Navier-Stokes Seminar: Caffarelli-Kohn-Nirenberg Theory

Dr. Jack Skipper
Universität Ulm, Summer 2019

Preface

These are lecture notes generated by the graduate seminar course on the Caffarelli-Kohn-Nirenberg Theory for the Navier-Stokes equations at the Universität of Ulm in the summer term of 2019. We mainly follow [**CKN82**] in a modern fashion. This work is aimed for enthusiastic Masters and PhD students.

I would like to thank everyone taking the seminar for typing parts of these notes. At the beginning of each chapter the according author will be named.

Corrections and suggestions should be sent to `jack.skipper@uni-ulm.de` or the associated author.

All typographical corrections should be sent to `jack.skipper@uni-ulm.de`.

Contents

CHAPTER 1

Talk 1: Introduction

By Dr. Jack Skipper

For this introduction we will use the original paper of [**CKN82**] and the excellent book [**RRS16**].

The Navier-Stokes equations are

$$\begin{aligned}\partial_t u(x,t) + (u\cdot\nabla)u(x,t) + \nabla p(x,t) - \Delta u(x,t) &= f\\ \operatorname{div} u(x,t) &= 0.\end{aligned}$$

Here, $(x,t)\in\Omega\times[0,T]$, where $\Omega\subset\mathbb{R}^3$ or $\mathbb{T}^3$ or $\mathbb{R}^3$ some domain. Initial data $u(x,0)=u_0$ and $u=0$ on $\partial\Omega\times(0,\infty)$. With compatibility conditions for u_0 and f we see that

$$-\Delta p = \partial_i\partial_j(u_iu_j) \quad \text{for } a.e\ t.$$

1.1. Outline: The Navier-Stokes Equations

1.1.1. Weak and Strong. Here we will give an overview of the important results currently known about the Navier-Stokes equations(NSE). The results here were taken from the book by Robinson, Rodrigo,

- (Leray 1934, $\mathbb{R}^3$) in [**Ler34**] and (Hopf 1951, Ω or $\mathbb{T}^3$) in [**Hop51**] showed that Leray-Hopf (LH) weak solutions exist globally in time. Here we assume that the initial data $u_0\in L^2_\sigma$ (in L^2 and weakly incompressible) and $u\in L^\infty(0,T;L^2_\sigma)\cap L^2(0,T;H^1)$ and satisfy the weak energy inequality, namely,

  $$\int_\Omega u^2(t)\,\mathrm{d}x + \int_s^t\int_\Omega |\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t \le \int_\Omega u(s)\,\mathrm{d}x$$

  for almost every t,s. We do not know about uniqueness here.
- (Leray 1934, $\mathbb{R}^3$) in [**Ler34**] and (Kiseler-Ladyzhenskaya 1857) in [**KL57**] showed that strong solutions (LH weak solutions with $u_0\in L^2_\sigma\cap H^1$ and $u\in L^\infty(0,T;H^1)\cap L^2(0,T;H^2)$) exist and are unique locally in time. They showed a lover bound on the potential "blow up" time $T=c\|\nabla u_0\|_{L^2}^{-4}$. Further, strong solutions are immediately smooth, even real analytic according to (Foias-Temam 1989) in [**FT89**].
- We have global existence of strong solutions for small data on Ω or $\mathbb{T}^3$ where we have an absolute constant $C(\Omega)$ or $\tilde{C}(\Omega)$ such that, for example,

  $$\|\nabla u_0\|_{L^2} < C \quad \|u_0\|_{L^2} < C\|\nabla u_0\|_{L^2} < \tilde{C}.$$

  For $\mathbb{R}^3$ we have a scaling $u_\lambda(x,t)=\lambda u(\lambda x,\lambda^2 t)$ is a solution. Thus if we want to talk about small data we need the norm to be invariant under this map, we say these spaces are critical spaces. $\dot{H}^{1/2}$, L^3, BMO^{-1} are invariant spaces where for small data we have strong solutions and for any data have local in time strong solutions.
- (Sather-Serrin 1963) see [**Ser63**] showed weak-strong uniqueness, that is, strong solutions are unique in the class of LH weak solutions. (Need the energy inequality) This suggests 2 possibilities u is strong always $\|\nabla u(t)\|_{L^2}<\infty$ for all $s>0$ or

there exists T^* the "blow-up" time where

$$\|\nabla u)(t)\|^2 \geq \frac{C(\Omega)}{\sqrt{(T^* - t)}}.$$

Can use similar techniques to show robustness of solutions "if initial data is close to a strong solution initial data then the solutions is strong for a while".

- Leary noticed that any global in time LH weak solution is eventually strong and for large time $\|u(t)\|_{L^2} \to 0$ as $t \to \infty$.

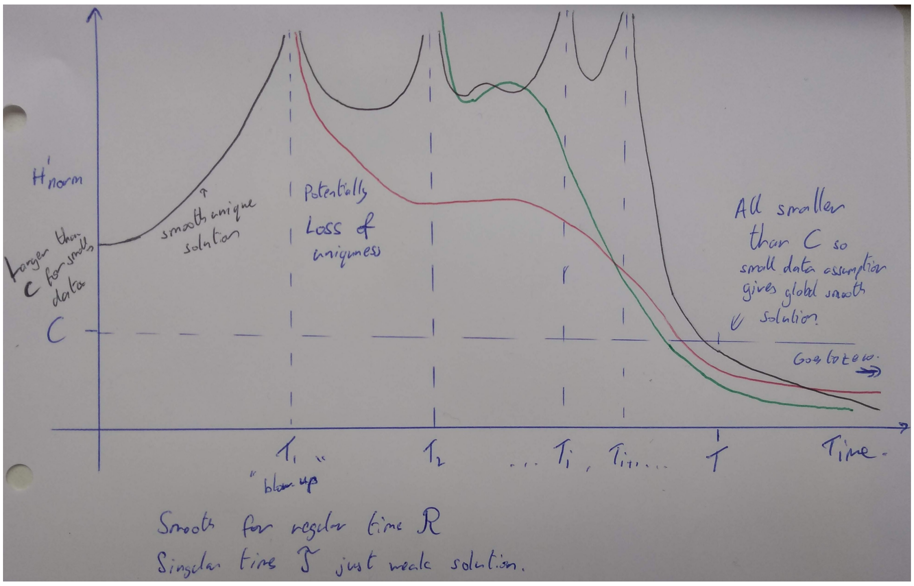

FIGURE 1. The H^1 norm of a potential solution to the Navier-Stokes equations.

1.1.2. Regularity. We can now look at the regularity of solutions and either we find conditions on how bad could the space of solutions be, or we find conditions on solutions that guarantee they are strong and smooth.

- (Scheffer 1976) in [**Sch76**] gave an upper bound on the size of the set of singular times. We say a time is regular and in the set $\mathcal{R}$ if $\|\nabla u(t)\|_{L^2}$ is essentially bounded. The singular times $\mathcal{T}$ a the rest. Here we see that the $\frac{1}{2}$ dimensional Hausdorff measure of the set $\mathcal{T}$ is zero. (Box counting measure is the same.)
- (Kato 1984) in [**Kat84**] showed that if
$$\int_0^T \|\nabla u(s)\|_{L^\infty} \, \mathrm{d}s < \infty$$
then u is strong on $(0, T]$.
- (Beal-Kato-Majda 1984) in [**BKM84**] showed that if
$$\int_0^T \|\operatorname{curl} u(s)\|_{L^\infty} \, \mathrm{d}s < \infty$$
then u is strong on $(0, T]$ and further if we have "blow-up" at T then
$$\lim_{t\to T} \int_0^t \|\operatorname{curl} u(s)\|_{L^\infty} \, \mathrm{d}s = \infty.$$
- Serrin see [**Ser63**] condition that
$$u \in L^r(0, T; L^s(\Omega)) \quad \frac{2}{r} + \frac{3}{s} = 1$$

gives a smooth solution on $(0,T]$. We note that we only unfortunately know that for a LH weak solution that

$\frac{2}{r}+\frac{3}{s}=\frac{3}{2}$.

Further, we have other Serrin type conditions, by (Beirão da Veiga 1995) in [**Bei95**]

$\nabla u \in L^r(0,T;L^s(\Omega)) \quad \frac{2}{r}+\frac{3}{s}=2 \quad \frac{3}{2}<s<\infty$

and by (Berselli-Galdi 2002) in [**BG02**] in

$p \in L^r(0,T;L^s(\Omega)) \quad \frac{2}{r}+\frac{3}{s}=2 \quad \frac{3}{2}<s.$

- (Serrin 1962) in [**Ser62**], for the (<) case, showed a local version of the Serrin condition that, on a sub-domain $U\times(t_1,t_2)$, if

  $u \in L^r(t_1,t_2;L^s(U)) \quad \frac{2}{r}+\frac{3}{s}=1$

  then u is smooth in space on $U\times(t_1,t_2)$ and α-Hölder continuous with $\alpha<\frac{1}{2}$ (Don't get smoothness in time as have problems with ∇p and $\partial_t u$ interacting locally.) The equality was worked out by (Fabes-Jones-Riviere 1972) see [**FJR72**], (Struwe 1988) see [**Str88**] and (Takahashi 1990) in [**Tak90**].

Leary thought that his solutions were turbulent solutions and that a self-similar construction would give a solution that would "blow-up", however, (Nečas-Růžička-Šverák 1996) in [**NRS96**] essentially disproved this. Further, for Euler equations non-uniqueness of weak solutions has been shown starting with the work of (Scheffer 1993) in [**Sch93**] then (De Lellis-Székelyhidi 2010) in [**DS10**] and finally with (Wiedemann 2011) in [**Wie11**].

We have a picture of how LH weak solutions are behaving and the interplay with strong solutions. Regularity results go down two lines where either we ask for extra conditions, we can't guarantee, from LH weak solutions so that then they are strong solutions an thus unique. Here, for the CKN result we want to keep with the regularity we know LH weak solutions can have and find upper bounds on how bad the set of "bad singular points" of the weak solutions can be. We will show that we get a bound of on the 1 dimensional Hausdorff measure and show that the size of the set in this measure is 0.

1.2. "Suitable" Weak Solutions

The CKN partial regularity result for suitable week solutions of the NSE. (How bad is the space-time set of blow-ups)

We know that for any $u_0 \in L^2_\sigma$ there us a LH weak solution of the NSE that satisfies the local energy inequality. (This modern result needs maximal regularity theory for the pressure p). (Sohr-von Wahl 1986) in [**SvW86**] showed that for any $\varepsilon>0$

$$p \in L^r(\varepsilon,T;L^s) \quad \text{for} \quad \frac{2}{r}+\frac{3}{s}=3 \quad (s>1)$$

or for the gradient of the pressure

$$\nabla p \in L^r(\varepsilon,T;L^s) \quad \text{for} \quad \frac{2}{r}+\frac{3}{s}=4 \quad (s>1)$$

and thus we obtain that $p \in L^{\frac{5}{3}}(\Omega\times(0,T]))$. CKN only knew that $p \in L^{\frac{5}{4}}(\Omega\times(0,T]))$ which adds extra technical difficulties.

DEFINITION 1.1. The pair (u,p) is a **suitable** weak solution of the NSE on $\Omega\times[0,T]$ with force f if the following are satisfied.

(1) Integrability:

(a) $f \in L^q(\Omega \times [0,T])$ for $q > \frac{5}{2}$,
(b) $p \in L^{\frac{5}{4}}(\Omega \times [0,T])$ [Modern times can get as high as $L^{\frac{5}{3}}(\Omega \times [0,T])$],
(c) $u \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$.

(2) Local energy inequality: For all $\phi \geq 0$, $\phi \in C_c^\infty$, then,

$$2\iint |\nabla u|^2 \phi \, \mathrm{d}x \, \mathrm{d}s \leq \iint |u|^2(\phi_t + \Delta\phi) + (|u|^2 + 2p)u \cdot \nabla\phi + 2(u \cdot f)\phi \, \mathrm{d}x \, \mathrm{d}s$$

(3) Weak solution: We need $u \in L^\infty(0,T;L^2_\sigma) \cap L^2(0,T;H^1_\sigma)$, $\nabla \cdot f = 0$, $-\Delta p = \partial_i\partial_j(u_i u_j)$ and for $a.e.t \in (a,b)$ and for all $\phi \in C^\infty_{\sigma,c}$

$$\int_{\Omega\times\{0\}} u_0 \cdot \phi(0) \, \mathrm{d}x = \int_0^T \int_\Omega \nabla u : \nabla\phi + (u\cdot\nabla)u\phi - u\cdot\partial_t\phi - f\cdot\phi \, \mathrm{d}x \, \mathrm{d}t.$$

For the CKN theory we do not need point 3 above, that is, the pair (u,p) does not actually need to be a LH weak solution of the NSE. The proof just deals with local energy inequality and interpolation inequalities as so points 1 and 2 are sufficient, the "suitable" bit.

As an interesting aside, it is important to note that in (Scheffer 1987) in [**Sch87**] he showed that the end result, that the one dimensional Hausedroff measure of the singular set of space-time points is zero, cannot be improved using the "suitable" criteria and the method would have to use (the equation) part 3 above. He showed that if you just pick a "suitable" pair (u,p) then for any $\gamma < 1$ there will exist at least one (u,p) pair where the γ- dimensional Hausdrof measure of the singular set is infinite.

1.3. Partial Regularity

We want to study "how bad" the set of "singular points" for u a suitable solution.

We denote $\mathcal{R}$ the set of regular points $(x,t) \in \mathcal{R}$ if there exists an open set $U \subset \Omega \times [0,T]$ with $(x,t) \in U$ and $u \in L^\infty(U)$. Let $\mathcal{S}$ be the set of singular points defined by $\mathcal{S} := \Omega \times [0,T] \setminus \mathcal{R}$, so the points where u is not L^∞_{loc} in any neighbourhood of (x,t). (Can also be defined similarly but with $\operatorname{curl} u$ or ∇u.) By "bad" we want an upper-bound on the dimension of $\mathcal{S}$ here using the Hausdroff measure.

THEOREM 1.2 (Main Theorem (B) in [**CKN82**]). *For any suitable weak solution of the NSE on an open set in space-time the asscoiated singular set $\mathcal{S}$ satisfies*

$$\mathcal{P}^1(S) = 0.$$

This condition is equivalent to $\mathcal{H}^1(S) = 0$ which denotes that the one dimensional Hausdroff measure of the singular set is 0.

Importantly this shows that there are no curves in space-time where the solution u is singular along the curve. If we have "blow-up" then this occurs at distinct points in space time and not on a continuum.

CKN also impose extra conditions to prove two other theorems. These results are more in the spirit of previous partial regularity results like Serrin conditions as discussed earlier.

Let E denote the initial "kinetic energy", the L^2 norm of for the initial data, that is,

$$E := \frac{1}{2}\int_{\mathbb{R}^3} |u_0|^2 \, \mathrm{d}x$$

and let G, be a weighted form of E where we want extra decay at infinity, that is,

$$G := \frac{1}{2}\int_{\mathbb{R}^3} |u_0|^2|x| \, \mathrm{d}x < \infty.$$

For initial data satisfying this condition one can show that a suitable weak solution of the NSE from this data satisfies

$$\frac{1}{2}\int_{\mathbb{R}^3\times\{t\}} |u|^2|x| \, \mathrm{d}x + \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2|x| \, \mathrm{d}x \, \mathrm{d}s < \infty$$

for every t, so obtain the following theorem showing that the solution is regular for large enough x.

THEOREM 1.3 (Theorem C in [**CKN82**]). *Suppose $u_0 \in L^2(\mathbb{R}^3)$ $\nabla \cdot u_0 = 0$ and $G < \infty$. Then there exists a weak solution of the NSE with $f = 0$ which is regular on the set*

$$\{(x,t) : |x|^2 t > K_1\}$$

where $K_1 = K_1(E,G)$ is a constant only depending on u_0 via E and G.

Here we see that G is a restriction that the initial data u_0 should decay sufficiently rapidly at infinity.

If instead we have a different condition where we ask for decay approaching zero, that is,

$$\int_{\mathbb{R}^3} |u_0|^2 |x|^{-1} \, \mathrm{d}x = L \le L_0$$

then we obtain

$$\sup_{\tau} \int_{\mathbb{R}^3 \times \{\tau\}} |u|^2 |x|^{-1} \, \mathrm{d}x < \infty, \qquad \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 |x|^{-1} \, \mathrm{d}x \, \mathrm{d}\tau < \infty$$

for each t. From this we obtain the following theorem where we see that u is regular in a parabola above the origin and the line $x = 0$ is regular for all t.

THEOREM 1.4 (Theorem D in [**CKN82**]). *There exists an absolute constant $L_0 > 0$ with the following properties. If $u_0 \in L^2(\mathbb{R}^3)$ $\nabla \cdot u_0 = 0$ and $L < L_0$ then there exists a weak solution of the NSE with $f = 0$ which is regular on the set*

$$\{(x,t) : |x|^2 < t(L_0 - L)\}.$$

1.4. Scale-invariant Quantities (Dimensionless Quantities)

On $\mathbb{R}^3$ if we have a solution to the NSE then by rescaling by λ, in the following way,

$$u(x,t) \mapsto \lambda u(\lambda x, \lambda^2 t)$$
$$p(x,t) \mapsto \lambda^2 p(\lambda x, \lambda^2 t)$$
$$f(x,t) \mapsto \lambda^3 f(\lambda x, \lambda^2 t)$$

we have another solution. Here we see that time scales quadratically and space linearly.

For local estimates it will be best to use, rather than balls, parabolic cylinders, that is,

$$Q_r(x,t) := \{(y,\tau) : |y - x| \le r,\ t - r^2 < \tau < t\}$$

or $Q_r^*(x,t) = Q_r(x, t - \frac{1}{8}r^2)$ (here (x,t) is the geometric centre of $Q_{\frac{r}{2}}(x, t + \frac{1}{8}r^2)$). The scaling that works on $\mathbb{R}^3$ also works on the parabolic cylinders where if (u,p) is a solution on $Q_r(x,t)$ then (u_λ, p_λ) will be a solution on $Q_{\frac{r}{\lambda}}(x,t)$.

We want to study "quantities" being "small" over parabolic cylinders and thus to have a sensible definition of a "smallness" assumption we should study scale invariant "quantities", that is, "quantities" whose value will not change after rescaling space and time as above. If the "quantities" we study did not have this property then under rescaling we could shrink or blow-up the values and could not compare the values. We will use factors of $\frac{1}{r}$ to make the scale invariant quantities we need.

For example,

$$\begin{aligned} \frac{1}{\left(\frac{r}{\lambda}\right)^2} \int_{Q_{\frac{r}{\lambda}}(0,0)} |u_\lambda|^3 \, \mathrm{d}x \, \mathrm{d}t &= \frac{\lambda^2}{r^2} \int_{Q_{\frac{r}{\lambda}}(0,0)} \lambda^3 |u(\lambda x, \lambda^2 t)|^3 \, \mathrm{d}x \, \mathrm{d}t \\ &= \frac{1}{r^2} \int_{Q_r(0,0)} |u(y,s)|^3 \, \mathrm{d}y \, \mathrm{d}s \end{aligned}$$

where we have a change of variable $y = \lambda x$, $s = \lambda^2 t$.

Some of the scale-invariant quantities we will use are

$$\frac{1}{r}\sup_{-r^2<t<0}\int_{B_r}|u(t)|^2\,\mathrm{d}x,\quad \frac{1}{r}\iint_{Q_r}|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t,\quad \frac{1}{r^2}\iint_{Q_r}|u|^3\,\mathrm{d}x\,\mathrm{d}t,\quad \frac{1}{r^2}\iint_{Q_r}|p|^{\frac{3}{2}}\,\mathrm{d}x\,\mathrm{d}t.$$

1.5. The Main Ideas

We need to show two main propositions that concern bounds on u for large radii giving properties for u on smaller radii.

PROPOSITION 1.5. *There are absolute constants $\varepsilon, C_1 > 0$ and constant $\varepsilon_2(q) > 0$ with the following properties. If (u,p) is a suitable weak solution of the NSE on $Q_1(0,0)$ with force $f \in L^q$, for some $q > \frac{5}{2}$ and*

$$\iint_{Q_1(0,0)}(|u|^3+|u||p|)\,\mathrm{d}x\,\mathrm{d}t+\int_{-1}^{0}\left(\int_{B_1}|p|\,\mathrm{d}x\right)^{\frac{5}{4}}\mathrm{d}t \le \varepsilon_1 \quad\text{and}\quad \iint_{Q_1(0,0)}|f|^q\,\mathrm{d}x\,\mathrm{d}t \le \varepsilon_2$$

then $u \in L^\infty(Q_{\frac{1}{2}}(0,0))$ with $\|u\|_{L^\infty(Q_{\frac{1}{2}}(0,0))} \le C_1$. ($u$ is regular on $Q_{\frac{1}{2}}(0,0)$).

With no force and modern $p \in L^{\frac{5}{3}}$ we can just assume that

$$\iint_{Q_1(0,0)}(|u|^3+|p|^{\frac{3}{2}})\,\mathrm{d}x\,\mathrm{d}t \le \varepsilon_1$$

and the proof is simplified.

We can shift and rescale this proposition to apply it to different $Q_r(x,t)$.

PROPOSITION 1.6. *There exists an absolute constant ε_3 such that if (u,p) is a suitable weak solution to the NSE on $Q_R(a,s)$ for some $R > 0$ and if*

$$\limsup_{r\to 0}\frac{1}{r}\int_{Q_r(as)}|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t \le \varepsilon_3$$

then $u \in L^\infty(Q_\rho(a,s))$ for some ρ with $0 < \rho < R$. (a,s) is a regular point.

We will now discuss a rough outline of the proof and the tools used.

- We have the local energy inequality,

  $$2\iint|\nabla u|^2\phi\,\mathrm{d}x\,\mathrm{d}s \le \iint|u|^2(\phi_t+\Delta\phi)+(|u|^2+2p)u\cdot\nabla\phi+2(u\cdot f)\phi\,\mathrm{d}x\,\mathrm{d}s.$$

  We use an approximation to the backwards heat equation for ϕ on a parabolic cylinder so it approximately solves $\phi_t + \Delta\phi = 0$ and get appropriate bounds on ϕ and $\nabla\phi$ as powers of $\frac{1}{r}$. This gives an inequality over parabolic cylinders with weighting in front of the remaining terms that means they are scaling invariant.
- We can use different interpolation inequalities over parabolic cylinders, for example,

  $$\frac{1}{r^2}\iint_{Q_r(a,s)}|u|^3\,\mathrm{d}x\,\mathrm{d}t \le C_0\left[\frac{1}{r}\sup_{s-r^2<t<s}\int_{B_r(a)}|u(t)|^2+\frac{1}{r}\iint_{Q_r(a,s)}|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t\right]^{\frac{3}{2}}.$$
- We can use these two inequalities. We see that the term on the RHS of the local energy inequality is quadratic in u and on the LHS they are all act cubic in u (with the assumed regularity on p and f) however this is the opposite for the interpolation inequality. We can thus iterate between these two inequalities to obtain inductive bounds on a solution u from the larger cylinder to a smaller cylinder that are shrinking and so can use Lebesgue differentiation theorem to get that the points (a,s) are regular on the smaller cylinder.

Bibliography

[BKM84] J.T. Beale, T. Kato and A. Majda. Remarks on the breakdown of smooth solutions for the 3-D Euler equations. *Comm. Math. Phys.*, 1984.

[Bei95] H. Beirão da Veiga. A new regularity class for the Navier-Stokes equations in $\mathbb{R}^n$. *Chinese Ann. Math. Ser. B.*, 1995.

[BG02] L.C. Berselli and G.P. Galdi. Regularity criteria involving the pressure for the weak solutions to the Navier-Stokes equations. *Proc. Amer. Math. Soc.*, 2002.

[CKN82] L. Caffarelli, R. Kohn and L. Nirenberg. Partial regularity of suitable weak solutions of the Navier-Stokes equations. *Communications on pure and applied mathematics*, 1982.

[DS10] C. De Lellis, and L. Székelyhidi. On admissibility criteria for weak solutions of the Euler equations. *Archive for rational mechanics and analysis*, 1982.

[FJR72] E.B. Fabes, B.F. Jones and M.N. Rivière. The initial value problem for the Navier-Stokes equations with data in L^p. *Arch. Ration. Mech. Anal.*, 1972.

[FT89] C. Foias and R. Temam. Gevrey class regularity for solutions of the Navier-Stokes equations. *J. Funct. Anal.*, 1989.

[Hop51] E. Hopf. Über die Aufgangswertaufgave für die hydrodynamischen Grundliechungen. *Math. Nachr.*, 1951.

[Kat84] T. Kato. Strong L^p-solutions of the Navier-Stokes equations in $\mathbb{R}^m$ with applications to weak solutions. *Math. Zeit.*, 1984.

[KL57] A.A. Kiselev and O.A. Ladyzhernskaya. On the existence and uniqueness of the solution of the nonstationary problem for a viscous, incompressible fluid. *Izv. Akad.Nauk SSSR. Ser. Mat.*, 1957.

[Ler34] J. Leray. Essai sur le mouvement d'un liquide visueux emplissant l'espace. *Acta Math*, 1934.

[NRS96] J. Nečas, M. Růžička and V. Šverák. On Leray's self-similar solutions of the Navier-Stokes equations. *Acta Math.*, 1996.

[RRS16] J.C. Robinson, J.L. Rodrigo and W. Sadowski. The Three-Dimensional Navier–Stokes Equations: Classical Theory. *Cambridge University Press.*, 2016.

[Sch76] V. Scheffer. Turbulence and Hausdorff dimension. *In Turbulence and Navier-Stokes equations, Orsay 1975* Springer lecture Notes in Mathematics **565**, 1976.

[Sch87] V. Scheffer. Nearly one dimensional singularities of solutions to the Navier-Stokes inequality. *Communications in Mathematical Physics*, 1987.

[Sch93] V. Scheffer. An inviscid flow with compact support in space-time . *J. Geom. Anal.* 1993.

[Ser62] J. Serrin. On the interior regularity of weak solutions for the Navier-Stokes equations. *Arch. Ration. Mech. Anal.*, 1962.

[Ser63] J. Serrin. The initial value problem for the Navier-Stokes equations. *In Nonlinear Problems (Proc. Sympos., Madison, Wis.)*, 1963.

[SvW86] H Sohr and W. von Wahl. On the regularity of the pressure of weak solutions of Navier-Stokes equations. *Archiv der Mathematik*, 1986.

[Str88] M. Struwe. On partial regularity results for the Navier-Stokes equations. *Communications on Pure and Applied Mathematics*, 1988.

[Tak90] S. Takahashi. On interior regularity criteria for weak solutions of the Navier-Stokes equations. *Manuscripta Mathematica*, 1990.

[Wie11] E. Wiedemann. Existence of weak solutions for the incompressible Euler equations. *In Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, 2011.