Navier-Stokes Seminar:
Caffarelli-Kohn-Nirenberg Theory

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Preface

These are lecture notes geberated by the seminar course on the Caffarelli-Kohn-Nirenberg Theory for the Navier-Stokes equations at the Universität Ulm in the summer term of 2019. We mainly follow the [**CKN82**] in a modern fashion. This work is aimed at enthusiastic Masters and PhD students.

I would like to thank everyone taking the seminar for typing parts of these notes.

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Contents

CHAPTER 1

Talk 8: The Blow-Up Estimate, Part 2

1.1. Introduction

In this talk, (u, p) will always denote a suitable weak solution in the sense of [**CKN82**], cf. Talk 1 and Talk 2.

Our goal is to finish up the proof of Proposition 2 of [**CKN82**] (cf. Talk 7). Proposition 2 roughly states that a certain L^2-control of the gradient $|\nabla u|$ in a neighborhood of a point (x, t) is sufficient for the regularity of (x, t). For the precise statement we refer to Talk 7.

In our computations, we will assume without loss of generality that $(x, t) = (0, 0)$. Another assumption that we make for the sake of simiplicity is that the force vanishes, i.e. $f \equiv 0$. This is less general than the situation in [**CKN82**], but it makes computations less lengthy and important concepts more obvious.

For the entire talk we set $Q_r^* := B_r(0) \times \left(-\frac{7}{8}r^2, \frac{1}{8}r^2\right)$, where the ball $B_r := B_r(0) \subset \mathbb{R}^3$ denotes a ball formed only in the $x-$variables. We leave out the integration measures in each integral as the integration set will always indicate clearly, whether the integral is over x or over t or even in both.

In Talk 7, Proposition 2 is shown once we accept Proposition 3, which will be proved in this talk as Proposition 1.7. We first recall some important quantities from Talk 7, which also have analogues in Section 3, cf. Talk 5.

Definition 1.1 (Some quantities, cf. Talk 7). Let u be a suitable weak solution of the Navier Stokes equations with $f \equiv 0$. With our fixed notation from above, we can define the following quantities

$$
\begin{aligned}
G_*(r) &:= \frac{1}{r^2} \int_{Q_r^*} |u|^3, \\
H_*(r) &:= \frac{1}{r^2} \int_{Q_r^*} |u| \left| |u|^2 - \fint_{B_r} |u|^2 \right|, \\
J_*(r) &:= \frac{1}{r^2} \int_{Q_r^*} |u||p|, \\
K_*(r) &:= \frac{1}{r^{\frac{13}{4}}} \int_{-\frac{7r^2}{8}}^{\frac{r^2}{8}} \left(\int_{B_r} |p| \right)^{\frac{5}{4}}, \\
M_*(r) &:= G_*^{\frac{2}{3}}(r) + H_*(r) + J_*(r) + K_*^{\frac{8}{5}}(r), \\
\delta_*(r) &:= \frac{1}{r} \int_{Q_{r*}} |\nabla u|^2 \\
A_*(r) &:= \sup_{t \in [-\frac{7r^2}{8}, \frac{r^2}{8}]} \int_{B_r} |u|^2.
\end{aligned}
$$

Remark 1.2. Proposition 2 in [**CKN82**] states that there is some $\epsilon_3 > 0$ such that the condition that $\limsup_{r \to 0} \delta_*(r) \leq \epsilon_3$ is sufficient for regularity of $(0, 0)$. If we imagine the condition to be satisfied we can think of δ_* as a small quantity. The estimates to come seem more natural once one keeps this in mind.

REMARK 1.3. To begin with, it may be unclear whether these quantities are finite. We refer to Remark 1 in Talk 7, where arguments for the finiteness of all quantities except for J_* and M_* are given. Finiteness of J_* will follow from Lemma 1.19, which will be proved in the sequel. Given this, one can easily infer that M_* is finite as a sum of finite quantities. One has to say that another method to deduce the finiteness of J_* is to use the integrability results on the top of page 783 in [**CKN82**] and Hölder's inequality. This computation is recommended as an exercise but not very insightful for our talk, since we cannot obtain appropriate control of J_* this way.

REMARK 1.4. Smallness of $M_*(r)$ implies regularity of u in $Q^*_{\frac{r}{2}}$ by Proposition 1 of [**CKN82**], cf. Talk 6. The strategy in the proof of Proposition 2 is therefore to show smallness of $M_*(r)$ for some sufficiently small $r > 0$ and apply Proposition 1.

Proposition 1 however requires actually a little bit less than the smallness of $M_*(r)$: It is enough if $G_*(r), J_*(r)$ and $K_*(r)$ are small, so no condition on $H_*(r)$ needs to be imposed.

This raises the question, whether H_* is actually needed in the definition of M_*, as no control of it is required for the regularity of $(0,0)$. We will justify its appearance during the proof.

REMARK 1.5. The quantity $A_*(r)$ seems to be unimportant for the proof of Proposition 2, since $M_*(r)$ does not contain it explicitly. It will however turn out to be of paramount importance since it behaves comparably to M_*. We can profit from this comparision since A_* is a quantity which is easier to handle than M_*.

We have seen part of this comparision result already in Talk 7, where Lukas prensented the inequality

$$H_*(r) \le C(G_*(r)^{\frac{2}{3}} + A_*(r)\delta_*(r)).$$

In a similar way, more quantities will be controllable by A_* and δ_*. We have already discussed in Remark 1.2 that control by δ_* is desirable. That control of quantities by A_* is also desirable will become clear when we observe an "interaction" between A_* and M_* in Lemma 1.9 and afterwards.

REMARK 1.6. The inequality we intend to prove is useful because it enables us to compare the values of M_* for different radii r and ρ. During this comparision process we will often use some obvious estimates for $r \le \rho$, for example

$$\delta_*(r) \le \frac{\rho}{r}\delta_*(\rho). \tag{1.1}$$

Indeed, this is easy to prove:

$$r\delta_*(r) = \int_{Q^*_r} |\nabla u|^2 \le \int_{Q^*_\rho} |\nabla u|^2 \le \rho\delta_*(\rho). \tag{1.2}$$

Later, the comparision with the half radius will be of particular importance, i.e. $\delta_*(\frac{\rho}{2}) \le 2\delta_*(\rho)$, which follows immediately from (1.1). Similar inequalities can be proved following the lines of (1.2) for other quantities. Let us point out one more such estimate:

$$K_*(r) \le \left(\frac{\rho}{r}\right)^{\frac{13}{4}} K_*(\rho),$$

i.e. $K_*(\frac{\rho}{2}) \le 2^{\frac{13}{4}} K_*(\rho)$. Deriving such inqualities for all given quantities we can infer that $M_*(\frac{\rho}{2}) \le CM_*(\rho)$ for some $C > 0$ independent of ρ. This will become important later.

1.2. Statement of Proposition 3

Just like in Talk 7, we state the version of Proposition 3 that we are going to prove:

Main Proposition 1.7 (Proposition 3 in [**CKN82**] with vanishing force). *Let $\rho > 0$ and let (u,p) be a suitable weak solution of the Navier Stokes System on Q^*_ρ with vanishing force $f \equiv 0$. If $\delta_*(\rho) \le 1$ then there exists a constant $C > 0$ such that*

$$M_*(r) \le C\left[\left(\frac{r}{\rho}\right)^{\frac{1}{5}} M_*(\rho) + \left(\frac{\rho}{r}\right)^2 \left(M_*^{\frac{1}{2}}(\rho)\delta_*^{\frac{1}{2}}(\rho) + M_*(\rho)\delta_*(\rho) + \delta_*(\rho)\right)\right], \tag{1.3}$$

for all $r \in \left(0, \frac{\rho}{4}\right)$.

The strategy of the proof will be the following: The structure of the equation can be used to relate the growth of the quantity M_* to the quantity A_*, which controls again all the quantities that contribute to M_*, possibly for a different radius.

Remark 1.8. In the case of $f \equiv 0$ (which is the only case we consider), we can acutally show an easier inequality, namely

$$M_*(r) \le C\left[\left(\frac{r}{\rho}\right)^{\frac{1}{5}} M_*(\rho) + \left(\frac{\rho}{r}\right)^2 \left(M_*^{\frac{1}{2}}(\rho)\delta_*^{\frac{1}{2}}(\rho) + M_*(\rho)\delta_*(\rho)\right)\right], \quad \forall r \in \left(0, \frac{\rho}{4}\right).$$

1.3. M_* and A_* interact because of the energy inequality

The energy inequality gives us an important relation between A_* and M_*, which we will prove now:

Lemma 1.9 (cf. Lemma 5.5 in [**CKN82**]). *There exists a constant $C_1 > 0$ such that for all $r \in (0, \frac{1}{2}\rho]$ one has*

$$A_*(r) \le C_1\left(\frac{\rho}{r}\right)\left(G_*^{\frac{2}{3}}(\rho) + H_*(\rho) + J_*(\rho)\right).$$

In particular,

$$A_*(r) \le C_1\left(\frac{\rho}{r}\right) M_*(\rho) \quad \forall r \in (0, \tfrac{1}{2}\rho]. \tag{1.4}$$

Proof. In this proof, C denotes a generic constant that can be chosen such that all the estimates are true. We use equation (2.17) in [**CKN82**] substantially, which is a slight improvment on the energy inequality. The equation reads as follows: If (u,p) is a suitable weak solution on a domain $\Omega \times (a,b)$ then each nonnegative $\phi \in C_0^\infty(\Omega \times (a,b))$ satisfies

$$\int_{\Omega\times\{t\}} |u|^2\phi + 2\int_{\Omega\times(a,t)} |\nabla u|^2\phi \le \int_{\Omega\times(a,t)} |u|^2(\phi_t + \Delta\phi) + (|u|^2 + 2p)u\cdot\nabla\phi \quad \forall t \in (a,b).$$

In our case we will choose $\Omega = B_\rho(0)$ as well as $a = -\frac{7}{8}\rho^2$ and $b = \frac{1}{8}\rho^2$. Note that $\Omega \times (a,b) = Q^*_\rho$. Choose as well $\phi \in C_0^\infty(Q^*_\rho)$ such that $0 \le \phi \le 1$ and $\phi \equiv 1$ on Q^*_r. Moreover we can require that $|\nabla\phi| \le \frac{C}{\rho}$ and $|\phi_t| + |\Delta\phi| \le \frac{C}{\rho^2}$, see (3.8) in [**CKN82**]. Now observe for arbitrary but fixed $t \in (-\frac{7}{8}r^2, \frac{1}{8}r^2)$ that

$$\begin{aligned}\int_{B_r\times\{t\}} |u|^2 \le \int_{B_\rho\times\{t\}} |u|^2\phi &\le \int_{B_\rho\times\{t\}} |u|^2\phi + 2\int_{B_\rho\times(-\frac{7}{8}\rho^2,t)} |\nabla u|^2\phi \\ &\le \int_{B_\rho\times(-\frac{7}{8}\rho^2,t)} |u|^2(\phi_t + \Delta\phi) + \int_{B_\rho\times(-\frac{7}{8}\rho^2,t)} (|u|^2 + 2p)u\cdot\nabla\phi.\end{aligned}$$

Now note that by (weak) divergence-freeness of u one has

$$\int_{B_\rho\times(-\frac{7}{8}\rho^2,t)} \left(⨍_{B_\rho} |u|^2\right) u\cdot\nabla\phi = \int_{-\frac{7}{8}\rho^2}^{t}\int_{B_\rho}\left(⨍_{B_\rho} |u|^2\right) u\cdot\nabla\phi \tag{1.5}$$

$$= \int_{-\frac{7}{8}\rho^2}^{t} \left(\fint_{B_\rho} |u|^2 \right) \underbrace{\int_{B_\rho} u \cdot \nabla\phi}_{=0\ a.e.} = 0.$$

Hence we can insert this term into the equation above to find

$$\begin{aligned}
\int_{B_r \times \{t\}} |u|^2 &\leq \int_{B_\rho \times (-\frac{7}{8}\rho^2, t)} |u|^2 (\phi_t + \Delta\phi) + \int_{B_\rho \times (-\frac{7}{8}\rho^2, t)} \left(|u|^2 - \fint_{B_\rho} |u|^2 \right) u \cdot \nabla\phi \\
&\qquad + 2 \int_{B_\rho \times (-\frac{7}{8}\rho^2, t)} p u \cdot \nabla\phi \\
&\leq \int_{B_\rho \times (-\frac{7}{8}\rho^2, t)} |u|^2 (|\phi_t| + |\Delta\phi|) + \int_{B_\rho \times (-\frac{7}{8}\rho^2, t)} \left| |u|^2 - \fint_{B_\rho} |u|^2 \right| |u|\, |\nabla\phi| \\
&\qquad + 2 \int_{B_\rho \times (-\frac{7}{8}\rho^2, t)} |p|\, |u|\, |\nabla\phi| \\
&\leq \int_{Q_\rho^*} |u|^2 (|\phi_t| + |\Delta\phi|) + \int_{Q_\rho^*} \left| |u|^2 - \fint_{B_\rho} |u|^2 \right| |u|\, |\nabla\phi| + 2 \int_{Q_\rho^*} |p|\, |u|\, |\nabla\phi|.
\end{aligned}$$

Using the estimates for ϕ and its derivatives we obtain

$$\begin{aligned}
\int_{B_r \times \{t\}} |u|^2 &\leq \frac{C}{\rho^2} \int_{Q_\rho^*} |u|^2 + \frac{C}{\rho} \int_{Q_\rho^*} \left(|u|^2 - \fint_{B_\rho} |u|^2 \right) |u| + 2 \frac{C}{\rho} \int_{Q_\rho} |p|\, |u| \\
&= \frac{C}{\rho^2} \int_{Q_\rho^*} |u|^2 + \rho H_*(\rho) + \rho J_*(\rho). \qquad (1.6)
\end{aligned}$$

Using Hölder's inequality with $q = \frac{3}{2}, q' = 3$ in the first term, we can estimate

$$\begin{aligned}
\frac{1}{\rho^2} \int_{Q_\rho^*} |u|^2 &\leq C \frac{1}{\rho^2} \rho^{\frac{5}{3}} \left(\int_{Q_\rho^*} |u|^3 \right)^{\frac{2}{3}} = C \frac{1}{\rho^{\frac{1}{3}}} \left(\int_{Q_\rho^*} |u|^3 \right)^{\frac{2}{3}} = C \frac{1}{\rho^{\frac{1}{3}}} \rho^{\frac{4}{3}} \left(\frac{1}{\rho^2} \int_{Q_\rho^*} |u|^3 \right)^{\frac{2}{3}} \\
&= C \rho G_*^{\frac{2}{3}}(\rho).
\end{aligned}$$

Plugging this into (1.6) we find

$$\int_{B_r \times \{t\}} |u|^2 \leq C\rho (G_*^{\frac{2}{3}}(\rho) + J_*(\rho) + H_*(\rho)) \quad \forall t \in (-\tfrac{7}{8} r^2, \tfrac{1}{8} r^2).$$

Dividing by r and taking the supremum over all $t \in (-\frac{7}{8} r^2, \frac{1}{8} r^2)$ we obtain the claim. □

Remark 1.10. The above computation reveals the reason why a summand in M_* is $G_*^{2/3}$ and not G_* to any other power. This differs from section 3. Moreover the power of $\frac{2}{3}$ is really needed, since higher powers of G_* in this estimate would lead to higher powers of $M_*(\rho)$ in the right hand side of (1.3) - at least if we can only use (1.8) to estimate G_*.

Remark 1.11. Without the trick in (1.5) we would not be able to bring H_* into play and therefore there would be no hope to control the third-power-of-u term with $G_*^{2/3}$ or with J_* (look at the scaling properties!). Hence H_* is really needed for the inequality we just proved.

Remark 1.12. An important special case is again $r = \frac{\rho}{2}$ for which one can deduce that there exists $C > 0$ such that

$$A_*(\tfrac{\rho}{2}) \leq C M_*(\rho). \qquad (1.7)$$

The main task for the rest of this section is to bound the quantities G_*, J_*, H_* and K_* in terms of M_* so that we get a converse inequality that bounds $M_*(r)$ in terms of $A_*(\rho)$, $\delta_*(\rho)$. H_* has already been bounded in Talk 7, see Remark 1.5. Before we bound the other quantities we state a general and recurrent proposition, which is a refinement of Sobolev's inequality, stating that for one certain exponent, the constant in the Sobolev inequality does not depend on the domain.

PROPOSITION 1.13 (Essentially Section 5.6.1. in [**EG92**]). *Let $\Omega \subset \mathbb{R}^n$ be a C^1-smooth domain or $\Omega = \mathbb{R}^n$. Then there exists a constant $D = D(n) > 0$ such that for each $f \in W^{1,1}(\Omega)$ one has*

$$\left(\int_{B_r(x)} \left| f - \fint_{B_r(x)} f \right|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \le D \int_{B_r(x)} |\nabla f| \quad \forall x \in \Omega, r > 0 : B_r(x) \subset \Omega.$$

REMARK 1.14. The fact that the constant D in the previous Proposition does not depend on Ω becomes clear once one proves the inequality for $f \in W^{1,1}(\mathbb{R}^n)$ and argues with the extension operator.

LEMMA 1.15 (Bounding G_*, cf. Lemma 5.2 in [**CKN82**]). *Suppose that $r \le \rho$. Then we have*

$$G_*(r) \le C_2 \left\{ \left(\frac{r}{\rho}\right)^3 A_*^{\frac{3}{2}}(\rho) + \left(\frac{\rho}{r}\right)^3 A_*^{\frac{3}{4}}(\rho) \delta_*^{\frac{3}{4}}(\rho) \right\}. \tag{1.8}$$

PROOF. In this proof, C will again be used as a generic constant that can be determined such that all estimates below are true. First of all recall equation (2.9) of [**CKN82**], which is a Sobolev inequality with explicit embeding constants in three dimensions. It reads:

$$\int_{B_r} |u|^q \le C \left(\int_{B_r} |\nabla u|^2\right)^a \left(\int_{B_r} |u|^2\right)^{\frac{q}{2}-a} + \frac{C}{r^{2a}} \left(\int_{B_r} |u|^2\right)^{\frac{q}{2}}, \tag{1.9}$$

for each $q \in [2,6]$ and $a = \frac{3}{4}(q-2)$. This connects for example the L^3-norms of u (which are relevant for G_*) to δ_* and the L^2-norms of u (which are crucial to compute A_*). This explains why $G_*(r)$ can be bounded by $A_*(r)$ and $\delta_*(r)$ and gives the desired inequality in the special case $r = \rho$. We however want to make a transition between different radii. For this we can use the following insightful estimate, employing the average integral and the Sobolev inequality in Proposition 1.13. For a fixed time t we can compute

$$\begin{aligned}
\int_{B_r} |u|^2 &= \int_{B_r} \left(|u|^2 - \fint_{B_\rho} |u|^2\right) + \int_{B_r} \fint_{B_\rho} |u|^2 \le \int_{B_r} \left| |u|^2 - \fint_{B_\rho} |u|^2 \right| + C\left(\frac{r}{\rho}\right)^3 \int_{B_\rho} |u|^2 \\
&\le \int_{B_\rho} \left| |u|^2 - \fint_{B_\rho} |u|^2 \right| + C\left(\frac{r}{\rho}\right)^3 \int_{B_\rho} |u|^2 \\
&\underset{\text{Hölder}}{\le} C(\rho^3)^{\frac{1}{3}} \left(\int_{B_\rho} \left| |u|^2 - \fint_{B_\rho} |u|^2 \right|^{\frac{3}{2}}\right)^{\frac{2}{3}} + C\left(\frac{r}{\rho}\right)^3 \int_{B_\rho} |u|^2 \\
&\underset{\text{Prop.1.13}}{\le} C\rho \int_{B_\rho} |\nabla |u|^2| + C\left(\frac{r}{\rho}\right)^3 \int_{B_\rho} |u|^2 \\
&\le C\rho \int_{B_\rho} |u||\nabla u| + C\left(\frac{r}{\rho}\right)^3 \int_{B_\rho} |u|^2 \\
&\le C\rho \left(\int_{B_\rho} |u|^2\right)^{\frac{1}{2}} \left(\int_{B_\rho} |\nabla u|^2\right)^{\frac{1}{2}} + C\left(\frac{r}{\rho}\right)^3 \int_{B_\rho} |u|^2.
\end{aligned}$$

Further, we estimate some terms with A_* to obtain

$$\int_{B_r} |u|^2 \le C\rho^{\frac{3}{2}} A_*(\rho)^{\frac{1}{2}} \left(\int_{B_\rho} |\nabla u|^2 \right)^{\frac{1}{2}} + C\frac{r^3}{\rho^2} A_*(\rho). \tag{1.10}$$

This gives us an estimate for the L^2-norm of u on B_r. Using (1.9) with $q = 3$ (which implies $a = \frac{3}{4}$) we find

$$\begin{aligned}
\int_{B_r} |u|^3 &\le C \left(\int_{B_r} |\nabla u|^2 \right)^{\frac{3}{4}} \left(\int_{B_r} |u|^2 \right)^{\frac{3}{4}} + \frac{C}{r^{\frac{3}{2}}} \left(\int_{B_r} |u|^2 \right)^{\frac{3}{2}} \\
&\le C \left(\int_{B_r} |\nabla u|^2 \right)^{\frac{3}{4}} \left(\int_{B_\rho} |u|^2 \right)^{\frac{3}{4}} + \frac{C}{r^{\frac{3}{2}}} \left(\int_{B_r} |u|^2 \right)^{\frac{3}{2}} \\
&\le C\rho^{\frac{3}{4}} A_*(\rho)^{\frac{3}{4}} \left(\int_{B_r} |\nabla u|^2 \right)^{\frac{3}{4}} + \frac{C}{r^{\frac{3}{2}}} \left(\int_{B_r} |u|^2 \right)^{\frac{3}{2}} \\
&\le C\rho^{\frac{3}{4}} A_*(\rho)^{\frac{3}{4}} \left(\int_{B_\rho} |\nabla u|^2 \right)^{\frac{3}{4}} + \frac{C}{r^{\frac{3}{2}}} \left(\int_{B_\rho} |u|^2 \right)^{\frac{3}{2}}
\end{aligned}$$

We can use (1.10) and the fact that for nonnegative a, b the expression $(a+b)^{\frac{3}{2}}$ is bounded by a constant multiple of $a^{\frac{3}{2}} + b^{\frac{3}{2}}$ to estimate

$$\begin{aligned}
\int_{B_r} |u|^3 &\le C\rho^{\frac{3}{4}} A_*(\rho)^{\frac{3}{4}} \left(\int_{B_r} |\nabla u|^2 \right)^{\frac{3}{4}} + \frac{C}{r^{\frac{3}{2}}} \left(C\rho^{\frac{3}{2}} A_*(\rho)^{\frac{1}{2}} \left(\int_{B_\rho} |\nabla u|^2 \right)^{\frac{1}{2}} + C\frac{r^3}{\rho^2} A_*(\rho) \right)^{\frac{3}{2}} \\
&\le C\rho^{\frac{3}{4}} A_*(\rho)^{\frac{3}{4}} \left(\int_{B_\rho} |\nabla u|^2 \right)^{\frac{3}{4}} + C\frac{\rho^{\frac{9}{4}}}{r^{\frac{3}{2}}} A_*(\rho)^{\frac{3}{4}} \left(\int_{B_\rho} |\nabla u|^2 \right)^{\frac{3}{4}} + C\left(\frac{r}{\rho}\right)^3 A_*(\rho)^{\frac{3}{2}}.
\end{aligned}$$

Integrating over $t \in (-\frac{7}{8}r^2, \frac{1}{8}r^2)$ and using Hölder's inequality with $q = \frac{3}{4}, q' = \frac{1}{4}$ we find that

$$\begin{aligned}
\int_{Q_r^*} |u|^3 &\le C\left(\rho^{\frac{3}{4}} + \frac{\rho^{\frac{9}{4}}}{r^{\frac{3}{2}}} \right) A_*(\rho)^{\frac{3}{4}} \int_{-\frac{7}{8}r^2}^{\frac{1}{8}r^2} \left(\int_{B_\rho} |\nabla u|^2 \right)^{\frac{3}{4}} + Cr^2 \left(\frac{r}{\rho}\right)^3 A_*(\rho)^{\frac{3}{2}} \\
&\le C\left(\rho^{\frac{3}{4}} + \frac{\rho^{\frac{9}{4}}}{r^{\frac{3}{2}}} \right) A_*(\rho)^{\frac{3}{4}} (r^2)^{\frac{1}{4}} \left(\int_{-\frac{7}{8}r^2}^{\frac{1}{8}r^2} \int_{B_\rho} |\nabla u|^2 \right)^{\frac{3}{4}} + Cr^2 \left(\frac{r}{\rho}\right)^3 A_*(\rho)^{\frac{3}{2}} \\
&\le C\left(\rho^{\frac{3}{4}} + \frac{\rho^{\frac{9}{4}}}{r^{\frac{3}{2}}} \right) A_*(\rho)^{\frac{3}{4}} r^{\frac{1}{2}} \left(\int_{-\frac{7}{8}\rho^2}^{\frac{1}{8}\rho^2} \int_{B_\rho} |\nabla u|^2 \right)^{\frac{3}{4}} + Cr^2 \left(\frac{r}{\rho}\right)^3 A_*(\rho)^{\frac{3}{2}} \\
&\le C\left(\rho^{\frac{3}{4}} + \frac{\rho^{\frac{9}{4}}}{r^{\frac{3}{2}}} \right) A_*(\rho)^{\frac{3}{4}} r^{\frac{1}{2}} \left(\int_{Q_\rho^*} |\nabla u|^2 \right)^{\frac{3}{4}} + Cr^2 \left(\frac{r}{\rho}\right)^3 A_*(\rho)^{\frac{3}{2}} \\
&\le C\left(\rho^{\frac{3}{2}} + \frac{\rho^3}{r^{\frac{3}{2}}} \right) r^{\frac{1}{2}} A_*(\rho)^{\frac{3}{4}} \delta_*(\rho)^{\frac{3}{4}} + Cr^2 \left(\frac{r}{\rho}\right)^3 A_*(\rho)^{\frac{3}{2}}.
\end{aligned}$$

Dividing by r^2 we finally obtain

$$G_*(r) \le C\left(\left(\frac{\rho}{r}\right)^{\frac{3}{2}} + \left(\frac{\rho}{r}\right)^3 \right) A_*(\rho)^{\frac{3}{4}} \delta_*(\rho)^{\frac{3}{4}} + C\left(\frac{r}{\rho}\right)^3 A_*(\rho)^{\frac{3}{2}}.$$

Due to the fact that $r \le \rho$ we can estimate $\left(\frac{\rho}{r}\right)^{\frac{3}{2}} \le \left(\frac{\rho}{r}\right)^3$ and conclude the claim. □

Before we can bound J_* we prove some useful estimates on the pressure, which can be deduced with the following splitting technique.

PROPOSITION 1.16 (Splitting Technique for the pressure, cf. p. 801 in [**CKN82**]). *Suppose that $\rho > 0$ and $\phi \in C_0^\infty(B_\rho)$ is such that $0 \le \phi \le 1$ and $\phi \equiv 1$ in $B_{\frac{3}{4}\rho}$ as well as $|\nabla\phi| \le \frac{C}{\rho}$ and $|\Delta\phi| \le \frac{C}{\rho^2}$ for some $C > 0$. Then for all $x \in B_{\frac{3}{4}\rho}$ and $t \in (0,T)$ one has*

$$p(x,t) = p_4(x,t) + p_5(x,t)$$

where

$$p_4(x,t) = \frac{3}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|x-y|} p(y,t)\Delta\phi(y)\,\mathrm{d}y + \frac{3}{2\pi}\int_{\mathbb{R}^3} \sum_{i=1}^3 \frac{x_i - y_i}{|x-y|^3}\partial_i\phi(y)p(y,t)\,\mathrm{d}y$$

and

$$p_5(x,t) = \frac{3}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|x-y|}\phi(y)\sum_{i,j=1}^3 \partial_{y_i}u^j(y,t)\partial_{y_j}u^i(y,t)\,\mathrm{d}y.$$

Moreover there exists a constant $C_3 > 0$ such that

$$|p_4(x,t)| \le C_3 \fint_{B_\rho} |p| \quad \forall x \in B_{\frac{\rho}{2}} \tag{1.11}$$

and

$$\int_{B_r} |p_5|^2 \le C_3\rho\left(\int_{B_\rho} |\nabla u|^2\right)^2 \quad \forall r \in (0, \tfrac{\rho}{2}]. \tag{1.12}$$

PROOF. Let ϕ be as in the statement. Recall that the fundamental solution of the Poisson equation is given by $k(z) := -\frac{3}{4\pi}\frac{1}{|z|}$. In the following we will leave out the t-argument. Moreover, integrals without a specified set are always over $\mathbb{R}^3$. In the following we will make extensive use of equation (2.12) in [**CKN82**] which reads

$$\Delta p = -\sum_{i,j=1}^3 \partial_i u^j \partial_j u^i, \tag{1.13}$$

in the sense of distributions. For the first we will assume, that p is a smooth function on B_ρ and (1.13) holds pointwise. We have to get rid of this assumption later. This assumption is restrictive but can be gotten rid of, as wee shall discuss in Proposition 1.17. With the fundamental solution property we infer for $x \in B_{\frac{\rho}{2}}$

$$\begin{aligned} p(x)\phi(x) &= -\frac{3}{4\pi}\int \frac{1}{|x-y|}\Delta_y(\phi p)\,\mathrm{d}y \\ &= -\frac{3}{4\pi}\int \frac{1}{|x-y|}(p\Delta\phi + 2(\nabla\phi,\nabla p) + \phi\Delta p)\,\mathrm{d}y. \end{aligned} \tag{1.14}$$

Now we split the integral into three summands and integrate by parts in the second one, more precisely we compute

$$\begin{aligned} -\frac{6}{4\pi}\int \frac{1}{|x-y|}(\nabla\phi,\nabla p)\,\mathrm{d}y &= -\sum_{i=1}^3 \frac{6}{4\pi}\int \frac{1}{|x-y|}\partial_{y_i}\phi\partial_{y_i}p\,\mathrm{d}y \\ &= \sum_{i=1}^3 \frac{6}{4\pi}\int \partial_{y_i}\left(\frac{1}{|x-y|}\partial_{y_i}\phi\right)p\,\mathrm{d}y \\ &= \frac{3}{2\pi}\int p\sum_{i=1}^3 \frac{x_i-y_i}{|x-y|^3}\partial_{y_i}\phi\,\mathrm{d}y + \frac{3}{2\pi}\int p\sum_{i=1}^3 \frac{1}{|x-y|}\partial^2_{y_iy_i}\phi\,\mathrm{d}y \\ &= \frac{3}{2\pi}\int p\sum_{i=1}^3 \frac{x_i-y_i}{|x-y|^3}\partial_{y_i}\phi\,\mathrm{d}y + \frac{3}{2\pi}\int \frac{1}{|x-y|}p\Delta\phi\,\mathrm{d}y. \end{aligned}$$

Plugging this back into (1.14) and using (2.12) in **[CKN82]** we obtain

$$\begin{aligned} p(x)\phi(x) &= \left(-\frac{3}{4\pi}+\frac{3}{2\pi}\right)\int \frac{1}{|x-y|}p\Delta\phi \,\mathrm{d}y + \frac{3}{2\pi}\int p\sum_{i=1}^{3}\frac{x_i-y_i}{|x-y|^3}\partial_{y_i}\phi \,\mathrm{d}y \\ &\quad -\frac{3}{4\pi}\int \frac{1}{|x-y|}\phi\Delta p\mathrm{d}y \\ &= \frac{3}{4\pi}\int \frac{1}{|x-y|}p\Delta\phi \,\mathrm{d}y + \frac{3}{2\pi}\int p\sum_{i=1}^{3}\frac{x_i-y_i}{|x-y|^3}\partial_{y_i}\phi \,\mathrm{d}y \\ &\quad +\frac{3}{4\pi}\int \frac{1}{|x-y|}\phi\sum_{i,j=1}^{3}\partial_i u^j\partial_j u^i \,\mathrm{d}y. \end{aligned}$$

If $x \in B_{\frac{3}{4}\rho}$ then $\phi(x) = 1$ and therefore we can infer the first sentence of the claim. For the pointwise estimate on p_4 in $B_{\frac{\rho}{2}}$ let $x \in B_{\frac{\rho}{2}}$ be arbitrary but fixed. We can estimate with the triangle inequality

$$|p_4(x)| \le \left|\frac{3}{4\pi}\int \frac{1}{|x-y|}p(y)\Delta\phi(y)\,\mathrm{d}y\right| + \left|\frac{3}{2\pi}\int \frac{x_i-y_i}{|x-y|^3}p(y)\partial_i\phi(y)\,\mathrm{d}y\right|. \tag{1.15}$$

Notice that $\nabla\phi \equiv 0, \Delta\phi \equiv 0$ on $B_{\frac{3}{4}\rho}$ since $\phi \equiv 1$ on $B_{\frac{3}{4}\rho}$. For the first summand we can estimate, using the properties of ϕ mentioned in the statement as well as the inverse triangle inequality

$$\begin{aligned} \left|\frac{3}{4\pi}\int \frac{1}{|x-y|}p(y)\Delta\phi(y)\,\mathrm{d}y\right| &\le \left|\frac{3}{4\pi}\int_{B_\rho\setminus B_{\frac{3}{4}\rho}} \frac{1}{|x-y|}p(y)\Delta\phi(y)\,\mathrm{d}y\right| \\ &\le \int_{B_\rho\setminus B_{\frac{3}{4}\rho}} \frac{1}{|x-y|}|p(y)||\Delta\phi(y)|\,\mathrm{d}y \\ &\le \frac{C}{\rho^2}\frac{3}{4\pi}\int_{B_\rho\setminus B_{\frac{3}{4}\rho}} \frac{1}{|x-y|}|p(y)|\,\mathrm{d}y \\ &\le \frac{C}{\rho^2}\frac{3}{4\pi}\int_{B_\rho\setminus B_{\frac{3}{4}\rho}} \frac{1}{|y|-|x|}|p(y)|\,\mathrm{d}y \\ &\le \frac{C}{\rho^2}\frac{3}{4\pi}\int_{B_\rho\setminus B_{\frac{3}{4}\rho}} \frac{1}{\frac{3}{4}\rho-\frac{1}{2}\rho}|p(y)|\,\mathrm{d}y \\ &\le \frac{3C}{\pi\rho^3}\int_{B_\rho}|p(y)|\,\mathrm{d}y \le \frac{C_3}{2}\fint_{B_\rho}|p(y)|\,\mathrm{d}y, \end{aligned}$$

for an appropriate choice of C_3. To estimate the second summand we use that $|x_i-y_i| \le |x-y|$ and otherwise the same techniques as above.

$$\begin{aligned} \left|\frac{3}{4\pi}\int \frac{x_i-y_i}{|x-y|^3}p(y)\partial_i\phi(y)\,\mathrm{d}y\right| &\le \frac{3}{4\pi}\int_{B_\rho\setminus B_{\frac{3}{4}\rho}} \frac{|x_i-y_i|}{|x-y|^3}|p(y)||\partial_i\phi(y)|\,\mathrm{d}y \\ &\le \frac{3C}{4\pi\rho}\int_{B_\rho\setminus B_{\frac{3}{4}\rho}} \frac{1}{|x-y|^2}|p(y)|\,\mathrm{d}y \\ &\le \frac{3C}{4\pi\rho}\int_{B_\rho\setminus B_{\frac{3}{4}\rho}} \frac{1}{|x-y|^2}|p(y)|\,\mathrm{d}y \\ &\le \frac{3C}{4\pi\rho}\int_{B_\rho\setminus B_{\frac{3}{4}\rho}} \frac{1}{(|y|-|x|)^2}|p(y)|\,\mathrm{d}y \end{aligned}$$

$$\begin{aligned}
&\le \frac{3C}{4\pi\rho}\int_{B_\rho\setminus B_{\frac{3}{4}\rho}} \frac{1}{(\frac{3}{4}\rho-\frac{1}{2}\rho)^2}|p(y)|\,\mathrm{d}y\\
&\le \frac{12C}{\pi\rho^3}\int_{B_\rho\setminus B_{\frac{3}{4}\rho}} |p(y)|\,\mathrm{d}y\\
&\le \frac{C_3}{2}\fint_{B_\rho}|p(y)|\,\mathrm{d}y,
\end{aligned}$$

by possibly increasing C_3. The two previous computations imply the pointwise estimate of p_4 together with (1.15). For the L^2-estimate on p_5 fix $r \le \frac{\varrho}{2}$. Estimating all derivatives of u by $|\nabla u|$ we get

$$\begin{aligned}
\int_{B_r}|p_5|^2 &= \int_{B_r}\left|\frac{3}{4\pi}\int\frac{1}{|x-y|}\phi(y)\sum_{i,j=1}^{3}\partial_i u^j\partial_j u^i\,\mathrm{d}y\right|^2\,\mathrm{d}x\\
&\le \frac{3}{4\pi}\int_{B_r}\left(\int\frac{1}{|x-y|}|\phi(y)|\sum_{i,j=1}^{3}|\partial_i u^j||\partial_j u^i|\,\mathrm{d}y\right)^2\,\mathrm{d}x\\
&\le \frac{243}{4\pi}\int_{B_r}\left(\int\frac{1}{|x-y|}|\phi(y)||\nabla u(y)|^2\,\mathrm{d}y\right)^2\,\mathrm{d}x\\
&= \frac{243}{4\pi}\int_{B_r}\left(\int\frac{1}{|x-y|}\sqrt{|\phi(y)|}|\nabla u(y)|\sqrt{|\phi(y)|}|\nabla u(y)|\,\mathrm{d}y\right)^2\,\mathrm{d}x\\
&\underset{\text{Hölder}}{\le} \frac{243}{4\pi}\int_{B_r}\left(\int\frac{1}{|x-y|^2}|\phi(y)||\nabla u(y)|^2\,\mathrm{d}y\right)\left(\int|\phi(z)||\nabla u(z)|^2\,\mathrm{d}z\right)\,\mathrm{d}x\\
&\le \frac{243}{4\pi}\left(\int|\phi(z)||\nabla u(z)|^2\,\mathrm{d}z\right)\int_{B_r(0)}\int_{B_\rho(0)}\frac{1}{|x-y|^2}|\phi(y)||\nabla u(y)|^2\,\mathrm{d}y\,\mathrm{d}x\\
&\underset{\text{Fubini}}{\le} \frac{243}{4\pi}\left(\int|\phi(z)||\nabla u(z)|^2\,\mathrm{d}z\right)\int_{B_\rho(0)}\left(\int_{B_r(0)}\frac{1}{|x-y|^2}\,\mathrm{d}x\right)|\phi(y)||\nabla u(y)|^2\,\mathrm{d}y.
\end{aligned}$$

Now note that for $y \in B_\rho$ one has $B_r(0) \subset B_{r+\rho}(y) \subset B_{2\rho}(y)$ and therefore

$$\begin{aligned}
\int_{B_r}|p_5|^2 &\le \frac{243}{4\pi}\left(\int|\phi(z)||\nabla u(z)|^2\,\mathrm{d}z\right)\int_{B_\rho(0)}\left(\int_{B_{2\rho}(y)}\frac{1}{|x-y|^2}\,\mathrm{d}x\right)|\phi(y)||\nabla u(y)|^2\,\mathrm{d}y\\
&\underset{\text{Subst.}w=x-y}{\le} \frac{243}{4\pi}\left(\int|\phi(z)||\nabla u(z)|^2\,\mathrm{d}z\right)\int_{B_\rho(0)}\left(\int_{B_{2\rho}(0)}\frac{1}{|w|^2}\,\mathrm{d}w\right)|\phi(y)||\nabla u(y)|^2\,\mathrm{d}y\\
&\le \frac{243}{4\pi}\left(\int|\phi(z)||\nabla u(z)|^2\,\mathrm{d}z\right)^2\int_{B_{2\rho}(0)}\frac{1}{|w|^2}\,\mathrm{d}w.
\end{aligned}$$

By radial integration one has

$$\int_{B_{2\rho}(0)}\frac{1}{|w|^2}\,\mathrm{d}w = \int_0^{2\rho}(4\pi s^2)\frac{1}{s^2}ds = 8\pi\rho$$

and hence we can conclude

$$\int_{B_r}|p_5|^2 \le C\rho\left(\int|\phi(z)||\nabla u(z)|^2\,\mathrm{d}z\right)^2 \le C\rho\left(\int_{B_\rho}|\nabla u(z)|^2\,\mathrm{d}z\right)^2.$$

Remark 1.17. In the fundamental solution argument in (1.14) we have used the additional assumption that p is smooth in B_ρ, which is not satisfied in general. If p is not smooth on B_ρ we follow the lines of the proof after (1.14), replacing p with $p * \phi_\epsilon$ for fixed

$\epsilon > 0$, where $(\phi_\epsilon)_{\epsilon>0}$ denotes the standard mollifier family. Note that by (1.13)

$$\Delta(p * \phi_\epsilon) = p * \Delta\phi_\epsilon = -\sum_{i,j=1}^{3} (\partial_i u^j \partial_j u^i) * \phi_\epsilon.$$

Using this and the fact that $p * \phi_\epsilon \to p$ almost everywhere by [**EG92**, Theorem 1 (iv), Section 4.2] one can possibly repeat the above computations and pass to the limit as $\epsilon \to 0$.

Remark 1.18. Possibly one can circumvent adjustments in the previous Remark with a maximal regularity argument for (1.13). For this however, more a-priori regularity of p and higher integrability of derivatives of u have to be shown first (in case that these are actually true).

Lemma 1.19 (Bounds for J_*, cf. Lemma 5.3 in [**CKN82**]). *For each $r \le \frac{\rho}{2}$ one has*

$$J_*(r) \le C_4 \left\{ \left(\frac{r}{\rho}\right)^{\frac{1}{5}} A_*^{\frac{1}{5}}(\rho) G_*^{\frac{1}{5}}(r) K_*^{\frac{4}{5}}(\rho) + \left(\frac{\rho}{r}\right)^2 A_*^{\frac{1}{2}}(\rho)\delta_*(\rho) \right\}.$$

Proof. We start using the splitting of p to get

$$J_*(r) = \frac{1}{r^2}\int_{Q_r^*} |u||p| \le \frac{1}{r^2}\int_{Q_r^*} |u||p_4| + \frac{1}{r^2}\int_{B_r} |u||p_5|. \tag{1.16}$$

We estimate both summands seperately, starting with the first one. As usual, we compute for a fixed time $t \in (-\frac{7}{8}r^2, \frac{1}{8}r^2)$ using (1.11)

$$\begin{aligned}
\int_{B_r} |u||p_4| &\le C\left(\int_{B_r}|u|\right)\left(\fint_{B_\rho}|p|\right) \le C\left(\int_{B_r}|u|\right)^{\frac{2}{5}}\left(\int_{B_r}|u|\right)^{\frac{3}{5}}\left(\fint_{B_\rho}|p|\right) \\
&\underset{\text{Hölder}}{\le} Cr^{\frac{3}{5}}\left(\int_{B_r}|u|^2\right)^{\frac{1}{5}}\left(\int_{B_r}|u|\right)^{\frac{3}{5}}\left(\fint_{B_\rho}|p|\right) \\
&\underset{\text{Hölder}}{\le} Cr^{\frac{3}{5}}\left(\int_{B_r}|u|^2\right)^{\frac{1}{5}} r^{\frac{6}{5}}\left(\int_{B_r}|u|^3\right)^{\frac{1}{5}}\left(\fint_{B_\rho}|p|\right) \\
&\le Cr^{\frac{9}{5}}\left(\int_{B_r}|u|^2\right)^{\frac{1}{5}}\left(\int_{B_r}|u|^3\right)^{\frac{1}{5}}\left(\fint_{B_\rho}|p|\right) \\
&\le Cr^{\frac{9}{5}}\left(\int_{B_\rho}|u|^2\right)^{\frac{1}{5}}\left(\int_{B_r}|u|^3\right)^{\frac{1}{5}}\left(\fint_{B_\rho}|p|\right) \\
&\le Cr^{\frac{9}{5}}(\rho A_*(\rho))^{\frac{1}{5}}\left(\int_{B_r}|u|^3\right)^{\frac{1}{5}}\left(\fint_{B_\rho}|p|\right).
\end{aligned}$$

Integrating in time we obtain

$$\begin{aligned}
\int_{Q_r^*} |u||p_4| &\le Cr^{\frac{9}{5}}\rho^{\frac{1}{5}}A_*^{\frac{1}{5}}(\rho)\int_{-\frac{7}{8}r^2}^{\frac{1}{8}r^2}\left(\int_{B_r}|u|^3\right)^{\frac{1}{5}}\left(\fint_{B_\rho}|p|\right) \\
&\le Cr^{\frac{9}{5}}\rho^{\frac{1}{5}}A_*^{\frac{1}{5}}(\rho)\left(\int_{-\frac{7}{8}r^2}^{\frac{1}{8}r^2}\int_{B_r}|u|^3\right)^{\frac{1}{5}}\left(\int_{-\frac{7}{8}r^2}^{\frac{1}{8}r^2}\left(\fint_{B_\rho}|p|\right)^{\frac{5}{4}}\right)^{\frac{4}{5}} \\
&\le Cr^{\frac{9}{5}}\rho^{\frac{1}{5}}A_*^{\frac{1}{5}}(\rho)\left(\int_{Q_r^*}|u|^3\right)^{\frac{1}{5}}\frac{1}{\rho^3}\left(\int_{-\frac{7}{8}r^2}^{\frac{1}{8}r^2}\left(\int_{B_\rho}|p|\right)^{\frac{5}{4}}\right)^{\frac{4}{5}} \\
&\le Cr^{\frac{9}{5}}\rho^{\frac{1}{5}}A_*^{\frac{1}{5}}(\rho)\left(\int_{Q_r^*}|u|^3\right)^{\frac{1}{5}}\frac{1}{\rho^3}\left(\int_{-\frac{7}{8}\rho^2}^{\frac{1}{8}\rho^2}\left(\int_{B_\rho}|p|\right)^{\frac{5}{4}}\right)^{\frac{4}{5}}
\end{aligned}$$

$$\le Cr^{\frac{9}{5}}\rho^{\frac{1}{5}}A_*^{\frac{1}{5}}(\rho)\left(\int_{Q_r^*}|u|^3\right)^{\frac{1}{5}}\frac{1}{\rho^3}\left(\rho^{\frac{13}{4}}K_*(\rho)\right)^{\frac{4}{5}}$$
$$\le Cr^{\frac{9}{5}}\rho^{\frac{1}{5}}A_*^{\frac{1}{5}}(\rho)\left(\int_{Q_r^*}|u|^3\right)^{\frac{1}{5}}\frac{1}{\rho^{\frac{2}{5}}}K_*(\rho)^{\frac{4}{5}}$$
$$\le Cr^{\frac{9}{5}}A_*^{\frac{1}{5}}(\rho)\frac{1}{\rho^{\frac{1}{5}}}(r^2G_*(r))^{\frac{1}{5}}K_*(\rho)^{\frac{4}{5}} = Cr^{\frac{11}{5}}\frac{1}{\rho^{\frac{1}{5}}}A_*^{\frac{1}{5}}(\rho)G_*(r)^{\frac{1}{5}}K_*(\rho)^{\frac{4}{5}}.$$

Dividing by r^2 we conclude

$$\frac{1}{r^2}\int_{Q_r^*}|u||p_4| \le C\left(\frac{r}{\rho}\right)^{\frac{1}{5}}A_*^{\frac{1}{5}}(\rho)G_*^{\frac{1}{5}}(\rho)K_*^{\frac{4}{5}}(\rho). \tag{1.17}$$

To estimate the second summand in (1.16) we compute with (1.12)

$$\int_{B_r}|u||p_5| \le \left(\int_{B_r}|u|^2\right)^{\frac{1}{2}}\left(\int_{B_r}|p_5|^2\right)^{\frac{1}{2}} \le \left(\int_{B_\rho}|u|^2\right)^{\frac{1}{2}}\rho^{\frac{1}{2}}\int_{B_\rho}|\nabla u|^2$$
$$\le C(\rho A_*(\rho))^{\frac{1}{2}}\rho^{\frac{1}{2}}\int_{B_\rho}|\nabla u|^2 = C\rho A_*(\rho)^{\frac{1}{2}}\int_{B_\rho}|\nabla u|^2.$$

Integrating with respect to t we obtain

$$\int_{Q_r^*}|u||p_5| \le C\rho A_*(\rho)^{\frac{1}{2}}\int_{Q_\rho^*}|\nabla u|^2 = C\rho^2A_*(\rho)^{\frac{1}{2}}\delta_*(\rho).$$

Dividing by r^2 we obtain

$$\frac{1}{r^2}\int_{Q_r^*}|u||p_5| \le C\left(\frac{\rho}{r}\right)^2 A_*(\rho)^{\frac{1}{2}}\delta_*(\rho).$$

This and (1.17) yield the claim. □

The proof of the following lemma will most likely be omitted in the talk, since it is somewhat technical. Nevertheless it is highly recommendable to read, since it presents useful refinements of the pressure estimate.

Lemma 1.20. *[An estimate for K_*, see Lemma 5.4 in [**CKN82**]] If $r \le \frac{1}{2}\rho$ then*

$$K_*(r) \le C\left\{\left(\frac{r}{\rho}\right)^{\frac{1}{2}}K_*(\rho) + \left(\frac{\rho}{r}\right)^{\frac{5}{4}}A_*^{\frac{5}{8}}(\rho)\delta_*^{\frac{5}{8}}(\rho)\right\}.$$

Before we can prove this lemma we have to prove another splitting property of the pressure

Proposition 1.21 (Refinement of the pressure splitting and L^1-control of p_5, cf. p. 803 in [**CKN82**]). *Let ρ, ϕ, p_4, p_5 be as in Proposition 1.16. Then p_5 can be split as follows*

$$p_5(x,t) = p_6(x,t) + p_7(x,t),$$

where

$$p_6(x,t) = -\frac{3}{4\pi}\int\sum_{i=1}^{3}\frac{x_i-y_i}{|x-y|^3}\phi(y)(u\cdot\nabla u^i)(y,t)\,\mathrm{d}y,$$

$$p_7(x,t) = -\frac{3}{4\pi}\int\frac{1}{|x-y|}\sum_{i=1}^{3}\partial_{y_i}\phi(y)(u\cdot\nabla u^i)(y).$$

Moreover, for each $r \le \frac{\rho}{2}$ one has

$$\int_{B_r} |p_5| \le C r \rho^{\frac{1}{2}} A_*(\rho)^{\frac{1}{2}} \left(\int_{B_\rho} |\nabla u|^2 \right)^{\frac{1}{2}}. \tag{1.18}$$

PROOF. Just like in the proof of Proposition 1.16, we leave out the t-argument. To simplify the following computation, we assume first that u is smooth in B_r, an assumption which is not at all justified but can be gotten rid of, as we will discuss after the computation. We integrate by parts in the expression for p_5 that we obtained in Proposition 1.16 to obtain

$$\begin{aligned}
p_5(x) &= \frac{3}{4\pi} \int \sum_{i,j=1}^{3} \frac{1}{|x-y|} \phi \partial_{y_j} u^i \partial_{y_i} u^j \, \mathrm{d}y \\
&= -\sum_{i,j=1}^{3} \frac{3}{4\pi} \int \partial_{y_i} \left(\frac{1}{|x-y|} \phi \partial_{y_j} u^i \right) u^j \, \mathrm{d}y. \\
&= -\sum_{i,j=1}^{3} \frac{3}{4\pi} \int \partial_{y_i} \left(\frac{1}{|x-y|} \phi \partial_{y_j} u^i \right) u^j \, \mathrm{d}y \\
&= -\sum_{i,j=1}^{3} \frac{3}{4\pi} \Bigg(\int \frac{x_i - y_i}{|x-y|^3} \phi \partial_{y_j} u^i u^j \, \mathrm{d}y + \int \frac{1}{|x-y|} \partial_{y_i} \phi u^j \partial_{y_j} u^i \, \mathrm{d}y \\
&\qquad + \int \frac{1}{|x-y|} \phi \partial^2_{y_i y_j} u^i u^j \Bigg) \\
&= p_6(x) + p_7(x) + \sum_{j=1}^{3} \int \frac{1}{|x-y|} u^j \sum_{i=1}^{3} \partial^2_{y_j y_i} u^i,
\end{aligned}$$

where we have rewritten the j–sums as with the dot product in the last step. Now observe that by Schwarz's Lemma (or Clairaut's Theorem)

$$\sum_{i=1}^{3} \partial^2_{y_i y_j} u^i = \partial_{y_j} \sum_{i=1}^{3} \partial_{y_i} u^i = \partial_{y_j} \operatorname{div}(u) = 0,$$

as u was assumed to be divergence-free. This implies that $p_5 = p_6 + p_7$ as claimed. The point where we apply Schwarz's Lemma is however exactly the point where the additional regularity assumption kicks in. We will now briefly comment on how we can overcome the unjustified regularity assumption. In the first step we rewrite

$$\frac{3}{4\pi} \int \sum_{i,j=1}^{3} \frac{1}{|x-y|} \phi \partial_{y_j} u^i \partial_{y_i} u^j \, \mathrm{d}y = \lim_{\epsilon \to 0} \frac{3}{4\pi} \int \sum_{i,j=1}^{3} \frac{1}{|x-y|} \phi (\partial_j u^i * \phi_\epsilon)(y) \partial_{y_i} u^j \, \mathrm{d}y,$$

where $(\phi_\epsilon)_{\epsilon > 0}$ is the standard mollifier family. Following the lines of the proof and using that $u(t,\cdot) \in W^{1,2}(\Omega)$ (which is true at least for almost every t) we obtain

$$p_5(x) = p_6(x) + p_7(x) + \lim_{\epsilon \to 0} \frac{-3}{4\pi} \sum_{j=1}^{3} \int \frac{1}{|x-y|} \phi u^j \sum_{i=1}^{3} \partial_{y_i} (\partial_j u^i * \phi_\epsilon)(y) \, \mathrm{d}y.$$

Now observe that

$$\begin{aligned}
\partial_{y_i} (\partial_j u^i * \phi_\epsilon) &= \partial_{y_i} \int \partial_{z_j} u^i(z) \phi_\epsilon(y-z) dz = -\partial_{y_i} \int u^i(z) \partial_{z_j} \phi_\epsilon(y-z) dz \\
&= -\int u^i(z) \partial_{y_i} \partial_{z_j} [\phi_\epsilon(y-z)] dz \underset{\text{Chain Rule}}{=} \int u^i(z) \partial^2_{z_i z_j} [\phi_\epsilon(y-z)] dz \\
&\underset{\text{Schwarz's Lemma}}{=} \int u^i(z) \partial_{z_i} [\partial_{z_j} \phi_\epsilon(y-z)] dz.
\end{aligned}$$

Summing over i and use the definition of the dot product we obtain

$$\sum_{i=1}^{3} \partial_{y_i}(\partial_j u^i * \phi_\epsilon)(y) = \int u(z) \cdot \nabla \partial_{z_j} \phi_\epsilon(y-z) dz = 0 \quad \forall \epsilon > 0,$$

since u is weakly divergence-free and hence L^2-orthogonal to $\nabla \partial_j \phi_\epsilon(y - \cdot)$ for each $\epsilon > 0$. We have shown the desired decomposition. To show (1.18) we show

$$\int_{B_r} |p_6| \leq C r \rho^{\frac{1}{2}} A_*(\rho)^{\frac{1}{2}} \left(\int_{B_\rho} |\nabla u|^2 \right)^{\frac{1}{2}}$$

and

$$\int_{B_r} |p_7| \leq C r \rho^{\frac{1}{2}} A_*(\rho)^{\frac{1}{2}} \left(\int_{B_\rho} |\nabla u|^2 \right)^{\frac{1}{2}}.$$

Given the two previous inequalities, the desired estimate follows easily with the triangle inequality. First we obtain the L^1-control for p_6:

$$\begin{aligned}
\int_{B_r} |p_6| &= \int_{B_r} \left| \int_{B_\rho} \frac{3}{4\pi} \sum_{i=1}^{3} \frac{x_i - y_i}{|x-y|^3} \phi(y)(u \cdot \nabla u^i)(y) \, \mathrm{d}y \right| \mathrm{d}x \\
&\leq C \int_{B_r} \int_{B_\rho} \frac{1}{|x-y|^2} |\phi(y)| \, |u(y)| \, |\nabla u(y)| \, \mathrm{d}y \, \mathrm{d}x \\
&\underset{\text{Fubini}}{=} C \int_{B_\rho} \left(\int_{B_r} \frac{1}{|x-y|^2} \mathrm{d}x \right) |\phi| \, |u| \, |\nabla u| \\
&= C \int_{B_{2r}} \left(\int_{B_r} \frac{1}{|x-y|^2} \mathrm{d}x \right) |\phi| \, |u| \, |\nabla u| \\
&\quad + C \int_{B_\rho \setminus B_{2r}} \left(\int_{B_r} \frac{1}{|x-y|^2} \mathrm{d}x \right) |\phi| \, |u| \, |\nabla u|.
\end{aligned}$$

We estimate both summands seperately. For the first summand we use that $y \in B_{2r}(0)$ implies $B_r(0) \subset B_{3r}(y)$ and hence

$$\begin{aligned}
\int_{B_{2r}} \left(\int_{B_r} \frac{1}{|x-y|^2} \mathrm{d}x \right) |\phi| \, |u| \, |\nabla u| &\leq \int_{B_{2r}} \left(\int_{B_{3r}(y)} \frac{1}{|x-y|^2} \mathrm{d}x \right) |\phi| \, |u| \, |\nabla u| \\
&\underset{z := x-y}{=} \left(\int_{B_{2r}} |\phi| \, |u| \, |\nabla u| \right) \left(\int_{B_{3r}(0)} \frac{1}{|z|^2} \, \mathrm{d}z \right) \\
&\underset{\text{Radial integration}}{=} \left(\int_{B_{2r}} |\phi| \, |u| \, |\nabla u| \right) \left(\int_0^{3r} (4\pi s^2) \frac{1}{s^2} \, \mathrm{d}s \right) \\
&\leq 12\pi r \left(\int_{B_{2r}} |\phi| \, |u| \, |\nabla u| \right).
\end{aligned} \tag{1.19}$$

For the other integral we use the inverse triangle inequality to estimate for $x \in B_r(0)$ and $|y| \geq 2r$

$$\frac{1}{|x-y|^2} \leq \frac{1}{(|y| - |x|)^2} \leq \frac{1}{(2r - r)^2} \leq \cdot \frac{1}{r^2}$$

Therefore

$$\begin{aligned}
\int_{B_\rho \setminus B_{2r}} \left(\int_{B_r} \frac{1}{|x-y|^2} \mathrm{d}x \right) |\phi| \, |u| \, |\nabla u| &\leq \int_{B_\rho \setminus B_{2r}} \frac{1}{r^2} |B_r(0)| \, |\phi| \, |u| \, |\nabla u| \\
&= \frac{4\pi}{3} r \int_{B_\rho \setminus B_{2r}} |\phi| \, |u| \, |\nabla u|,
\end{aligned}$$

where we have used that $|B_r(0)| = \frac{4}{3}\pi r^3$ is the volume of $B_r(0)$. Plugging both considerations back into (1.19) we obtain

$$\begin{aligned}\int_{B_r}|p_6| &\leq Cr\int_{B_\rho}|\phi|\,|u|\,|\nabla u| \leq Cr\int_{B_\rho}|u|\,|\nabla u| \leq Cr\left(\int_{B_\rho}|u|^2\right)^{\frac{1}{2}}\left(\int_{B_\rho}|\nabla u|^2\right)^{\frac{1}{2}}\\ &\leq Cr\rho^{\frac{1}{2}}A_*^{\frac{1}{2}}(\rho)\left(\int_{B_\rho}|\nabla u|^2\right)^{\frac{1}{2}}, \qquad (1.20)\end{aligned}$$

which is the desired estimate for p_6. Now for the estimation of p_7 fix $r \leq \frac{\rho}{2}$ and $x \in B_r(0)$ to estimate with the properties of ϕ (cf. statement of Proposition 1.16)

$$\begin{aligned}|p_7(x)| &\leq \frac{3}{4\pi}\int\frac{1}{|x-y|}|\nabla\phi(y)|\,|u(y)|\,|\nabla u(y)|\,\mathrm{d}y\\ &\underset{\text{Choice of }\phi}{=} \frac{3}{4\pi}\int_{B_\rho\setminus B_{\frac{3}{4}\rho}}\frac{1}{|x-y|}|\nabla\phi(y)|\,|u(y)|\,|\nabla u(y)|\,\mathrm{d}y\\ &\underset{\text{Choice of }\phi}{\leq} \frac{C}{\rho}\int_{B_\rho\setminus B_{\frac{3}{4}\rho}}\frac{1}{|x-y|}|u(y)|\,|\nabla u(y)|\,\mathrm{d}y\\ &\underset{\text{Inv. triangle inequality}}{\leq} \frac{C}{\rho}\int_{B_\rho\setminus B_{\frac{3}{4}\rho}}\frac{1}{|y|-|x|}|u(y)|\,|\nabla u(y)|\,\mathrm{d}y\\ &\underset{x\in B_{\frac{\rho}{2}}}{\leq} \frac{C}{\rho}\int_{B_\rho\setminus B_{\frac{3}{4}\rho}}\frac{1}{\frac{3}{4}\rho-\frac{1}{2}\rho}|u(y)|\,|\nabla u(y)|\,\mathrm{d}y\\ &\leq \frac{C}{\rho^2}\int_{B_\rho\setminus B_{\frac{3}{4}\rho}}|u(y)|\,|\nabla u(y)|\,\mathrm{d}y \leq \frac{C}{\rho^2}\int_{B_\rho}|u(y)|\,|\nabla u(y)|\,\mathrm{d}y.\end{aligned}$$

Integrating over $x \in B_r(0)$ we obtain

$$\begin{aligned}\int_{B_r}|p_7| &\leq \frac{C}{\rho^2}r^3\int_{B_\rho}|u(y)|\,|\nabla u(y)|\,\mathrm{d}y \leq C\left(\frac{r}{\rho}\right)^2 r\left(\int_{B_\rho}|u|^2\right)^{\frac{1}{2}}\left(\int_{B_\rho}|\nabla u|^2\right)^{\frac{1}{2}}\\ &\leq Cr\rho^{\frac{1}{2}}A_*(\rho)^{\frac{1}{2}}\left(\int_{B_\rho}|\nabla u|^2\right)^{\frac{1}{2}}, \qquad (1.21)\end{aligned}$$

where we used that $\frac{r}{\rho} < 1$ in the last step. As we discussed before, the claim follows from (1.20) and (1.21). □

Proof of Lemma 1.20. Let r, ρ be as in the statement. By (1.11) we conclude that

$$\int_{B_r}|p_4| \leq C\left(\frac{r}{\rho}\right)^3\int_{B_\rho}|p|,$$

in particular

$$\left(\int_{B_r}|p_4|\right)^{\frac{5}{4}} \leq C\left(\frac{r}{\rho}\right)^{\frac{15}{4}}\left(\int_{B_\rho}|p|\right)^{\frac{5}{4}}.$$

Integrating over $t \in (-\frac{7}{8}r^2, \frac{1}{8}r^2)$ we obtain

$$\begin{aligned}\int_{-\frac{7}{8}r^2}^{\frac{r^2}{8}}\left(\int_{B_r}|p_4|\right)^{\frac{5}{4}} &\leq C\left(\frac{r}{\rho}\right)^{\frac{15}{4}}\int_{-\frac{7}{8}r^2}^{\frac{r^2}{8}}\left(\int_{B_\rho}|p|\right)^{\frac{5}{4}} \leq C\left(\frac{r}{\rho}\right)^{\frac{15}{4}}\int_{-\frac{7}{8}\rho^2}^{\frac{\rho^2}{8}}\left(\int_{B_\rho}|p|\right)^{\frac{5}{4}}\\ &= C\left(\frac{r}{\rho}\right)^{\frac{15}{4}}\rho^{\frac{13}{4}}A_*(\rho) = Cr^{\frac{13}{4}}\left(\frac{r}{\rho}\right)^{\frac{1}{2}}K_*(\rho).\end{aligned}$$

Dividing by $r^{\frac{13}{4}}$ yields

$$\frac{1}{r^{\frac{13}{4}}}\int_{-\frac{7}{8}r^2}^{\frac{r^2}{8}}\left(\int_{B_r}|p_4|\right)^{\frac{5}{4}} \le C\left(\frac{r}{\rho}\right)^{\frac{1}{2}} K_*(\rho). \tag{1.22}$$

Furthermore, using (1.18) we find

$$\begin{aligned}
\int_{-\frac{7}{8}r^2}^{\frac{r^2}{8}}\left(\int_{B_r}|p_5|\right)^{\frac{5}{4}} &\le C\int_{-\frac{7}{8}r^2}^{\frac{r^2}{8}} r^{\frac{5}{4}}\rho^{\frac{5}{8}}A_*^{\frac{5}{8}}(\rho)\left(\int_{B_\rho}|\nabla u|^2\right)^{\frac{5}{8}}\\
&\le Cr^{\frac{5}{4}}\rho^{\frac{5}{8}}A_*^{\frac{5}{8}}(\rho)\int_{-\frac{7}{8}r^2}^{\frac{r^2}{8}}\left(\int_{B_\rho}|\nabla u|^2\right)^{\frac{5}{8}}\\
&\underset{\text{Hölder } p=\frac{8}{5},q=\frac{8}{3}}{\le} Cr^{\frac{5}{4}}\rho^{\frac{5}{8}}A_*^{\frac{5}{8}}(\rho)(r^2)^{\frac{3}{8}}\left(\int_{-\frac{7}{8}r^2}^{\frac{r^2}{8}}\int_{B_\rho}|\nabla u|^2\right)^{\frac{5}{8}}\\
&\le Cr^2\rho^{\frac{5}{8}}A_*^{\frac{5}{8}}(\rho)\left(\int_{-\frac{7}{8}\rho^2}^{\frac{\rho^2}{8}}\int_{B_\rho}|\nabla u|^2\right)^{\frac{5}{8}} \le Cr^2\rho^{\frac{5}{8}}A_*^{\frac{5}{8}}(\rho)(\rho\delta_*(\rho))^{\frac{5}{8}}\\
&= Cr^2\rho^{\frac{5}{4}}A_*^{\frac{5}{8}}(\rho)\delta_*^{\frac{5}{8}}(\rho).
\end{aligned}$$

Dividing by $r^{\frac{13}{4}}$ we infer

$$\frac{1}{r^{\frac{13}{4}}}\int_{-\frac{7}{8}r^2}^{\frac{r^2}{8}}\left(\int_{B_r}|p_5|\right)^{\frac{5}{4}} \le C\left(\frac{\rho}{r}\right)^{\frac{5}{4}} A_*^{\frac{5}{8}}(\rho)\delta_*^{\frac{5}{8}}(\rho). \tag{1.23}$$

Given (1.22) and (1.23) we conclude

$$\begin{aligned}
K_*(r) = \frac{1}{r^{\frac{13}{4}}}\int_{-\frac{7}{8}r^2}^{\frac{r^2}{8}}\left(\int_{B_r}|p|\right)^{\frac{5}{4}} &\le \frac{1}{r^{\frac{13}{4}}}\int_{-\frac{7}{8}r^2}^{\frac{r^2}{8}}\left(\int_{B_r}|p_4| + \int_{B_r}|p_5|\right)^{\frac{5}{4}}\\
&\le C\left(\frac{1}{r^{\frac{13}{4}}}\int_{-\frac{7}{8}r^2}^{\frac{r^2}{8}}\left(\int_{B_r}|p_4|\right)^{\frac{5}{4}} + \frac{1}{r^{\frac{13}{4}}}\int_{-\frac{7}{8}r^2}^{\frac{r^2}{8}}\left(\int_{B_r}|p_5|\right)^{\frac{5}{4}}\right)\\
&\le C\left\{\left(\frac{r}{\rho}\right)^{\frac{1}{2}} K_*(\rho) + \left(\frac{\rho}{r}\right)^{\frac{5}{4}} A_*^{\frac{5}{8}}(\rho)\delta_*^{\frac{5}{8}}(\rho)\right\}.
\end{aligned}$$

□

1.4. Proof of the main proposition

PROOF. Let $r \le \frac{1}{4}\rho$. Recall that the statement imposes the condition $\delta_*(\rho) \le 1$. We bound $G_*^{\frac{2}{3}}(r), H_*(r), J_*(r)$ and $K_*(r)$ seperately in terms of M_* and δ_*. Again, we use C to denote a generic constant which we possibly have to increase after each estimate.

Step 1: Estimating G_*. First we can use (1.8) with input parameters $\widetilde{r} := r$ and $\widetilde{\rho} := \frac{\rho}{2}$ to obtain

$$\begin{aligned}
G_*^{\frac{2}{3}}(r) &\underset{(1.8)}{\le} \left\{\left(\frac{r}{\rho/2}\right)^3 A_*^{\frac{3}{2}}(\tfrac{\rho}{2}) + \left(\frac{\rho/2}{r}\right)^3 A_*^{\frac{3}{4}}(\tfrac{\rho}{2})\delta_*^{\frac{3}{4}}(\tfrac{\rho}{2})\right\}^{\frac{2}{3}}\\
&\le C\left\{\left(\frac{r}{\rho/2}\right)^2 A_*(\tfrac{\rho}{2}) + \left(\frac{\rho/2}{r}\right)^2 A_*(\tfrac{\rho}{2})^{\frac{1}{2}}\delta_*(\tfrac{\rho}{2})^{\frac{1}{2}}\right\}
\end{aligned}$$

$$
\begin{aligned}
&\le C\left\{4\left(\frac{r}{\rho}\right)^2 A_*(\tfrac{\rho}{2}) + \frac{1}{4}\left(\frac{\rho}{r}\right)^2 A_*(\tfrac{\rho}{2})^{\frac12}\delta_*(\tfrac{\rho}{2})^{\frac12}\right\}\\
&\le C\left\{\left(\frac{r}{\rho}\right)^2 A_*(\tfrac{\rho}{2}) + \left(\frac{\rho}{r}\right)^2 A_*(\tfrac{\rho}{2})^{\frac12}\delta_*(\tfrac{\rho}{2})^{\frac12}\right\}\\
&\underset{(1.4)}{\le} \left\{\left(\frac{r}{\rho}\right)^2 M_*(\rho) + \left(\frac{\rho}{r}\right)^2 M_*(\rho)^{\frac12}\delta_*(\tfrac{\rho}{2})^{\frac12}\right\}\\
&\underset{\text{Remark 1.6}}{\le} \left\{\left(\frac{r}{\rho}\right)^2 M_*(\rho) + \left(\frac{\rho}{r}\right)^2 M_*(\rho)^{\frac12}\delta_*(\rho)^{\frac12}\right\}.
\end{aligned}
\tag{1.24}
$$

Step 2: Estimating H_*.

$$
\begin{aligned}
H_*(r) &\underset{\text{Remark 1.5}}{\le} C(G_*^{\frac23}(r) + A_*(r)\delta_*(r)) \underset{\text{Remark 1.6}}{\le} C\left\{G_*^{\frac23}(r) + \left(\frac{\rho}{r}\right) A_*(r)\delta_*(\rho)\right\}\\
&\underset{(1.4)}{\le} C\left\{G_*^{\frac23}(r) + \left(\frac{\rho}{r}\right)^2 M_*(\rho)\delta_*(\rho)\right\}\\
&\underset{(1.24)}{\le} C\left\{\left(\frac{r}{\rho}\right)^2 M_*(\rho) + \left(\frac{\rho}{r}\right)^2\left[M_*(\rho)\delta_*(\rho) + M_*^{\frac12}(\rho)\delta_*^{\frac12}(\rho)\right]\right\}.
\end{aligned}
$$

We can merge the estimates in Step 1 and Step 2 to get

$$
\begin{aligned}
G_*^{\frac23}(r) + H_*(r) &\le C\left\{\left(\frac{r}{\rho}\right)^2 M_*(\rho) + \left(\frac{\rho}{r}\right)^2\left[M_*(\rho)\delta_*(\rho) + M_*^{\frac12}(\rho)\delta_*^{\frac12}(\rho)\right]\right\}\\
&\le C\left\{\left(\frac{r}{\rho}\right)^{\frac15} M_*(\rho) + \left(\frac{\rho}{r}\right)^2\left[M_*(\rho)\delta_*(\rho) + M_*^{\frac12}(\rho)\delta_*^{\frac12}(\rho)\right]\right\},
\end{aligned}
\tag{1.25}
$$

where we used in the last estimate that $\frac{r}{\rho} \le 1$.

Step 3: Estimating J_*. By Lemma 1.19 and similar techniques as in the first three estimates of Step 1 we obtain

$$
\begin{aligned}
J_*(r) &\le C\left\{\left(\frac{r}{\rho}\right)^{\frac15} A_*(\tfrac{\rho}{2})^{\frac15} G_*(r)^{\frac15} K_*(\tfrac{\rho}{2})^{\frac45} + \left(\frac{\rho}{r}\right)^2 A_*(\tfrac{\rho}{2})^{\frac12}\delta_*(\tfrac{\rho}{2})\right\}\\
&\underset{(1.4),\text{Remark 1.6}}{\le} C\left\{\left(\frac{r}{\rho}\right)^{\frac15} A_*(\tfrac{\rho}{2})^{\frac15} G_*(r)^{\frac15} K_*(\tfrac{\rho}{2})^{\frac45} + \left(\frac{\rho}{r}\right)^2 M_*(\rho)^{\frac12}\delta_*(\rho)\right\}\\
&\underset{\delta_*(\rho)\le 1}{\le} C\left\{\left(\frac{r}{\rho}\right)^{\frac15} A_*(\tfrac{\rho}{2})^{\frac15} G_*(r)^{\frac15} K_*(\tfrac{\rho}{2})^{\frac45} + \left(\frac{\rho}{r}\right)^2 M_*(\rho)^{\frac12}\delta_*(\rho)^{\frac12}\right\}.
\end{aligned}
\tag{1.26}
$$

We can now use the generalized Young inequality $abc \le C(a^{p_1} + b^{p_2} + c^{p_3})$ whenever $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$ to estimate $A_*(\frac{\rho}{2})^{\frac15} G_*(r)^{\frac15} K_*(\frac{\rho}{2})^{\frac45}$. Here the choice $p_1 = 5, p_2 = \frac{10}{3}, p_3 = 2$ yields

$$
\begin{aligned}
A_*(\tfrac{\rho}{2})^{\frac15} G_*(r)^{\frac15} K_*(\tfrac{\rho}{2})^{\frac45} &\le C\left(A_*(\tfrac{\rho}{2}) + G_*^{\frac23}(r) + K_*^{\frac85}(\tfrac{\rho}{2})\right)\\
&\le C\left(M_*(\rho) + K_*^{\frac85}(\rho) + G_*^{\frac23}(r)\right)
\end{aligned}
$$

where we have used (1.4) and Remark 1.6 in the last step. Notice that one can also estimate $K_*^{\frac85}(\rho) \le M_*(\rho)$ to simplify

$$
A_*(\tfrac{\rho}{2})^{\frac15} G_*(r)^{\frac15} K_*(\tfrac{\rho}{2})^{\frac45} \le C\left(M_*(\rho) + G_*^{\frac23}(r)\right).
$$

Plugging this into (1.26) we obtain

$$
\begin{aligned}
J_*(r) &\leq C\left\{\left(\frac{r}{\rho}\right)^{\frac{1}{5}} M_*(\rho) + \left(\frac{r}{\rho}\right)^{\frac{1}{5}} G_*^{\frac{2}{3}}(r) + \left(\frac{\rho}{r}\right)^{2} M_*(\rho)^{\frac{1}{2}}\delta_*(\rho)^{\frac{1}{2}}\right\} \\
&\underset{(1.24)}{\leq} C\left\{\left(\frac{r}{\rho}\right)^{\frac{1}{5}} M_*(\rho) + \left(\frac{r}{\rho}\right)^{\frac{1}{5}} \left(\left(\frac{r}{\rho}\right)^{2} M_*(\rho) + \left(\frac{\rho}{r}\right)^{2} M_*(\rho)^{\frac{1}{2}}\delta_*(\rho)^{\frac{1}{2}}\right)\right. \\
&\qquad \left. + \left(\frac{\rho}{r}\right)^{2} M_*(\rho)^{\frac{1}{2}}\delta_*(\rho)^{\frac{1}{2}}\right\} \\
&\leq \left\{\left(\left(\frac{r}{\rho}\right)^{\frac{1}{5}} + \left(\frac{r}{\rho}\right)^{\frac{11}{5}}\right) M_*(\rho) + \left(\left(\frac{\rho}{r}\right)^{2} + \left(\frac{\rho}{r}\right)^{\frac{9}{5}}\right) M_*(\rho)^{\frac{1}{2}}\delta_*(\rho)^{\frac{1}{2}}\right\}.
\end{aligned}
$$

Using that $r < \rho$ we can determine which power of $\frac{r}{\rho}$ or $\frac{\rho}{r}$ respectively dominates and infer

$$
J_*(r) \leq \left\{\left(\frac{r}{\rho}\right)^{\frac{1}{5}} M_*(\rho) + \left(\frac{\rho}{r}\right)^{2} M_*(\rho)^{\frac{1}{2}}\delta_*(\rho)^{\frac{1}{2}}\right\}. \tag{1.27}
$$

Step 4: Estimating $K_*^{\frac{8}{5}}$. By Lemma 1.20 and $(a+b)^{\frac{8}{5}} \leq C(a^{\frac{8}{5}} + b^{\frac{8}{5}})$ we obtain that

$$
\begin{aligned}
K_*^{\frac{8}{5}}(r) &\leq \left(C\left\{\left(\frac{r}{\rho}\right)^{\frac{1}{2}} K_*(\tfrac{\rho}{2}) + \left(\frac{\rho}{r}\right)^{\frac{5}{4}} A_*^{\frac{5}{8}}(\tfrac{\rho}{2})\delta_*^{\frac{5}{8}}(\tfrac{\rho}{2})\right\}\right)^{\frac{8}{5}} \\
&\leq C\left\{\left(\frac{r}{\rho}\right)^{\frac{4}{5}} K_*^{\frac{8}{5}}(\tfrac{\rho}{2}) + \left(\frac{\rho}{r}\right)^{2} A_*(\tfrac{\rho}{2})\delta_*(\tfrac{\rho}{2})\right\} \\
&\leq C\left\{\left(\frac{r}{\rho}\right)^{\frac{4}{5}} K_*^{\frac{8}{5}}(\rho) + \left(\frac{\rho}{r}\right)^{2} A_*(\tfrac{\rho}{2})\delta_*(\rho)\right\}
\end{aligned}
$$

We can use that by definition of M_* one has $K_*^{\frac{8}{5}}(\rho) \leq M_*(\rho)$ as well as (1.7) to obtain

$$
\begin{aligned}
K_*^{\frac{8}{5}}(r) &\leq C\left\{\left(\frac{r}{\rho}\right)^{\frac{4}{5}} M_*(\rho) + \left(\frac{\rho}{r}\right)^{2} M_*(\rho)\delta_*(\rho)\right\} \\
&\leq C\left\{\left(\frac{r}{\rho}\right)^{\frac{1}{5}} M_*(\rho) + \left(\frac{\rho}{r}\right)^{2} M_*(\rho)\delta_*(\rho)\right\},
\end{aligned} \tag{1.28}
$$

where the last step uses again that $r < \rho$.

Step 5: The claim follows now by adding up (1.25), (1.27) and (1.28), all of which consist only of terms that appear on the right hand side of the desired inequality. □

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