# Navier-Stokes Seminar:
# Caffarelli-Kohn-Nirenberg Theory

Edited: Jack Skipper

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# Preface

These are lecture notes geberated by the seminar course on the Caffarelli-Kohn-Nirenberg Theory for the Navier-Stokes equations at the Universität Ulm in the summer term of 2019. We mainly follow the [**CKN82**] in a modern fashion. This work is aimed at enthusiastic Masters and PhD students.

I would like to thank everyone taking the seminar for typing parts of these notes.

Corrections and suggestions should be sent to `jack.skipper@uni-ulm.de`.

# Contents

CHAPTER 1

# Talk 1: Introduction

## By Dr. Jack Skipper

For this introduction we will use the original paper of [**CKN82**] and the excellent book [**RRS16**].

The three-dimensional Navier-Stokes equations are

$$\begin{aligned} \partial_t u(x,t) + (u\cdot\nabla)u(x,t) + \nabla p(x,t) - \Delta u(x,t) &= f(x,t) \\ \operatorname{div} u(x,t) &= 0. \end{aligned} \tag{1.1}$$

Here, $(x,t) \in \Omega \times [0,T]$, where $\Omega \subset \mathbb{R}^3$ or $\mathbb{T}^3$ or $\mathbb{R}^3$ some domain, and we have the unknown velocity field

$$u\colon \Omega \times [0,T] \to \mathbb{R}^3;$$

the unknown pressure field

$$p\colon \Omega \times [0,T] \to \mathbb{R};$$

and the given force $f\colon \Omega \times [0,T] \to \mathbb{R}^3$ with $\operatorname{div} f = 0$ in $\Omega \times [0,T]$. Together with initial data and boundary data, (1.1) turns into an initial boundary value problem

$$\begin{aligned} u(x,0) &= u_0(x), && x \in \Omega, \\ u(x,t) &= 0, && x \in \partial\Omega \quad \text{for} \quad 0 < t < T. \end{aligned} \tag{1.2}$$

With compatibility conditions for $u_0$ and $f$ we see that

$$-\Delta p = \partial_i \partial_j (u_i u_j) \quad \text{for } a.e\ t.$$

### 1.1. Outline: The Navier-Stokes Equations

**1.1.1. Weak and Strong.** Here we will give an overview of the important results currently known about the Navier-Stokes equations(NSE). The results here were taken from the book by Robinson, Rodrigo,

- (Leray 1934, $\mathbb{R}^3$) in [**Ler34**] and (Hopf 1951, $\Omega$ or $\mathbb{T}^3$) in [**Hop51**] showed that Leray-Hopf (LH) weak solutions exist globally in time. Here we assume that the initial data $u_0 \in L^2_\sigma$ (in $L^2$ and weakly incompressible) and $u \in L^\infty(0,T;L^2_\sigma) \cap L^2(0,T;H^1)$ and satisfy the weak energy inequality, namely,

$$\int_\Omega u^2(t)\,\mathrm{d}x + \int_s^t \int_\Omega |\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t \le \int_\Omega u(s)\,\mathrm{d}x$$

  for almost every $t, s$. We do not know about uniqueness here.
- (Leray 1934, $\mathbb{R}^3$) in [**Ler34**] and (Kiseler-Ladyzhenskaya 1857) in [**KL57**] showed that strong solutions (LH weak solutions with $u_0 \in L^2_\sigma \cap H^1$ and $u \in L^\infty(0,T;H^1) \cap L^2(0,T;H^2)$) exist and are unique locally in time. They showed a lover bound on the potential "blow up" time $T = c\|\nabla u_0\|_{L^2}^{-4}$. Further, strong solutions are immediately smooth, even real analytic according to (Foias-Temam 1989) in [**FT89**].

- We have global existence of strong solutions for small data on $\Omega$ or $\mathbb{T}^3$ where we have an absolute constant $C(\Omega)$ or $\tilde{C}(\Omega)$ such that, for example,

  $$\|\nabla u_0\|_{L^2} < C \quad \|u_0\|_{L^2} < C\|\nabla u_0\|_{L^2} < \tilde{C}.$$

  For $\mathbb{R}^3$ we have a scaling $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$ is a solution. Thus if we want to talk about small data we need the norm to be invariant under this map, we say these spaces are critical spaces. $\dot{H}^{1/2}$, $L^3$, $BMO^{-1}$ are invariant spaces where for small data we have strong solutions and for any data have local in time strong solutions.
- (Sather-Serrin 1963) see **[Ser63]** showed weak-strong uniqueness, that is, strong solutions are unique in the class of LH weak solutions. (Need the energy inequality) This suggests 2 possibilities $u$ is strong always $\|\nabla u(t)\|_{L^2} < \infty$ for all $s > 0$ or there exists $T^*$ the "blow-up" time where

  $$\|\nabla u)(t)\|^2 \geq \frac{C(\Omega)}{\sqrt{(T^* - t)}}.$$

  Can use similar techniques to show robustness of solutions "if initial data is close to a strong solution initial data then the solutions is strong for a while".
- Leary noticed that any global in time LH weak solution is eventually strong and for large time $\|u(t)\|_{L^2} \to 0$ as $t \to \infty$.

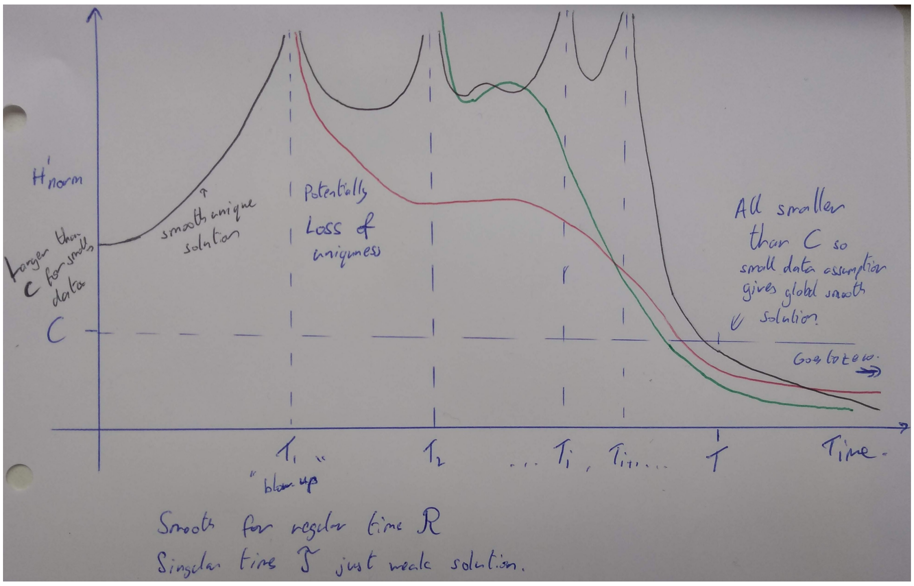


FIGURE 1. The $H^1$ norm of a potential solution to the Navier-Stokes equations.

**1.1.2. Regularity.** We can now look at the regularity of solutions and either we find conditions on how bad could the space of solutions be, or we find conditions on solutions that guarantee they are strong and smooth.

- (Scheffer 1976) in **[Sch76]** gave an upper bound on the size of the set of singular times. We say a time is regular and in the set $\mathcal{R}$ if $\|\nabla u(t)\|_{L^2}$ is essentially bounded. The singular times $\mathcal{T}$ a the rest. Here we see that the $\frac{1}{2}$ dimensional Hausdorff measure of the set $\mathcal{T}$ is zero. (Box counting measure is the same.)
- (Kato 1984) in **[Kat84]** showed that if

  $$\int_0^T \|\nabla u(s)\|_{L^\infty} \, \mathrm{d}s < \infty$$

then $u$ is strong on $(0,T]$.

- (Beal-Kato-Majda 1984) in [**BKM84**] showed that if

  $$\int_0^T \|\operatorname{curl} u(s)\|_{L^\infty}\,\mathrm{d}s < \infty$$

  then $u$ is strong on $(0,T]$ and further if we have "blow-up" at $T$ then

  $$\lim_{t\to T}\int_0^t \|\operatorname{curl} u(s)\|_{L^\infty}\,\mathrm{d}s = \infty.$$

- Serrin see [**Ser63**] condition that

  $$u \in L^r(0,T;L^s(\Omega)) \quad \frac{2}{r}+\frac{3}{s}=1$$

  gives a smooth solution on $(0,T]$. We note that we only unfortunately know that for a LH weak solution that

  $$\frac{2}{r}+\frac{3}{s}=\frac{3}{2}.$$

  Further, we have other Serrin type conditions, by (Beirão da Veiga 1995) in [**Bei95**]

  $$\nabla u \in L^r(0,T;L^s(\Omega)) \quad \frac{2}{r}+\frac{3}{s}=2 \quad \frac{3}{2}<s<\infty$$

  and by (Berselli-Galdi 2002) in [**BG02**] in

  $$p \in L^r(0,T;L^s(\Omega)) \quad \frac{2}{r}+\frac{3}{s}=2 \quad \frac{3}{2}<s.$$

- (Serrin 1962) in [**Ser62**], for the (<) case, showed a local version of the Serrin condition that, on a sub-domain $U\times(t_1,t_2)$, if

  $$u \in L^r(t_1,t_2;L^s(U)) \quad \frac{2}{r}+\frac{3}{s}=1$$

  then $u$ is smooth in space on $U\times(t_1,t_2)$ and $\alpha$-Hölder continuous with $\alpha<\frac{1}{2}$ (Don't get smoothness in time as have problems with $\nabla p$ and $\partial_t u$ interacting locally.) The equality was worked out by (Fabes-Jones-Riviere 1972) see [**FJR72**], (Struwe 1988) see [**Str88**] and (Takahashi 1990) in [**Tak90**].

Leary thought that his solutions were turbulent solutions and that a self-similar construction would give a solution that would "blow-up", however, (Nečas-Růžička-Šverák 1996) in [**NRS96**] essentially disproved this. Further, for Euler equations non-uniqueness of weak solutions has been shown starting with the work of (Scheffer 1993) in [**Sch93**] then (De Lellis-Székelyhidi 2010) in [**DS10**] and finally with (Wiedemann 2011) in [**Wie11**].

We have a picture of how LH weak solutions are behaving and the interplay with strong solutions. Regularity results go down two lines where either we ask for extra conditions, we can't guarantee, from LH weak solutions so that then they are strong solutions an thus unique. Here, for the CKN result we want to keep with the regularity we know LH weak solutions can have and find upper bounds on how bad the set of "bad singular points" of the weak solutions can be. We will show that we get a bound of on the 1 dimensional Hausdorff measure and show that the size of the set in this measure is 0.

### 1.2. "Suitable" Weak Solutions

The CKN partial regularity result for suitable week solutions of the NSE. (How bad is the space-time set of blow-ups)

We know that for any $u_0 \in L^2_\sigma$ there us a LH weak solution of the NSE that satisfies the local energy inequality. (This modern result needs maximal regularity theory for the pressure $p$). (Sohr-von Wahl 1986) in [**SvW86**] showed that for any $\varepsilon > 0$

$$p \in L^r(\varepsilon, T; L^s) \quad \text{for} \quad \frac{2}{r} + \frac{3}{s} = 3 \quad (s > 1)$$

or for the gradient of the pressure

$$\nabla p \in L^r(\varepsilon, T; L^s) \quad \text{for} \quad \frac{2}{r} + \frac{3}{s} = 4 \quad (s > 1)$$

and thus we obtain that $p \in L^{\frac{5}{3}}(\Omega \times (0,T]))$. CKN only knew that $p \in L^{\frac{5}{4}}(\Omega \times (0,T]))$ which adds extra technical difficulties.

Definition 1.1. The pair $(u,p)$ is a **suitable** weak solution of the NSE on $\Omega \times [0,T]$ with force $f$ if the following are satisfied.

(1) Integrability:
  (a) $f \in L^q(\Omega \times [0,T])$ for $q > \frac{5}{2}$,
  (b) $p \in L^{\frac{5}{4}}(\Omega \times [0,T])$ [Modern times can get as high as $L^{\frac{5}{3}}(\Omega \times [0,T])$],
  (c) $u \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$.

(2) Local energy inequality: For all $\phi \geq 0$, $\phi \in C_c^\infty$, then,

$$2 \iint |\nabla u|^2 \phi \, \mathrm{d}x \, \mathrm{d}s \leq \iint |u|^2(\phi_t + \Delta\phi) + (|u|^2 + 2p) u \cdot \nabla\phi + 2(u \cdot f)\phi \, \mathrm{d}x \, \mathrm{d}s$$

(3) Weak solution: We need $u \in L^\infty(0,T;L^2_\sigma) \cap L^2(0,T;H^1_\sigma)$, $\nabla \cdot f = 0$, $-\Delta p = \partial_i \partial_j (u_i u_j)$ and for $a.e. t \in (a,b)$ and for all $\phi \in C^\infty_{\sigma,c}$

$$\int_{\Omega \times \{0\}} u_0 \cdot \phi(0) \, \mathrm{d}x = \int_0^T \int_\Omega \nabla u : \nabla\phi + (u \cdot \nabla) u \phi - u \cdot \partial_t \phi - f \cdot \phi \, \mathrm{d}x \, \mathrm{d}t.$$

For the CKN theory we do not need point 3 above, that is, the pair $(u,p)$ does not actually need to be a LH weak solution of the NSE. The proof just deals with local energy inequality and interpolation inequalities as so points 1 and 2 are sufficient, the "suitable" bit.

As an interesting aside, it is important to note that in (Scheffer 1987) in [**Sch87**] he showed that the end result, that the one dimensional Hausedroff measure of the singular set of space-time points is zero, cannot be improved using the "suitable" criteria and the method would have to use (the equation) part 3 above. He showed that if you just pick a "suitable" pair $(u,p)$ then for any $\gamma < 1$ there will exist at least one $(u,p)$ pair where the $\gamma$- dimensional Hausdrof measure of the singular set is infinite.

### 1.3. Partial Regularity

We want to study "how bad" the set of "singular points" for $u$ a suitable solution.

We denote $\mathcal{R}$ the set of regular points $(x,t) \in \mathcal{R}$ if there exists an open set $U \subset \Omega \times [0,T]$ with $(x,t) \in U$ and $u \in L^\infty(U)$. Let $\mathcal{S}$ be the set of singular points defined by $\mathcal{S} := \Omega \times [0,T] \setminus \mathcal{R}$, so the points where $u$ is not $L^\infty_{loc}$ in any neighbourhood of $(x,t)$. (Can also be defined similarly but with $\operatorname{curl} u$ or $\nabla u$.) By "bad" we want an upper-bound on the dimension of $\mathcal{S}$ here using the Hausdroff measure.

Theorem 1.2 (Main Theorem (B) in [**CKN82**]). *For any suitable weak solution of the NSE on an open set in space-time the asscoiated singular set $\mathcal{S}$ satisfies*

$$\mathcal{P}^1(S) = 0.$$

This condition is equivalent to $\mathcal{H}^1(S) = 0$ which denotes that the one dimensional Hausdroff measure of the singular set is 0.

Importantly this shows that there are no curves in space-time where the solution $u$ is singular along the curve. If we have "blow-up" then this occurs at distinct points in space time and not on a continuum.

CKN also impose extra conditions to prove two other theorems. These results are more in the spirit of previous partial regularity results like Serrin conditions as discussed earlier.

Let $E$ denote the initial "kinetic energy", the $L^2$ norm of for the initial data, that is,

$$E := \frac{1}{2}\int_{\mathbb{R}^3} |u_0|^2 \,\mathrm{d}x$$

and let $G$, be a weighted form of $E$ where we want extra decay at infinity, that is,

$$G := \frac{1}{2}\int_{\mathbb{R}^3} |u_0|^2|x| \,\mathrm{d}x < \infty.$$

For initial data satisfying this condition one can show that a suitable weak solution of the NSE from this data satisfies

$$\frac{1}{2}\int_{\mathbb{R}^3\times\{t\}} |u|^2|x| \,\mathrm{d}x + \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2|x| \,\mathrm{d}x \,\mathrm{d}s < \infty$$

for every $t$, so obtain the following theorem showing that the solution is regular for large enough $x$.

THEOREM 1.3 (Theorem C in [**CKN82**]). *Suppose $u_0 \in L^2(\mathbb{R}^3)$ $\nabla\cdot u_0 = 0$ and $G < \infty$. Then there exists a weak solution of the NSE with $f = 0$ which is regular on the set*

$$\{(x,t)\colon |x|^2 t > K_1\}$$

*where $K_1 = K_1(E,G)$ is a constant only depending on $u_0$ via $E$ and $G$.*

Here we see that $G$ is a restriction that the initial data $u_0$ should decay sufficiently rapidly at infinity.

If instead we have a different condition where we ask for decay approaching zero, that is,

$$\int_{\mathbb{R}^3} |u_0|^2|x|^{-1} \,\mathrm{d}x = L \le L_0$$

then we obtain

$$\sup_\tau \int_{\mathbb{R}^3\times\{\tau\}} |u|^2|x|^{-1} \,\mathrm{d}x < \infty, \qquad \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2|x|^{-1} \,\mathrm{d}x \,\mathrm{d}\tau < \infty$$

for each $t$. From this we obtain the following theorem where we see that $u$ is regular in a parabola above the origin and the line $x = 0$ is regular for all $t$.

THEOREM 1.4 (Theorem D in [**CKN82**]). *There exists an absolute constant $L_0 > 0$ with the following properties. If $u_0 \in L^2(\mathbb{R}^3)$ $\nabla\cdot u_0 = 0$ and $L < L_0$ then there exists a weak solution of the NSE with $f = 0$ which is regular on the set*

$$\{(x,t)\colon |x|^2 < t(L_0 - L)\}.$$

### 1.4. Scale-invariant Quantities (Dimensionless Quantities)

On $\mathbb{R}^3$ if we have a solution to the NSE then by rescaling by $\lambda$, in the following way,

$$u(x,t) \mapsto \lambda u(\lambda x, \lambda^2 t)$$
$$p(x,t) \mapsto \lambda^2 p(\lambda x, \lambda^2 t)$$
$$f(x,t) \mapsto \lambda^3 f(\lambda x, \lambda^2 t)$$

we have another solution. Here we see that time scales quadratically and space linearly.

For local estimates it will be best to use, rather than balls, parabolic cylinders, that is,

$$Q_r(x,t) := \{(y,\tau): |y-x| \le r,\ t-r^2 < \tau < t\}$$

or $Q_r^*(x,t) = Q_r(x, t-\frac{1}{8}r^2)$ (here $(x,t)$ is the geometric centre of $Q_{\frac{r}{2}}(x, t+\frac{1}{8}r^2)$). The scaling that works on $\mathbb{R}^3$ also works on the parabolic cylinders where if $(u,p)$ is a solution on $Q_r(x,t)$ then $(u_\lambda, p_\lambda)$ will be a solution on $Q_{\frac{r}{\lambda}}(x,t)$.

We want to study "quantities" being "small" over parabolic cylinders and thus to have a sensible definition of a "smallness" assumption we should study scale invariant "quantities", that is, "quantities" whose value will not change after rescaling space and time as above. If the "quantities" we study did not have this property then under rescaling we could shrink or blow-up the values and could not compare the values. We will use factors of $\frac{1}{r}$ to make the scale invariant quantities we need.

For example,

$$\begin{aligned}\frac{1}{\left(\frac{r}{\lambda}\right)^2}\int_{Q_{\frac{r}{\lambda}}(0,0)} |u_\lambda|^3\,\mathrm{d}x\,\mathrm{d}t &= \frac{\lambda^2}{r^2}\int_{Q_{\frac{r}{\lambda}}(0,0)} \lambda^3|u(\lambda x, \lambda^2 t)|^3\,\mathrm{d}x\,\mathrm{d}t \\ &= \frac{1}{r^2}\int_{Q_r(0,0)} |u(y,s)|^3\,\mathrm{d}y\,\mathrm{d}s\end{aligned}$$

where we have a change of variable $y = \lambda x,\ s = \lambda^2 t$.

Some of the scale-invariant quantities we will use are

$$\frac{1}{r}\sup_{-r^2<t<0}\int_{B_r}|u(t)|^2\,\mathrm{d}x, \quad \frac{1}{r}\iint_{Q_r}|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t, \quad \frac{1}{r^2}\iint_{Q_r}|u|^3\,\mathrm{d}x\,\mathrm{d}t, \quad \frac{1}{r^2}\iint_{Q_r}|p|^{\frac{3}{2}}\,\mathrm{d}x\,\mathrm{d}t.$$

## 1.5. The Main Ideas

We need to show two main propositions that concern bounds on $u$ for large radii giving properties for $u$ on smaller radii.

Proposition 1.5. *There are absolute constants $\varepsilon, C_1 > 0$ and constant $\varepsilon_2(q) > 0$ with the following properties. If $(u,p)$ is a suitable weak solution of the NSE on $Q_1(0,0)$ with force $f \in L^q$, for some $q > \frac{5}{2}$ and*

$$\iint_{Q_1(0,0)} (|u|^3 + |u||p|)\,\mathrm{d}x\,\mathrm{d}t + \int_{-1}^{0}\left(\int_{B_1}|p|\,\mathrm{d}x\right)^{\frac{5}{4}}\mathrm{d}t \le \varepsilon_1 \quad \text{and} \quad \iint_{Q_1(0,0)} |f|^q\,\mathrm{d}x\,\mathrm{d}t \le \varepsilon_2$$

*then $u \in L^\infty(Q_{\frac{1}{2}}(0,0))$ with $\|u\|_{L^\infty(Q_{\frac{1}{2}}(0,0))} \le C_1$. ($u$ is regular on $Q_{\frac{1}{2}}(0,0)$).*

With no force and modern $p \in L^{\frac{5}{3}}$ we can just assume that

$$\iint_{Q_1(0,0)} (|u|^3 + |p|^{\frac{3}{2}})\,\mathrm{d}x\,\mathrm{d}t \le \varepsilon_1$$

and the proof is simplified.

We can shift and rescale this proposition to apply it to different $Q_r(x,t)$.

Proposition 1.6. *There exists an absolute constant $\varepsilon_3$ such that if $(u,p)$ is a suitable weak solution to the NSE on $Q_R(a,s)$ for some $R > 0$ and if*

$$\limsup_{r\to 0}\frac{1}{r}\int_{Q_r(as)}|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t \le \varepsilon_3$$

*then $u \in L^\infty(Q_\rho(a,s))$ for some $\rho$ with $0 < \rho < R$. $(a,s)$ is a regular point.*

We will now discuss a rough outline of the proof and the tools used.

- We have the local energy inequality,

$$2\iint |\nabla u|^2\phi \, \mathrm{d}x \, \mathrm{d}s \le \iint |u|^2(\phi_t + \Delta\phi) + (|u|^2 + 2p)u \cdot \nabla\phi + 2(u \cdot f)\phi \, \mathrm{d}x \, \mathrm{d}s.$$

  We use an approximation to the backwards heat equation for $\phi$ on a parabolic cylinder so it approximately solves $\phi_t + \Delta\phi = 0$ and get appropriate bounds on $\phi$ and $\nabla\phi$ as powers of $\frac{1}{r}$. This gives an inequality over parabolic cylinders with weighting in front of the remaining terms that means they are scaling invariant.
- We can use different interpolation inequalities over parabolic cylinders, for example,

$$\frac{1}{r^2}\iint\limits_{Q_r(a,s)} |u|^3 \, \mathrm{d}x \, \mathrm{d}t \le C_0 \left[\frac{1}{r}\sup_{s-r^2<t<s}\int_{B_r(a)} |u(t)|^2 + \frac{1}{r}\iint\limits_{Q_r(a,s)} |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}t\right]^{\frac{3}{2}}.$$

- We can use these two inequalities. We see that the term on the RHS of the local energy inequality is quadratic in $u$ and on the LHS they are all act cubic in $u$ (with the assumed regularity on $p$ and $f$) however this is the opposite for the interpolation inequality. We can thus iterate between these two inequalities to obtain inductive bounds on a solution $u$ from the larger cylinder to a smaller cylinder that are shrinking and so can use Lebesgue differentiation theorem to get that the points $(a, s)$ are regular on the smaller cylinder.

CHAPTER 2

# Talk 2: Suitable weak solutions: part 1

## By Farid Mohamed

We introduce the spaces for $\Omega \subset \mathbb{R}^3$

$$\mathcal{V} = \{u \in C_0^\infty(\Omega), \text{div } u = 0\},$$
$$V = \overline{\mathcal{V}}^{\|\cdot\|_{H_0^1(\Omega)}} \text{ and}$$
$$H = \overline{\mathcal{V}}^{\|\cdot\|_{L^2(\Omega)}}.$$

The space $H$ is equipped with the norm $\|\cdot\|_{L^2(\Omega)}$ and we write

$$(u,v)_{L^2(\Omega)} := \int_\Omega uv\,dx$$

for the generating scalar product. In the case of $V$ we need to distinguish two cases. If $\Omega$ is bounded we set $\|u\|_V := \|\nabla u\|_{L^2(\Omega)}$ and if $\Omega$ is unbounded we define $\|u\|_V := \|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}$. We observe that $V \hookrightarrow H \hookrightarrow V'$, where we identify $H$ and $H'$ in the sense that for every $u \in H$ we set

$$\langle u, f\rangle = T_u(f) = \int_\Omega uf dx$$

for $f \in H$. In this case we see that $\langle u, f\rangle = (u,f)_{L^2(\Omega)}$.
We assume for this section that

$$\Omega = \mathbb{R}^3,$$
$$f \in L^2(0,T;H^{-1}(\mathbb{R}^3)) \text{ and } \nabla\cdot f = 0,$$
$$u_0 \in H$$

or

$$\Omega \text{ is a smooth, bounded, open and connected set in } \mathbb{R}^3$$
$$f \in L^2(\Omega\times(0,T)) \text{ and } \nabla\cdot f = 0,$$
$$u_0 \in H \cap W_{5/4}^{2/5}(\Omega).$$

It follows directly that the spaces $L^2(0,T;H)$ and $L^2(0,T;V)$ are reflexive and $L^\infty(0,T;H)$ and $L^\infty(0,T;V)$ are the duals of separable Banach spaces, see for example [?], Theorem 1.29.

Definition 2.1. We call the pair $(u,p)$ a suitable weak solution of the Navier-Stokes system on an open set $D = \Omega\times(0,T) \subset \mathbb{R}^3\times\mathbb{R}$ with force $f$ if:

i) $u, p$ and $f$ are measurable functions on $D$,
ii) $f \in L^q(D)$ for $q > 5/2$, $\nabla\cdot f = 0$ and $p \in L^{5/4}(D)$,

iii) the solution $u$ is bounded in the following sense

$$E_0(u) :=_{0<t<T} \int_\Omega |u(x,t)|^2 dx < \infty \text{ and } E_1(u) := \iint_D |\nabla u|^2 dxdt < \infty,$$

iv) $u, p$ and $f$ solve

$$\partial_t u(x,t) + (u \cdot \nabla) u(x,t) + \nabla p(x,t) - \Delta u(x,t) = f \text{ in } \Omega,$$
$$\operatorname{div} u(x,t) = 0 \text{ on } \partial\Omega \text{ for all } 0 < t < T$$

in the sense of distributions in $D$, i.e. $u \in L^2(0,T;V)$ and for all $v \in V$ we have

$$\frac{d}{dt}\int_\Omega u(x,t)v(x)\,dx + \int_\Omega (u \cdot \nabla)u(x,t)v(x)\,dx + \int_\Omega \nabla u \cdot \nabla v\,dx = \int_\Omega f(t,x)v(x)dx$$

in the distributional sense on $(0,T)$.

v) for all $\varphi \in C_0^\infty(D)$, $\varphi \geq 0$ it holds

$$2\iint_D |\nabla u|^2 \varphi dxdt \leq \iint_D (|u|^2(\varphi_t + \Delta\varphi) + (|u|^2 + 2p)u \cdot \nabla\varphi + 2(u \cdot f)\varphi)dxdt.$$

The goal of this chapter is to show that for every $f \in L^q(D)$ there exists a suitable weak solution in the sense of Defintion 2.1.

The first step is to show that the equation

$$u_t + (w \cdot \nabla)u - \Delta u + \nabla p = f$$

has a solution for suitable $f$ and $w$, where we use the following lemma.

Lemma 2.2 (see [**Tem79**], Lemma 1.2). *Suppose* $f \in L^2(0,T;V')$, $u \in L^2(0,T;V)$, $p$ *is a distribution and*

$$u_t - \Delta u + \nabla p = f$$

*in the sense of distributions on* $D$. *Then*

$$u_t \in L^2(0,T;V'),$$
$$\frac{d}{dt}\int_\Omega |u|^2 = 2(u_t, u)_{L^2(\Omega)}$$

*in the sense of distributions on* $(0,T)$ *and*

$$u \in C([0,T], H)$$

*after modification on a set of measure zero. Solutions of* (2.2) *are unique in the space* $L^2(0,T;V)$ *for given initial data* $u_0 \in H$.

Proof. Here we give the main ideas of the proof.
Let the function $\hat{u} : \mathbb{R} \to V$ be equal to $u$ on $[0,T]$ and to 0 outside this interval. We see by [**LM72**], Theorem 4.3 a sequence $(u_m)_{m\in\mathbb{N}}$ such that

$$\forall m, u_m \text{ is infinitly differentiable from } [0,T] \text{ onto } V, \text{ as } m \to \infty$$
$$u_m \to u \text{ in } L^2_{loc}(0,T;V),$$
$$u'_m \to u' \text{ in } L^2_{loc}(0,T;V').$$

It follows directly

$$\frac{d}{dt}\int_\Omega |u_m(t)|^2 = 2(u'_m(t), u_m(t))_{L^2(\Omega)}$$

and as $m \to \infty$ we get

$$\|u_m\|^2_{L^2(\Omega)} \to \|u\|^2_{L^2(\Omega)} \text{ in } L^1_{loc}((0,T))$$
$$(u'_m, u_m)_{L^2(\Omega)} \to (u', u)_{L^2(\Omega)} \text{ in } L^1_{loc}((0,T)).$$

These convergences also hold in the distribution sense. So by passing to the limit we get

$$\frac{d}{dt}\int_\Omega |u|^2 = 2(u_t, u)_{L^2(\Omega)}$$

and by (2) we see that $u \in L^\infty(0,T;H)$. We conclude by [**Tem79**], Lemma 1.4 that $u \in C([0,T];H)$. Uniqueness will follow by the next lemma. □

LEMMA 2.3. *Let $f \in L^2(0,T;V')$, $u_0 \in H$ and $w \in C^\infty(\overline{D},\mathbb{R}^3)$ with $\nabla \cdot w = 0$. Then there exists a unique function $u$ and a distribution $p$ such that*

$$\begin{aligned} &u \in C([0,T],H) \cap L^2(0,T;V),\\ &\quad u_t + (w\cdot\nabla)u - \Delta u + \nabla p = f \end{aligned}$$

*in the sense of distributions on $D$, with $u(0) = u_0$.*

PROOF. We will follow [**Tem79**], Theorem 1.1 by constructing the solution. Let $\{x_n\}_{n\in\mathbb{N}} \subset V$ be a sequence of linearly indepedent vectors such that $\overline{\operatorname{span}((x_n)_{n\in\mathbb{N}})} = V$, which exists as $V$ is separable. We set $V_n := \operatorname{span}(x_1,\dots,x_n)$ and $u_n := \sum_{i=1}^n g_{in}(t)x_i$, where $(g_{in})_{i=1}^n$ is a solution of the system

$$\begin{aligned} \sum_{i=1}^n g'_{in}(t)(x_i,x_j)_{L^2(\Omega)} + \sum_{i=1}^n g_{in}(t)(((w\cdot\nabla)x_i,x_j)_{L^2(\Omega)} + (\nabla x_i,\nabla x_j)_{L^2(\Omega)}) = \langle f, x_j\rangle\\ g_{jn}(0) = P_{V_n}(x_0)_j \end{aligned}$$

for $j = 1,\dots,n$. Then $u_n$ solves the equation

$$(u'_n, v)_{L^2(\Omega)} + ((w\cdot\nabla)u_n, v) + (\nabla u_n, \nabla v)_{L^2(\Omega)} = \langle f, v\rangle$$

for all $v \in V_n$. Observe by partial integration that

$$((w\cdot\nabla)u_n, u_n)_{L^2(\Omega)} = -(u_n, (w\cdot\nabla)u_n)_{L^2(\Omega)} = 0$$

and one obtains

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u_n\|^2_{L^2(\Omega)} =&(u'_n, u_n)_{L^2(\Omega)}\\ =&\langle f, u_n\rangle - (\nabla u_n, \nabla u_n)_{L^2(\Omega)}\\ \le&\frac{1}{2}\|f\|^2_{V'} + \frac{1}{2}\|u_n\|^2_{L^2(\Omega)} - \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)}\\ \le&\frac{1}{2}\|f\|^2_{V'} + \frac{1}{2}\|u_n\|^2_{L^2(\Omega)}, \end{aligned}$$

whch follows by

$$\langle f, u_n\rangle \le \frac{1}{2}\|f\|^2_{V'} + \frac{1}{2}\|u_n\|^2_V \le \frac{1}{2}\|f\|^2_{V'} + \frac{1}{2}\|u_n\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u_n\|^2_{L^2(\Omega)}.$$

The continuity of the projection and Gronwall's inequality imply that

$$\|u_n(t)\|^2_{L^2(\Omega)} \le \left(\|u_0\|^2_{L^2(\Omega)} + \int_0^T \|f(s)\|^2_{V'}\,ds\right)e^T < \infty,$$

which implies that $(u_n)_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty(0,T;H)$. Furthermore, we see by integrating (2)

$$\begin{aligned}
&\|u_n(t)\|^2_{L^2(\Omega)} + \int\limits_0^t \|\nabla u_n(s)\|^2_{L^2(\Omega)} ds \\
\le&\|u_n(0)\|^2_{L^2(\Omega)} + \int\limits_0^t \|f(s)\|^2_{V'} ds + \int_0^T \|u_n(s)\|^2_{L^2(\Omega)}\, ds \\
\le&\Big(\|u(0)\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(0,T;V')}\Big)(1+Te^T)
\end{aligned}$$

and we conclude that $(u_n)_{n\in\mathbb{N}}$ is uniformly bounded in $L^2(0,T;V)$. One infers that there exists a subsequence $(u_n)_{n\in\mathbb{N}} \subset L^2(0,T;V)\cap L^\infty(0,T;H)$ such that there exists an $u \in L^2(0,T;V)\cap L^\infty(0,T;H)$

$$u_n \rightharpoonup u \text{ for } n\to\infty \text{ in } L^2(0,T;V) \text{ and}$$
$$u_n \overset{*}{\rightharpoonup} u \text{ for } n\to\infty \text{ in } L^\infty(0,T;H).$$

We conclude for every $\varphi \in C^1([0,T])$ with $\varphi(T)=0$ that

$$\begin{aligned}
0 =& \int_0^T ((u_n'(t),\varphi(t)x_j)_{L^2(\Omega)} + ((w\cdot\nabla)u_n(t),\varphi(t)x_j) + (\nabla u_n(t),\nabla x_j\varphi(t))_{L^2(\Omega)} \\
&- \langle f(t),\varphi(t)x_j\rangle)\, dt \\
=& \int_0^T (-(u_n(t),\varphi'(t)x_j)_{L^2(\Omega)} + ((w\cdot\nabla)u_n(t),\varphi(t)x_j) + (\nabla u_n(t),\nabla x_j\varphi(t))_{L^2(\Omega)} \\
&- \langle f(t),\varphi(t)x_j\rangle dt - (u_n(0),x_j)_{L^2(\Omega)}\varphi(0)) \\
\to& \int_0^T (-(u(t),\varphi'(t)x_j)_{L^2(\Omega)} + ((w\cdot\nabla)u(t),\varphi(t)x_j) + (\nabla u(t),\nabla x_j\varphi(t))_{L^2(\Omega)} \\
&- \langle f(t),\varphi(t)x_j\rangle dt - (u(0),x_j)_{L^2(\Omega)}\varphi(0))
\end{aligned}$$

for $n\to\infty$ for every $j\in\mathbb{N}$. Moreover, the equality holds for every finite combination of the $(x_j)$ and by continuity even for all $v\in V$. We obtain that

$$\frac{d}{dt}(u,v)_{L^2(\Omega)} + ((w\cdot\nabla)u,v) + (\nabla u,\nabla v)_{L^2(\Omega)} = \langle f,v\rangle$$

in the sense of distributions on $(0,T)$.
In order to see that $u(0)=u_0$ we use that

$$\int_0^T \frac{d}{dt}(u(t),v)_{L^2(\Omega)}\varphi(t)dt = -\int_0^T (u(t),v)\varphi'(t)\,dt + (u(0),v)\varphi(0),$$

which implies that

$$\begin{aligned}
&-\int_0^T (u(t),v)\varphi'(t)dt + \int_0^T (\nabla u,\nabla v)_{L^2(\Omega)}\varphi(t)\,dt + \int_0^T ((w\cdot\nabla)u,v)_{L^2(\Omega)}\varphi(t)\,dt \\
=&(u(0),v)\varphi(0) + \int_0^T \langle f(t),v\rangle\varphi(t)dt
\end{aligned}$$

By comparison with the above equality we see that

$$(u_0 - u(0),v)\varphi(0) = 0.$$

As $v$ was arbitrary we conclude that $u_0 = u(0)$.
To show uniqueness assume that we have two solutions $u_1$ and $u_2$ with some initial data

and force $f$. We know that $u_1 - u_2$ solves (2) with $f = 0$. We conclude by (2) that

$$\frac{1}{2}\frac{d}{dt}\|u_1 - u_2\|^2_{L^2(\Omega)} \le -(\nabla(u_1 - u_2), \nabla(u_1 - u_2))_{L^2(\Omega)} \le 0.$$

As $u_1(0) = u_2(0)$ we conclude that $u_1 = u_2$. □

A solution of the Poisson equation $-\Delta u = f$ for $f \in L^q(\mathbb{R}^3)$ for some $1 < q < \infty$ can be written as

$$u(x) := (-\Delta)^{-1} f(x) := c_3 \int_{\mathbb{R}^3} \frac{1}{|x-y|} f(y) dy,$$

where $c_3 \in \mathbb{R}$ can be given explicitly. We use the following theorem, which can be shown by the Calderón-Zygmund theorem.

Theorem 2.4 (see [?], Theorem B.7). *The linear operator $T_{jk}$ defined by*

$$T_{jk} f := \partial_j \partial_k (-\Delta)^{-1} f$$

*is a bounded linear operator from $L^q(\mathbb{R}^3)$ into $L^q(\mathbb{R}^3)$ for all $1 < q < \infty$, i.e.*

$$\|T_{jk} f\|_{L^q(\mathbb{R}^3)} \le C \|f\|_{L^q(\mathbb{R}^3)}$$

*for some constant $C > 0$.*

Lemma 2.5. *Let $\Omega = \mathbb{R}^3$, $f \in L^2(0,T; H^{-1}(\mathbb{R}^3))$,* div $f = 0$ *and $u_0 \in H$. Then it holds that*

$$\Delta p = -\sum_{i,j} \partial_i \partial_j (w_i u_j),$$

*in the sense of distribution. Hence, we obtain*

$$\iint_D |p|^{5/3} dx dt \le C \iint_D |w|^{5/3} \cdot |u|^{5/3} dx dt.$$

Remark 2.6. For general $\Omega$ (if $\Omega$ is bounded) it is also possible to show that $p \in L^{5/3}(D)$.

Proof. We follow [?] to show that $p$ is given by (2.5). At first, observe that

$$\{\varphi \in [\mathcal{S}(\mathbb{R}^3)]^3 : \operatorname{div} \varphi = 0\}$$

is a dense subset of $V$. Furthermore, for every $h \in [\mathcal{S}(\mathbb{R}^3)]^3$ there exists a $\varphi \in [\mathcal{S}(\mathbb{R}^3)]^3$ and $\psi \in \mathcal{S}(\mathbb{R}^3)$ such that $h = \varphi + \nabla\psi$ and $\nabla \cdot \varphi = 0$, see for example [?], Exercise 5.2. Now let $\xi \in C_0^\infty((0,T))$. As $u$ is the solution of (2) we obtain by partial integration

$$\begin{aligned}
&-\int_0^T (u,h)_{L^2(\mathbb{R}^3)} \xi'(t)\, dt - \int_0^T (u, \Delta h)_{L^2(\mathbb{R}^3)} \xi(t)\, dt \\
&-\int_0^T (u \otimes w, \nabla h)_{L^2(\mathbb{R}^3)} \xi(t)\, dt - \int_0^T \langle f, h\rangle \xi(t) dt \\
=& -\int_0^T (u,\varphi)_{L^2(\mathbb{R}^3)} \xi'(t)\, dt + \int_0^T (\nabla u, \nabla\varphi)_{L^2(\mathbb{R}^3)} \xi(t)\, dt \\
&+\int_0^T ((w\cdot\nabla)u, \varphi)_{L^2(\mathbb{R}^3)} \xi(t)\, dt - \int_0^T \sum_{i,j} (u_i w_j, \partial_i\partial_j \psi)_{L^2(\mathbb{R}^3)} \xi(t)\, dt \\
&-\int_0^T \langle f, \varphi\rangle \xi(t) dt \\
=& -\int_0^T \sum_{i,j} (u_i w_j, \partial_i \partial_j \psi)_{L^2(\mathbb{R}^3)} \xi(t)\, dt.
\end{aligned}$$

As $u \in V$, we conclude that $\Delta p = -\sum_{i,j} \partial_i\partial_j(w_i u_j)$, where we used that $\nabla \cdot h = \Delta\psi$. By taking the Fourier transform we see that we can interchange the Laplace operator and $\partial_i\partial_j$ and we obtain

$$p = (-\Delta)^{-1}(-\Delta)p = \sum_{i,j}(-\Delta)^{-1}\partial_i\partial_j w_i u_j = \sum_{i,j}\partial_i\partial_j(-\Delta)^{-1}w_i u_j,$$

and one infers by Theorem 2.4 that $\|p\|_{L^{5/3}(\mathbb{R}^3)} \le C\||w|\cdot|u|\|_{L^{5/3}}$. □

Later on we want to estimate the pressure $p$ by using following inequality

$$\int_{\mathbb{R}^3} |u|^q dx \le C\left(\int_{\mathbb{R}} |\nabla u|^2\, dx\right)^{\frac{3}{4}(q-2)} \left(\int_{\mathbb{R}} |u|^2\, dx\right)^{\frac{1}{4}(6-q)}$$

for $2 \le q \le 6$, which is a special case of the Gagliardo-Nirenberg interpolation inequality

$$\|D^j u\|_{L^q(\mathbb{R}^3)} \le C\|D^m u\|^{\alpha}_{L^r(\mathbb{R}^3)}\|u\|^{1-\alpha}_{L^p(\mathbb{R}^3)}$$

where $1 < q,p,r < \infty$ and $m,j \in \mathbb{N}$. $\alpha$ is chosen is such a way that $\frac{1}{q} = \frac{j}{3} + \left(\frac{1}{r} - \frac{m}{3}\right)\alpha + \frac{1-\alpha}{p}$ and $\frac{j}{m} \le \alpha \le 1$. By choosing $j = 0$, $m = 1$, $r = p = 2$ and $\alpha = 3(\frac{1}{2} - \frac{1}{q})$ we obtain (2). We recall that we denote by

$$E_0(u) :=_{0<t<T} \int_{\Omega} |u(x,t)|^2 dx \text{ and } E_1(u) := \iint_D |\nabla u|^2 dxdt.$$

LEMMA 2.7. *For $u, w \in L^2(0,T;H^1(\mathbb{R}^3))$,*

$$\|u\|_{L^{10/3}(0,T;L^{10/3}(\mathbb{R}^3))} \le CE_1^{3/10}(u)E_0^{1/5}(u),$$
$$\|w\cdot\nabla u\|_{L^{5/4}(0,T;L^{5/4}(\mathbb{R}^3))} \le CE_1^{1/2}(u)E_1^{3/10}(w)E_0^{1/5}(w),$$
$$\|u\|_{L^5(0,T;L^{5/2}(\mathbb{R}^3))} \le CT^{1/20}E_0^{7/20}(u)E_1^{3/20}(u).$$

PROOF. For (2.7) we use (2) and obtain

$$\int_{\mathbb{R}^3} |u|^{10/3}\, dx \le C\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)\left(\int_{\mathbb{R}^3} |u|^2 dx\right)^{2/3} \le C\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)E_0(u)^{2/3}$$

for almost all $t \in (0,T)$. Integrating over $(0,T)$ gives the result. For (2.7) we see by Hölder's inequality that

$$\int_0^T \int_{\mathbb{R}^3} |w\cdot\nabla u|^{5/4} dxdt \le \left(\int_0^T \int_{\mathbb{R}^3} |w|^{10/3}\, dxdt\right)^{3/8} E_1(u)^{\frac{5}{8}}$$
$$= \|w\|^{5/4}_{L^{10/3}(0,T;L^{10/3}(\mathbb{R}^3))}E_1(u)^{\frac{5}{8}}.$$

By applying (2.7) we obtain (2.7). Furthermore, we see by (2) and Hölder's inequality that

$$\int_0^T \left(\int_{\mathbb{R}^3} |u|^{5/2}\, dx\right)^2 dt \le C\int_0^T \left(\int_{\mathbb{R}} |\nabla u|^2\, dx\right)^{3/4} \left(\int_{\mathbb{R}^3} |u|^2\, dx\right)^{7/4} dt$$
$$\le CE_0(u)^{7/4}\int_0^T \left(\int_{\mathbb{R}} |\nabla u|^2\, dx\right)^{3/4} dt$$
$$\le CE_0(u)^{7/4}T^{1/4}\left(\int_0^T \int_{\mathbb{R}} |\nabla u|^2\, dxdt\right)^{3/4}.$$

We conclude that (2.7) holds true. □

CHAPTER 3

# Talk 3: Suitable weak solutions: part 2

**By David Berger**

LEMMA 3.1 (see [**GS91**], Theorem 2.8). *Assume that $\Omega, f$ and $u_0$ satisfy the assumptions of Lemma 2.3. Let $\Omega$ be bounded, $4 = 3/q + 2/s$ and $w \cdot \nabla u, f \in L^s(0,T;L^q(\Omega))$ and $u_0 \in W_s^{2-2/s}(\Omega)$. Then the solution $(u,p)$ constructed in Lemma 2.3 satisfies*

$$\begin{aligned}&\|\nabla p\|^s_{L^s(0,T;L^q(\Omega))} + \|u_t\|^s_{L^s(0,T;L^q(\Omega))} + \|\nabla^2 u\|^s_{L^s((0,T;L^q(\Omega))}\\ \leq &C(\|u_0\|^s_{W_s^{2-2/s}(\Omega)} + \|w\cdot\nabla u\|^s_{L^s(0,T;L^q(\Omega))} + \|f\|^s_{L^s(0,T;L^q(\Omega))}).\end{aligned}$$

*Furthermore, by normalizing $p$ such that $\int_\Omega p = 0$ for all $t$ we obtain*

$$\|p\|_{L^{5/3}(0,T;L^{5/3}(\Omega))} < \infty.$$

LEMMA 3.2. *Let $\Omega, u_0$ and $f$ satisfy the assumption of Chapter 2 and let $w \in C^\infty(\bar{D},\mathbb{R}^3)$ with $\nabla\cdot w = 0$. Let $(u,p)$ be the solution of Lemma 2.3. Then, for every $\varphi \in C^\infty(\bar{D})$ with $\varphi = 0$ near $\partial\Omega \times (0,T)$, and for every $t$, $0 < t \leq T$,*

$$\begin{aligned}\int_\Omega |u(x,t)|^2\varphi(x,t)dx + 2\iint_D |\nabla u|^2\varphi = \int_\Omega |u_0|^2\varphi(x,0) + \iint_D |u|^2(\varphi_t + \Delta\varphi)\\ + \iint_D (|u|^2 w + 2pu)\cdot\nabla\varphi + 2\iint_D (u\cdot f)\varphi\end{aligned}$$

PROOF. We assume that $\Omega$ is bounded. Suppose for the moment that $\varphi$ vanishes near $t = 0$, choose $\Omega_1$, so that $\Omega_1 \subset \Omega$ and supp$\varphi \subset \Omega_1 \times (0,T)$. Writing $F = f - w\cdot\nabla u$, we have

$$u_t - \Delta u + \nabla p = F \text{ on } D.$$

Mollifying in $\mathbb{R}^4$ each term of the equation above, we obtain sequences of smooth functions $u_m$, $p_m$ and $F_m$, $m = 1, 2, \ldots$, such that

$$\frac{du_m}{dt} - \Delta u_m + \nabla p_m = F_m \qquad \nabla\cdot u_m = 0$$

in a neighborhood of supp$\Phi$, and such that

$$\begin{aligned}u_m &\to u && \text{in } L^5(0,T;L^{\frac{5}{2}}(\Omega)\cap L^2(D)),\\ \nabla u_m &\to \nabla u && \text{in } L^2(D),\\ p_m &\to p && \text{in } L^{\frac{5}{4}}(0,T;L^{\frac{5}{3}}(\Omega_1)),\\ F_m &\to F && \text{in } L^2(D).\end{aligned}$$

Taking the inner product of 3 with $2u_m\Phi$ and integrating by parts yields

$$2\iint_D |\nabla u_m|^2\varphi = \iint_D |u_m|^2(\varphi_t + \Delta\varphi) + 2\iint_D p_m(u_m\cdot\nabla\varphi) + 2\iint_D (u_m\cdot F_m)\varphi.$$

We pass to the limit as $m \to \infty$, to conclude for $u, p$ and $F$, with $F = f - w\cdot\nabla u$,

$$2\iint_D (u\cdot F)\varphi = 2\iint_D (u\cdot f)\varphi + \iint_D |u|^2 w\cdot\nabla\varphi.$$

This gives the proof when $\varphi \in C_0^\infty(D)$ and $t = T$. For the more general case use a cutoff function in time and the continuity of $u$ in $H$ at 0. □

The goal of this chapter is to use the results shown in Chapter 2 to prove the existence of the weak solution. Therefore, we will introduce the mollyfing operator

$$\Psi_\delta(u)(x,t) := (\delta^{-4}\psi(\cdot/\delta)) * u(x,t) = \delta^{-4} \iint\limits_{\mathbb{R}^4} \psi\left(\frac{y}{\delta},\frac{\tau}{\delta}\right) \tilde{u}(x-y,t-\tau)dyd\tau,$$

where $\psi \in C^\infty(\mathbb{R}^4)$, $\psi \geq 0$, $\iint_{\mathbb{R}^4} \psi(x,t)dxdt = 1$ and $\operatorname{supp}\psi \subset \{(x,t) : |x|^2 < t, 1 < t < 2\}$ and $\tilde{u}$ is the extension of $u$ on $\mathbb{R}^4$, i.e. $\tilde{u}(x,t) = u(x,t)$ on $D$ and elsewhere 0. We see by [**Gra14**], Theorem 1.2.19 that $\psi_\delta$ is an approximating identity on $\mathbb{R}^4$.

LEMMA 3.3. *For any $u \in L^\infty(0,T;H) \cap L^2(0,T;V)$ it holds*

$$\nabla \cdot \psi_\delta(u) = 0,$$

$$\sup_{0\leq t\leq T} \int_\Omega |\psi_\delta(u)|^2 dx \leq CE_0(u),$$

$$\iint\limits_D |\nabla\psi_\delta(u)|^2 dxdt \leq CE_1(u),$$

*for some $C > 0$ independent of $u$ and $\delta$.*

PROOF. It is easy to see that

$$\begin{aligned}\nabla \cdot \Psi_\delta(u) &= \delta^{-4} \iint\limits_{\mathbb{R}^4} \nabla\psi\left(\frac{y}{\delta},\frac{\tau}{\delta}\right) \cdot \tilde{u}(x-y,t-\tau)dyd\tau \\ &= \delta^{-4} \iint\limits_{\Omega} \nabla\psi\left(\frac{y}{\delta},\frac{\tau}{\delta}\right) \cdot u(x-y,t-\tau)dyd\tau = 0.\end{aligned}$$

Furthemore, we obtain (3.3) by Hölder's and Young's inequality

$$\begin{aligned}\int_\Omega |\psi_\delta(u)_j|^2 dx &= \int_\Omega \left(\int_\delta^{2\delta} \int_{\mathbb{R}^3} \psi_\delta\left(y,\tau\right) \tilde{u}_j(x-y,t-\tau)dyd\tau\right)^2 dx \\ &\leq \delta \int_\delta^{2\delta} \int_\Omega \left(\int_{\mathbb{R}^3} \psi_\delta\left(y,\tau\right) \tilde{u}_j(x-y,t-\tau)dy\right)^2 dxd\tau \\ &\leq \int_{\mathbb{R}} \delta^{-1} \|\psi(\cdot,\tau/\delta)\|^2_{L^1(\mathbb{R}^3)} \|u(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau \\ &\leq E_0(u) \int_{\mathbb{R}} \|\psi(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau.\end{aligned}$$

The inequality (3.3) is a direct consequence of Young's inequality

$$\begin{aligned}\iint\limits_D |\nabla_j\psi_\delta(u)_i|^2 dxdt &\leq \iint\limits_{\mathbb{R}^4} \left|\delta^{-4} \iint\limits_{\mathbb{R}^4} \psi\left(\frac{y}{\delta},\frac{\tau}{\delta}\right) \nabla_j\tilde{u}_i(x-y,t-\tau)dyd\tau\right|^2 dxdt \\ &\leq \|\psi\|^2_{L^1(\mathbb{R}^4)} \|\nabla_j u_i\|^2_{L^2(\mathbb{R}^3)}.\end{aligned}$$

□

In the proof of the main theorem we will use the following theorem, which gives a sufficient condition that a sequence $(x_n)_{n\in\mathbb{N}} \cap L^2(0,T;L^2(\Omega))$ is relatively compact.

THEOREM 3.4 (see [**Tem79**], Theorem 1). *Let $X_0 \subset X \subset X_1$ be threee Banach spaces such that $X_0$ is compact in $X$, and $X_0$ and $X_1$ are reflexive. Then the space*

$$Y = \left\{v \in L^{\alpha_0}(0,T;X_0), \frac{d}{dt}v \in L^{\alpha_1}(0,T;X_1)\right\}$$

*with* $\alpha_0, \alpha_1 > 1$ *is compact in* $L^{\alpha_0}(0,T;X)$.

THEOREM 3.5. *Assume that* $\Omega, u_0$ *and* $f$ *satisfy the assumptions from Chapter 2. Then there exists a weak solution* $(u,p)$ *of the Navier-Stokes system such that*

$$\begin{aligned} u &\in L^2(0,T;V) \cap L^\infty(0,T;H), \\ u(t) &\rightharpoonup u_0 \text{ in } H \text{ as } t \to 0, \\ p &\in L^{5/3}(D) \text{ if } \Omega = \mathbb{R}^3, \\ \nabla p &\in L^{5/4}(D) \text{ if } \Omega \text{ is bounded and} \end{aligned}$$

*for all* $\varphi \in C_0^\infty(D)$, $\varphi \geq 0$ *and* $\varphi = 0$ *near* $\partial\Omega \times (0,T)$ *it holds*

$$\begin{aligned} &\int_\Omega |u(x,t)|^2 \varphi(x,t)dx + 2\int_0^t \int_\Omega |\nabla u|^2 \varphi dx dt \\ &\leq \int_\Omega |u_0|^2 \varphi(x,0)dx + \int_0^t \int_\Omega (|u|^2(\varphi_t + \Delta\varphi) + (|u|^2 + 2p)u \cdot \nabla\varphi + 2(u\cdot f)\varphi)dxdt. \end{aligned}$$

Let $N \in \mathbb{N}$ and $\delta = T/N$. $u_N \in L^2(0,T;V) \cap C([0,T];H)$ is the solution of the equation

$$\frac{d}{dt}u_N + (\psi_\delta(u_N)\cdot\nabla)u_N - \Delta u_N + \nabla p_N = f, u_N(0) = u_0,$$

which exists by applying Lemma 2.3 on each time interval $(\delta m, \delta(m+1))$ for each $m = 0,\dots,N-1$ separately. By using (2), (2) and (2) we obtain

$$\int_\Omega |u_N(t,x)|^2 dx + \int_0^t \int_\Omega |\nabla u_N|^2 dxdt \leq C\left(\int_\Omega |u_0|^2 dx + \int_0^t \|f(t)\|_{V'} dt\right),$$

for some constant $C > 0$ which implies that $u_N$ is bounded in $L^\infty(0,T;H) \cap L^2(0,T;V)$. Morever, by [**Tem79**], Lemma 4.2 we conclude that $\frac{d}{dt}u_n$ is bounded in $L^2(0,T;V_2')$, hence $(u_N)_{N\in\mathbb{N}}$ is relatively compact in $L^2(D)$ by Theorem 3.4. We obtain a subsequence $(u_n)$ such that $u_n \to u_*$ in $L^2(D)$, $u_n \rightharpoonup u_*$ in $L^2(0,T;V)$ and $u_n \overset{*}{\rightharpoonup} u_*$ in $L^\infty(0,T;H)$. Moreover, as $(u_N)$ is bounded in $L^{10/3}(D)$ we see easily by an interpolation argument that $u_n \to u_*$ in $L^s(D)$ for every $2 \leq s < 10/3$. Using the above inequalities it is possible to show that $u_*$ solves the Navier-Stokes equation. We will only prove the convergence of the term $\int_0^t \varphi(t)((\psi_\delta(u_N)\cdot\nabla)u_N, v)_{L^2(\Omega)}dt$, as the other parts are trivial. As $v \in H^1(\Omega)$, we see that $\|u_i v_j\|_{L^2(\mathbb{R}^3)} < \infty$, which follows by the Sobolev embedding theorem. We conclude that

$$\begin{aligned} &\left|\int_0^t \int_\Omega ((\psi_\delta(u_N)\cdot\nabla)u_N, v)\varphi(t)dxdt - \int_0^t \int_\Omega ((u\cdot\nabla)u, v)\varphi(t)dxdt\right| \\ &\leq \left|\int_0^t \int_\Omega ((\psi_\delta(u_N)\cdot\nabla)u_N, v)\varphi(t)dxdt - \int_0^t \int_\Omega ((u\cdot\nabla)u_N, v)\varphi(t)dxdt\right| \\ &\quad + \left|\int_0^t \int_\Omega ((u\cdot\nabla)u_N, v)\varphi(t)dxdt - \int_0^t \int_\Omega ((u\cdot\nabla)u, v)\varphi(t)dxdt\right| \\ &\quad \to 0 \text{ for } N \to \infty, \end{aligned}$$

where we use for the first term that $\psi_\delta(u_N) \to u$ in $L^3(\mathbb{R}^3)$ and in the second term that $u_n \rightharpoonup u$ in $L^2(0,T;V)$.

In the case that $\Omega$ is bounded, we use Lemma 3.1. Let $\{\Omega_j\}_{j\in\mathbb{N}}$ be a sequence of subdomains such that $\overline{\Omega}_j \subset \Omega_{j+1}$ and $\cup_{j\in\mathbb{N}}\Omega_j = \Omega$. We see that $\nabla p_N$ is bounded in $L^{5/4}(D)$ and $p_n$ in $L^{5/4}(0,T;L^{5/3}(\Omega_j))$. We obtain for every $j$ a subsequence $p_N \rightharpoonup p_*$ in $L^{5/4}(0,T;L^{5/3}(\Omega_j))$. Moreover, we see that $u_N \to u_*$ in $L^5(0,T;L^{5/2}(\Omega))$. The proof follows the same arguments as in the case of $\Omega = \mathbb{R}^3$.

CHAPTER 4

# Talk 4: Background and Definitions

## By Fabian Rupp

### 4.1. On the initial boundary value problem

First, note that the condition $\operatorname{div} f = 0$ is not a restriction at all. Indeed, suppose we want to solve (1.1) for a general force $f \in L^q(\Omega)$ with $1 < q < \infty$. We may apply a $L^q$-Helmholtz decomposition to write $f = \nabla\Phi + f_1$ with $\operatorname{div} f_1 = 0$ and $\|f_1\|_{L^q(\Omega\times[0,T])} \leq C(q,\Omega)\,\|f\|_{L^q(\Omega\times[0,T])}$. If $(u,p)$ is a solution of (1.1) with the force term $f_1$, it is easy to see that $(u, p+\Phi)$ is a solution to (1.1) with the right hand side $\nabla\Phi + f_1 = f$ as desired.

To obtain an existence theory for arbitrary time intervals, we study weak solutions of (1.1) for which the energy

$$\operatorname{ess\,sup}_{0<t<T} \int_\Omega |u|^2 \,\mathrm{d}x + \int_0^T \int_\Omega |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}t < \infty, \tag{4.1}$$

is finite, where $|\nabla u|^2 \coloneqq \sum_{i,j} \left|\partial_i u^j\right|^2$. This choice is motivated by multiplying (1.1) by $u$, integration and using integration by parts. (4.1) justifies why requiring a solution $u$ to have space derivatives of first order is a somewhat physical assumption.

If one instead multiplies (1.1) by $2u\phi$ for some $\phi \in \mathcal{C}^\infty(\Omega\times[0,T])$ and integrates one obtains

$$\int_0^t \int_\Omega 2\partial_t u\cdot u\phi + 2\left((u\cdot\nabla)u\right)\cdot u\phi - 2\Delta u\cdot u\phi + 2\nabla p\cdot u\phi \,\mathrm{d}x = \int_0^t \int_\Omega 2f\cdot u\phi \,\mathrm{d}x. \tag{4.2}$$

Since $u|_{\partial\Omega} = 0$ by (1.2), we may use integration by parts without creating any boundary terms. For the first term, we use $\partial_t |u|^2 = 2\partial_t u\cdot u$, so

$$\begin{aligned}\int_0^t \int_\Omega 2\partial_t u\cdot u\phi \,\mathrm{d}x\,\mathrm{d}t &= \int_0^t \partial_t \int_\Omega |u|^2\phi \,\mathrm{d}x\,\mathrm{d}t - \int_\Omega |u|^2 \partial_t\phi \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_\Omega |u(t)|^2\phi\,\mathrm{d}x - \int_\Omega |u(0)|^2\phi\,\mathrm{d}x - \int_\Omega |u|^2\partial_t\phi\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{4.3}$$

For the second part, integration by parts yields, using summation convention,

$$\begin{aligned}\int_0^t \int_\Omega 2u^i\partial_i u^j u^j\phi \,\mathrm{d}x\,\mathrm{d}t &= -\int_0^t \int_\Omega |u|^2 \partial_i u^i\phi\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega |u|^2 u^i\partial_i\phi\,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_0^t \int_\Omega |u|^2 u\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{4.4}$$

since $\partial_i |u|^2 = 2\partial_i u^j u^j$ and $\operatorname{div} u = 0$ by (1.1). For the third term, we get using $\partial_i |u|^2 = 2\partial_i u^j u^j$ again

$$\begin{aligned}-2\int_0^t \int_\Omega \partial_i\partial_i u^j u^j\phi\,\mathrm{d}x &= 2\int_0^t \int_\Omega |\nabla u|^2\phi\,\mathrm{d}x\,\mathrm{d}t + 2\int_0^t\int_\Omega \partial_i u^j u^j\partial_i\phi\,\mathrm{d}x\,\mathrm{d}t \\ &= 2\int_0^t \int_\Omega |\nabla u|^2\phi\,\mathrm{d}x\,\mathrm{d}t - \int_0^t\int_\Omega |u|^2\partial_i\partial_i\phi\,\mathrm{d}x\,\mathrm{d}t \\ &= 2\int_0^t \int_\Omega |\nabla u|^2\phi\,\mathrm{d}x\,\mathrm{d}t - \int_0^t\int_\Omega |u|^2\Delta\phi\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{4.5}$$

Finally, for the last term, using $\operatorname{div} u = 0$, we have

$$\begin{aligned} 2\int_0^t\int_\Omega \partial_i p u^i \phi \,\mathrm{d}x\,\mathrm{d}t &= -2\int_0^t\int_\Omega p\partial_i u^i\phi\,\mathrm{d}x\,\mathrm{d}t - 2\int_0^t\int_\Omega pu^i\partial_i\phi\,\mathrm{d}x\,\mathrm{d}t \\ &= -2\int_0^t\int_\Omega pu\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{4.6}$$

Combining, (4.2),(4.3),(4.4),(4.5) and (4.6), we get

$$\begin{aligned} \int_\Omega |u(t)|^2\phi\,\mathrm{d}x + 2\int_0^t\int_\Omega |\nabla u|^2\phi\,\mathrm{d}x\,\mathrm{d}t &= \int_\Omega |u_0|^2\phi\,\mathrm{d}x \\ &+ \int_0^t\int_\Omega |u|^2(\partial_t\phi+\Delta\phi)\,\mathrm{d}x\,\mathrm{d}t + \int_0^t\int_\Omega (|u|^2+2p)u\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t \\ &+ 2\int_0^t\int_\Omega f\cdot u\phi\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{4.7}$$

Pluggin in $\phi \equiv 1$ in (4.7) we obtain

$$\int_\Omega |u(t)|^2\,\mathrm{d}x + 2\int_0^t\int_\Omega |\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega |u_0|^2 + 2\int_0^t\int_\Omega f\cdot u\,\mathrm{d}x. \tag{4.8}$$

Note that for $f \equiv 0$ in (4.8), we may formally conclude (4.1) with an explicit bound depending on the initial date $u_0 \in L^2(\Omega)$. The key point in proving existence of weak *Leray-Hopf solutions* is the *energy inequality*, an inequality form of (4.8).

$$\int_\Omega |u(t)|^2\,\mathrm{d}x + 2\int_0^t\int_\Omega |\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t \le \int_\Omega |u_0|^2 + 2\int_0^t\int_\Omega f\cdot u\,\mathrm{d}x, \tag{4.9}$$

for almost every $t$.

For the main result, the localized version of (4.9) is crucial. Taking any $\phi \ge 0$ with compact support in $\Omega \times (0,T)$ in (4.7), one may conclude the following *generalized energy inequality* by estimating the first term by zero

$$2\int_0^T\int_\Omega |\nabla u|^2\phi\,\mathrm{d}x\,\mathrm{d}t \le \int_0^T\int_\Omega \left[|u|^2(\partial_t\phi+\Delta\phi) + (|u|^2+2p)u\cdot\nabla\phi + 2u\cdot f\phi\right]\mathrm{d}x\,\mathrm{d}t. \tag{4.10}$$

By definition, any *suitable weak solution* satisfies (4.10). Last week, we saw that such a suitable weak solution in fact exists (cf. David's talk Lemma 2.2, Theorem 2.5, Farid's talk Lemma 1.3).

DEFINITION 4.1. We call a pair $(u,p)$ a *suitable weak solution* to the Navier-Stokes equation with force $f$ on $\Omega \times (0,T)$ if the following conditions are satisfied.

(1) $u, p, f$ are measureable on $\Omega \times (0,T)$ and
  (a) $f \in L^q(\Omega\times(0,T))$ for $q > \frac{5}{2}$ and $\operatorname{div} f = 0$,
  (b) $p \in L^{\frac{5}{4}}(\Omega\times(0,T))$
  (c) for some $E_0, E_1 < \infty$ we have

$$\int_\Omega |u|^2\,\mathrm{d}x \le E_0 \text{ for almost every } t \in (0,T), \text{ and} \tag{4.11}$$

$$\int_0^T\int_\Omega |\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t \le E_1. \tag{4.12}$$

(2) $u, p$ and $f$ satisfy (1.1) in the sense of distributions on $\Omega\times(0,T)$.
(3) For each $\phi \in \mathcal{C}_0^\infty(\Omega\times(0,T))$ with $\phi \ge 0$, inequality (4.10) holds.

Even for a suitable weak solution, it is not immediately clear that the right hand side of (4.10) is well, defined, i.e. it is not obvious that the integrals

$$\int_0^T \int_\Omega |u|^2 u \cdot \nabla\phi \,\mathrm{d}x\,\mathrm{d}t \quad \text{and} \quad \int_0^T \int_\Omega pu \cdot \nabla\phi \,\mathrm{d}x\,\mathrm{d}t$$

do exist. We will prove that this is the case.

## 4.2. Higher Regularity

Recall that a point $(x,t)$ in space-time is *regular* if $u \in L^\infty_{loc}(U)$ for an open neighborhood $U$ of $(x,t)$. This is justified by the following result due to Serrin [**Ser63**]. If $u$ is a weak solution of (1.1) on a cylinder $\Omega \times (a,b)$ satisfying

$$\int_a^b \left(\int_\Omega |u|^q \,\mathrm{d}x\right)^{\frac{s}{q}} \mathrm{d}t < \infty \text{ with } \frac{3}{q} + \frac{2}{s} < 1, \tag{4.13}$$

then $u$ us necessarily $\mathcal{C}^{m+2,\beta}$ in space on compact subsets of $\Omega$, provided $f$ is $\mathcal{C}^{m,\beta}$ in space with $m \geq 0$ and $0 < \beta < 1$. In particular if $f$ is $\mathcal{C}^\infty$ in space and (4.13) is satisfied, then $u$ is $\mathcal{C}^\infty$ in space. Regularity in time is more difficult. If $u \in L^\infty(0,T;L^3(U))$, then $u$ is Hölder continuous in time. From this, if $u \in L^\infty_{loc}(U)$ in a neighborhood $U$ of $(x,t)$, then (4.13) clearly holds, so $u$ is smooth in space, provided $f$ is smooth in space.

## 4.3. Recurrent Themes

The following three observations will be used frequently.

**4.3.1. Interpolation inequalities for $u$ and $p$.** If $B_r \subset \mathbb{R}^3$ be a ball of radius $r > 0$ and let $u \in H^1(B_r)$. Then, the *Gagliardo-Nirenberg-Sobolev inequality* yields

$$\int_{B_r} |u|^q \,\mathrm{d}x \leq C\left(\int_{B_r} |\nabla u|^2 \,\mathrm{d}x\right)^a \left(\int_{B_r} |u|^2 \,\mathrm{d}x\right)^{\frac{q}{2}-a} + \frac{C}{r^{2a}}\left(\int_{B_r} |u|^2 \,\mathrm{d}x\right)^{\frac{q}{2}}, \tag{4.14}$$

where $C > 0$, $2 \leq q \leq 6$ and $a = \frac{3}{4}(q-2)$. If $B_r$ is replaced by $\mathbb{R}^3$ the second term on the right in (4.14) can be omitted. Inequality (4.14) follows from the classical Gagliardo-Nirenberg-Sobolev inequality [**Nir59**] by applying an extension operator to $u \in H^1(B_r)$. The term $\frac{1}{r^{2a}}$ makes (4.14) scaling invariant with respect to $r > 0$.

We will now use (4.14) to interpolate between (4.11) and (4.12). Take $q = \frac{10}{3}$ so $a = 1$ in (4.14) and integrate in time. Then

$$\int_0^T \int_{B_r} |u|^{\frac{10}{3}} \,\mathrm{d}x\,\mathrm{d}t \leq C\left(E_0^{\frac{2}{3}} E_1 + r^{-2} E_0^{\frac{5}{3}} T\right). \tag{4.15}$$

A particular consequence is that $u \in L^3(\Omega \times (0,T))$, hence

$$\left|\int_0^T \int_\Omega |u|^2 u \cdot \nabla\phi \,\mathrm{d}x\,\mathrm{d}t\right| \leq \|\nabla\phi\|_{L^\infty(\Omega\times(0,T))} \|u\|_{L^3(\Omega\times(0,T))} < \infty,$$

so the corresponding term in (4.10) is in fact finite if $u$ is a suitable weak solution and $\phi \in \mathcal{C}^\infty(\Omega \times (0,T))$. Moreover, if $q = \frac{5}{2}$, so $a = \frac{3}{8}$ we get

$$\int_0^T \left(\int_{B_r} |u|^{\frac{5}{2}} \,\mathrm{d}x\right)^{\frac{8}{3}} \mathrm{d}t \leq C(E_0^{\frac{7}{3}} E_1 + r^{-2} E_0^{\frac{10}{3}} T). \tag{4.16}$$

If we take the (distributional) divergence of (1.1), we get

$$0 = \Delta p + \partial_i\left(u^j \partial_j u^i\right) = \Delta p + \partial_i\partial_j(u^j u^i),$$

hence

$$\Delta p = -\partial_i\partial_j(u^i u^j) \text{ on } \Omega \times (0,T) \text{ in the sense of distributions.} \tag{4.17}$$

In addition, any solution $u \in \mathcal{C}^1(0,T;\mathcal{C}^2(\overline{\Omega}))$ of (1.1) on $\overline{\Omega} \times (0,T)$ for $f \equiv 0$ satisfying (1.2) has to fulfill

$$\nu \cdot \nabla p = \nu \cdot \Delta u \text{ on } \partial\Omega \times (0,T),$$

by simply restricting (1.1) to $\partial\Omega$ and multiplying with $\nu$.
Recall that in $\mathbb{R}^3$, the unique solution to $-\Delta v = f$, with $f \in L^q(\mathbb{R}^3)$ is given by

$$v(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|x-y|} f(y)\,\mathrm{d}y.$$

We may thus rewrite (4.17) as $p = (-\Delta)^{-1}\partial_i\partial_j(u^i u^j)$.

First, we consider the case $\Omega = \mathbb{R}^3$. For $u$ smooth enough, we have

$$p(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|x-y|}\partial_{y_i}\partial_{y_j}(u^i u^j)\,\mathrm{d}y = \alpha_{ij} u^i(x) u^j(x) + \frac{1}{4\pi}\int_{\mathbb{R}^3} \partial_{y_i}\partial_{y_j}\left(\frac{1}{|x-y|}\right) u^i u^j\,\mathrm{d}y,$$

where the latter has to be understood as a singular integral, i.e. a principal value

$$\lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon}.$$

Also note that $\alpha_{ij} = 0$ if $i \neq j$.

We now use standard Calderón-Zygmund theory, see for instance [**Ste70**]. To that end, fix $i,j \in \{1,\ldots,3\}$ and consider the convolution operator

$$S_{ij} f = \frac{1}{4\pi}\int_{\mathbb{R}^3} \partial_{y_j}\partial_{y_i}\left(\frac{1}{|x-y|}\right) f\,\mathrm{d}y.$$

A computation yields $\partial_{y_j}\partial_{y_i}\left(\frac{1}{|x-y|}\right) = -\frac{\delta_{ij}}{|x-y|^3} + 3\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^5}$. We may write

$$S_{ij} f(x) = \int_{\mathbb{R}^3} \frac{\Omega(x-y)}{|x-y|^3} f(y)\,\mathrm{d}y,$$

with $\Omega(y) = -\delta_{ij} + 3\frac{y_i y_j}{|y|^2}$. Note that $\Omega$ is homogeneous of degree 0 and a computation shows $\int_{\mathbb{S}^2}\Omega(y)\,\mathrm{d}S(y) = 0$ for all $i,j$. Clearly, $\Omega$ is Lipschitz on $\mathbb{S}^2$. Thus, by Calderón-Zygmund theory [**Ste70**, §4.3, Theorem 3],

$$S_{ij}\colon L^q(\mathbb{R}^3) \to L^q(\mathbb{R}^3) \text{ is bounded for any } 1 < q < \infty, i,j = 1,\ldots,3. \tag{4.18}$$

As a consequence

$$\|p\|_{L^q(\mathbb{R}^3)} = \left\|(-\Delta)^{-1}\partial_i\partial_j(u^i u^j)\right\|_{L^q(\mathbb{R}^3)} \le C\sum_{i,j}\left\|u^i u^j\right\|_{L^q(\mathbb{R}^3)},$$

for some $C = C(q) > 0$ and

$$\left\|u^i u^j\right\|^q_{L^q(\mathbb{R}^3)} = \int_{\mathbb{R}^3}\left|u^i u^j\right|^q\,\mathrm{d}x \le \int_{\mathbb{R}^3}|u|^{2q}\,\mathrm{d}x.$$

This yields

$$\int_{\mathbb{R}^3}|p|^q\,\mathrm{d}x \le C\int_{\mathbb{R}^3}|u|^{2q}\,\mathrm{d}x.$$

In particular, if $(u,p)$ is a suitable weak solution of (1.1) on $\mathbb{R}^3 \times (0,T)$ we have

$$\int_0^T\int_{\mathbb{R}^3}|p|^{\frac{5}{3}}\,\mathrm{d}x\,\mathrm{d}t \le C\int_0^T\int_{\mathbb{R}^3}|u|^{\frac{10}{3}}\,\mathrm{d}x\,\mathrm{d}t \le C E_0^{\frac{2}{3}} E_1$$

by (4.15) using that we don't need the second term in (4.14) since we are in the whole space $\mathbb{R}^3$.

For general $\Omega \subset \mathbb{R}^3$ bounded, let $\overline{\Omega}_1 \subset \Omega$ and $\phi \in \mathcal{C}_0^\infty(\Omega)$ with $\phi \equiv 1$ in a neighborhood $U$ of $\overline{\Omega}_1$. Then for $t$ fixed we have using

$$\begin{aligned} \phi(x)p(x,t) &= -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|x-y|}\Delta_y(\phi p)\,\mathrm{d}y \qquad (4.19)\\ &= -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|x-y|}\left[p\Delta\phi + 2\langle\nabla\phi,\nabla p\rangle + \phi\Delta p\right]\,\mathrm{d}y. \end{aligned}$$

We plug in (4.17) for $\Delta p$ in (4.19) and obtain using summation convention

$$\phi p = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|x-y|}\left[p\Delta\phi + 2\langle\nabla\phi,\nabla p\rangle - \phi\partial_i\partial_j(u^i u^j)\right]\,\mathrm{d}y. \qquad (4.20)$$

Now, we integrate by parts to remove all derivatives on $p$ and $u$. Note that in order to do this in a precise way, you have to cut out a ball $B_\varepsilon$ of radius $\varepsilon$ and do integration by parts there. However, since $\partial_{y_i}\left(\frac{1}{|x-y|}\right)$ is $L^1_{loc}(\mathbb{R}^3)$, the boundary terms will vanish as $\varepsilon \to 0$. We have

$$\int_{\mathbb{R}^3}\frac{1}{|x-y|}\langle\nabla\phi,\nabla p\rangle\,\mathrm{d}y = -\int_{\mathbb{R}^3}\partial_{y_i}\left(\frac{1}{|x-y|}\right)\partial_i\phi p\,\mathrm{d}y - \int_{\mathbb{R}^3}\frac{1}{|x-y|}\Delta\phi p\,\mathrm{d}y. \qquad (4.21)$$

For the last term in (4.20) we have

$$\begin{aligned} \int_{\mathbb{R}^3}\frac{1}{|x-y|}\phi\partial_i\partial_j(u^iu^j)\,\mathrm{d}y &= -\int_{\mathbb{R}^3}\partial_{y_i}\left(\frac{1}{|x-y|}\right)\phi\partial_j(u^iu^j)\,\mathrm{d}y \qquad (4.22)\\ &\quad -\int_{\mathbb{R}^3}\frac{1}{|x-y|}\partial_i\phi\partial_j(u^iu^j)\,\mathrm{d}y\\ &= \int_{\mathbb{R}^3}\partial_{y_j}\partial_{y_i}\left(\frac{1}{|x-y|}\right)\phi u^iu^j\,\mathrm{d}y + \int_{\mathbb{R}^3}\partial_{y_i}\left(\frac{1}{|x-y|}\right)\partial_j\phi u^iu^j\,\mathrm{d}y\\ &\quad + \int_{\mathbb{R}^3}\partial_{y_j}\left(\frac{1}{|x-y|}\right)\partial_i\phi u^iu^j\,\mathrm{d}y + \int_{\mathbb{R}^3}\frac{1}{|x-y|}\partial_i\partial_j\phi u^iu^j\,\mathrm{d}y\\ &= \int_{\mathbb{R}^3}\partial_{y_j}\partial_{y_i}\left(\frac{1}{|x-y|}\right)\phi u^iu^j\,\mathrm{d}y + \int_{\mathbb{R}^3}\frac{x_i-y_i}{|x-y|^3}\partial_j\phi u^iu^j\,\mathrm{d}y\\ &\quad + \int_{\mathbb{R}^3}\frac{x_j-y_j}{|x-y|^3}\partial_i\phi u^iu^j\,\mathrm{d}y + \int_{\mathbb{R}^3}\frac{1}{|x-y|}\partial_i\partial_j\phi u^iu^j\,\mathrm{d}y\\ &= \int_{\mathbb{R}^3}\partial_{y_j}\partial_{y_i}\left(\frac{1}{|x-y|}\right)\phi u^iu^j\,\mathrm{d}y + 2\int_{\mathbb{R}^3}\frac{x_i-y_i}{|x-y|^3}\partial_j\phi u^iu^j\,\mathrm{d}y\\ &\quad + \int_{\mathbb{R}^3}\frac{1}{|x-y|}\partial_i\partial_j\phi u^iu^j\,\mathrm{d}y \end{aligned}$$

Therefore, combining (4.19), (4.20), (4.21) and (4.22) we get

$$p\phi = \tilde{p} + p_3 + p_4 \qquad (4.23)$$

with

$$\begin{aligned} \tilde{p} &= \alpha_{ij}u^i(x)u^j(x) + \frac{1}{4\pi}\int_{\mathbb{R}^3}\partial_{y_j}\partial_{y_i}\left(\frac{1}{|x-y|}\right)\phi u^iu^j\,\mathrm{d}y\\ p_3 &= \frac{1}{2\pi}\int_{\mathbb{R}^3}\frac{x_i-y_i}{|x-y|^3}\partial_j\phi u^iu^j\,\mathrm{d}y + \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|x-y|}\partial_i\partial_j\phi u^iu^j\,\mathrm{d}y\\ p_4 &= \left(-\frac{1}{4\pi} + \frac{2}{4\pi}\right)\int_{\mathbb{R}^3}\frac{1}{|x-y|}p\Delta\phi\,\mathrm{d}y + \frac{1}{2\pi}\int_{\mathbb{R}^3}\frac{x_i-y_i}{|x-y|^3}\partial_i\phi p\,\mathrm{d}y. \end{aligned}$$

Note that we have for $x \in \Omega_1$, using $\phi \equiv 1$ on $U$ and $\phi \equiv 0$ on $\mathbb{R}^3 \setminus \Omega$

$$\begin{aligned} |p_3|(x,t) &\le \left| \frac{1}{2\pi} \int_{\mathbb{R}^3} \frac{x_i - y_i}{|x-y|^3} \partial_j \phi u^i u^j \, \mathrm{d}y \right| + \left| \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|} \partial_i \partial_j \phi u^i u^j \, \mathrm{d}y \right| \\ &\le \frac{1}{2\pi} \int_{\Omega \setminus U} \frac{1}{|x-y|^2} |\partial_j \phi| \, |u|^2 \, \mathrm{d}y + \frac{1}{4\pi} \int_{\Omega \setminus U} \frac{1}{|x-y|} |\partial_i \partial_j \phi| \, |u|^2 \, \mathrm{d}y \\ &\le \frac{\|\phi\|_{\mathcal{C}^1}}{2\pi\delta^2} \int_\Omega |u|^2 \, \mathrm{d}y + \frac{\|\phi\|_{\mathcal{C}^2}}{4\pi\delta} \int_\Omega |u|^2 \, \mathrm{d}y, \end{aligned}$$

where $\delta := d(\overline{\Omega}_1, \partial U) > 0$ gives lower bounds on $|x-y|$. Similarly for $p_4$, we have for $x \in \Omega_1$

$$\begin{aligned} |p_4|(x,t) &\le \frac{1}{4\pi} \int_{\Omega \setminus U} \frac{1}{|x-y|} |p| \, |\Delta \phi| \, \mathrm{d}y + \frac{1}{2\pi} \int_{\Omega \setminus U} \frac{1}{|x-y|^2} |\partial_i \phi| \, |p| \, \mathrm{d}y \\ &\le \frac{\|\phi\|_{\mathcal{C}^2}}{4\pi\delta} \int_\Omega |p| \, \mathrm{d}y + \frac{\|\phi\|_{\mathcal{C}^1}}{2\pi\delta^2} \int_\Omega |p| \, \mathrm{d}y. \end{aligned}$$

Consequently,

$$|p_3|(x,t) + |p_4|(x,t) \le C \int_\Omega \left( |p| + |u|^2 \right) \mathrm{d}y, \text{ for } x \in \Omega_1. \tag{4.24}$$

Since the operators $S_{ij}$ are bounded by (4.18), there exists $C > 0$ such that

$$\int_{\mathbb{R}^3} |\tilde{p}|^{5/3} \, \mathrm{d}x \le \sum_{i,j} \int_{\mathbb{R}^3} \left| S_{ij}(\phi u^i u^j) \right|^{5/3} \mathrm{d}x \le C \sum_{i,j} \int_{\mathbb{R}^3} \left| \phi u^i u^j \right|^{5/3} \mathrm{d}x,$$

and consequently

$$\int_{\Omega_1} |\tilde{p}|^{5/3} \, \mathrm{d}x \le C \sum_{i,j} \int_{\mathbb{R}^3} \left| \phi u^i u^j \right|^{5/3} \mathrm{d}x \le C \|\phi\|_{L^\infty} \int_\Omega |u|^{10/3} \, \mathrm{d}x. \tag{4.25}$$

From (4.24) and (4.25), we may deduce $p \in L^{5/4}(0,T; L^{5/3}(\Omega_1)))$.

We have using (4.15) and (4.25)

$$\begin{aligned} \int_0^T \left( \int_{\Omega_1} |\tilde{p}|^{5/3} \, \mathrm{d}x \right)^{3/5 \cdot 5/4} \mathrm{d}t &\le C \int_0^T \left( \int_\Omega |u|^{10/3} \, \mathrm{d}x + 1 \right)^{3/4} \mathrm{d}t \\ &\le C \left( \int_0^T \int_\Omega |u|^{10/3} \, \mathrm{d}x \, \mathrm{d}t + T \right) \\ &\le C(E_0^{2/3} E_1 + E_0^{5/3} T + T), \end{aligned} \tag{4.26}$$

where the constant $C > 0$ changes from line to line. For the remaining terms in (4.23), we have using (4.24) and Jensen's inequality

$$\begin{aligned} \int_0^T \left( \int_{\Omega_1} (|p_3| + |p_4|)^{5/3} \, \mathrm{d}x \right)^{3/4} \mathrm{d}t &\le C |\Omega_1| \int_0^T \left( \int_\Omega (|p| + |u|^2) \, \mathrm{d}x \right)^{5/3 \cdot 3/4} \mathrm{d}t \\ &\le C \int_0^T \left( \left( \int_\Omega |p| \, \mathrm{d}x \right)^{5/4} + \left( \int_\Omega |u|^2 \, \mathrm{d}x \right)^{5/4} \right) \mathrm{d}t \\ &\le C \int_0^T \int_\Omega |p|^{5/4} \, \mathrm{d}x \, \mathrm{d}t + CTE_0^{5/4} \\ &= C \|p\|_{L^{5/4}(\Omega \times (0,T))} + CTE_0^{5/4}. \end{aligned} \tag{4.27}$$

Therefore, combining (4.26) and (4.27) we get using $p = \phi p$ for a.e. $t$ and $x \in \Omega_1$

$$\|p\|_{L^{5/4}(0,T;L^{5/3}(\Omega_1))} \le \|\tilde{p}\|_{L^{5/4}(0,T;L^{5/3}(\Omega_1))} + \||p_3| + |p_4|\|_{L^{5/4}(0,T;L^{5/3}(\Omega_1))} < \infty, \tag{4.28}$$

if $(u,p)$ is a suitable weak solution. Thus, we have proven the following

LEMMA 4.2. *If* $(u,p)$ *is a suitable weak solution of* (1.1) *on* $\Omega \times (0,T)$ *and* $\overline{B}_r \times (a,b) \subset \Omega \times (0,T)$*, then* $p \in L^{5/4}(a,b;L^{5/3}(B_r))$ *and* $u \in L^5(a,b;L^{5/2}(B_r))$*.*

PROOF. This follows from (4.28) and (4.16). □

In particular, the term $\int\int p(u\cdot\nabla\phi)$ in (4.10) is integrable, since if $\operatorname{supp}\phi \subset \Omega_1$ we have

$$\begin{aligned}\int_0^T \int_\Omega |pu\cdot\nabla\phi| \,\mathrm{d}x\,\mathrm{d}t &\leq C\int_{0^T} \|u(t)\|_{L^{5/2}(\Omega_1)} \|p(t)\|_{L^{5/3}(\Omega_1)} \,\mathrm{d}t \\ &\leq C\left(\int_0^T \|u(t)\|^5_{L^{5/2}(\Omega_1)} \,\mathrm{d}t\right)^{1/5} \left(\int_0^T \|p(t)\|^{5/4}_{L^{5/3}(\Omega_1)} \,\mathrm{d}t\right)^{4/5} \\ &= C\|u\|_{L^5(0,T;L^{5/2}(\Omega_1))} \|p\|_{L^{5/4}(0,T;L^{5/3}(\Omega_1))},\end{aligned}$$

by Hölder's inequality and since $\frac{3}{5}+\frac{2}{5}=\frac{4}{5}+\frac{1}{5}=1$. Thus, we have shown that for any suitable weak solution of (1.1), the right hand side of (4.9) exists.

**4.3.2. Weak continuity.** It can be shown, that any suitable weak solution $u$ of (1.1) is weakly continuous in time with values in $L^2(\Omega)$, i.e. for any $w \in L^2(\Omega)$ we have

$$\int_\Omega u(x,t)w(x)\,\mathrm{d}x \to \int_\Omega u(x,t_0)w(x)\,\mathrm{d}x \text{ as } t \to t_0.$$

For a proof of this property we refer to [**Tem79**, p. 281-282]. This has some important consequences.

(i) We can evaluate $u$ at times $t$ and it makes sense to impose the initial condition $u(0) = u_0$ in the sense that $u(t) \rightharpoonup u_0$ in $L^2(\Omega)$ as $t \to 0$, i.e. $u$ extends weakly continously to $[0,T)$.
(ii) The integrability condition (4.11) holds for every $t \in (0,T)$. If $t_0 \in (0,T)$, then there exist $t_n \to t_0$ with $\int_\Omega |u(t_n)|^2 \,\mathrm{d}x \leq E_0$, otherwise (4.11) would not hold almost everywhere. But since the $L^2(\Omega)$-norm is weakly lower semicontinuous and as $u(t_n) \to u(t_0)$ as $n \to \infty$, we conclude $\int_\Omega |u(t_0)|^2 \,\mathrm{d}x \leq E_0$.
(iii) If $(u,p)$ is a suitable weak solution of (1.1) on $\Omega \times (a,b)$, then for each $a < t_0 < b$ and $\phi \in \mathcal{C}_0^\infty(\Omega \times (a,b))$ with $\phi \geq 0$ we have

$$\begin{aligned}\int_\Omega |u(t_0)|^2 \phi(t_0)\,\mathrm{d}x + 2\int_a^{t_0}\int_\Omega |\nabla u|^2 \phi \,\mathrm{d}x\,\mathrm{d}t \qquad (4.29)\\ \leq \int_a^{t_0}\int_\Omega \left[|u|^2(\partial_t\phi + \Delta\phi) + (|u|^2+2p)u\cdot\nabla\phi + 2u\cdot f\phi\right]\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

This follows from (4.10), by choosing the positive test function $\phi(x,t)\chi\left((t_0-t)/\varepsilon\right)$, where $\varepsilon > 0$ and $\chi$ is a smooth function with $0 \leq \chi \leq 1$, $\chi(s) \equiv 0$ for $s \leq 0$ and $\chi(s) \equiv 1$ for $s \geq 1$. Then (4.10) yields

$$\begin{aligned}2\int_a^{t_0}\int_\Omega |\nabla u|^2 \phi\chi\left((t_0-t)/\varepsilon\right)\mathrm{d}x\,\mathrm{d}t \leq \int_a^{t_0}\int_\Omega \Big[|u|^2\left(\partial_t\left(\phi\chi\left((t_0-t)/\varepsilon\right)\right)\right. \qquad (4.30)\\ \left.+\Delta\phi\chi\left((t_0-t)/\varepsilon\right)\right) + (|u|^2+2p)u\cdot\nabla\phi\chi\left((t_0-t)/\varepsilon\right)\\ + 2u\cdot f\phi\chi\left((t_0-t)/\varepsilon\right)\Big]\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Note that for $t \le t_0$, $\chi\left((t_0-t)/\varepsilon\right) \to 1$ as $\varepsilon \to 0$. Since $0 \le \chi \le 1$, the dominated convergence theorem yields that as $\varepsilon \to 0$ in (4.30)

$$2\int_a^{t_0}\int_\Omega |\nabla u|^2\,\phi \,\mathrm{d}x\,\mathrm{d}t \le \int_a^{t_0}\int_\Omega \Big[|u|^2\left(\partial_t\phi + \Delta\phi + (|u|^2+2p)u\cdot\nabla\phi + 2u\cdot f\phi\right]\,\mathrm{d}x\,\mathrm{d}t \tag{4.31}$$
$$+\lim_{\varepsilon\to 0}\int_a^{t_0}\int_\Omega |u|^2\,\phi\partial_t\left(\chi\left((t_0-t)/\varepsilon\right)\right)\,\mathrm{d}x\,\mathrm{d}t,$$

since all terms in $u$ and $p$ are integrable. Taking a closer look at the last term, we observe that for $u$ smooth enough

$$\begin{aligned}\int_a^{t_0}\int_\Omega |u|^2\,\phi\partial_t\left(\chi\left((t_0-t)/\varepsilon\right)\right)\,\mathrm{d}x\,\mathrm{d}t &= \int_\Omega\int_a^{t_0} |u|^2\,\phi\partial_t\left(\chi\left((t_0-t)/\varepsilon\right)\right)\,\mathrm{d}t\,\mathrm{d}x\\
&= \int_\Omega |u(t_0)|^2\,\phi(t_0)\chi(0)\,\mathrm{d}x - \int_\Omega |u(a)|^2\,\phi(a)\chi\left((t_0-a)/\varepsilon\right)\,\mathrm{d}x\\
&- \int_a^{t_0}\int_\Omega \partial_t\,|u|^2\,\phi\chi\left((t_0-t)/\varepsilon\right)\,\mathrm{d}x\,\mathrm{d} - \int_a^{t_0}\int_\Omega |u|^2\,\partial_t\phi\chi\left((t_0-t)/\varepsilon\right)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

If we let $\varepsilon \to 0$ we obtain

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_a^{t_0}\int_\Omega |u|^2\,\phi\partial_t\left(\chi\left((t_0-t)/\varepsilon\right)\right)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad= -\int_\Omega |u(a)|^2\,\phi(a)\,\mathrm{d}x - \int_a^{t_0}\int_\Omega \partial_t\,|u|^2\,\phi\,\mathrm{d}x\,\mathrm{d}t - \int_a^{t_0}\int_\Omega |u|^2\,\partial_t\phi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad= -\int_\Omega |u(a)|^2\,\phi(a)\,\mathrm{d}x - \int_a^{t_0}\int_\Omega \partial_t\left(|u|^2\,\phi\right)\,\mathrm{d}x\,\mathrm{d}t = -\int_\Omega |u(t_0)|^2\,\phi(t_0)\,\mathrm{d}x,\end{aligned}$$

which together with (4.31) proves (4.29). If $u$ is not smooth in time, we can approximate, so (4.29) holds for a.e. $t_0$ and any suitable weak solution $(u,p)$. But by weak continuity this implies that (4.29) has to hold for all $t_0$. Like in (ii), for any $t_0 \in (a,b)$ we may find $t_n$ such that (4.29) holds along $t_n$. By dominated convergence, all double integrals in (4.29) will then converge in the correct way as $t_n \to t_0$ since the involved functions are integrable on $\Omega \times (a,b)$ as $(u,p)$ is a suitable weak solution. Moreover, for the single integral, we have using weak continuity and the Cauchy-Schwarz inequality

$$\begin{aligned}\int_\Omega |u(t_0)|^2\,\phi(t_0)\,\mathrm{d}x &= \lim_{n\to\infty}\int_\Omega u(t_n)\sqrt{\phi(t_n)}\cdot u(t_0)\sqrt{\phi(t_0)}\,\mathrm{d}x\\
&\le \liminf_{n\to\infty}\left(\int_\Omega |u(t_n)|^2\,\phi(t_n)\,\mathrm{d}x\right)^{1/2}\left(\int_\Omega |u(t_0)|^2\,\phi(t_0)\,\mathrm{d}x\right)^{1/2},\end{aligned}$$

hence $\int_\Omega |u(t_0)|^2\,\phi(t_0)\,\mathrm{d}x \le \liminf_{n\to\infty}\int_\Omega |u(t_n)|^2\,\phi(t_n)\,\mathrm{d}x$. Here we used that for any $v \in L^2(\Omega)$

$$\begin{aligned}&\int_\Omega \left(u(t_n)\sqrt{\phi(t_n)} - u(t_0)\sqrt{\phi(t_0)}\right)v\,\mathrm{d}x\\
&\quad= \int_\Omega u(t_n)\left(\sqrt{\phi(t_n)} - \sqrt{\phi(t_0)}\right)v\,\mathrm{d}x + \int_\Omega \left(u(t_n)-u(t_0)\right)\sqrt{\phi(t_0)}v\,\mathrm{d}x \to 0,\end{aligned}$$

as $n \to \infty$ since $\|u(t_n)\|_{L^2(\Omega)}$ is bounded. This proves (4.29) for all $t_0 \in (a,b)$.

**4.3.3. The measures $\mathscr{H}^k$ and $\mathscr{P}^k$.** Recall that the *k-dimensional Hausdorff measure in* $\mathbb{R}^d$ of a set $X \subset \mathbb{R}^d$ is given by

$$\mathscr{H}^k(X) := \lim_{\delta\to 0^+} \mathscr{H}^k_\delta(X) = \sup_{\delta>0} \mathscr{H}^k_\delta(X),$$

where

$$\mathscr{H}^k_\delta(X) := \inf\left\{ \sum_{\ell=1}^\infty \alpha(k)(\operatorname{diam} U_\ell)^k \,\middle|\, U_\ell \subset \mathbb{R}^d \text{ closed}, \ X \subset \bigcup_{\ell=1}^\infty U_\ell, \operatorname{diam} U_\ell < \delta \right\},$$

where $\alpha(k)$ is chosen such that $\mathscr{H}^k([0,1]^k \times \{0\}^{d-k}) = 1$. In a completely analogous manner, we define a *"parabolic" Hausdorff meausre* via

$$\mathscr{P}^k(X) := \lim_{\delta\to 0^+} \mathscr{P}^k_\delta(X) = \sup_{\delta>0} \mathscr{P}^k_\delta(X),$$

with

$$\mathscr{P}^k_\delta(X) := \inf\left\{ \sum_{\ell=1}^\infty r_\ell^k \,\middle|\, Q_{r_\ell} \subset \mathbb{R}^3 \times \mathbb{R}, X \subset \bigcup_{\ell=1}^\infty Q_{r_\ell}, r_\ell < \delta \right\},$$

where the supremum is taken over any parabolic cylinders, i.e. any sets

$$Q_{r,x_0,t} := \{(y,\tau) \in \mathbb{R}^3 \times \mathbb{R} \mid |y - x_0| \le r, t - r^2 \le \tau \le t\}.$$

Like for $\mathscr{H}^k$, one can show that $\mathscr{P}^k$ is an outer measure for which all Borel sets are measurable and a Borel regular measure on the $\sigma$-algebra of measurable sets.

LEMMA 4.3. *There exists $C(k) > 0$ such that $\mathscr{H}^k \le C(k)\mathscr{P}^k$.*

PROOF. Let $0 < \delta < 1$ and let $Q_\ell = Q_{r_\ell,x_\ell,t_\ell}$ be parabolic cylinders with $r_\ell < \delta$. Let $d_\ell := \operatorname{diam} Q_\ell$. Then, clearly $r_\ell \le d_\ell$. Moreover, by the Pythagorean theorem $d_\ell \le \sqrt{r_\ell + r_\ell^2} \le \sqrt{2} r_\ell$, since $r_\ell < \delta < 1$. Thus, for $X \subset \mathbb{R}^3 \times \mathbb{R}$, we have

$$\begin{aligned}
\mathscr{H}^k_\delta(X) &\le \inf\left\{ \sum_{\ell=1}^\infty \alpha(k)(d_\ell)^k \,\middle|\, Q_\ell \subset \mathbb{R}^3 \times \mathbb{R} \text{ parabolic cylinders }, X \subset \bigcup_{\ell=1}^\infty Q_\ell, d_\ell < \delta \right\} \\
&\le \alpha(k)\sqrt{2}^k \inf\left\{ \sum_{\ell=1}^\infty (r_\ell)^k \,\middle|\, Q_\ell \subset \mathbb{R}^3 \times \mathbb{R} \text{ parabolic cylinders}, X \subset \bigcup_{\ell=1}^\infty Q_\ell, r_\ell < \frac{\delta}{\sqrt{2}} \right\} \\
&= \alpha(k)\sqrt{2}^k \mathscr{P}^k_{\delta/\sqrt{2}}(X).
\end{aligned}$$

Taking $\delta \to 0$ finishes the proof. □

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