CHAPTER 1

Talk 2: Suitable weak solutions: part 1

By Farid Mohamed

We introduce the spaces for $\Omega \subset \mathbb{R}^3$

$$\mathcal{V} = \{u \in C_0^\infty(\Omega), \text{div } u = 0\},$$
$$V = \overline{\mathcal{V}}^{\|\cdot\|_{H_0^1(\Omega)}} \text{ and}$$
$$H = \overline{\mathcal{V}}^{\|\cdot\|_{L^2(\Omega)}}.$$

The space H is equipped with the norm $\|\cdot\|_{L^2(\Omega)}$ and we write

$$(u,v)_{L^2(\Omega)} := \int_\Omega uv\,dx$$

for the generating scalar product. In the case of V we need to distinguish two cases. If Ω is bounded we set $\|u\|_V := \|\nabla u\|_{L^2(\Omega)}$ and if Ω is unbounded we define $\|u\|_V := \|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}$. We observe that $V \hookrightarrow H \hookrightarrow V'$, where we identify H and H' in the sense that for every $u \in H$ we set

$$\langle u, f\rangle = T_u(f) = \int_\Omega uf dx$$

for $f \in H$. In this case we see that $\langle u, f\rangle = (u,f)_{L^2(\Omega)}$.
We assume for this section that

$$\Omega = \mathbb{R}^3,$$
$$f \in L^2(0,T;H^{-1}(\mathbb{R}^3)) \text{ and } \nabla\cdot f = 0,$$
$$u_0 \in H$$

or

$$\Omega \text{ is a smooth, bounded, open and connected set in } \mathbb{R}^3$$
$$f \in L^2(\Omega\times(0,T)) \text{ and } \nabla\cdot f = 0,$$
$$u_0 \in H \cap W_{5/4}^{2/5}(\Omega).$$

It follows directly that the spaces $L^2(0,T;H)$ and $L^2(0,T;V)$ are reflexive and $L^\infty(0,T;H)$ and $L^\infty(0,T;V)$ are the duals of separable Banach spaces, see for example [**4**], Theorem 1.29.

Definition 1.1. We call the pair (u,p) a suitable weak solution of the Navier-Stokes system on an open set $D = \Omega\times(0,T) \subset \mathbb{R}^3\times\mathbb{R}$ with force f if:

i) u, p and f are measurable functions on D,
ii) $f \in L^q(D)$ for $q > 5/2$, $\nabla\cdot f = 0$ and $p \in L^{5/4}(D)$,

iii) the solution u is bounded in the following sense

$$E_0(u) := \operatorname*{ess\,sup}_{0<t<T} \int_\Omega |u(x,t)|^2 dx < \infty \text{ and } E_1(u) := \iint_D |\nabla u|^2 dxdt < \infty, \tag{1.1}$$

iv) u, p and f solve

$$\partial_t u(x,t) + (u\cdot\nabla)u(x,t) + \nabla p(x,t) - \Delta u(x,t) = f \text{ in } \Omega, \tag{1.2}$$

$$\operatorname{div} u(x,t) = 0 \text{ on } \partial\Omega \text{ for all } 0<t<T \tag{1.3}$$

in the sense of distributions in D, i.e. $u \in L^2(0,T;V)$ and for all $v \in V$ we have

$$\frac{d}{dt}\int_\Omega u(x,t)v(x)\,dx + \int_\Omega (u\cdot\nabla)u(x,t)v(x)\,dx + \int_\Omega \nabla u\cdot\nabla v\,dx = \int_\Omega f(t,x)v(x)dx$$

in the distributional sense on $(0,T)$.

v) for all $\varphi \in C_0^\infty(D)$, $\varphi \geq 0$ it holds

$$2\iint_D |\nabla u|^2\varphi dxdt \leq \iint_D (|u|^2(\varphi_t + \Delta\varphi) + (|u|^2+2p)u\cdot\nabla\varphi + 2(u\cdot f)\varphi)dxdt.$$

The goal of this chapter is to show that for every $f \in L^q(D)$ there exists a suitable weak solution in the sense of Defintion 1.1.

The first step is to show that the equation

$$u_t + (w\cdot\nabla)u - \Delta u + \nabla p = f$$

has a solution for suitable f and w, where we use the following lemma.

LEMMA 1.2 (see [**7**], Lemma 1.2). *Suppose $f \in L^2(0,T;V')$, $u \in L^2(0,T;V)$, p is a distribution and*

$$u_t - \Delta u + \nabla p = f \tag{1.4}$$

in the sense of distributions on D. Then

$$u_t \in L^2(0,T;V'),$$

$$\frac{d}{dt}\int_\Omega |u|^2 = 2(u_t,u)_{L^2(\Omega)}$$

in the sense of distributions on $(0,T)$ and

$$u \in C([0,T],H)$$

after modification on a set of measure zero. Solutions of (1.4) are unique in the space $L^2(0,T;V)$ for given initial data $u_0 \in H$.

PROOF. Here we give the main ideas of the proof.
Let the function $\hat{u} : \mathbb{R} \to V$ be equal to u on $[0,T]$ and to 0 outside this interval. We see by [**3**], Theorem 4.3 a sequence $(u_m)_{m\in\mathbb{N}}$ such that

$$\forall m, u_m \text{ is infinitly differentiable from } [0,T] \text{ onto } V, \text{ as } m\to\infty$$

$$u_m \to u \text{ in } L^2_{loc}(0,T;V),$$

$$u_m' \to u' \text{ in } L^2_{loc}(0,T;V').$$

It follows directly

$$\frac{d}{dt}\int_\Omega |u_m(t)|^2 = 2(u_m'(t),u_m(t))_{L^2(\Omega)}$$

and as $m\to\infty$ we get

$$\|u_m\|^2_{L^2(\Omega)} \to \|u\|^2_{L^2(\Omega)} \text{ in } L^1_{loc}((0,T))$$

$$(u_m',u_m)_{L^2(\Omega)} \to (u',u)_{L^2(\Omega)} \text{ in } L^1_{loc}((0,T)).$$

These convergences also hold in the distribution sense. So by passing to the limit we get

$$\frac{d}{dt}\int_\Omega |u|^2 = 2(u_t, u)_{L^2(\Omega)} \tag{1.5}$$

and by (1.5) we see that $u \in L^\infty(0,T;H)$. We conclude by [7], Lemma 1.4 that $u \in C([0,T];H)$. Uniqueness will follow by the next lemma. □

LEMMA 1.3. *Let $f \in L^2(0,T;V')$, $u_0 \in H$ and $w \in C^\infty(\overline{D},\mathbb{R}^3)$ with $\nabla \cdot w = 0$. Then there exists a unique function u and a distribution p such that*

$$\begin{aligned} &u \in C([0,T],H) \cap L^2(0,T;V),\\ &\quad u_t + (w\cdot\nabla)u - \Delta u + \nabla p = f \end{aligned}$$

in the sense of distributions on D, with $u(0) = u_0$.

PROOF. We will follow [7], Theorem 1.1 by constructing the solution. Let $\{x_n\}_{n\in\mathbb{N}} \subset V$ be a sequence of linearly indepedent vectors such that $\overline{\operatorname{span}((x_n)_{n\in\mathbb{N}})} = V$, which exists as V is separable. We set $V_n := \operatorname{span}(x_1,\dots,x_n)$ and $u_n := \sum_{i=1}^n g_{in}(t)x_i$, where $(g_{in})_{i=1}^n$ is a solution of the system

$$\begin{aligned} \sum_{i=1}^n g_{in}'(t)(x_i,x_j)_{L^2(\Omega)} + \sum_{i=1}^n g_{in}(t)\big(((w\cdot\nabla)x_i,x_j)_{L^2(\Omega)} + (\nabla x_i,\nabla x_j)_{L^2(\Omega)}\big) = \langle f,x_j\rangle\\ g_{jn}(0) = P_{V_n}(x_0)_j \end{aligned}$$

for $j = 1,\dots,n$. Then u_n solves the equation

$$(u_n',v)_{L^2(\Omega)} + ((w\cdot\nabla)u_n,v) + (\nabla u_n,\nabla v)_{L^2(\Omega)} = \langle f,v\rangle$$

for all $v \in V_n$. Observe by partial integration that

$$((w\cdot\nabla)u_n,u_n)_{L^2(\Omega)} = -(u_n,(w\cdot\nabla)u_n)_{L^2(\Omega)} = 0$$

and one obtains

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u_n\|_{L^2(\Omega)}^2 =&(u_n',u_n)_{L^2(\Omega)}\\ =&\langle f,u_n\rangle - (\nabla u_n,\nabla u_n)_{L^2(\Omega)}\\ \le&\frac{1}{2}\|f\|_{V'}^2 + \frac{1}{2}\|u_n\|_{L^2(\Omega)}^2 - \frac{1}{2}\|\nabla u\|_{L^2(\Omega)}^2\\ \le&\frac{1}{2}\|f\|_{V'}^2 + \frac{1}{2}\|u_n\|_{L^2(\Omega)}^2, \end{aligned} \tag{1.6}$$

whch follows by

$$\langle f,u_n\rangle \le \frac{1}{2}\|f\|_{V'}^2 + \frac{1}{2}\|u_n\|_V^2 \le \frac{1}{2}\|f\|_{V'}^2 + \frac{1}{2}\|u_n\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\nabla u_n\|_{L^2(\Omega)}^2.$$

The continuity of the projection and Gronwall's inequality imply that

$$\|u_n(t)\|_{L^2(\Omega)}^2 \le \left(\|u_0\|_{L^2(\Omega)}^2 + \int_0^T \|f(s)\|_{V'}^2\,ds\right)e^T < \infty, \tag{1.7}$$

which implies that $(u_n)_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty(0,T;H)$. Furthermore, we see by integrating (1.6)

$$\begin{aligned}&\|u_n(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla u_n(s)\|^2_{L^2(\Omega)} ds \\ \le&\|u_n(0)\|^2_{L^2(\Omega)} + \int_0^t \|f(s)\|^2_{V'} ds + \int_0^T \|u_n(s)\|^2_{L^2(\Omega)}\, ds \\ \le&\left(\|u(0)\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(0,T;V')}\right)(1+Te^T)\end{aligned}$$

and we conclude that $(u_n)_{n\in\mathbb{N}}$ is uniformly bounded in $L^2(0,T;V)$. One infers that there exists a subsequence $(u_n)_{n\in\mathbb{N}} \subset L^2(0,T;V)\cap L^\infty(0,T;H)$ such that there exists an $u \in L^2(0,T;V)\cap L^\infty(0,T;H)$

$$u_n \rightharpoonup u \text{ for } n\to\infty \text{ in } L^2(0,T;V) \text{ and} \tag{1.8}$$

$$u_n \overset{*}{\rightharpoonup} u \text{ for } n\to\infty \text{ in } L^\infty(0,T;H). \tag{1.9}$$

We conclude for every $\varphi \in C^1([0,T])$ with $\varphi(T) = 0$ that

$$\begin{aligned}0 =&\int_0^T ((u_n'(t),\varphi(t)x_j)_{L^2(\Omega)} + ((w\cdot\nabla)u_n(t),\varphi(t)x_j) + (\nabla u_n(t),\nabla x_j\varphi(t))_{L^2(\Omega)} \\ &- \langle f(t),\varphi(t)x_j\rangle)\, dt \\ =&\int_0^T (-(u_n(t),\varphi'(t)x_j)_{L^2(\Omega)} + ((w\cdot\nabla)u_n(t),\varphi(t)x_j) + (\nabla u_n(t),\nabla x_j\varphi(t))_{L^2(\Omega)} \\ &- \langle f(t),\varphi(t)x_j\rangle dt - (u_n(0),x_j)_{L^2(\Omega)}\varphi(0)) \\ \to&\int_0^T (-(u(t),\varphi'(t)x_j)_{L^2(\Omega)} + ((w\cdot\nabla)u(t),\varphi(t)x_j) + (\nabla u(t),\nabla x_j\varphi(t))_{L^2(\Omega)} \\ &- \langle f(t),\varphi(t)x_j\rangle dt - (u(0),x_j)_{L^2(\Omega)}\varphi(0))\end{aligned}$$

for $n\to\infty$ for every $j\in\mathbb{N}$. Moreover, the equality holds for every finite combination of the (x_j) and by continuity even for all $v\in V$. We obtain that

$$\frac{d}{dt}(u,v)_{L^2(\Omega)} + ((w\cdot\nabla)u,v) + (\nabla u,\nabla v)_{L^2(\Omega)} = \langle f,v\rangle \tag{1.10}$$

in the sense of distributions on $(0,T)$.
In order to see that $u(0)=u_0$ we use that

$$\int_0^T \frac{d}{dt}(u(t),v)_{L^2(\Omega)}\varphi(t)dt = -\int_0^T (u(t),v)\varphi'(t)\,dt + (u(0),v)\varphi(0),$$

which implies that

$$\begin{aligned}&-\int_0^T (u(t),v)\varphi'(t)dt + \int_0^T (\nabla u,\nabla v)_{L^2(\Omega)}\varphi(t)\,dt + \int_0^T ((w\cdot\nabla)u,v)_{L^2(\Omega)}\varphi(t)\,dt \\ =&(u(0),v)\varphi(0) + \int_0^T \langle f(t),v\rangle\varphi(t)dt\end{aligned}$$

By comparison with the above equality we see that

$$(u_0 - u(0),v)\varphi(0) = 0.$$

As v was arbitrary we conclude that $u_0 = u(0)$.
To show uniqueness assume that we have two solutions u_1 and u_2 with some initial data

and force f. We know that $u_1 - u_2$ solves (1.10) with $f = 0$. We conclude by (1.6) that

$$\frac{1}{2}\frac{d}{dt}\|u_1 - u_2\|^2_{L^2(\Omega)} \le -(\nabla(u_1 - u_2), \nabla(u_1 - u_2))_{L^2(\Omega)} \le 0.$$

As $u_1(0) = u_2(0)$ we conclude that $u_1 = u_2$. □

A solution of the Poisson equation $-\Delta u = f$ for $f \in L^q(\mathbb{R}^3)$ for some $1 < q < \infty$ can be written as

$$u(x) := (-\Delta)^{-1} f(x) := c_3 \int\limits_{\mathbb{R}^3} \frac{1}{|x-y|} f(y)dy,$$

where $c_3 \in \mathbb{R}$ can be given explicitly. We use the following theorem, which can be shown by the Calderón-Zygmund theorem.

THEOREM 1.4 (see [**4**], Theorem B.7). *The linear operator T_{jk} defined by*

$$T_{jk} f := \partial_j \partial_k (-\Delta)^{-1} f$$

is a bounded linear operator from $L^q(\mathbb{R}^3)$ into $L^q(\mathbb{R}^3)$ for all $1 < q < \infty$, i.e.

$$\|T_{jk} f\|_{L^q(\mathbb{R}^3)} \le C \|f\|_{L^q(\mathbb{R}^3)}$$

for some constant $C > 0$.

LEMMA 1.5. *Let $\Omega = \mathbb{R}^3$, $f \in L^2(0,T; H^{-1}(\mathbb{R}^3))$,* div $f = 0$ *and $u_0 \in H$. Then it holds that*

$$\Delta p = -\sum_{i,j} \partial_i \partial_j (w_i u_j), \tag{1.11}$$

in the sense of distribution. Hence, we obtain

$$\iint\limits_D |p|^{5/3} dxdt \le C \iint\limits_D |w|^{5/3} \cdot |u|^{5/3} dxdt.$$

REMARK 1.6. For general Ω (if Ω is bounded) it is also possible to show that $p \in L^{5/3}(D)$.

PROOF. We follow [**4**] to show that p is given by (1.11). At first, observe that

$$\{\varphi \in [\mathcal{S}(\mathbb{R}^3)]^3 : \text{div } \varphi = 0\}$$

is a dense subset of V. Furthermore, for every $h \in [\mathcal{S}(\mathbb{R}^3)]^3$ there exists a $\varphi \in [\mathcal{S}(\mathbb{R}^3)]^3$ and $\psi \in \mathcal{S}(\mathbb{R}^3)$ such that $h = \varphi + \nabla\psi$ and $\nabla \cdot \varphi = 0$, see for example [**4**], Exercise 5.2. Now let $\xi \in C_0^\infty((0,T))$. As u is the solution of (1.10) we obtain by partial integration

$$-\int_0^T (u,h)_{L^2(\mathbb{R}^3)} \xi'(t)\, dt - \int_0^T (u, \Delta h)_{L^2(\mathbb{R}^3)} \xi(t)\, dt \tag{1.12}$$

$$-\int_0^T (u \otimes w, \nabla h)_{L^2(\mathbb{R}^3)} \xi(t)\, dt - \int_0^T \langle f, h\rangle \xi(t) dt \tag{1.13}$$

$$= -\int_0^T (u,\varphi)_{L^2(\mathbb{R}^3)} \xi'(t)\, dt + \int_0^T (\nabla u, \nabla\varphi)_{L^2(\mathbb{R}^3)} \xi(t)\, dt$$

$$+ \int_0^T ((w\cdot\nabla)u, \varphi)_{L^2(\mathbb{R}^3)} \xi(t)\, dt - \int_0^T \sum_{i,j} (u_i w_j, \partial_i\partial_j \psi)_{L^2(\mathbb{R}^3)} \xi(t)\, dt \tag{1.14}$$

$$-\int_0^T \langle f, \varphi\rangle \xi(t) dt \tag{1.15}$$

$$= -\int_0^T \sum_{i,j} (u_i w_j, \partial_i\partial_j\psi)_{L^2(\mathbb{R}^3)} \xi(t)\, dt. \tag{1.16}$$

As $u \in V$, we conclude that $\Delta p = -\sum_{i,j} \partial_i \partial_j (w_i u_j)$, where we used that $\nabla \cdot h = \Delta\psi$. By taking the Fourier transform we see that we can interchange the Laplace operator and $\partial_i\partial_j$ and we obtain

$$p = (-\Delta)^{-1}(-\Delta)p = \sum_{i,j}(-\Delta)^{-1}\partial_i\partial_j w_i u_j = \sum_{i,j}\partial_i\partial_j(-\Delta)^{-1}w_i u_j,$$

and one infers by Theorem 1.4 that $\|p\|_{L^{5/3}(\mathbb{R}^3)} \leq C\||w|\cdot|u|\|_{L^{5/3}}$. □

Later on we want to estimate the pressure p by using following inequality

$$\int_{\mathbb{R}^3} |u|^q dx \leq C\left(\int_{\mathbb{R}} |\nabla u|^2\, dx\right)^{\frac{3}{4}(q-2)}\left(\int_{\mathbb{R}} |u|^2\, dx\right)^{\frac{1}{4}(6-q)} \tag{1.17}$$

for $2 \leq q \leq 6$, which is a special case of the Gagliardo-Nirenberg interpolation inequality

$$\|D^j u\|_{L^q(\mathbb{R}^3)} \leq C\|D^m u\|^{\alpha}_{L^r(\mathbb{R}^3)}\|u\|^{1-\alpha}_{L^p(\mathbb{R}^3)}$$

where $1 < q, p, r < \infty$ and $m, j \in \mathbb{N}$. α is chosen is such a way that $\frac{1}{q} = \frac{j}{3} + \left(\frac{1}{r} - \frac{m}{3}\right)\alpha + \frac{1-\alpha}{p}$ and $\frac{j}{m} \leq \alpha \leq 1$. By choosing $j = 0$, $m = 1$, $r = p = 2$ and $\alpha = 3(\frac{1}{2} - \frac{1}{q})$ we obtain (1.17). We recall that we denote by

$$E_0(u) \coloneqq \operatorname*{ess\,sup}_{0<t<T}\int_{\Omega} |u(x,t)|^2 dx \text{ and } E_1(u) \coloneqq \iint_D |\nabla u|^2 dxdt.$$

LEMMA 1.7. *For $u, w \in L^2(0,T;H^1(\mathbb{R}^3))$,*

$$\|u\|_{L^{10/3}(0,T;L^{10/3}(\mathbb{R}^3))} \leq CE_1^{3/10}(u)E_0^{1/5}(u), \tag{1.18}$$

$$\|w\cdot\nabla u\|_{L^{5/4}(0,T;L^{5/4}(\mathbb{R}^3))} \leq CE_1^{1/2}(u)E_1^{3/10}(w)E_0^{1/5}(w), \tag{1.19}$$

$$\|u\|_{L^5(0,T;L^{5/2}(\mathbb{R}^3))} \leq CT^{1/20}E_0^{7/20}(u)E_1^{3/20}(u). \tag{1.20}$$

PROOF. For (1.18) we use (1.17) and obtain

$$\int_{\mathbb{R}^3} |u|^{10/3}\, dx \leq C\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)\left(\int_{\mathbb{R}^3} |u|^2 dx\right)^{2/3} \leq C\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)E_0(u)^{2/3}$$

for almost all $t \in (0,T)$. Integrating over $(0,T)$ gives the result. For (1.19) we see by Hölder's inequality that

$$\int_0^T\int_{\mathbb{R}^3} |w\cdot\nabla u|^{5/4} dxdt \leq \left(\int_0^T\int_{\mathbb{R}^3} |w|^{10/3}\, dxdt\right)^{3/8} E_1(u)^{\frac{5}{8}}$$
$$= \|w\|^{5/4}_{L^{10/3}(0,T;L^{10/3}(\mathbb{R}^3))}E_1(u)^{\frac{5}{8}}.$$

By applying (1.18) we obtain (1.19). Furthermore, we see by (1.17) and Hölder's inequality that

$$\begin{aligned}\int_0^T \left(\int_{\mathbb{R}^3} |u|^{5/2}\,dx\right)^2 dt \leq & C\int_0^T \left(\int\limits_{\mathbb{R}} |\nabla u|^2\,dx\right)^{3/4} \left(\int_{\mathbb{R}^3} |u|^2\,dx\right)^{7/4} dt \\ \leq & CE_0(u)^{7/4}\int_0^T \left(\int\limits_{\mathbb{R}} |\nabla u|^2\,dx\right)^{3/4} dt \\ \leq & CE_0(u)^{7/4}T^{1/4}\left(\int_0^T \int\limits_{\mathbb{R}} |\nabla u|^2\,dxdt\right)^{3/4}.\end{aligned}$$

We conclude that (1.20) holds true. □

CHAPTER 2

Talk 3: Suitable weak solutions: part 2

By David Berger

LEMMA 2.1 (see [**2**], Theorem 2.8). *Assume that* Ω, f *and* u_0 *satisfy the assumptions of Lemma 1.3. Let* Ω *be bounded,* $4 = 3/q + 2/s$ *and* $w \cdot \nabla u, f \in L^s(0,T;L^q(\Omega))$ *and* $u_0 \in W_s^{2-2/s}(\Omega)$. *Then the solution* (u,p) *constructed in Lemma 1.3 satisfies*

$$\begin{aligned}&\|\nabla p\|^s_{L^s(0,T;L^q(\Omega))} + \|u_t\|^s_{L^s(0,T;L^q(\Omega))} + \|\nabla^2 u\|^s_{L^s((0,T;L^q(\Omega))}\\ \leq &C(\|u_0\|^s_{W_s^{2-2/s}(\Omega)} + \|w\cdot\nabla u\|^s_{L^s(0,T;L^q(\Omega))} + \|f\|^s_{L^s(0,T;L^q(\Omega))}).\end{aligned}$$

Furthermore, by normalizing p *such that* $\int_\Omega p = 0$ *for all* t *we obtain*

$$\|p\|_{L^{5/3}(0,T;L^{5/3}(\Omega))} < \infty. \tag{2.1}$$

LEMMA 2.2. *Let* Ω, u_0 *and* f *satisfy the assumption of Chapter 1 and let* $w \in C^\infty(\bar{D}, \mathbb{R}^3)$ *with* $\nabla \cdot w = 0$. *Let* (u,p) *be the solution of Lemma 1.3. Then, for every* $\varphi \in C^\infty(\bar{D})$ *with* $\varphi = 0$ *near* $\partial\Omega \times (0,T)$, *and for every* t, $0 < t \leq T$,

$$\begin{aligned}\int_\Omega |u(x,t)|^2\varphi(x,t)dx + 2\iint_D |\nabla u|^2\varphi = &\int_\Omega |u_0|^2\varphi(x,0) + \iint_D |u|^2(\varphi_t + \Delta\varphi)\\ &+ \iint_D (|u|^2 w + 2pu)\cdot\nabla\varphi + 2\iint_D (u\cdot f)\varphi\end{aligned}$$

PROOF. We assume that Ω is bounded. Suppose for the moment that φ vanishes near $t = 0$, choose Ω_1, so that $\Omega_1 \subset \Omega$ and $\operatorname{supp}\varphi \subset \Omega_1 \times (0,T)$. Writing $F = f - w\cdot\nabla u$, we have

$$u_t - \Delta u + \nabla p = F \text{ on } D.$$

Mollifying in $\mathbb{R}^4$ each term of the equation above, we obtain sequences of smooth functions u_m, p_m and F_m, $m = 1, 2, \ldots$, such that

$$\frac{du_m}{dt} - \Delta u_m + \nabla p_m = F_m \qquad \nabla\cdot u_m = 0 \tag{2.2}$$

in a neighborhood of $\operatorname{supp}\Phi$, and such that

$$\begin{aligned} u_m &\to u && \text{in } L^5(0,T;L^{\frac{5}{2}}(\Omega)\cap L^2(D)),\\ \nabla u_m &\to \nabla u && \text{in } L^2(D),\\ p_m &\to p && \text{in } L^{\frac{5}{4}}(0,T;L^{\frac{5}{3}}(\Omega_1)),\\ F_m &\to F && \text{in } L^2(D).\end{aligned}$$

Taking the inner product of 2.2 with $2u_m\Phi$ and integrating by parts yields

$$2\iint_D |\nabla u_m|^2\varphi = \iint_D |u_m|^2(\varphi_t + \Delta\varphi) + 2\iint_D p_m(u_m\cdot\nabla\varphi) + 2\iint_D (u_m\cdot F_m)\varphi.$$

We pass to the limit as $m \to \infty$, to conclude for u, p and F, with $F = f - w\cdot\nabla u$,

$$2\iint_D (u\cdot F)\varphi = 2\iint_D (u\cdot f)\varphi + \iint_D |u|^2 w\cdot\nabla\varphi.$$

This gives the proof when $\varphi \in C_0^\infty(D)$ and $t = T$. For the more general case use a cutoff function in time and the continuity of u in H at 0. □

The goal of this chapter is to use the results shown in Chapter 1 to prove the existence of the weak solution. Therefore, we will introduce the mollyfing operator

$$\Psi_\delta(u)(x,t) := (\delta^{-4}\psi(\cdot/\delta)) * u(x,t) = \delta^{-4} \iint\limits_{\mathbb{R}^4} \psi\left(\frac{y}{\delta},\frac{\tau}{\delta}\right)\tilde{u}(x-y,t-\tau)dyd\tau,$$

where $\psi \in C^\infty(\mathbb{R}^4)$, $\psi \geq 0$, $\iint_{\mathbb{R}^4}\psi(x,t)dxdt = 1$ and supp $\psi \subset \{(x,t) : |x|^2 < t, 1 < t < 2\}$ and $\tilde{u}$ is the extension of u on $\mathbb{R}^4$, i.e. $\tilde{u}(x,t) = u(x,t)$ on D and elsewhere 0. We see by [**5**], Theorem 1.2.19 that ψ_δ is an approximating identity on $\mathbb{R}^4$.

Lemma 2.3. *For any $u \in L^\infty(0,T;H) \cap L^2(0,T;V)$ it holds*

$$\nabla \cdot \psi_\delta(u) = 0, \tag{2.3}$$

$$\sup_{0\leq t\leq T}\int_\Omega |\psi_\delta(u)|^2 dx \leq CE_0(u), \tag{2.4}$$

$$\iint\limits_D |\nabla\psi_\delta(u)|^2 dxdt \leq CE_1(u), \tag{2.5}$$

for some $C > 0$ independent of u and δ.

Proof. It is easy to see that

$$\begin{aligned}\nabla\cdot\Psi_\delta(u) &= \delta^{-4}\iint\limits_{\mathbb{R}^4}\nabla\psi\left(\frac{y}{\delta},\frac{\tau}{\delta}\right)\cdot\tilde{u}(x-y,t-\tau)dyd\tau\\ &= \delta^{-4}\iint\limits_{\Omega}\nabla\psi\left(\frac{y}{\delta},\frac{\tau}{\delta}\right)\cdot u(x-y,t-\tau)dyd\tau = 0.\end{aligned}$$

Furthemore, we obtain (2.4) by Hölder's and Young's inequality

$$\begin{aligned}\int_\Omega |\psi_\delta(u)_j|^2 dx =& \int_\Omega\left(\int_\delta^{2\delta}\int_{\mathbb{R}^3}\psi_\delta\left(y,\tau\right)\tilde{u}_j(x-y,t-\tau)dyd\tau\right)^2 dx\\ \leq&\delta\int_\delta^{2\delta}\int_\Omega\left(\int_{\mathbb{R}^3}\psi_\delta\left(y,\tau\right)\tilde{u}_j(x-y,t-\tau)dy\right)^2 dxd\tau\\ \leq&\int_{\mathbb{R}}\delta^{-1}\|\psi(\cdot,\tau/\delta)\|^2_{L^1(\mathbb{R}^3)}\|u(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau\\ \leq& E_0(u)\int_{\mathbb{R}}\|\psi(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau.\end{aligned}$$

The inequality (2.5) is a direct consequence of Young's inequality

$$\begin{aligned}\iint\limits_D |\nabla_j\psi_\delta(u)_i|^2 dxdt \leq& \iint\limits_{\mathbb{R}^4}\left|\delta^{-4}\iint\limits_{\mathbb{R}^4}\psi\left(\frac{y}{\delta},\frac{\tau}{\delta}\right)\nabla_j\tilde{u}_i(x-y,t-\tau)dyd\tau\right|^2 dxdt\\ \leq& \|\psi\|^2_{L^1(\mathbb{R}^4)}\|\nabla_j u_i\|^2_{L^2(\mathbb{R}^3)}.\end{aligned}$$

□

In the proof of the main theorem we will use the following theorem, which gives a sufficient condition that a sequence $(x_n)_{n\in\mathbb{N}} \cap L^2(0,T;L^2(\Omega))$ is relatively compact.

Theorem 2.4 (see [**7**], Theorem 1). *Let $X_0 \subset X \subset X_1$ be threee Banach spaces such that X_0 is compact in X, and X_0 and X_1 are reflexive. Then the space*

$$Y = \left\{v \in L^{\alpha_0}(0,T;X_0), \frac{d}{dt}v \in L^{\alpha_1}(0,T;X_1)\right\}$$

with $\alpha_0, \alpha_1 > 1$ *is compact in* $L^{\alpha_0}(0,T;X)$.

THEOREM 2.5. *Assume that* Ω, u_0 *and* f *satisfy the assumptions from Chapter 1. Then there exists a weak solution* (u,p) *of the Navier-Stokes system such that*

$$\begin{aligned} u &\in L^2(0,T;V) \cap L^\infty(0,T;H), \\ u(t) &\rightharpoonup u_0 \text{ in } H \text{ as } t \to 0, \\ p &\in L^{5/3}(D) \text{ if } \Omega = \mathbb{R}^3, \\ \nabla p &\in L^{5/4}(D) \text{ if } \Omega \text{ is bounded and} \end{aligned}$$

for all $\varphi \in C_0^\infty(D)$, $\varphi \geq 0$ *and* $\varphi = 0$ *near* $\partial\Omega \times (0,T)$ *it holds*

$$\begin{aligned} &\int_\Omega |u(x,t)|^2 \varphi(x,t)dx + 2\int_0^t \int_\Omega |\nabla u|^2 \varphi dxdt \\ &\leq \int_\Omega |u_0|^2 \varphi(x,0)dx + \int_0^t \int_\Omega (|u|^2(\varphi_t + \Delta\varphi) + (|u|^2 + 2p)u \cdot \nabla\varphi + 2(u \cdot f)\varphi)dxdt. \end{aligned}$$

Let $N \in \mathbb{N}$ and $\delta = T/N$. $u_N \in L^2(0,T;V) \cap C([0,T];H)$ is the solution of the equation

$$\frac{d}{dt}u_N + (\psi_\delta(u_N)\cdot\nabla)u_N - \Delta u_N + \nabla p_N = f, u_N(0) = u_0,$$

which exists by applying Lemma 1.3 on each time interval $(\delta m, \delta(m+1))$ for each $m = 0, \ldots, N-1$ separately. By using (1.7), (1.8) and (1.9) we obtain

$$\int_\Omega |u_N(t,x)|^2 dx + \int_0^t \int_\Omega |\nabla u_N|^2 dxdt \leq C\left(\int_\Omega |u_0|^2 dx + \int_0^t \|f(t)\|_{V'} dt\right),$$

for some constant $C > 0$ which implies that u_N is bounded in $L^\infty(0,T;H) \cap L^2(0,T;V)$. Morever, by [7], Lemma 4.2 we conclude that $\frac{d}{dt}u_n$ is bounded in $L^2(0,T;V_2')$, hence $(u_N)_{N\in\mathbb{N}}$ is relatively compact in $L^2(D)$ by Theorem 2.4. We obtain a subsequence (u_n) such that $u_n \to u_*$ in $L^2(D)$, $u_n \rightharpoonup u_*$ in $L^2(0,T;V)$ and $u_n \overset{*}{\rightharpoonup} u_*$ in $L^\infty(0,T;H)$. Moreover, as (u_N) is bounded in $L^{10/3}(D)$ we see easily by an interpolation argument that $u_n \to u_*$ in $L^s(D)$ for every $2 \leq s < 10/3$. Using the above inequalities it is possible to show that u_* solves the Navier-Stokes equation. We will only prove the convergence of the term $\int_0^t \varphi(t)((\psi_\delta(u_N)\cdot\nabla)u_N, v)_{L^2(\Omega)}dt$, as the other parts are trivial. As $v \in H^1(\Omega)$, we see that $\|u_i v_j\|_{L^2(\mathbb{R}^3)} < \infty$, which follows by the Sobolev embedding theorem. We conclude that

$$\begin{aligned} &\left|\int_0^t \int_\Omega ((\psi_\delta(u_N)\cdot\nabla)u_N, v)\varphi(t)dxdt - \int_0^t \int_\Omega ((u\cdot\nabla)u, v)\varphi(t)dxdt\right| \\ &\leq \left|\int_0^t \int_\Omega ((\psi_\delta(u_N)\cdot\nabla)u_N, v)\varphi(t)dxdt - \int_0^t \int_\Omega ((u\cdot\nabla)u_N, v)\varphi(t)dxdt\right| \\ &+ \left|\int_0^t \int_\Omega ((u\cdot\nabla)u_N, v)\varphi(t)dxdt - \int_0^t \int_\Omega ((u\cdot\nabla)u, v)\varphi(t)dxdt\right| \\ &\to 0 \text{ for } N \to \infty, \end{aligned}$$

where we use for the first term that $\psi_\delta(u_N) \to u$ in $L^3(\mathbb{R}^3)$ and in the second term that $u_n \rightharpoonup u$ in $L^2(0,T;V)$.

In the case that Ω is bounded, we use Lemma 2.1. Let $\{\Omega_j\}_{j\in\mathbb{N}}$ be a sequence of subdomains such that $\overline{\Omega}_j \subset \Omega_{j+1}$ and $\cup_{j\in\mathbb{N}}\Omega_j = \Omega$. We see that ∇p_N is bounded in $L^{5/4}(D)$ and p_n in $L^{5/4}(0,T;L^{5/3}(\Omega_j))$. We obtain for every j a subsequence $p_N \rightharpoonup p_*$ in $L^{5/4}(0,T;L^{5/3}(\Omega_j))$. Moreover, we see that $u_N \to u_*$ in $L^5(0,T;L^{5/2}(\Omega))$. The proof follows the same arguments as in the case of $\Omega = \mathbb{R}^3$.

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