence of (1.1), we get

$$0 = \Delta p + \partial_i\left(u^j \partial_j u^i\right) = \Delta p + \partial_i \partial_j (u^j u^i),$$

hence

$$\Delta p = -\partial_i \partial_j (u^i u^j) \text{ on } \Omega \times (0,T) \text{ in the sense of distributions.} \tag{2.17}$$

In addition, any solution $u \in \mathcal{C}^1(0,T;\mathcal{C}^2(\overline{\Omega}))$ of (1.1) on $\overline{\Omega} \times (0,T)$ for $f \equiv 0$ satisfying (1.2) has to fulfill

$$\nu \cdot \nabla p = \nu \cdot \Delta u \text{ on } \partial\Omega \times (0,T),$$

by simply restricting (1.1) to $\partial\Omega$ and multiplying with ν.
Recall that in $\mathbb{R}^3$, the unique solution to $-\Delta v = f$, with $f \in L^q(\mathbb{R}^3)$ is given by

$$v(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|} f(y) \, \mathrm{d}y.$$

We may thus rewrite (2.17) as $p = (-\Delta)^{-1}\partial_i\partial_j(u^i u^j)$.

First, we consider the case $\Omega = \mathbb{R}^3$. For u smooth enough, we have

$$p(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|} \partial_{y_i}\partial_{y_j}(u^i u^j) \, \mathrm{d}y = \alpha_{ij} u^i(x) u^j(x) + \frac{1}{4\pi} \int_{\mathbb{R}^3} \partial_{y_i}\partial_{y_j} \left(\frac{1}{|x-y|} \right) u^i u^j \, \mathrm{d}y,$$

where the latter has to be understood as a singular integral, i.e. a principal value $\lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon}$.
Also note that $\alpha_{ij} = 0$ if $i \neq j$.

We now use standard Calderón-Zygmund theory, see for instance [**5**]. To that end, fix $i,j \in \{1,\ldots,3\}$ and consider the convolution operator

$$S_{ij} f = \frac{1}{4\pi} \int_{\mathbb{R}^3} \partial_{y_j}\partial_{y_i} \left(\frac{1}{|x-y|} \right) f \, \mathrm{d}y.$$

A computation yields $\partial_{y_j}\partial_{y_i}\left(\frac{1}{|x-y|}\right) = -\frac{\delta_{ij}}{|x-y|^3} + 3\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^5}$. We may write

$$S_{ij} f(x) = \int_{\mathbb{R}^3} \frac{\Omega(x-y)}{|x-y|^3} f(y) \, \mathrm{d}y,$$

with $\Omega(y) = -\delta_{ij} + 3\frac{y_i y_j}{|y|^2}$. Note that Ω is homogeneous of degree 0 and a computation shows $\int_{\mathbb{S}^2} \Omega(y) \, \mathrm{d}S(y) = 0$ for all i,j. Clearly, Ω is Lipschitz on $\mathbb{S}^2$. Thus, by Calderón-Zygmund theory [**5**, §4.3, Theorem 3],

$$S_{ij} \colon L^q(\mathbb{R}^3) \to L^q(\mathbb{R}^3) \text{ is bounded for any } 1 < q < \infty, i,j = 1,\ldots,3. \tag{2.18}$$

As a consequence

$$\|p\|_{L^q(\mathbb{R}^3)} = \left\|(-\Delta)^{-1}\partial_i\partial_j(u^i u^j)\right\|_{L^q(\mathbb{R}^3)} \leq C \sum_{i,j} \left\|u^i u^j\right\|_{L^q(\mathbb{R}^3)},$$

for some $C = C(q) > 0$ and

$$\left\|u^i u^j\right\|_{L^q(\mathbb{R}^3)}^q = \int_{\mathbb{R}^3} \left|u^i u^j\right|^q \, \mathrm{d}x \leq \int_{\mathbb{R}^3} |u|^{2q} \, \mathrm{d}x.$$

This yields

$$\int_{\mathbb{R}^3} |p|^q \, \mathrm{d}x \leq C \int_{\mathbb{R}^3} |u|^{2q} \, \mathrm{d}x.$$

In particular, if (u,p) is a suitable weak solution of (1.1) on $\mathbb{R}^3 \times (0,T)$ we have

$$\int_0^T \int_{\mathbb{R}^3} |p|^{5/3} \, \mathrm{d}x \, \mathrm{d}t \leq C \int_0^T \int_{\mathbb{R}^3} |u|^{10/3} \, \mathrm{d}x \, \mathrm{d}t \leq C E_0^{2/3} E_1$$

by (2.15) using that we don't need the second term in (2.14) since we are in the whole space $\mathbb{R}^3$.

For general $\Omega \subset \mathbb{R}^3$ bounded, let $\overline{\Omega}_1 \subset \Omega$ and $\phi \in \mathcal{C}_0^\infty(\Omega)$ with $\phi \equiv 1$ in a neighborhood U of $\overline{\Omega}_1$. Then for t fixed we have using

$$\begin{aligned} \phi(x)p(x,t) &= -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|x-y|}\Delta_y(\phi p)\,\mathrm{d}y \qquad (2.19)\\ &= -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|x-y|}\left[p\Delta\phi + 2\langle\nabla\phi,\nabla p\rangle + \phi\Delta p\right]\,\mathrm{d}y. \end{aligned}$$

We plug in (2.17) for Δp in (2.19) and obtain using summation convention

$$\phi p = -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|x-y|}\left[p\Delta\phi + 2\langle\nabla\phi,\nabla p\rangle - \phi\partial_i\partial_j(u^iu^j)\right]\,\mathrm{d}y. \qquad (2.20)$$

Now, we integrate by parts to remove all derivatives on p and u. Note that in order to do this in a precise way, you have to cut out a ball B_ε of radius ε and do integration by parts there. However, since $\partial_{y_i}\left(\frac{1}{|x-y|}\right)$ is $L^1_{loc}(\mathbb{R}^3)$, the boundary terms will vanish as $\varepsilon \to 0$. We have

$$\int_{\mathbb{R}^3}\frac{1}{|x-y|}\langle\nabla\phi,\nabla p\rangle\,\mathrm{d}y = -\int_{\mathbb{R}^3}\partial_{y_i}\left(\frac{1}{|x-y|}\right)\partial_i\phi p\,\mathrm{d}y - \int_{\mathbb{R}^3}\frac{1}{|x-y|}\Delta\phi p\,\mathrm{d}y. \qquad (2.21)$$

For the last term in (2.20) we have

$$\begin{aligned} \int_{\mathbb{R}^3}\frac{1}{|x-y|}\phi\partial_i\partial_j(u^iu^j)\,\mathrm{d}y &= -\int_{\mathbb{R}^3}\partial_{y_i}\left(\frac{1}{|x-y|}\right)\phi\partial_j(u^iu^j)\,\mathrm{d}y \qquad (2.22)\\ &\quad -\int_{\mathbb{R}^3}\frac{1}{|x-y|}\partial_i\phi\partial_j(u^iu^j)\,\mathrm{d}y\\ &= \int_{\mathbb{R}^3}\partial_{y_j}\partial_{y_i}\left(\frac{1}{|x-y|}\right)\phi u^iu^j\,\mathrm{d}y + \int_{\mathbb{R}^3}\partial_{y_i}\left(\frac{1}{|x-y|}\right)\partial_j\phi u^iu^j\,\mathrm{d}y\\ &\quad + \int_{\mathbb{R}^3}\partial_{y_j}\left(\frac{1}{|x-y|}\right)\partial_i\phi u^iu^j\,\mathrm{d}y + \int_{\mathbb{R}^3}\frac{1}{|x-y|}\partial_i\partial_j\phi u^iu^j\,\mathrm{d}y\\ &= \int_{\mathbb{R}^3}\partial_{y_j}\partial_{y_i}\left(\frac{1}{|x-y|}\right)\phi u^iu^j\,\mathrm{d}y + \int_{\mathbb{R}^3}\frac{x_i-y_i}{|x-y|^3}\partial_j\phi u^iu^j\,\mathrm{d}y\\ &\quad + \int_{\mathbb{R}^3}\frac{x_j-y_j}{|x-y|^3}\partial_i\phi u^iu^j\,\mathrm{d}y + \int_{\mathbb{R}^3}\frac{1}{|x-y|}\partial_i\partial_j\phi u^iu^j\,\mathrm{d}y\\ &= \int_{\mathbb{R}^3}\partial_{y_j}\partial_{y_i}\left(\frac{1}{|x-y|}\right)\phi u^iu^j\,\mathrm{d}y + 2\int_{\mathbb{R}^3}\frac{x_i-y_i}{|x-y|^3}\partial_j\phi u^iu^j\,\mathrm{d}y\\ &\quad + \int_{\mathbb{R}^3}\frac{1}{|x-y|}\partial_i\partial_j\phi u^iu^j\,\mathrm{d}y \end{aligned}$$

Therefore, combining (2.19), (2.20), (2.21) and (2.22) we get

$$p\phi = \tilde{p} + p_3 + p_4 \qquad (2.23)$$

with

$$\begin{aligned} \tilde{p} &= \alpha_{ij}u^i(x)u^j(x) + \frac{1}{4\pi}\int_{\mathbb{R}^3}\partial_{y_j}\partial_{y_i}\left(\frac{1}{|x-y|}\right)\phi u^iu^j\,\mathrm{d}y\\ p_3 &= \frac{1}{2\pi}\int_{\mathbb{R}^3}\frac{x_i-y_i}{|x-y|^3}\partial_j\phi u^iu^j\,\mathrm{d}y + \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|x-y|}\partial_i\partial_j\phi u^iu^j\,\mathrm{d}y\\ p_4 &= \left(-\frac{1}{4\pi} + \frac{2}{4\pi}\right)\int_{\mathbb{R}^3}\frac{1}{|x-y|}p\Delta\phi\,\mathrm{d}y + \frac{1}{2\pi}\int_{\mathbb{R}^3}\frac{x_i-y_i}{|x-y|^3}\partial_i\phi p\,\mathrm{d}y. \end{aligned}$$

Note that we have for $x \in \Omega_1$, using $\phi \equiv 1$ on U and $\phi \equiv 0$ on $\mathbb{R}^3 \setminus \Omega$

$$\begin{aligned}|p_3|(x,t) &\le \left|\frac{1}{2\pi}\int_{\mathbb{R}^3} \frac{x_i - y_i}{|x-y|^3}\partial_j \phi u^i u^j \, \mathrm{d}y\right| + \left|\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|x-y|}\partial_i\partial_j \phi u^i u^j \, \mathrm{d}y\right| \\ &\le \frac{1}{2\pi}\int_{\Omega\setminus U} \frac{1}{|x-y|^2}|\partial_j\phi|\,|u|^2 \, \mathrm{d}y + \frac{1}{4\pi}\int_{\Omega\setminus U}\frac{1}{|x-y|}|\partial_i\partial_j\phi|\,|u|^2 \, \mathrm{d}y \\ &\le \frac{\|\phi\|_{\mathcal{C}^1}}{2\pi\delta^2}\int_\Omega |u|^2 \, \mathrm{d}y + \frac{\|\phi\|_{\mathcal{C}^2}}{4\pi\delta}\int_\Omega |u|^2 \, \mathrm{d}y,\end{aligned}$$

where $\delta := d(\overline{\Omega}_1, \partial U) > 0$ gives lower bounds on $|x-y|$. Similarly for p_4, we have for $x \in \Omega_1$

$$\begin{aligned}|p_4|(x,t) &\le \frac{1}{4\pi}\int_{\Omega\setminus U}\frac{1}{|x-y|}|p|\,|\Delta\phi| \, \mathrm{d}y + \frac{1}{2\pi}\int_{\Omega\setminus U}\frac{1}{|x-y|^2}|\partial_i\phi|\,|p| \, \mathrm{d}y \\ &\le \frac{\|\phi\|_{\mathcal{C}^2}}{4\pi\delta}\int_\Omega |p| \, \mathrm{d}y + \frac{\|\phi\|_{\mathcal{C}^1}}{2\pi\delta^2}\int_\Omega |p| \, \mathrm{d}y.\end{aligned}$$

Consequently,

$$|p_3|(x,t) + |p_4|(x,t) \le C\int_\Omega \left(|p| + |u|^2\right) \mathrm{d}y, \text{ for } x \in \Omega_1. \tag{2.24}$$

Since the operators S_{ij} are bounded by (2.18), there exists $C > 0$ such that

$$\int_{\mathbb{R}^3} |\tilde{p}|^{5/3} \, \mathrm{d}x \le \sum_{i,j}\int_{\mathbb{R}^3} \left|S_{ij}(\phi u^i u^j)\right|^{5/3} \mathrm{d}x \le C\sum_{i,j}\int_{\mathbb{R}^3}\left|\phi u^i u^j\right|^{5/3} \mathrm{d}x,$$

and consequently

$$\int_{\Omega_1} |\tilde{p}|^{5/3} \, \mathrm{d}x \le C\sum_{i,j}\int_{\mathbb{R}^3}\left|\phi u^i u^j\right|^{5/3} \mathrm{d}x \le C\,\|\phi\|_{L^\infty}\int_\Omega |u|^{10/3} \, \mathrm{d}x. \tag{2.25}$$

From (2.24) and (2.25), we may deduce $p \in L^{5/4}(0,T;L^{5/3}(\Omega_1)))$.

We have using (2.15) and (2.25)

$$\begin{aligned}\int_0^T \left(\int_{\Omega_1} |\tilde{p}|^{5/3} \, \mathrm{d}x\right)^{3/5 \cdot 5/4} \mathrm{d}t &\le C\int_0^T \left(\int_\Omega |u|^{10/3} \, \mathrm{d}x + 1\right)^{3/4} \mathrm{d}t \\ &\le C\left(\int_0^T\int_\Omega |u|^{10/3} \, \mathrm{d}x\,\mathrm{d}t + T\right) \le C(E_0^{2/3}E_1 + E_0^{5/3}T + T),\end{aligned} \tag{2.26}$$

where the constant $C > 0$ changes from line to line. For the remaining terms in (2.23), we have using (2.24) and Jensen's inequality

$$\begin{aligned}\int_0^T \left(\int_{\Omega_1} (|p_3| + |p_4|)^{5/3} \, \mathrm{d}x\right)^{3/4} \mathrm{d}t &\le C\,|\Omega_1|\int_0^T \left(\int_\Omega (|p| + |u|^2) \, \mathrm{d}x\right)^{5/3 \cdot 3/4} \mathrm{d}t \\ &\le C\int_0^T \left(\left(\int_\Omega |p| \, \mathrm{d}x\right)^{5/4} + \left(\int_\Omega |u|^2 \, \mathrm{d}x\right)^{5/4}\right) \mathrm{d}t \\ &\le C\int_0^T\int_\Omega |p|^{5/4} \, \mathrm{d}x\,\mathrm{d}t + CTE_0^{5/4} \\ &= C\,\|p\|_{L^{5/4}(\Omega\times(0,T))} + CTE_0^{5/4}.\end{aligned} \tag{2.27}$$

Therefore, combining (2.26) and (2.27) we get using $p = \phi p$ for a.e. t and $x \in \Omega_1$

$$\|p\|_{L^{5/4}(0,T;L^{5/3}(\Omega_1))} \le \|\tilde{p}\|_{L^{5/4}(0,T;L^{5/3}(\Omega_1))} + \||p_3| + |p_4|\|_{L^{5/4}(0,T;L^{5/3}(\Omega_1))} < \infty, \tag{2.28}$$

if (u,p) is a suitable weak solution. Thus, we have proven the following

LEMMA 2.2. *If* (u,p) *is a suitable weak solution of* (1.1) *on* $\Omega\times(0,T)$ *and* $\overline{B}_r\times(a,b)\subset\Omega\times(0,T)$, *then* $p\in L^{5/4}(a,b;L^{5/3}(B_r))$ *and* $u\in L^5(a,b;L^{5/2}(B_r))$.

PROOF. This follows from (2.28) and (2.16). □

In particular, the term $\int\int p(u\cdot\nabla\phi)$ in (2.10) is integrable, since if $\operatorname{supp}\phi\subset\Omega_1$ we have

$$\begin{aligned}\int_0^T\int_\Omega |pu\cdot\nabla\phi|\ \mathrm{d}x\ \mathrm{d}t &\le C\int_{0^T}\|u(t)\|_{L^{5/2}(\Omega_1)}\|p(t)\|_{L^{5/3}(\Omega_1)}\ \mathrm{d}t\\ &\le C\left(\int_0^T\|u(t)\|^5_{L^{5/2}(\Omega_1)}\ \mathrm{d}t\right)^{1/5}\left(\int_0^T\|p(t)\|^{5/4}_{L^{5/3}(\Omega_1)}\ \mathrm{d}t\right)^{4/5}\\ &= C\|u\|_{L^5(0,T;L^{5/2}(\Omega_1))}\|p\|_{L^{5/4}(0,T;L^{5/3}(\Omega_1))},\end{aligned}$$

by Hölder's inequality and since $\frac{3}{5}+\frac{2}{5}=\frac{4}{5}+\frac{1}{5}=1$. Thus, we have shown that for any suitable weak solution of (1.1), the right hand side of (2.9) exists.

2.3.2. Weak continuity. It can be shown, that any suitable weak solution u of (1.1) is weakly continuous in time with values in $L^2(\Omega)$, i.e. for any $w\in L^2(\Omega)$ we have

$$\int_\Omega u(x,t)w(x)\ \mathrm{d}x\to\int_\Omega u(x,t_0)w(x)\ \mathrm{d}x \text{ as } t\to t_0.$$

For a proof of this property we refer to [**6**, p. 281-282]. This has some important consequences.

(i) We can evaluate u at times t and it makes sense to impose the initial condition $u(0)=u_0$ in the sense that $u(t)\rightharpoonup u_0$ in $L^2(\Omega)$ as $t\to 0$, i.e. u extends weakly continously to $[0,T)$.
(ii) The integrability condition (2.11) holds for every $t\in(0,T)$. If $t_0\in(0,T)$, then there exist $t_n\to t_0$ with $\int_\Omega|u(t_n)|^2\ \mathrm{d}x\le E_0$, otherwise (2.11) would not hold almost everywhere. But since the $L^2(\Omega)$-norm is weakly lower semicontinuous and as $u(t_n)\to u(t_0)$ as $n\to\infty$, we conclude $\int_\Omega|u(t_0)|^2\ \mathrm{d}x\le E_0$.
(iii) If (u,p) is a suitable weak solution of (1.1) on $\Omega\times(a,b)$, then for each $a<t_0<b$ and $\phi\in\mathcal{C}_0^\infty(\Omega\times(a,b))$ with $\phi\ge 0$ we have

$$\begin{aligned}\int_\Omega|u(t_0)|^2\phi(t_0)\ \mathrm{d}x+2\int_a^{t_0}\int_\Omega|\nabla u|^2\phi\ \mathrm{d}x\ \mathrm{d}t\\ \le\int_a^{t_0}\int_\Omega\left[|u|^2(\partial_t\phi+\Delta\phi)+(|u|^2+2p)u\cdot\nabla\phi+2u\cdot f\phi\right]\ \mathrm{d}x\ \mathrm{d}t.\end{aligned}\tag{2.29}$$

This follows from (2.10), by choosing the positive test function $\phi(x,t)\chi\left((t_0-t)/\varepsilon\right)$, where $\varepsilon>0$ and χ is a smooth function with $0\le\chi\le 1$, $\chi(s)\equiv 0$ for $s\le 0$ and $\chi(s)\equiv 1$ for $s\ge 1$. Then (2.10) yields

$$\begin{aligned}2\int_a^{t_0}\int_\Omega|\nabla u|^2\phi\chi\left((t_0-t)/\varepsilon\right)\ \mathrm{d}x\ \mathrm{d}t\le\int_a^{t_0}\int_\Omega\Big[|u|^2\left(\partial_t\left(\phi\chi\left((t_0-t)/\varepsilon\right)\right)\right.\\ \left.+\Delta\phi\chi\left((t_0-t)/\varepsilon\right)\right)+(|u|^2+2p)u\cdot\nabla\phi\chi\left((t_0-t)/\varepsilon\right)\\ +2u\cdot f\phi\chi\left((t_0-t)/\varepsilon\right)\Big]\ \mathrm{d}x\ \mathrm{d}t.\end{aligned}\tag{2.30}$$

Note that for $t \le t_0$, $\chi\left({(t_0-t)}/{\varepsilon}\right) \to 1$ as $\varepsilon \to 0$. Since $0 \le \chi \le 1$, the dominated convergence theorem yields that as $\varepsilon \to 0$ in (2.30)

$$2\int_a^{t_0}\int_\Omega |\nabla u|^2\,\phi\;\mathrm{d}x\;\mathrm{d}t \le \int_a^{t_0}\int_\Omega \Big[|u|^2\,(\partial_t\phi + \Delta\phi + (|u|^2+2p)u\cdot\nabla\phi + 2u\cdot f\phi\Big]\;\mathrm{d}x\;\mathrm{d}t \tag{2.31}$$

$$+\lim_{\varepsilon\to 0}\int_a^{t_0}\int_\Omega |u|^2\,\phi\partial_t\left(\chi\left({(t_0-t)}/{\varepsilon}\right)\right)\;\mathrm{d}x\;\mathrm{d}t,$$

since all terms in u and p are integrable. Taking a closer look at the last term, we observe that for u smooth enough

$$\begin{aligned}\int_a^{t_0}\int_\Omega |u|^2\,\phi\partial_t\left(\chi\left({(t_0-t)}/{\varepsilon}\right)\right)\;\mathrm{d}x\;\mathrm{d}t &= \int_\Omega\int_a^{t_0} |u|^2\,\phi\partial_t\left(\chi\left({(t_0-t)}/{\varepsilon}\right)\right)\;\mathrm{d}t\;\mathrm{d}x\\
&= \int_\Omega |u(t_0)|^2\,\phi(t_0)\chi(0)\;\mathrm{d}x - \int_\Omega |u(a)|^2\,\phi(a)\chi\left({(t_0-a)}/{\varepsilon}\right)\;\mathrm{d}x\\
&\quad - \int_a^{t_0}\int_\Omega \partial_t\,|u|^2\,\phi\chi\left({(t_0-t)}/{\varepsilon}\right)\;\mathrm{d}x\;\mathrm{d} - \int_a^{t_0}\int_\Omega |u|^2\,\partial_t\phi\chi\left({(t_0-t)}/{\varepsilon}\right)\;\mathrm{d}x\;\mathrm{d}t.\end{aligned}$$

If we let $\varepsilon \to 0$ we obtain

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_a^{t_0}\int_\Omega |u|^2\,\phi\partial_t\left(\chi\left({(t_0-t)}/{\varepsilon}\right)\right)\;\mathrm{d}x\;\mathrm{d}t\\
&\quad= -\int_\Omega |u(a)|^2\,\phi(a)\;\mathrm{d}x - \int_a^{t_0}\int_\Omega \partial_t\,|u|^2\,\phi\;\mathrm{d}x\;\mathrm{d}t - \int_a^{t_0}\int_\Omega |u|^2\,\partial_t\phi\;\mathrm{d}x\;\mathrm{d}t\\
&\quad= -\int_\Omega |u(a)|^2\,\phi(a)\;\mathrm{d}x - \int_a^{t_0}\int_\Omega \partial_t\left(|u|^2\,\phi\right)\;\mathrm{d}x\;\mathrm{d}t = -\int_\Omega |u(t_0)|^2\,\phi(t_0)\;\mathrm{d}x,\end{aligned}$$

which together with (2.31) proves (2.29). If u is not smooth in time, we can approximate, so (2.29) holds for a.e. t_0 and any suitable weak solution (u,p). But by weak continuity this implies that (2.29) has to hold for all t_0. Like in (ii), for any $t_0 \in (a,b)$ we may find t_n such that (2.29) holds along t_n. By dominated convergence, all double integrals in (2.29) will then converge in the correct way as $t_n \to t_0$ since the involved functions are integrable on $\Omega \times (a,b)$ as (u,p) is a suitable weak solution. Moreover, for the single integral, we have using weak continuity and the Cauchy-Schwarz inequality

$$\begin{aligned}\int_\Omega |u(t_0)|^2\,\phi(t_0)\;\mathrm{d}x &= \lim_{n\to\infty}\int_\Omega u(t_n)\sqrt{\phi(t_n)}\cdot u(t_0)\sqrt{\phi(t_0)}\;\mathrm{d}x\\
&\le \liminf_{n\to\infty}\left(\int_\Omega |u(t_n)|^2\,\phi(t_n)\;\mathrm{d}x\right)^{1/2}\left(\int_\Omega |u(t_0)|^2\,\phi(t_0)\;\mathrm{d}x\right)^{1/2},\end{aligned}$$

hence $\int_\Omega |u(t_0)|^2\,\phi(t_0)\;\mathrm{d}x \le \liminf_{n\to\infty}\int_\Omega |u(t_n)|^2\,\phi(t_n)\;\mathrm{d}x$. Here we used that for any $v \in L^2(\Omega)$

$$\begin{aligned}&\int_\Omega\left(u(t_n)\sqrt{\phi(t_n)} - u(t_0)\sqrt{\phi(t_0)}\right)v\;\mathrm{d}x\\
&\quad= \int_\Omega u(t_n)\left(\sqrt{\phi(t_n)} - \sqrt{\phi(t_0)}\right)v\;\mathrm{d}x + \int_\Omega\left(u(t_n)-u(t_0)\right)\sqrt{\phi(t_0)}v\;\mathrm{d}x \to 0,\end{aligned}$$

as $n \to \infty$ since $\|u(t_n)\|_{L^2(\Omega)}$ is bounded. This proves (2.29) for all $t_0 \in (a,b)$.

2.3.3. The measures $\mathscr{H}^k$ and $\mathscr{P}^k$. Recall that the *k-dimensional Hausdorff measure in $\mathbb{R}^d$* of a set $X \subset \mathbb{R}^d$ is given by

$$\mathscr{H}^k(X) := \lim_{\delta\to 0^+} \mathscr{H}^k_\delta(X) = \sup_{\delta>0} \mathscr{H}^k_\delta(X),$$

where

$$\mathscr{H}^k_\delta(X) := \inf\left\{ \sum_{\ell=1}^\infty \alpha(k)(\operatorname{diam} U_\ell)^k \,\middle|\, U_\ell \subset \mathbb{R}^d \text{ closed, } X \subset \bigcup_{\ell=1}^\infty U_\ell, \operatorname{diam} U_\ell < \delta \right\},$$

where $\alpha(k)$ is chosen such that $\mathscr{H}^k([0,1]^k \times \{0\}^{d-k}) = 1$. In a completely analogous manner, we define a *"parabolic" Hausdorff meausre* via

$$\mathscr{P}^k(X) := \lim_{\delta\to 0^+} \mathscr{P}^k_\delta(X) = \sup_{\delta>0} \mathscr{P}^k_\delta(X),$$

with

$$\mathscr{P}^k_\delta(X) := \inf\left\{ \sum_{\ell=1}^\infty r_\ell^k \,\middle|\, Q_{r_\ell} \subset \mathbb{R}^3 \times \mathbb{R}, X \subset \bigcup_{\ell=1}^\infty Q_{r_\ell}, r_\ell < \delta \right\},$$

where the supremum is taken over any parabolic cylinders, i.e. any sets

$$Q_{r,x_0,t} := \{(y,\tau) \in \mathbb{R}^3 \times \mathbb{R} \mid |y - x_0| \le r, t - r^2 \le \tau \le t\}.$$

Like for $\mathscr{H}^k$, one can show that $\mathscr{P}^k$ is an outer measure for which all Borel sets are measurable and a Borel regular measure on the σ-algebra of measurable sets.

LEMMA 2.3. *There exists $C(k) > 0$ such that $\mathscr{H}^k \le C(k)\mathscr{P}^k$.*

PROOF. Let $0 < \delta < 1$ and let $Q_\ell = Q_{r_\ell,x_\ell,t_\ell}$ be parabolic cylinders with $r_\ell < \delta$. Let $d_\ell := \operatorname{diam} Q_\ell$. Then, clearly $r_\ell \le d_\ell$. Moreover, by the Pythagorean theorem $d_\ell \le \sqrt{r_\ell + r_\ell^2} \le \sqrt{2} r_\ell$, since $r_\ell < \delta < 1$. Thus, for $X \subset \mathbb{R}^3 \times \mathbb{R}$, we have

$$\begin{aligned}
\mathscr{H}^k_\delta(X) &\le \inf\left\{ \sum_{\ell=1}^\infty \alpha(k)(d_\ell)^k \,\middle|\, Q_\ell \subset \mathbb{R}^3 \times \mathbb{R} \text{ parabolic cylinders }, X \subset \bigcup_{\ell=1}^\infty Q_\ell, d_\ell < \delta \right\} \\
&\le \alpha(k)\sqrt{2}^k \inf\left\{ \sum_{\ell=1}^\infty (r_\ell)^k \,\middle|\, Q_\ell \subset \mathbb{R}^3 \times \mathbb{R} \text{ parabolic cylinders }, X \subset \bigcup_{\ell=1}^\infty Q_\ell, r_\ell < \frac{\delta}{\sqrt{2}} \right\} \\
&= \alpha(k)\sqrt{2}^k \mathscr{P}^k_{\delta/\sqrt{2}}(X).
\end{aligned}$$

Taking $\delta \to 0$ finishes the proof. □

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