Navier-Stokes Seminar: Caffarelli-Kohn-Nirenberg Theory

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Preface

These are lecture notes geberated by the seminar course on the Caffarelli-Kohn-Nirenberg Theory for the Navier-Stokes equations at the Universität Ulm in the summer term of 2019. We mainly follow the [**CKN82**] in a modern fashion. This work is aimed at enthusiastic Masters and PhD students.

I would like to thank everyone taking the seminar for typing parts of these notes.

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CHAPTER 1

Talk 1: Introduction

By Dr. Jack Skipper

For this introduction we will use the original paper of [**CKN82**] and the excellent book [**RRS16**].

The three-dimensional Navier-Stokes equations are

$$\begin{aligned} \partial_t u(x,t) + (u \cdot \nabla) u(x,t) + \nabla p(x,t) - \Delta u(x,t) &= f(x,t) \\ \operatorname{div} u(x,t) &= 0. \end{aligned} \tag{1.1}$$

Here, $(x,t) \in \Omega \times [0,T]$, where $\Omega \subset \mathbb{R}^3$ or $\mathbb{T}^3$ or $\mathbb{R}^3$ some domain, and we have the unknown velocity field

$$u \colon \Omega \times [0,T] \to \mathbb{R}^3;$$

the unknown pressure field

$$p \colon \Omega \times [0,T] \to \mathbb{R};$$

and the given force $f \colon \Omega \times [0,T] \to \mathbb{R}^3$ with $\operatorname{div} f = 0$ in $\Omega \times [0,T]$. Together with initial data and boundary data, (1.1) turns into an initial boundary value problem

$$\begin{aligned} u(x,0) &= u_0(x), && x \in \Omega, \\ u(x,t) &= 0, && x \in \partial\Omega \quad \text{for} \quad 0 < t < T. \end{aligned} \tag{1.2}$$

With compatibility conditions for u_0 and f we see that

$$-\Delta p = \partial_i \partial_j (u_i u_j) \quad \text{for } a.e\ t.$$

1.1. Outline: The Navier-Stokes Equations

1.1.1. Weak and Strong. Here we will give an overview of the important results currently known about the Navier-Stokes equations(NSE). The results here were taken from the book by Robinson, Rodrigo,

- (Leray 1934, $\mathbb{R}^3$) in [**Ler34**] and (Hopf 1951, Ω or $\mathbb{T}^3$) in [**Hop51**] showed that Leray-Hopf (LH) weak solutions exist globally in time. Here we assume that the initial data $u_0 \in L^2_\sigma$ (in L^2 and weakly incompressible) and $u \in L^\infty(0,T;L^2_\sigma) \cap L^2(0,T;H^1)$ and satisfy the weak energy inequality, namely,

  $$\int_\Omega u^2(t)\,\mathrm{d}x + \int_s^t \int_\Omega |\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t \le \int_\Omega u(s)\,\mathrm{d}x$$

  for almost every t, s. We do not know about uniqueness here.
- (Leray 1934, $\mathbb{R}^3$) in [**Ler34**] and (Kiseler-Ladyzhenskaya 1857) in [**KL57**] showed that strong solutions (LH weak solutions with $u_0 \in L^2_\sigma \cap H^1$ and $u \in L^\infty(0,T;H^1) \cap L^2(0,T;H^2)$) exist and are unique locally in time. They showed a lover bound on the potential "blow up" time $T = c\|\nabla u_0\|^{-4}_{L^2}$. Further, strong solutions are immediately smooth, even real analytic according to (Foias-Temam 1989) in [**FT89**].

- We have global existence of strong solutions for small data on Ω or $\mathbb{T}^3$ where we have an absolute constant $C(\Omega)$ or $\tilde{C}(\Omega)$ such that, for example,

  $\|\nabla u_0\|_{L^2} < C \quad \|u_0\|_{L^2} < C\|\nabla u_0\|_{L^2} < \tilde{C}.$

  For $\mathbb{R}^3$ we have a scaling $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$ is a solution. Thus if we want to talk about small data we need the norm to be invariant under this map, we say these spaces are critical spaces. $\dot{H}^{1/2}$, L^3, BMO^{-1} are invariant spaces where for small data we have strong solutions and for any data have local in time strong solutions.
- (Sather-Serrin 1963) see **[Ser63]** showed weak-strong uniqueness, that is, strong solutions are unique in the class of LH weak solutions. (Need the energy inequality) This suggests 2 possibilities u is strong always $\|\nabla u(t)\|_{L^2} < \infty$ for all $s > 0$ or there exists T^* the "blow-up" time where

  $$\|\nabla u)(t)\|^2 \geq \frac{C(\Omega)}{\sqrt{(T^* - t)}}.$$

  Can use similar techniques to show robustness of solutions "if initial data is close to a strong solution initial data then the solutions is strong for a while".
- Leary noticed that any global in time LH weak solution is eventually strong and for large time $\|u(t)\|_{L^2} \to 0$ as $t \to \infty$.

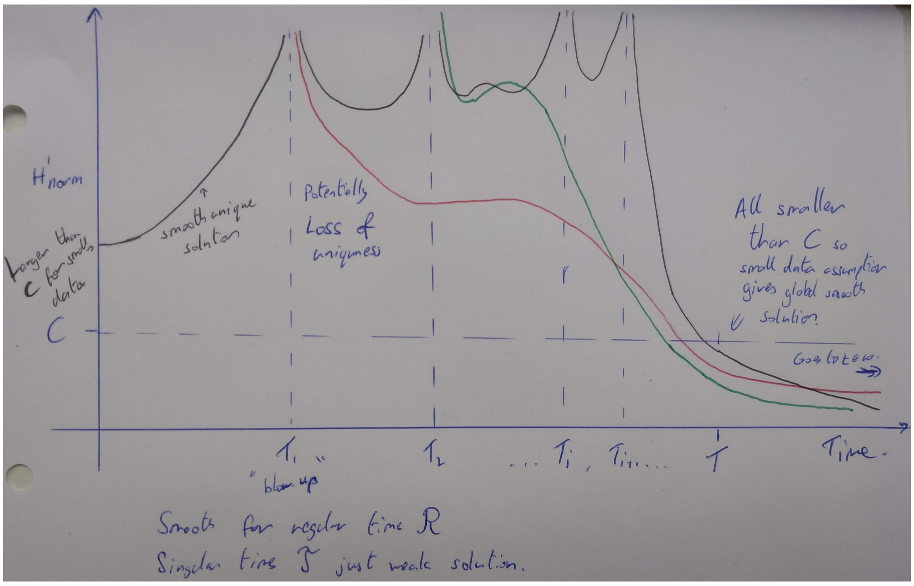

FIGURE 1. The H^1 norm of a potential solution to the Navier-Stokes equations.

1.1.2. Regularity. We can now look at the regularity of solutions and either we find conditions on how bad could the space of solutions be, or we find conditions on solutions that guarantee they are strong and smooth.

- (Scheffer 1976) in **[Sch76]** gave an upper bound on the size of the set of singular times. We say a time is regular and in the set $\mathcal{R}$ if $\|\nabla u(t)\|_{L^2}$ is essentially bounded. The singular times $\mathcal{T}$ a the rest. Here we see that the $\frac{1}{2}$ dimensional Hausdorff measure of the set $\mathcal{T}$ is zero. (Box counting measure is the same.)
- (Kato 1984) in **[Kat84]** showed that if

  $$\int_0^T \|\nabla u(s)\|_{L^\infty}\, \mathrm{d}s < \infty$$

then u is strong on $(0,T]$.

- (Beal-Kato-Majda 1984) in [**BKM84**] showed that if

$$\int_0^T \|\operatorname{curl} u(s)\|_{L^\infty}\,\mathrm{d}s < \infty$$

then u is strong on $(0,T]$ and further if we have "blow-up" at T then

$$\lim_{t\to T}\int_0^t \|\operatorname{curl} u(s)\|_{L^\infty}\,\mathrm{d}s = \infty.$$

- Serrin see [**Ser63**] condition that

$$u \in L^r(0,T;L^s(\Omega)) \quad \frac{2}{r}+\frac{3}{s}=1$$

gives a smooth solution on $(0,T]$. We note that we only unfortunately know that for a LH weak solution that

$$\frac{2}{r}+\frac{3}{s}=\frac{3}{2}.$$

Further, we have other Serrin type conditions, by (Beirão da Veiga 1995) in [**Bei95**]

$$\nabla u \in L^r(0,T;L^s(\Omega)) \quad \frac{2}{r}+\frac{3}{s}=2 \quad \frac{3}{2}<s<\infty$$

and by (Berselli-Galdi 2002) in [**BG02**] in

$$p \in L^r(0,T;L^s(\Omega)) \quad \frac{2}{r}+\frac{3}{s}=2 \quad \frac{3}{2}<s.$$

- (Serrin 1962) in [**Ser62**], for the (<) case, showed a local version of the Serrin condition that, on a sub-domain $U\times(t_1,t_2)$, if

$$u \in L^r(t_1,t_2;L^s(U)) \quad \frac{2}{r}+\frac{3}{s}=1$$

then u is smooth in space on $U\times(t_1,t_2)$ and α-Hölder continuous with $\alpha<\frac{1}{2}$ (Don't get smoothness in time as have problems with ∇p and $\partial_t u$ interacting locally.) The equality was worked out by (Fabes-Jones-Riviere 1972) see [**FJR72**], (Struwe 1988) see [**Str88**] and (Takahashi 1990) in [**Tak90**].

Leary thought that his solutions were turbulent solutions and that a self-similar construction would give a solution that would "blow-up", however, (Nečas-Růžička-Šverák 1996) in [**NRS96**] essentially disproved this. Further, for Euler equations non-uniqueness of weak solutions has been shown starting with the work of (Scheffer 1993) in [**Sch93**] then (De Lellis-Székelyhidi 2010) in [**DS10**] and finally with (Wiedemann 2011) in [**Wie11**].

We have a picture of how LH weak solutions are behaving and the interplay with strong solutions. Regularity results go down two lines where either we ask for extra conditions, we can't guarantee, from LH weak solutions so that then they are strong solutions an thus unique. Here, for the CKN result we want to keep with the regularity we know LH weak solutions can have and find upper bounds on how bad the set of "bad singular points" of the weak solutions can be. We will show that we get a bound of on the 1 dimensional Hausdorff measure and show that the size of the set in this measure is 0.

1.2. "Suitable" Weak Solutions

The CKN partial regularity result for suitable week solutions of the NSE. (How bad is the space-time set of blow-ups)

We know that for any $u_0 \in L^2_\sigma$ there us a LH weak solution of the NSE that satisfies the local energy inequality. (This modern result needs maximal regularity theory for the pressure p). (Sohr-von Wahl 1986) in [**SvW86**] showed that for any $\varepsilon > 0$

$$p \in L^r(\varepsilon, T; L^s) \quad \text{for} \quad \frac{2}{r} + \frac{3}{s} = 3 \quad (s > 1)$$

or for the gradient of the pressure

$$\nabla p \in L^r(\varepsilon, T; L^s) \quad \text{for} \quad \frac{2}{r} + \frac{3}{s} = 4 \quad (s > 1)$$

and thus we obtain that $p \in L^{\frac{5}{3}}(\Omega \times (0,T]))$. CKN only knew that $p \in L^{\frac{5}{4}}(\Omega \times (0,T]))$ which adds extra technical difficulties.

Definition 1.1. The pair (u,p) is a **suitable** weak solution of the NSE on $\Omega \times [0,T]$ with force f if the following are satisfied.

(1) Integrability:
   (a) $f \in L^q(\Omega \times [0,T])$ for $q > \frac{5}{2}$,
   (b) $p \in L^{\frac{5}{4}}(\Omega \times [0,T])$ [Modern times can get as high as $L^{\frac{5}{3}}(\Omega \times [0,T])$],
   (c) $u \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$.
(2) Local energy inequality: For all $\phi \geq 0$, $\phi \in C_c^\infty$, then,
$$2 \iint |\nabla u|^2 \phi \, \mathrm{d}x \, \mathrm{d}s \leq \iint |u|^2(\phi_t + \Delta\phi) + (|u|^2 + 2p) u \cdot \nabla\phi + 2(u \cdot f)\phi \, \mathrm{d}x \, \mathrm{d}s$$
(3) Weak solution: We need $u \in L^\infty(0,T;L^2_\sigma) \cap L^2(0,T;H^1_\sigma)$, $\nabla \cdot f = 0$, $-\Delta p = \partial_i \partial_j (u_i u_j)$ and for $a.e. t \in (a,b)$ and for all $\phi \in C^\infty_{\sigma,c}$
$$\int_{\Omega \times \{0\}} u_0 \cdot \phi(0) \, \mathrm{d}x = \int_0^T \int_\Omega \nabla u : \nabla\phi + (u \cdot \nabla)u\phi - u \cdot \partial_t \phi - f \cdot \phi \, \mathrm{d}x \, \mathrm{d}t.$$

For the CKN theory we do not need point 3 above, that is, the pair (u,p) does not actually need to be a LH weak solution of the NSE. The proof just deals with local energy inequality and interpolation inequalities as so points 1 and 2 are sufficient, the "suitable" bit.

As an interesting aside, it is important to note that in (Scheffer 1987) in [**Sch87**] he showed that the end result, that the one dimensional Hausedroff measure of the singular set of space-time points is zero, cannot be improved using the "suitable" criteria and the method would have to use (the equation) part 3 above. He showed that if you just pick a "suitable" pair (u,p) then for any $\gamma < 1$ there will exist at least one (u,p) pair where the γ- dimensional Hausdrof measure of the singular set is infinite.

1.3. Partial Regularity

We want to study "how bad" the set of "singular points" for u a suitable solution.

We denote $\mathcal{R}$ the set of regular points $(x,t) \in \mathcal{R}$ if there exists an open set $U \subset \Omega \times [0,T]$ with $(x,t) \in U$ and $u \in L^\infty(U)$. Let $\mathcal{S}$ be the set of singular points defined by $\mathcal{S} := \Omega \times [0,T] \setminus \mathcal{R}$, so the points where u is not L^∞_{loc} in any neighbourhood of (x,t). (Can also be defined similarly but with $\operatorname{curl} u$ or ∇u.) By "bad" we want an upper-bound on the dimension of $\mathcal{S}$ here using the Hausdroff measure.

Theorem 1.2 (Main Theorem (B) in [**CKN82**]). *For any suitable weak solution of the NSE on an open set in space-time the asscoiated singular set $\mathcal{S}$ satisfies*

$$\mathcal{P}^1(S) = 0.$$

This condition is equivalent to $\mathcal{H}^1(S) = 0$ which denotes that the one dimensional Hausdroff measure of the singular set is 0.

Importantly this shows that there are no curves in space-time where the solution u is singular along the curve. If we have "blow-up" then this occurs at distinct points in space time and not on a continuum.

CKN also impose extra conditions to prove two other theorems. These results are more in the spirit of previous partial regularity results like Serrin conditions as discussed earlier.

Let E denote the initial "kinetic energy", the L^2 norm of for the initial data, that is,

$$E := \frac{1}{2}\int_{\mathbb{R}^3} |u_0|^2 \,\mathrm{d}x$$

and let G, be a weighted form of E where we want extra decay at infinity, that is,

$$G := \frac{1}{2}\int_{\mathbb{R}^3} |u_0|^2|x| \,\mathrm{d}x < \infty.$$

For initial data satisfying this condition one can show that a suitable weak solution of the NSE from this data satisfies

$$\frac{1}{2}\int_{\mathbb{R}^3\times\{t\}} |u|^2|x| \,\mathrm{d}x + \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2|x| \,\mathrm{d}x \,\mathrm{d}s < \infty$$

for every t, so obtain the following theorem showing that the solution is regular for large enough x.

Theorem 1.3 (Theorem C in [**CKN82**]). *Suppose $u_0 \in L^2(\mathbb{R}^3)$ $\nabla\cdot u_0 = 0$ and $G < \infty$. Then there exists a weak solution of the NSE with $f = 0$ which is regular on the set*

$$\{(x,t) : |x|^2 t > K_1\}$$

where $K_1 = K_1(E,G)$ is a constant only depending on u_0 via E and G.

Here we see that G is a restriction that the initial data u_0 should decay sufficiently rapidly at infinity.

If instead we have a different condition where we ask for decay approaching zero, that is,

$$\int_{\mathbb{R}^3} |u_0|^2|x|^{-1} \,\mathrm{d}x = L \le L_0$$

then we obtain

$$\sup_\tau \int_{\mathbb{R}^3\times\{\tau\}} |u|^2|x|^{-1} \,\mathrm{d}x < \infty, \qquad \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2|x|^{-1} \,\mathrm{d}x \,\mathrm{d}\tau < \infty$$

for each t. From this we obtain the following theorem where we see that u is regular in a parabola above the origin and the line $x = 0$ is regular for all t.

Theorem 1.4 (Theorem D in [**CKN82**]). *There exists an absolute constant $L_0 > 0$ with the following properties. If $u_0 \in L^2(\mathbb{R}^3)$ $\nabla\cdot u_0 = 0$ and $L < L_0$ then there exists a weak solution of the NSE with $f = 0$ which is regular on the set*

$$\{(x,t) : |x|^2 < t(L_0 - L)\}.$$

1.4. Scale-invariant Quantities (Dimensionless Quantities)

On $\mathbb{R}^3$ if we have a solution to the NSE then by rescaling by λ, in the following way,

$$u(x,t) \mapsto \lambda u(\lambda x, \lambda^2 t)$$
$$p(x,t) \mapsto \lambda^2 p(\lambda x, \lambda^2 t)$$
$$f(x,t) \mapsto \lambda^3 f(\lambda x, \lambda^2 t)$$

we have another solution. Here we see that time scales quadratically and space linearly.

For local estimates it will be best to use, rather than balls, parabolic cylinders, that is,

$$Q_r(x,t) := \{(y,\tau) : |y-x| \le r,\ t-r^2 < \tau < t\}$$

or $Q_r^*(x,t) = Q_r(x, t-\frac{1}{8}r^2)$ (here (x,t) is the geometric centre of $Q_{\frac{r}{2}}(x, t+\frac{1}{8}r^2)$). The scaling that works on $\mathbb{R}^3$ also works on the parabolic cylinders where if (u,p) is a solution on $Q_r(x,t)$ then (u_λ, p_λ) will be a solution on $Q_{\frac{r}{\lambda}}(x,t)$.

We want to study "quantities" being "small" over parabolic cylinders and thus to have a sensible definition of a "smallness" assumption we should study scale invariant "quantities", that is, "quantities" whose value will not change after rescaling space and time as above. If the "quantities" we study did not have this property then under rescaling we could shrink or blow-up the values and could not compare the values. We will use factors of $\frac{1}{r}$ to make the scale invariant quantities we need.

For example,

$$\begin{aligned}\frac{1}{\left(\frac{r}{\lambda}\right)^2} \int_{Q_{\frac{r}{\lambda}}(0,0)} |u_\lambda|^3 \, \mathrm{d}x \, \mathrm{d}t &= \frac{\lambda^2}{r^2} \int_{Q_{\frac{r}{\lambda}}(0,0)} \lambda^3 |u(\lambda x, \lambda^2 t)|^3 \, \mathrm{d}x \, \mathrm{d}t \\ &= \frac{1}{r^2} \int_{Q_r(0,0)} |u(y,s)|^3 \, \mathrm{d}y \, \mathrm{d}s\end{aligned}$$

where we have a change of variable $y = \lambda x,\ s = \lambda^2 t$.

Some of the scale-invariant quantities we will use are

$$\frac{1}{r} \sup_{-r^2<t<0} \int_{B_r} |u(t)|^2 \, \mathrm{d}x, \quad \frac{1}{r} \iint_{Q_r} |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}t, \quad \frac{1}{r^2} \iint_{Q_r} |u|^3 \, \mathrm{d}x \, \mathrm{d}t, \quad \frac{1}{r^2} \iint_{Q_r} |p|^{\frac{3}{2}} \, \mathrm{d}x \, \mathrm{d}t.$$

1.5. The Main Ideas

We need to show two main propositions that concern bounds on u for large radii giving properties for u on smaller radii.

PROPOSITION 1.5. *There are absolute constants $\varepsilon, C_1 > 0$ and constant $\varepsilon_2(q) > 0$ with the following properties. If (u,p) is a suitable weak solution of the NSE on $Q_1(0,0)$ with force $f \in L^q$, for some $q > \frac{5}{2}$ and*

$$\iint_{Q_1(0,0)} (|u|^3 + |u||p|) \, \mathrm{d}x \, \mathrm{d}t + \int_{-1}^{0} \left(\int_{B_1} |p| \, \mathrm{d}x \right)^{\frac{5}{4}} \mathrm{d}t \le \varepsilon_1 \quad \text{and} \quad \iint_{Q_1(0,0)} |f|^q \, \mathrm{d}x \, \mathrm{d}t \le \varepsilon_2$$

then $u \in L^\infty(Q_{\frac{1}{2}}(0,0))$ with $\|u\|_{L^\infty(Q_{\frac{1}{2}}(0,0))} \le C_1$. ($u$ is regular on $Q_{\frac{1}{2}}(0,0)$).

With no force and modern $p \in L^{\frac{5}{3}}$ we can just assume that

$$\iint_{Q_1(0,0)} (|u|^3 + |p|^{\frac{3}{2}}) \, \mathrm{d}x \, \mathrm{d}t \le \varepsilon_1$$

and the proof is simplified.

We can shift and rescale this proposition to apply it to different $Q_r(x,t)$.

PROPOSITION 1.6. *There exists an absolute constant ε_3 such that if (u,p) is a suitable weak solution to the NSE on $Q_R(a,s)$ for some $R > 0$ and if*

$$\limsup_{r \to 0} \frac{1}{r} \int_{Q_r(as)} |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}t \le \varepsilon_3$$

then $u \in L^\infty(Q_\rho(a,s))$ for some ρ with $0 < \rho < R$. (a,s) is a regular point.

We will now discuss a rough outline of the proof and the tools used.

- We have the local energy inequality,
$$2\iint |\nabla u|^2\phi \,\mathrm{d}x\,\mathrm{d}s \le \iint |u|^2(\phi_t + \Delta\phi) + (|u|^2 + 2p)u\cdot\nabla\phi + 2(u\cdot f)\phi \,\mathrm{d}x\,\mathrm{d}s.$$
We use an approximation to the backwards heat equation for ϕ on a parabolic cylinder so it approximately solves $\phi_t + \Delta\phi = 0$ and get appropriate bounds on ϕ and $\nabla\phi$ as powers of $\frac{1}{r}$. This gives an inequality over parabolic cylinders with weighting in front of the remaining terms that means they are scaling invariant.
- We can use different interpolation inequalities over parabolic cylinders, for example,
$$\frac{1}{r^2}\iint\limits_{Q_r(a,s)} |u|^3 \,\mathrm{d}x\,\mathrm{d}t \le C_0\left[\frac{1}{r}\sup_{s-r^2<t<s}\int_{B_r(a)} |u(t)|^2 + \frac{1}{r}\iint\limits_{Q_r(a,s)} |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}t\right]^{\frac{3}{2}}.$$
- We can use these two inequalities. We see that the term on the RHS of the local energy inequality is quadratic in u and on the LHS they are all act cubic in u (with the assumed regularity on p and f) however this is the opposite for the interpolation inequality. We can thus iterate between these two inequalities to obtain inductive bounds on a solution u from the larger cylinder to a smaller cylinder that are shrinking and so can use Lebesgue differentiation theorem to get that the points (a,s) are regular on the smaller cylinder.

CHAPTER 2

Talk 2: Suitable weak solutions: part 1

By Farid Mohamed

We introduce the spaces for $\Omega \subset \mathbb{R}^3$

$$\mathcal{V} = \{u \in C_0^\infty(\Omega), \text{div } u = 0\},$$
$$V = \overline{\mathcal{V}}^{\|\cdot\|_{H_0^1(\Omega)}} \text{ and}$$
$$H = \overline{\mathcal{V}}^{\|\cdot\|_{L^2(\Omega)}}.$$

The space H is equipped with the norm $\|\cdot\|_{L^2(\Omega)}$ and we write

$$(u, v)_{L^2(\Omega)} := \int_\Omega uv\, dx$$

for the generating scalar product. In the case of V we need to distinguish two cases. If Ω is bounded we set $\|u\|_V := \|\nabla u\|_{L^2(\Omega)}$ and if Ω is unbounded we define $\|u\|_V := \|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}$. We observe that $V \hookrightarrow H \hookrightarrow V'$, where we identify H and H' in the sense that for every $u \in H$ we set

$$\langle u, f\rangle = T_u(f) = \int_\Omega uf dx$$

for $f \in H$. In this case we see that $\langle u, f\rangle = (u, f)_{L^2(\Omega)}$.
We assume for this section that

$$\Omega = \mathbb{R}^3,$$
$$f \in L^2(0, T; H^{-1}(\mathbb{R}^3)) \text{ and } \nabla \cdot f = 0,$$
$$u_0 \in H$$

or

$$\Omega \text{ is a smooth, bounded, open and connected set in } \mathbb{R}^3$$
$$f \in L^2(\Omega \times (0, T)) \text{ and } \nabla \cdot f = 0,$$
$$u_0 \in H \cap W^{2/5}_{5/4}(\Omega).$$

It follows directly that the spaces $L^2(0, T; H)$ and $L^2(0, T; V)$ are reflexive and $L^\infty(0, T; H)$ and $L^\infty(0, T; V)$ are the duals of separable Banach spaces, see for example [?], Theorem 1.29.

Definition 2.1. We call the pair (u, p) a suitable weak solution of the Navier-Stokes system on an open set $D = \Omega \times (0, T) \subset \mathbb{R}^3 \times \mathbb{R}$ with force f if:

i) u, p and f are measurable functions on D,
ii) $f \in L^q(D)$ for $q > 5/2$, $\nabla \cdot f = 0$ and $p \in L^{5/4}(D)$,

iii) the solution u is bounded in the following sense

$$E_0(u) :=_{0<t<T} \int_\Omega |u(x,t)|^2 dx < \infty \text{ and } E_1(u) := \iint_D |\nabla u|^2 dxdt < \infty,$$

iv) u, p and f solve

$$\partial_t u(x,t) + (u \cdot \nabla) u(x,t) + \nabla p(x,t) - \Delta u(x,t) = f \text{ in } \Omega,$$
$$\operatorname{div} u(x,t) = 0 \text{ on } \partial\Omega \text{ for all } 0 < t < T$$

in the sense of distributions in D, i.e. $u \in L^2(0,T;V)$ and for all $v \in V$ we have

$$\frac{d}{dt}\int_\Omega u(x,t)v(x)\,dx + \int_\Omega (u \cdot \nabla)u(x,t)v(x)\,dx + \int_\Omega \nabla u \cdot \nabla v\,dx = \int_\Omega f(t,x)v(x)dx$$

in the distributional sense on $(0,T)$.

v) for all $\varphi \in C_0^\infty(D)$, $\varphi \geq 0$ it holds

$$2\iint_D |\nabla u|^2 \varphi dxdt \leq \iint_D (|u|^2(\varphi_t + \Delta\varphi) + (|u|^2 + 2p)u \cdot \nabla\varphi + 2(u \cdot f)\varphi)dxdt.$$

The goal of this chapter is to show that for every $f \in L^q(D)$ there exists a suitable weak solution in the sense of Defintion 2.1.

The first step is to show that the equation

$$u_t + (w \cdot \nabla)u - \Delta u + \nabla p = f$$

has a solution for suitable f and w, where we use the following lemma.

Lemma 2.2 (see [**Tem79**], Lemma 1.2). *Suppose $f \in L^2(0,T;V')$, $u \in L^2(0,T;V)$, p is a distribution and*

$$u_t - \Delta u + \nabla p = f$$

in the sense of distributions on D. Then

$$u_t \in L^2(0,T;V'),$$
$$\frac{d}{dt}\int_\Omega |u|^2 = 2(u_t, u)_{L^2(\Omega)}$$

in the sense of distributions on $(0,T)$ and

$$u \in C([0,T], H)$$

after modification on a set of measure zero. Solutions of (2.2) are unique in the space $L^2(0,T;V)$ for given initial data $u_0 \in H$.

Proof. Here we give the main ideas of the proof.
Let the function $\hat{u} : \mathbb{R} \to V$ be equal to u on $[0,T]$ and to 0 outside this interval. We see by [**LM72**], Theorem 4.3 a sequence $(u_m)_{m\in\mathbb{N}}$ such that

$$\forall m, u_m \text{ is infinitly differentiable from } [0,T] \text{ onto } V, \text{ as } m \to \infty$$
$$u_m \to u \text{ in } L^2_{loc}(0,T;V),$$
$$u'_m \to u' \text{ in } L^2_{loc}(0,T;V').$$

It follows directly

$$\frac{d}{dt}\int_\Omega |u_m(t)|^2 = 2(u'_m(t), u_m(t))_{L^2(\Omega)}$$

and as $m \to \infty$ we get

$$\|u_m\|^2_{L^2(\Omega)} \to \|u\|^2_{L^2(\Omega)} \text{ in } L^1_{loc}((0,T))$$
$$(u'_m, u_m)_{L^2(\Omega)} \to (u', u)_{L^2(\Omega)} \text{ in } L^1_{loc}((0,T)).$$

These convergences also hold in the distribution sense. So by passing to the limit we get

$$\frac{d}{dt}\int_\Omega |u|^2 = 2(u_t, u)_{L^2(\Omega)}$$

and by (2) we see that $u \in L^\infty(0,T;H)$. We conclude by [**Tem79**], Lemma 1.4 that $u \in C([0,T];H)$. Uniqueness will follow by the next lemma. □

LEMMA 2.3. *Let $f \in L^2(0,T;V')$, $u_0 \in H$ and $w \in C^\infty(\overline{D},\mathbb{R}^3)$ with $\nabla\cdot w = 0$. Then there exists a unique function u and a distribution p such that*

$$\begin{aligned}
&u \in C([0,T],H)\cap L^2(0,T;V),\\
&\quad u_t + (w\cdot\nabla)u - \Delta u + \nabla p = f
\end{aligned}$$

in the sense of distributions on D, with $u(0) = u_0$.

PROOF. We will follow [**Tem79**], Theorem 1.1 by constructing the solution. Let $\{x_n\}_{n\in\mathbb{N}} \subset V$ be a sequence of linearly indepedent vectors such that $\overline{\operatorname{span}((x_n)_{n\in\mathbb{N}})} = V$, which exists as V is separable. We set $V_n := \operatorname{span}(x_1,\dots,x_n)$ and $u_n := \sum_{i=1}^n g_{in}(t)x_i$, where $(g_{in})_{i=1}^n$ is a solution of the system

$$\begin{aligned}
\sum_{i=1}^n g_{in}'(t)(x_i,x_j)_{L^2(\Omega)} + \sum_{i=1}^n g_{in}(t)(((w\cdot\nabla)x_i,x_j)_{L^2(\Omega)} + (\nabla x_i,\nabla x_j)_{L^2(\Omega)}) = \langle f, x_j\rangle\\
g_{jn}(0) = P_{V_n}(x_0)_j
\end{aligned}$$

for $j = 1,\dots,n$. Then u_n solves the equation

$$(u_n', v)_{L^2(\Omega)} + ((w\cdot\nabla)u_n, v) + (\nabla u_n, \nabla v)_{L^2(\Omega)} = \langle f, v\rangle$$

for all $v \in V_n$. Observe by partial integration that

$$((w\cdot\nabla)u_n, u_n)_{L^2(\Omega)} = -(u_n,(w\cdot\nabla)u_n)_{L^2(\Omega)} = 0$$

and one obtains

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_n\|^2_{L^2(\Omega)} =&(u_n', u_n)_{L^2(\Omega)}\\
=&\langle f, u_n\rangle - (\nabla u_n, \nabla u_n)_{L^2(\Omega)}\\
\le&\frac{1}{2}\|f\|^2_{V'} + \frac{1}{2}\|u_n\|^2_{L^2(\Omega)} - \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)}\\
\le&\frac{1}{2}\|f\|^2_{V'} + \frac{1}{2}\|u_n\|^2_{L^2(\Omega)},
\end{aligned}$$

whch follows by

$$\langle f, u_n\rangle \le \frac{1}{2}\|f\|^2_{V'} + \frac{1}{2}\|u_n\|^2_V \le \frac{1}{2}\|f\|^2_{V'} + \frac{1}{2}\|u_n\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u_n\|^2_{L^2(\Omega)}.$$

The continuity of the projection and Gronwall's inequality imply that

$$\|u_n(t)\|^2_{L^2(\Omega)} \le \left(\|u_0\|^2_{L^2(\Omega)} + \int_0^T \|f(s)\|^2_{V'}\,ds\right)e^T < \infty,$$

which implies that $(u_n)_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty(0,T;H)$. Furthermore, we see by integrating (2)

$$\begin{aligned}
&\|u_n(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla u_n(s)\|^2_{L^2(\Omega)} ds \\
\leq &\|u_n(0)\|^2_{L^2(\Omega)} + \int_0^t \|f(s)\|^2_{V'} ds + \int_0^T \|u_n(s)\|^2_{L^2(\Omega)}\, ds \\
\leq &\left(\|u(0)\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(0,T;V')}\right)(1+Te^T)
\end{aligned}$$

and we conclude that $(u_n)_{n\in\mathbb{N}}$ is uniformly bounded in $L^2(0,T;V)$. One infers that there exists a subsequence $(u_n)_{n\in\mathbb{N}} \subset L^2(0,T;V)\cap L^\infty(0,T;H)$ such that there exists an $u \in L^2(0,T;V)\cap L^\infty(0,T;H)$

$$\begin{aligned}
&u_n \rightharpoonup u \text{ for } n\to\infty \text{ in } L^2(0,T;V) \text{ and} \\
&u_n \overset{*}{\rightharpoonup} u \text{ for } n\to\infty \text{ in } L^\infty(0,T;H).
\end{aligned}$$

We conclude for every $\varphi \in C^1([0,T])$ with $\varphi(T) = 0$ that

$$\begin{aligned}
0 = &\int_0^T ((u_n'(t), \varphi(t)x_j)_{L^2(\Omega)} + ((w\cdot\nabla)u_n(t), \varphi(t)x_j) + (\nabla u_n(t), \nabla x_j\varphi(t))_{L^2(\Omega)} \\
&- \langle f(t), \varphi(t)x_j\rangle)\, dt \\
= &\int_0^T (-(u_n(t), \varphi'(t)x_j)_{L^2(\Omega)} + ((w\cdot\nabla)u_n(t), \varphi(t)x_j) + (\nabla u_n(t), \nabla x_j\varphi(t))_{L^2(\Omega)} \\
&- \langle f(t), \varphi(t)x_j\rangle dt - (u_n(0), x_j)_{L^2(\Omega)}\varphi(0)) \\
\to &\int_0^T (-(u(t), \varphi'(t)x_j)_{L^2(\Omega)} + ((w\cdot\nabla)u(t), \varphi(t)x_j) + (\nabla u(t), \nabla x_j\varphi(t))_{L^2(\Omega)} \\
&- \langle f(t), \varphi(t)x_j\rangle dt - (u(0), x_j)_{L^2(\Omega)}\varphi(0))
\end{aligned}$$

for $n\to\infty$ for every $j\in\mathbb{N}$. Moreover, the equality holds for every finite combination of the (x_j) and by continuity even for all $v\in V$. We obtain that

$$\frac{d}{dt}(u,v)_{L^2(\Omega)} + ((w\cdot\nabla)u, v) + (\nabla u, \nabla v)_{L^2(\Omega)} = \langle f, v\rangle$$

in the sense of distributions on $(0,T)$.
In order to see that $u(0) = u_0$ we use that

$$\int_0^T \frac{d}{dt}(u(t),v)_{L^2(\Omega)}\varphi(t)dt = -\int_0^T (u(t),v)\varphi'(t)\,dt + (u(0),v)\varphi(0),$$

which implies that

$$\begin{aligned}
&-\int_0^T (u(t),v)\varphi'(t)dt + \int_0^T (\nabla u, \nabla v)_{L^2(\Omega)}\varphi(t)\,dt + \int_0^T ((w\cdot\nabla)u, v)_{L^2(\Omega)}\varphi(t)\,dt \\
=&(u(0),v)\varphi(0) + \int_0^T \langle f(t), v\rangle\varphi(t)dt
\end{aligned}$$

By comparison with the above equality we see that

$$(u_0 - u(0), v)\varphi(0) = 0.$$

As v was arbitrary we conclude that $u_0 = u(0)$.
To show uniqueness assume that we have two solutions u_1 and u_2 with some initial data

and force f. We know that $u_1 - u_2$ solves (2) with $f = 0$. We conclude by (2) that

$$\frac{1}{2}\frac{d}{dt}\|u_1 - u_2\|^2_{L^2(\Omega)} \le -(\nabla(u_1 - u_2), \nabla(u_1 - u_2))_{L^2(\Omega)} \le 0.$$

As $u_1(0) = u_2(0)$ we conclude that $u_1 = u_2$. □

A solution of the Poisson equation $-\Delta u = f$ for $f \in L^q(\mathbb{R}^3)$ for some $1 < q < \infty$ can be written as

$$u(x) := (-\Delta)^{-1} f(x) := c_3 \int\limits_{\mathbb{R}^3} \frac{1}{|x-y|} f(y) dy,$$

where $c_3 \in \mathbb{R}$ can be given explicitly. We use the following theorem, which can be shown by the Calderón-Zygmund theorem.

THEOREM 2.4 (see [?], Theorem B.7). *The linear operator T_{jk} defined by*

$$T_{jk} f := \partial_j \partial_k (-\Delta)^{-1} f$$

is a bounded linear operator from $L^q(\mathbb{R}^3)$ into $L^q(\mathbb{R}^3)$ for all $1 < q < \infty$, i.e.

$$\|T_{jk} f\|_{L^q(\mathbb{R}^3)} \le C \|f\|_{L^q(\mathbb{R}^3)}$$

for some constant $C > 0$.

LEMMA 2.5. *Let $\Omega = \mathbb{R}^3$, $f \in L^2(0,T; H^{-1}(\mathbb{R}^3))$, $\operatorname{div} f = 0$ and $u_0 \in H$. Then it holds that*

$$\Delta p = -\sum_{i,j} \partial_i \partial_j (w_i u_j),$$

in the sense of distribution. Hence, we obtain

$$\iint\limits_D |p|^{5/3} dx dt \le C \iint\limits_D |w|^{5/3} \cdot |u|^{5/3} dx dt.$$

REMARK 2.6. For general Ω (if Ω is bounded) it is also possible to show that $p \in L^{5/3}(D)$.

PROOF. We follow [?] to show that p is given by (2.5). At first, observe that

$$\{\varphi \in [\mathcal{S}(\mathbb{R}^3)]^3 : \operatorname{div} \varphi = 0\}$$

is a dense subset of V. Furthermore, for every $h \in [\mathcal{S}(\mathbb{R}^3)]^3$ there exists a $\varphi \in [\mathcal{S}(\mathbb{R}^3)]^3$ and $\psi \in \mathcal{S}(\mathbb{R}^3)$ such that $h = \varphi + \nabla \psi$ and $\nabla \cdot \varphi = 0$, see for example [?], Exercise 5.2. Now let $\xi \in C_0^\infty((0,T))$. As u is the solution of (2) we obtain by partial integration

$$\begin{aligned}
&-\int_0^T (u,h)_{L^2(\mathbb{R}^3)} \xi'(t)\, dt - \int_0^T (u, \Delta h)_{L^2(\mathbb{R}^3)} \xi(t)\, dt \\
&-\int_0^T (u \otimes w, \nabla h)_{L^2(\mathbb{R}^3)} \xi(t)\, dt - \int_0^T \langle f, h\rangle \xi(t) dt \\
=& -\int_0^T (u,\varphi)_{L^2(\mathbb{R}^3)} \xi'(t)\, dt + \int_0^T (\nabla u, \nabla \varphi)_{L^2(\mathbb{R}^3)} \xi(t)\, dt \\
&+\int_0^T ((w \cdot \nabla) u, \varphi)_{L^2(\mathbb{R}^3)} \xi(t)\, dt - \int_0^T \sum_{i,j} (u_i w_j, \partial_i \partial_j \psi)_{L^2(\mathbb{R}^3)} \xi(t)\, dt \\
&-\int_0^T \langle f, \varphi\rangle \xi(t) dt \\
=& -\int_0^T \sum_{i,j} (u_i w_j, \partial_i \partial_j \psi)_{L^2(\mathbb{R}^3)} \xi(t)\, dt.
\end{aligned}$$

As $u \in V$, we conclude that $\Delta p = -\sum_{i,j} \partial_i \partial_j (w_i u_j)$, where we used that $\nabla \cdot h = \Delta \psi$. By taking the Fourier transform we see that we can interchange the Laplace operator and $\partial_i \partial_j$ and we obtain

$$p = (-\Delta)^{-1}(-\Delta)p = \sum_{i,j}(-\Delta)^{-1}\partial_i\partial_j w_i u_j = \sum_{i,j}\partial_i\partial_j(-\Delta)^{-1} w_i u_j,$$

and one infers by Theorem 2.4 that $\|p\|_{L^{5/3}(\mathbb{R}^3)} \le C\||w|\cdot|u|\|_{L^{5/3}}$. □

Later on we want to estimate the pressure p by using following inequality

$$\int\limits_{\mathbb{R}^3} |u|^q dx \le C\left(\int\limits_{\mathbb{R}} |\nabla u|^2\, dx\right)^{\frac{3}{4}(q-2)} \left(\int_{\mathbb{R}} |u|^2\, dx\right)^{\frac{1}{4}(6-q)}$$

for $2 \le q \le 6$, which is a special case of the Gagliardo-Nirenberg interpolation inequality

$$\|D^j u\|_{L^q(\mathbb{R}^3)} \le C\|D^m u\|^{\alpha}_{L^r(\mathbb{R}^3)}\|u\|^{1-\alpha}_{L^p(\mathbb{R}^3)}$$

where $1 < q,p,r < \infty$ and $m,j \in \mathbb{N}$. α is chosen is such a way that $\frac{1}{q} = \frac{j}{3} + \left(\frac{1}{r} - \frac{m}{3}\right)\alpha + \frac{1-\alpha}{p}$ and $\frac{j}{m} \le \alpha \le 1$. By choosing $j = 0$, $m = 1$, $r = p = 2$ and $\alpha = 3(\frac{1}{2} - \frac{1}{q})$ we obtain (2). We recall that we denote by

$$E_0(u) :=_{0<t<T} \int\limits_{\Omega} |u(x,t)|^2 dx \text{ and } E_1(u) := \iint\limits_{D} |\nabla u|^2 dx dt.$$

LEMMA 2.7. *For $u, w \in L^2(0,T;H^1(\mathbb{R}^3))$,*

$$\|u\|_{L^{10/3}(0,T;L^{10/3}(\mathbb{R}^3))} \le CE_1^{3/10}(u)E_0^{1/5}(u),$$
$$\|w\cdot\nabla u\|_{L^{5/4}(0,T;L^{5/4}(\mathbb{R}^3))} \le CE_1^{1/2}(u)E_1^{3/10}(w)E_0^{1/5}(w),$$
$$\|u\|_{L^5(0,T;L^{5/2}(\mathbb{R}^3))} \le CT^{1/20}E_0^{7/20}(u)E_1^{3/20}(u).$$

PROOF. For (2.7) we use (2) and obtain

$$\int\limits_{\mathbb{R}^3} |u|^{10/3}\, dx \le C\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)\left(\int_{\mathbb{R}^3} |u|^2 dx\right)^{2/3} \le C\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)E_0(u)^{2/3}$$

for almost all $t \in (0,T)$. Integrating over $(0,T)$ gives the result. For (2.7) we see by Hölder's inequality that

$$\int_0^T \int_{\mathbb{R}^3} |w\cdot\nabla u|^{5/4} dx dt \le \left(\int_0^T \int_{\mathbb{R}^3} |w|^{10/3}\, dx dt\right)^{3/8} E_1(u)^{\frac{5}{8}}$$
$$= \|w\|^{5/4}_{L^{10/3}(0,T;L^{10/3}(\mathbb{R}^3))} E_1(u)^{\frac{5}{8}}.$$

By applying (2.7) we obtain (2.7). Furthermore, we see by (2) and Hölder's inequality that

$$\int_0^T \left(\int_{\mathbb{R}^3} |u|^{5/2}\, dx\right)^2 dt \le C\int_0^T \left(\int\limits_{\mathbb{R}} |\nabla u|^2\, dx\right)^{3/4} \left(\int_{\mathbb{R}^3} |u|^2\, dx\right)^{7/4} dt$$
$$\le CE_0(u)^{7/4}\int_0^T \left(\int\limits_{\mathbb{R}} |\nabla u|^2\, dx\right)^{3/4} dt$$
$$\le CE_0(u)^{7/4}T^{1/4}\left(\int_0^T \int\limits_{\mathbb{R}} |\nabla u|^2\, dx dt\right)^{3/4}.$$

We conclude that (2.7) holds true. □

CHAPTER 3

Talk 3: Suitable weak solutions: part 2

By David Berger

LEMMA 3.1 (see [**GS91**], Theorem 2.8). *Assume that Ω, f and u_0 satisfy the assumptions of Lemma 2.3. Let Ω be bounded, $4 = 3/q + 2/s$ and $w \cdot \nabla u, f \in L^s(0,T;L^q(\Omega))$ and $u_0 \in W_s^{2-2/s}(\Omega)$. Then the solution (u,p) constructed in Lemma 2.3 satisfies*

$$\begin{aligned}&\|\nabla p\|^s_{L^s(0,T;L^q(\Omega))} + \|u_t\|^s_{L^s(0,T;L^q(\Omega))} + \|\nabla^2 u\|^s_{L^s((0,T;L^q(\Omega))}\\ \le& C(\|u_0\|^s_{W_s^{2-2/s}(\Omega)} + \|w\cdot\nabla u\|^s_{L^s(0,T;L^q(\Omega))} + \|f\|^s_{L^s(0,T;L^q(\Omega))}).\end{aligned}$$

Furthermore, by normalizing p such that $\int_\Omega p = 0$ for all t we obtain

$$\|p\|_{L^{5/3}(0,T;L^{5/3}(\Omega))} < \infty.$$

LEMMA 3.2. *Let Ω, u_0 and f satisfy the assumption of Chapter 2 and let $w \in C^\infty(\bar{D},\mathbb{R}^3)$ with $\nabla\cdot w = 0$. Let (u,p) be the solution of Lemma 2.3. Then, for every $\varphi \in C^\infty(\bar{D})$ with $\varphi = 0$ near $\partial\Omega \times (0,T)$, and for every t, $0 < t \le T$,*

$$\begin{aligned}\int_\Omega |u(x,t)|^2\varphi(x,t)dx + 2\iint_D |\nabla u|^2\varphi =& \int_\Omega |u_0|^2\varphi(x,0) + \iint_D |u|^2(\varphi_t + \Delta\varphi)\\ &+ \iint_D (|u|^2 w + 2pu)\cdot\nabla\varphi + 2\iint_D (u\cdot f)\varphi\end{aligned}$$

PROOF. We assume that Ω is bounded. Suppose for the moment that φ vanishes near $t = 0$, choose Ω_1, so that $\Omega_1 \subset \Omega$ and supp$\varphi \subset \Omega_1 \times (0,T)$. Writing $F = f - w\cdot\nabla u$, we have

$$u_t - \Delta u + \nabla p = F \text{ on } D.$$

Mollifying in $\mathbb{R}^4$ each term of the equation above, we obtain sequences of smooth functions u_m, p_m and F_m, $m = 1, 2, \ldots$, such that

$$\frac{du_m}{dt} - \Delta u_m + \nabla p_m = F_m \qquad \nabla\cdot u_m = 0$$

in a neighborhood of suppΦ, and such that

$$\begin{aligned}u_m &\to u && \text{in } L^5(0,T;L^{\frac{5}{2}}(\Omega)\cap L^2(D)),\\ \nabla u_m &\to \nabla u && \text{in } L^2(D),\\ p_m &\to p && \text{in } L^{\frac{5}{4}}(0,T;L^{\frac{5}{3}}(\Omega_1)),\\ F_m &\to F && \text{in } L^2(D).\end{aligned}$$

Taking the inner product of 3 with $2u_m\Phi$ and integrating by parts yields

$$2\iint_D |\nabla u_m|^2\varphi = \iint_D |u_m|^2(\varphi_t + \Delta\varphi) + 2\iint_D p_m(u_m\cdot\nabla\varphi) + 2\iint_D (u_m\cdot F_m)\varphi.$$

We pass to the limit as $m \to \infty$, to conclude for u, p and F, with $F = f - w\cdot\nabla u$,

$$2\iint_D (u\cdot F)\varphi = 2\iint_D (u\cdot f)\varphi + \iint_D |u|^2 w\cdot\nabla\varphi.$$

This gives the proof when $\varphi \in C_0^\infty(D)$ and $t = T$. For the more general case use a cutoff function in time and the continuity of u in H at 0. □

The goal of this chapter is to use the results shown in Chapter 2 to prove the existence of the weak solution. Therefore, we will introduce the mollyfing operator

$$\Psi_\delta(u)(x,t) := (\delta^{-4}\psi(\cdot/\delta)) * u(x,t) = \delta^{-4} \iint\limits_{\mathbb{R}^4} \psi\left(\frac{y}{\delta},\frac{\tau}{\delta}\right)\tilde{u}(x-y,t-\tau)dyd\tau,$$

where $\psi \in C^\infty(\mathbb{R}^4)$, $\psi \geq 0$, $\iint_{\mathbb{R}^4}\psi(x,t)dxdt = 1$ and $\operatorname{supp}\psi \subset \{(x,t) : |x|^2 < t, 1 < t < 2\}$ and $\tilde{u}$ is the extension of u on $\mathbb{R}^4$, i.e. $\tilde{u}(x,t) = u(x,t)$ on D and elsewhere 0. We see by [**Gra14**], Theorem 1.2.19 that ψ_δ is an approximating identity on $\mathbb{R}^4$.

Lemma 3.3. *For any $u \in L^\infty(0,T;H) \cap L^2(0,T;V)$ it holds*

$$\nabla\cdot\psi_\delta(u) = 0,$$

$$\sup_{0\leq t\leq T}\int_\Omega |\psi_\delta(u)|^2dx \leq CE_0(u),$$

$$\iint\limits_D |\nabla\psi_\delta(u)|^2dxdt \leq CE_1(u),$$

for some $C > 0$ independent of u and δ.

Proof. It is easy to see that

$$\begin{aligned}\nabla\cdot\Psi_\delta(u) &= \delta^{-4}\iint\limits_{\mathbb{R}^4}\nabla\psi\left(\frac{y}{\delta},\frac{\tau}{\delta}\right)\cdot\tilde{u}(x-y,t-\tau)dyd\tau\\ &= \delta^{-4}\iint\limits_{\Omega}\nabla\psi\left(\frac{y}{\delta},\frac{\tau}{\delta}\right)\cdot u(x-y,t-\tau)dyd\tau = 0.\end{aligned}$$

Furthemore, we obtain (3.3) by Hölder's and Young's inequality

$$\begin{aligned}\int_\Omega |\psi_\delta(u)_j|^2dx =& \int_\Omega\left(\int_\delta^{2\delta}\int_{\mathbb{R}^3}\psi_\delta(y,\tau)\,\tilde{u}_j(x-y,t-\tau)dyd\tau\right)^2dx\\ \leq&\delta\int_\delta^{2\delta}\int_\Omega\left(\int_{\mathbb{R}^3}\psi_\delta(y,\tau)\,\tilde{u}_j(x-y,t-\tau)dy\right)^2dxd\tau\\ \leq&\int_{\mathbb{R}}\delta^{-1}\|\psi(\cdot,\tau/\delta)\|^2_{L^1(\mathbb{R}^3)}\|u(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau\\ \leq&E_0(u)\int_{\mathbb{R}}\|\psi(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau.\end{aligned}$$

The inequality (3.3) is a direct consequence of Young's inequality

$$\begin{aligned}\iint\limits_D|\nabla_j\psi_\delta(u)_i|^2dxdt &\leq \iint\limits_{\mathbb{R}^4}\left|\delta^{-4}\iint\limits_{\mathbb{R}^4}\psi\left(\frac{y}{\delta},\frac{\tau}{\delta}\right)\nabla_j\tilde{u}_i(x-y,t-\tau)dyd\tau\right|^2dxdt\\ &\leq \|\psi\|^2_{L^1(\mathbb{R}^4)}\|\nabla_ju_i\|^2_{L^2(\mathbb{R}^3)}.\end{aligned}$$

□

In the proof of the main theorem we will use the following theorem, which gives a sufficient condition that a sequence $(x_n)_{n\in\mathbb{N}} \cap L^2(0,T;L^2(\Omega))$ is relatively compact.

Theorem 3.4 (see [**Tem79**], Theorem 1). *Let $X_0 \subset X \subset X_1$ be threee Banach spaces such that X_0 is compact in X, and X_0 and X_1 are reflexive. Then the space*

$$Y = \left\{v \in L^{\alpha_0}(0,T;X_0), \frac{d}{dt}v \in L^{\alpha_1}(0,T;X_1)\right\}$$

with $\alpha_0, \alpha_1 > 1$ *is compact in* $L^{\alpha_0}(0,T;X)$.

THEOREM 3.5. *Assume that* Ω, u_0 *and* f *satisfy the assumptions from Chapter 2. Then there exists a weak solution* (u,p) *of the Navier-Stokes system such that*

$$\begin{aligned} u &\in L^2(0,T;V)\cap L^\infty(0,T;H), \\ u(t) &\rightharpoonup u_0 \text{ in } H \text{ as } t\to 0, \\ p &\in L^{5/3}(D) \text{ if } \Omega = \mathbb{R}^3, \\ \nabla p &\in L^{5/4}(D) \text{ if } \Omega \text{ is bounded and} \end{aligned}$$

for all $\varphi \in C_0^\infty(D)$, $\varphi \geq 0$ *and* $\varphi = 0$ *near* $\partial\Omega \times (0,T)$ *it holds*

$$\begin{aligned} &\int_\Omega |u(x,t)|^2\varphi(x,t)dx + 2\int_0^t\int_\Omega |\nabla u|^2\varphi dxdt \\ &\leq \int_\Omega |u_0|^2\varphi(x,0)dx + \int_0^t\int_\Omega (|u|^2(\varphi_t + \Delta\varphi) + (|u|^2 + 2p)u\cdot\nabla\varphi + 2(u\cdot f)\varphi)dxdt. \end{aligned}$$

Let $N \in \mathbb{N}$ and $\delta = T/N$. $u_N \in L^2(0,T;V)\cap C([0,T];H)$ is the solution of the equation

$$\frac{d}{dt}u_N + (\psi_\delta(u_N)\cdot\nabla)u_N - \Delta u_N + \nabla p_N = f, u_N(0) = u_0,$$

which exists by applying Lemma 2.3 on each time interval $(\delta m, \delta(m+1))$ for each $m = 0,\dots,N-1$ separately. By using (2), (2) and (2) we obtain

$$\int_\Omega |u_N(t,x)|^2dx + \int_0^t\int_\Omega |\nabla u_N|^2dxdt \leq C\left(\int_\Omega |u_0|^2dx + \int_0^t \|f(t)\|_{V'}dt\right),$$

for some constant $C > 0$ which implies that u_N is bounded in $L^\infty(0,T;H)\cap L^2(0,T;V)$. Morever, by [**Tem79**], Lemma 4.2 we conclude that $\frac{d}{dt}u_n$ is bounded in $L^2(0,T;V_2')$, hence $(u_N)_{N\in\mathbb{N}}$ is relatively compact in $L^2(D)$ by Theorem 3.4. We obtain a subsequence (u_n) such that $u_n \to u_*$ in $L^2(D)$, $u_n \rightharpoonup u_*$ in $L^2(0,T;V)$ and $u_n \overset{*}{\rightharpoonup} u_*$ in $L^\infty(0,T;H)$. Moreover, as (u_N) is bounded in $L^{10/3}(D)$ we see easily by an interpolation argument that $u_n \to u_*$ in $L^s(D)$ for every $2 \leq s < 10/3$. Using the above inequalities it is possible to show that u_* solves the Navier-Stokes equation. We will only prove the convergence of the term $\int_0^t \varphi(t)((\psi_\delta(u_N)\cdot\nabla)u_N, v)_{L^2(\Omega)}dt$, as the other parts are trivial. As $v \in H^1(\Omega)$, we see that $\|u_iv_j\|_{L^2(\mathbb{R}^3)} < \infty$, which follows by the Sobolev embedding theorem. We conclude that

$$\begin{aligned} &\left|\int_0^t\int_\Omega ((\psi_\delta(u_N)\cdot\nabla)u_N, v)\varphi(t)dxdt - \int_0^t\int_\Omega ((u\cdot\nabla)u, v)\varphi(t)dxdt\right| \\ \leq &\left|\int_0^t\int_\Omega ((\psi_\delta(u_N)\cdot\nabla)u_N, v)\varphi(t)dxdt - \int_0^t\int_\Omega ((u\cdot\nabla)u_N, v)\varphi(t)dxdt\right| \\ &+\left|\int_0^t\int_\Omega ((u\cdot\nabla)u_N, v)\varphi(t)dxdt - \int_0^t\int_\Omega ((u\cdot\nabla)u, v)\varphi(t)dxdt\right| \\ &\to 0 \text{ for } N\to\infty, \end{aligned}$$

where we use for the first term that $\psi_\delta(u_N) \to u$ in $L^3(\mathbb{R}^3)$ and in the second term that $u_n \rightharpoonup u$ in $L^2(0,T;V)$.

In the case that Ω is bounded, we use Lemma 3.1. Let $\{\Omega_j\}_{j\in\mathbb{N}}$ be a sequence of subdomains such that $\overline{\Omega}_j \subset \Omega_{j+1}$ and $\cup_{j\in\mathbb{N}}\Omega_j = \Omega$. We see that ∇p_N is bounded in $L^{5/4}(D)$ and p_n in $L^{5/4}(0,T;L^{5/3}(\Omega_j))$. We obtain for every j a subsequence $p_N \rightharpoonup p_*$ in $L^{5/4}(0,T;L^{5/3}(\Omega_j))$. Moreover, we see that $u_N \to u_*$ in $L^5(0,T;L^{5/2}(\Omega))$. The proof follows the same arguments as in the case of $\Omega = \mathbb{R}^3$.

CHAPTER 4

Talk 4: Background and Definitions

By Fabian Rupp

4.1. On the initial boundary value problem

First, note that the condition $\operatorname{div} f = 0$ is not a restriction at all. Indeed, suppose we want to solve (1.1) for a general force $f \in L^q(\Omega)$ with $1 < q < \infty$. We may apply a L^q-Helmholtz decomposition to write $f = \nabla\Phi + f_1$ with $\operatorname{div} f_1 = 0$ and $\|f_1\|_{L^q(\Omega\times[0,T])} \leq C(q,\Omega)\|f\|_{L^q(\Omega\times[0,T])}$. If (u,p) is a solution of (1.1) with the force term f_1, it is easy to see that $(u, p+\Phi)$ is a solution to (1.1) with the right hand side $\nabla\Phi + f_1 = f$ as desired.

To obtain an existence theory for arbitrary time intervals, we study weak solutions of (1.1) for which the energy

$$\operatorname{ess\,sup}_{0<t<T} \int_\Omega |u|^2 \,\mathrm{d}x + \int_0^T \int_\Omega |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}t < \infty, \tag{4.1}$$

is finite, where $|\nabla u|^2 := \sum_{i,j} \left|\partial_i u^j\right|^2$. This choice is motivated by multiplying (1.1) by u, integration and using integration by parts. (4.1) justifies why requiring a solution u to have space derivatives of first order is a somewhat physical assumption.

If one instead multiplies (1.1) by $2u\phi$ for some $\phi \in \mathcal{C}^\infty(\Omega\times[0,T])$ and integrates one obtains

$$\int_0^t \int_\Omega 2\partial_t u\cdot u\phi + 2\left((u\cdot\nabla)u\right)\cdot u\phi - 2\Delta u\cdot u\phi + 2\nabla p\cdot u\phi \,\mathrm{d}x = \int_0^t\int_\Omega 2f\cdot u\phi\,\mathrm{d}x. \tag{4.2}$$

Since $u|_{\partial\Omega} = 0$ by (1.2), we may use integration by parts without creating any boundary terms. For the first term, we use $\partial_t |u|^2 = 2\partial_t u\cdot u$, so

$$\begin{aligned}\int_0^t \int_\Omega 2\partial_t u\cdot u\phi\,\mathrm{d}x\,\mathrm{d}t &= \int_0^t \partial_t \int_\Omega |u|^2\phi\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega |u|^2\partial_t\phi\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_\Omega |u(t)|^2\phi\,\mathrm{d}x - \int_\Omega |u(0)|^2\phi\,\mathrm{d}x - \int_\Omega |u|^2\partial_t\phi\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{4.3}$$

For the second part, integration by parts yields, using summation convention,

$$\begin{aligned}\int_0^t\int_\Omega 2u^i\partial_i u^j u^j\phi\,\mathrm{d}x\,\mathrm{d}t &= -\int_0^t\int_\Omega |u|^2\partial_i u^i\phi\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega |u|^2 u^i\partial_i\phi\,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_0^t\int_\Omega |u|^2 u\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{4.4}$$

since $\partial_i |u|^2 = 2\partial_i u^j u^j$ and $\operatorname{div} u = 0$ by (1.1). For the third term, we get using $\partial_i |u|^2 = 2\partial_i u^j u^j$ again

$$\begin{aligned}-2\int_0^t\int_\Omega \partial_i\partial_i u^j u^j\phi\,\mathrm{d}x &= 2\int_0^t\int_\Omega |\nabla u|^2\phi\,\mathrm{d}x\,\mathrm{d}t + 2\int_0^t\int_\Omega \partial_i u^j u^j\partial_i\phi\,\mathrm{d}x\,\mathrm{d}t \\ &= 2\int_0^t\int_\Omega |\nabla u|^2\phi\,\mathrm{d}x\,\mathrm{d}t - \int_0^t\int_\Omega |u|^2\partial_i\partial_i\phi\,\mathrm{d}x\,\mathrm{d}t \\ &= 2\int_0^t\int_\Omega |\nabla u|^2\phi\,\mathrm{d}x\,\mathrm{d}t - \int_0^t\int_\Omega |u|^2\Delta\phi\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{4.5}$$

Finally, for the last term, using $\operatorname{div} u = 0$, we have

$$\begin{aligned} 2\int_0^t \int_\Omega \partial_i p u^i \phi \, \mathrm{d}x \, \mathrm{d}t &= -2\int_0^t \int_\Omega p \partial_i u^i \phi \, \mathrm{d}x \, \mathrm{d}t - 2\int_0^t \int_\Omega p u^i \partial_i \phi \, \mathrm{d}x \, \mathrm{d}t \\ &= -2\int_0^t \int_\Omega p u \cdot \nabla\phi \, \mathrm{d}x \, \mathrm{d}t. \end{aligned} \tag{4.6}$$

Combining, (10.6),(10.10),(4.4),(4.5) and (4.6), we get

$$\begin{aligned} \int_\Omega |u(t)|^2 \phi \, \mathrm{d}x + 2\int_0^t \int_\Omega |\nabla u|^2 \phi \, \mathrm{d}x \, \mathrm{d}t &= \int_\Omega |u_0|^2 \phi \, \mathrm{d}x \\ &+ \int_0^t \int_\Omega |u|^2 (\partial_t \phi + \Delta\phi) \, \mathrm{d}x \, \mathrm{d}t + \int_0^t \int_\Omega (|u|^2 + 2p) u \cdot \nabla\phi \, \mathrm{d}x \, \mathrm{d}t \\ &+ 2\int_0^t \int_\Omega f \cdot u\phi \, \mathrm{d}x \, \mathrm{d}t. \end{aligned} \tag{4.7}$$

Pluggin in $\phi \equiv 1$ in (4.7) we obtain

$$\int_\Omega |u(t)|^2 \, \mathrm{d}x + 2\int_0^t \int_\Omega |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}t = \int_\Omega |u_0|^2 + 2\int_0^t \int_\Omega f \cdot u \, \mathrm{d}x. \tag{4.8}$$

Note that for $f \equiv 0$ in (4.8), we may formally conclude (4.1) with an explicit bound depending on the initial date $u_0 \in L^2(\Omega)$. The key point in proving existence of weak *Leray-Hopf solutions* is the *energy inequality*, an inequality form of (4.8).

$$\int_\Omega |u(t)|^2 \, \mathrm{d}x + 2\int_0^t \int_\Omega |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}t \le \int_\Omega |u_0|^2 + 2\int_0^t \int_\Omega f \cdot u \, \mathrm{d}x, \tag{4.9}$$

for almost every t.

For the main result, the localized version of (4.9) is crucial. Taking any $\phi \ge 0$ with compact support in $\Omega \times (0,T)$ in (4.7), one may conclude the following *generalized energy inequality* by estimating the first term by zero

$$2\int_0^T \int_\Omega |\nabla u|^2 \phi \, \mathrm{d}x \, \mathrm{d}t \le \int_0^T \int_\Omega \left[|u|^2 (\partial_t \phi + \Delta\phi) + (|u|^2 + 2p) u \cdot \nabla\phi + 2u \cdot f\phi \right] \mathrm{d}x \, \mathrm{d}t. \tag{4.10}$$

By definition, any *suitable weak solution* satisfies (4.10). Last week, we saw that such a suitable weak solution in fact exists (cf. David's talk Lemma 2.2, Theorem 2.5, Farid's talk Lemma 1.3).

Definition 4.1. We call a pair (u,p) a *suitable weak solution* to the Navier-Stokes equation with force f on $\Omega \times (0,T)$ if the following conditions are satisfied.

(1) u, p, f are measureable on $\Omega \times (0,T)$ and
   (a) $f \in L^q(\Omega \times (0,T))$ for $q > \frac{5}{2}$ and $\operatorname{div} f = 0$,
   (b) $p \in L^{\frac{5}{4}}(\Omega \times (0,T))$
   (c) for some $E_0, E_1 < \infty$ we have

$$\int_\Omega |u|^2 \, \mathrm{d}x \le E_0 \text{ for almost every } t \in (0,T), \text{ and} \tag{4.11}$$

$$\int_0^T \int_\Omega |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}t \le E_1. \tag{4.12}$$

(2) u, p and f satisfy (1.1) in the sense of distributions on $\Omega \times (0,T)$.
(3) For each $\phi \in \mathcal{C}_0^\infty(\Omega \times (0,T))$ with $\phi \ge 0$, inequality (4.10) holds.

Even for a suitable weak solution, it is not immediately clear that the right hand side of (4.10) is well, defined, i.e. it is not obvious that the integrals

$$\int_0^T \int_\Omega |u|^2 u \cdot \nabla\phi \,\mathrm{d}x\,\mathrm{d}t \quad \text{and} \quad \int_0^T \int_\Omega pu \cdot \nabla\phi \,\mathrm{d}x\,\mathrm{d}t$$

do exist. We will prove that this is the case.

4.2. Higher Regularity

Recall that a point (x,t) in space-time is *regular* if $u \in L^\infty_{loc}(U)$ for an open neighborhood U of (x,t). This is justified by the following result due to Serrin [**Ser63**]. If u is a weak solution of (1.1) on a cylinder $\Omega \times (a,b)$ satisfying

$$\int_a^b \left(\int_\Omega |u|^q \,\mathrm{d}x\right)^{\frac{s}{q}} \mathrm{d}t < \infty \text{ with } \frac{3}{q} + \frac{2}{s} < 1, \tag{4.13}$$

then u us necessarily $\mathcal{C}^{m+2,\beta}$ in space on compact subsets of Ω, provided f is $\mathcal{C}^{m,\beta}$ in space with $m \geq 0$ and $0 < \beta < 1$. In particular if f is $\mathcal{C}^\infty$ in space and (4.13) is satisfied, then u is $\mathcal{C}^\infty$ in space. Regularity in time is more difficult. If $u \in L^\infty(0,T;L^3(U))$, then u is Hölder continuous in time. From this, if $u \in L^\infty_{loc}(U)$ in a neighborhood U of (x,t), then (4.13) clearly holds, so u is smooth in space, provided f is smooth in space.

4.3. Recurrent Themes

The following three observations will be used frequently.

4.3.1. Interpolation inequalities for u and p. If $B_r \subset \mathbb{R}^3$ be a ball of radius $r > 0$ and let $u \in H^1(B_r)$. Then, the *Gagliardo-Nirenberg-Sobolev inequality* yields

$$\int_{B_r} |u|^q \,\mathrm{d}x \leq C\left(\int_{B_r} |\nabla u|^2 \,\mathrm{d}x\right)^a \left(\int_{B_r} |u|^2 \,\mathrm{d}x\right)^{\frac{q}{2}-a} + \frac{C}{r^{2a}}\left(\int_{B_r} |u|^2 \,\mathrm{d}x\right)^{\frac{q}{2}}, \tag{4.14}$$

where $C > 0$, $2 \leq q \leq 6$ and $a = \frac{3}{4}(q-2)$. If B_r is replaced by $\mathbb{R}^3$ the second term on the right in (4.14) can be omitted. Inequality (4.14) follows from the classical Gagliardo-Nirenberg-Sobolev inequality [**Nir59**] by applying an extension operator to $u \in H^1(B_r)$. The term $\frac{1}{r^{2a}}$ makes (4.14) scaling invariant with respect to $r > 0$.

We will now use (4.14) to interpolate between (4.11) and (4.12). Take $q = \frac{10}{3}$ so $a = 1$ in (4.14) and integrate in time. Then

$$\int_0^T \int_{B_r} |u|^{\frac{10}{3}} \,\mathrm{d}x\,\mathrm{d}t \leq C\left(E_0^{\frac{2}{3}} E_1 + r^{-2} E_0^{\frac{5}{3}} T\right). \tag{4.15}$$

A particular consequence is that $u \in L^3(\Omega \times (0,T))$, hence

$$\left|\int_0^T \int_\Omega |u|^2 u \cdot \nabla\phi \,\mathrm{d}x\,\mathrm{d}t\right| \leq \|\nabla\phi\|_{L^\infty(\Omega\times(0,T))} \|u\|_{L^3(\Omega\times(0,T))} < \infty,$$

so the corresponding term in (4.10) is in fact finite if u is a suitable weak solution and $\phi \in \mathcal{C}^\infty(\Omega \times (0,T))$. Moreover, if $q = \frac{5}{2}$, so $a = \frac{3}{8}$ we get

$$\int_0^T \left(\int_{B_r} |u|^{\frac{5}{2}} \,\mathrm{d}x\right)^{\frac{8}{3}} \mathrm{d}t \leq C(E_0^{\frac{7}{3}} E_1 + r^{-2} E_0^{\frac{10}{3}} T). \tag{4.16}$$

If we take the (distributional) divergence of (1.1), we get

$$0 = \Delta p + \partial_i \left(u^j \partial_j u^i\right) = \Delta p + \partial_i \partial_j (u^j u^i),$$

hence

$$\Delta p = -\partial_i \partial_j (u^i u^j) \text{ on } \Omega \times (0,T) \text{ in the sense of distributions.} \tag{4.17}$$

In addition, any solution $u \in \mathcal{C}^1(0,T;\mathcal{C}^2(\overline{\Omega}))$ of (1.1) on $\overline{\Omega} \times (0,T)$ for $f \equiv 0$ satisfying (1.2) has to fulfill

$$\nu \cdot \nabla p = \nu \cdot \Delta u \text{ on } \partial\Omega \times (0,T),$$

by simply restricting (1.1) to $\partial\Omega$ and multiplying with ν.
Recall that in $\mathbb{R}^3$, the unique solution to $-\Delta v = f$, with $f \in L^q(\mathbb{R}^3)$ is given by

$$v(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|} f(y) \, \mathrm{d}y.$$

We may thus rewrite (4.17) as $p = (-\Delta)^{-1} \partial_i \partial_j (u^i u^j)$.

First, we consider the case $\Omega = \mathbb{R}^3$. For u smooth enough, we have

$$p(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|} \partial_{y_i} \partial_{y_j} (u^i u^j) \, \mathrm{d}y = \alpha_{ij} u^i(x) u^j(x) + \frac{1}{4\pi} \int_{\mathbb{R}^3} \partial_{y_i} \partial_{y_j} \left(\frac{1}{|x-y|} \right) u^i u^j \, \mathrm{d}y,$$

where the latter has to be understood as a singular integral, i.e. a principal value

$$\lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} .$$

Also note that $\alpha_{ij} = 0$ if $i \neq j$.

We now use standard Calderón-Zygmund theory, see for instance [**Ste70**]. To that end, fix $i, j \in \{1, \ldots, 3\}$ and consider the convolution operator

$$S_{ij} f = \frac{1}{4\pi} \int_{\mathbb{R}^3} \partial_{y_j} \partial_{y_i} \left(\frac{1}{|x-y|} \right) f \, \mathrm{d}y.$$

A computation yields $\partial_{y_j} \partial_{y_i} \left(\frac{1}{|x-y|} \right) = -\frac{\delta_{ij}}{|x-y|^3} + 3 \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^5}$. We may write

$$S_{ij} f(x) = \int_{\mathbb{R}^3} \frac{\Omega(x-y)}{|x-y|^3} f(y) \, \mathrm{d}y,$$

with $\Omega(y) = -\delta_{ij} + 3 \frac{y_i y_j}{|y|^2}$. Note that Ω is homogeneous of degree 0 and a computation shows $\int_{\mathbb{S}^2} \Omega(y) \, \mathrm{d}S(y) = 0$ for all i, j. Clearly, Ω is Lipschitz on $\mathbb{S}^2$. Thus, by Calderón-Zygmund theory [**Ste70**, §4.3, Theorem 3],

$$S_{ij} \colon L^q(\mathbb{R}^3) \to L^q(\mathbb{R}^3) \text{ is bounded for any } 1 < q < \infty, i, j = 1, \ldots, 3. \tag{4.18}$$

As a consequence

$$\|p\|_{L^q(\mathbb{R}^3)} = \left\| (-\Delta)^{-1} \partial_i \partial_j (u^i u^j) \right\|_{L^q(\mathbb{R}^3)} \le C \sum_{i,j} \left\| u^i u^j \right\|_{L^q(\mathbb{R}^3)},$$

for some $C = C(q) > 0$ and

$$\left\| u^i u^j \right\|^q_{L^q(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \left| u^i u^j \right|^q \, \mathrm{d}x \le \int_{\mathbb{R}^3} |u|^{2q} \, \mathrm{d}x.$$

This yields

$$\int_{\mathbb{R}^3} |p|^q \, \mathrm{d}x \le C \int_{\mathbb{R}^3} |u|^{2q} \, \mathrm{d}x.$$

In particular, if (u, p) is a suitable weak solution of (1.1) on $\mathbb{R}^3 \times (0,T)$ we have

$$\int_0^T \int_{\mathbb{R}^3} |p|^{\frac{5}{3}} \, \mathrm{d}x \, \mathrm{d}t \le C \int_0^T \int_{\mathbb{R}^3} |u|^{\frac{10}{3}} \, \mathrm{d}x \, \mathrm{d}t \le C E_0^{\frac{2}{3}} E_1$$

by (4.15) using that we don't need the second term in (4.14) since we are in the whole space $\mathbb{R}^3$.

For general $\Omega \subset \mathbb{R}^3$ bounded, let $\overline{\Omega}_1 \subset \Omega$ and $\phi \in \mathcal{C}_0^\infty(\Omega)$ with $\phi \equiv 1$ in a neighborhood U of $\overline{\Omega}_1$. Then for t fixed we have using

$$\begin{aligned}\phi(x)p(x,t) &= -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|x-y|}\Delta_y(\phi p)\,\mathrm{d}y \\ &= -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|x-y|}\left[p\Delta\phi + 2\langle\nabla\phi,\nabla p\rangle + \phi\Delta p\right]\,\mathrm{d}y.\end{aligned} \tag{4.19}$$

We plug in (4.17) for Δp in (4.19) and obtain using summation convention

$$\phi p = -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|x-y|}\left[p\Delta\phi + 2\langle\nabla\phi,\nabla p\rangle - \phi\partial_i\partial_j(u^iu^j)\right]\,\mathrm{d}y. \tag{4.20}$$

Now, we integrate by parts to remove all derivatives on p and u. Note that in order to do this in a precise way, you have to cut out a ball B_ε of radius ε and do integration by parts there. However, since $\partial_{y_i}\left(\frac{1}{|x-y|}\right)$ is $L^1_{loc}(\mathbb{R}^3)$, the boundary terms will vanish as $\varepsilon \to 0$. We have

$$\int_{\mathbb{R}^3}\frac{1}{|x-y|}\langle\nabla\phi,\nabla p\rangle\,\mathrm{d}y = -\int_{\mathbb{R}^3}\partial_{y_i}\left(\frac{1}{|x-y|}\right)\partial_i\phi p\,\mathrm{d}y - \int_{\mathbb{R}^3}\frac{1}{|x-y|}\Delta\phi p\,\mathrm{d}y. \tag{4.21}$$

For the last term in (4.20) we have

$$\begin{aligned}\int_{\mathbb{R}^3}\frac{1}{|x-y|}\phi\partial_i\partial_j(u^iu^j)\,\mathrm{d}y &= -\int_{\mathbb{R}^3}\partial_{y_i}\left(\frac{1}{|x-y|}\right)\phi\partial_j(u^iu^j)\,\mathrm{d}y \\ &\qquad -\int_{\mathbb{R}^3}\frac{1}{|x-y|}\partial_i\phi\partial_j(u^iu^j)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^3}\partial_{y_j}\partial_{y_i}\left(\frac{1}{|x-y|}\right)\phi u^iu^j\,\mathrm{d}y + \int_{\mathbb{R}^3}\partial_{y_i}\left(\frac{1}{|x-y|}\right)\partial_j\phi u^iu^j\,\mathrm{d}y \\ &\quad + \int_{\mathbb{R}^3}\partial_{y_j}\left(\frac{1}{|x-y|}\right)\partial_i\phi u^iu^j\,\mathrm{d}y + \int_{\mathbb{R}^3}\frac{1}{|x-y|}\partial_i\partial_j\phi u^iu^j\,\mathrm{d}y \\ &= \int_{\mathbb{R}^3}\partial_{y_j}\partial_{y_i}\left(\frac{1}{|x-y|}\right)\phi u^iu^j\,\mathrm{d}y + \int_{\mathbb{R}^3}\frac{x_i-y_i}{|x-y|^3}\partial_j\phi u^iu^j\,\mathrm{d}y \\ &\quad + \int_{\mathbb{R}^3}\frac{x_j-y_j}{|x-y|^3}\partial_i\phi u^iu^j\,\mathrm{d}y + \int_{\mathbb{R}^3}\frac{1}{|x-y|}\partial_i\partial_j\phi u^iu^j\,\mathrm{d}y \\ &= \int_{\mathbb{R}^3}\partial_{y_j}\partial_{y_i}\left(\frac{1}{|x-y|}\right)\phi u^iu^j\,\mathrm{d}y + 2\int_{\mathbb{R}^3}\frac{x_i-y_i}{|x-y|^3}\partial_j\phi u^iu^j\,\mathrm{d}y \\ &\quad + \int_{\mathbb{R}^3}\frac{1}{|x-y|}\partial_i\partial_j\phi u^iu^j\,\mathrm{d}y\end{aligned} \tag{4.22}$$

Therefore, combining (4.19), (4.20), (4.21) and (4.22) we get

$$p\phi = \tilde{p} + p_3 + p_4 \tag{4.23}$$

with

$$\begin{aligned}\tilde{p} &= \alpha_{ij}u^i(x)u^j(x) + \frac{1}{4\pi}\int_{\mathbb{R}^3}\partial_{y_j}\partial_{y_i}\left(\frac{1}{|x-y|}\right)\phi u^iu^j\,\mathrm{d}y \\ p_3 &= \frac{1}{2\pi}\int_{\mathbb{R}^3}\frac{x_i-y_i}{|x-y|^3}\partial_j\phi u^iu^j\,\mathrm{d}y + \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|x-y|}\partial_i\partial_j\phi u^iu^j\,\mathrm{d}y \\ p_4 &= \left(-\frac{1}{4\pi} + \frac{2}{4\pi}\right)\int_{\mathbb{R}^3}\frac{1}{|x-y|}p\Delta\phi\,\mathrm{d}y + \frac{1}{2\pi}\int_{\mathbb{R}^3}\frac{x_i-y_i}{|x-y|^3}\partial_i\phi p\,\mathrm{d}y.\end{aligned}$$

Note that we have for $x \in \Omega_1$, using $\phi \equiv 1$ on U and $\phi \equiv 0$ on $\mathbb{R}^3 \setminus \Omega$

$$\begin{aligned}
|p_3|(x,t) &\le \left| \frac{1}{2\pi} \int_{\mathbb{R}^3} \frac{x_i - y_i}{|x-y|^3} \partial_j \phi u^i u^j \, \mathrm{d}y \right| + \left| \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|} \partial_i \partial_j \phi u^i u^j \, \mathrm{d}y \right| \\
&\le \frac{1}{2\pi} \int_{\Omega \setminus U} \frac{1}{|x-y|^2} |\partial_j \phi| |u|^2 \, \mathrm{d}y + \frac{1}{4\pi} \int_{\Omega \setminus U} \frac{1}{|x-y|} |\partial_i \partial_j \phi| |u|^2 \, \mathrm{d}y \\
&\le \frac{\|\phi\|_{\mathcal{C}^1}}{2\pi\delta^2} \int_\Omega |u|^2 \, \mathrm{d}y + \frac{\|\phi\|_{\mathcal{C}^2}}{4\pi\delta} \int_\Omega |u|^2 \, \mathrm{d}y,
\end{aligned}$$

where $\delta := d(\overline{\Omega}_1, \partial U) > 0$ gives lower bounds on $|x-y|$. Similarly for p_4, we have for $x \in \Omega_1$

$$\begin{aligned}
|p_4|(x,t) &\le \frac{1}{4\pi} \int_{\Omega \setminus U} \frac{1}{|x-y|} |p| |\Delta\phi| \, \mathrm{d}y + \frac{1}{2\pi} \int_{\Omega \setminus U} \frac{1}{|x-y|^2} |\partial_i \phi| |p| \, \mathrm{d}y \\
&\le \frac{\|\phi\|_{\mathcal{C}^2}}{4\pi\delta} \int_\Omega |p| \, \mathrm{d}y + \frac{\|\phi\|_{\mathcal{C}^1}}{2\pi\delta^2} \int_\Omega |p| \, \mathrm{d}y.
\end{aligned}$$

Consequently,

$$|p_3|(x,t) + |p_4|(x,t) \le C \int_\Omega \left(|p| + |u|^2\right) \mathrm{d}y, \text{ for } x \in \Omega_1. \tag{4.24}$$

Since the operators S_{ij} are bounded by (4.18), there exists $C > 0$ such that

$$\int_{\mathbb{R}^3} |\tilde{p}|^{5/3} \, \mathrm{d}x \le \sum_{i,j} \int_{\mathbb{R}^3} \left| S_{ij}(\phi u^i u^j) \right|^{5/3} \mathrm{d}x \le C \sum_{i,j} \int_{\mathbb{R}^3} \left| \phi u^i u^j \right|^{5/3} \mathrm{d}x,$$

and consequently

$$\int_{\Omega_1} |\tilde{p}|^{5/3} \, \mathrm{d}x \le C \sum_{i,j} \int_{\mathbb{R}^3} \left| \phi u^i u^j \right|^{5/3} \mathrm{d}x \le C \|\phi\|_{L^\infty} \int_\Omega |u|^{10/3} \, \mathrm{d}x. \tag{4.25}$$

From (4.24) and (4.25), we may deduce $p \in L^{5/4}(0,T; L^{5/3}(\Omega_1)))$.

We have using (4.15) and (4.25)

$$\begin{aligned}
\int_0^T \left(\int_{\Omega_1} |\tilde{p}|^{5/3} \, \mathrm{d}x \right)^{3/5 \cdot 5/4} \mathrm{d}t &\le C \int_0^T \left(\int_\Omega |u|^{10/3} \, \mathrm{d}x + 1 \right)^{3/4} \mathrm{d}t \\
&\le C \left(\int_0^T \int_\Omega |u|^{10/3} \, \mathrm{d}x \, \mathrm{d}t + T \right) \\
&\le C(E_0^{2/3} E_1 + E_0^{5/3} T + T),
\end{aligned} \tag{4.26}$$

where the constant $C > 0$ changes from line to line. For the remaining terms in (4.23), we have using (4.24) and Jensen's inequality

$$\begin{aligned}
\int_0^T \left(\int_{\Omega_1} (|p_3| + |p_4|)^{5/3} \, \mathrm{d}x \right)^{3/4} \mathrm{d}t &\le C |\Omega_1| \int_0^T \left(\int_\Omega (|p| + |u|^2) \, \mathrm{d}x \right)^{5/3 \cdot 3/4} \mathrm{d}t \\
&\le C \int_0^T \left(\left(\int_\Omega |p| \, \mathrm{d}x \right)^{5/4} + \left(\int_\Omega |u|^2 \, \mathrm{d}x \right)^{5/4} \right) \mathrm{d}t \\
&\le C \int_0^T \int_\Omega |p|^{5/4} \, \mathrm{d}x \, \mathrm{d}t + CTE_0^{5/4} \\
&= C \|p\|_{L^{5/4}(\Omega \times (0,T))} + CTE_0^{5/4}.
\end{aligned} \tag{4.27}$$

Therefore, combining (4.26) and (4.27) we get using $p = \phi p$ for a.e. t and $x \in \Omega_1$

$$\|p\|_{L^{5/4}(0,T;L^{5/3}(\Omega_1))} \le \|\tilde{p}\|_{L^{5/4}(0,T;L^{5/3}(\Omega_1))} + \||p_3| + |p_4|\|_{L^{5/4}(0,T;L^{5/3}(\Omega_1))} < \infty, \tag{4.28}$$

if (u,p) is a suitable weak solution. Thus, we have proven the following

Lemma 4.2. *If* (u,p) *is a suitable weak solution of* (1.1) *on* $\Omega\times(0,T)$ *and* $\overline{B}_r\times(a,b)\subset\Omega\times(0,T)$*, then* $p\in L^{5/4}(a,b;L^{5/3}(B_r))$ *and* $u\in L^5(a,b;L^{5/2}(B_r))$.

Proof. This follows from (4.28) and (4.16). □

In particular, the term $\int\int p(u\cdot\nabla\phi)$ in (4.10) is integrable, since if $\operatorname{supp}\phi\subset\Omega_1$ we have

$$\begin{aligned}\int_0^T\int_\Omega |pu\cdot\nabla\phi|\ \mathrm{d}x\ \mathrm{d}t &\le C\int_{0^T}\|u(t)\|_{L^{5/2}(\Omega_1)}\|p(t)\|_{L^{5/3}(\Omega_1)}\ \mathrm{d}t\\ &\le C\left(\int_0^T\|u(t)\|^5_{L^{5/2}(\Omega_1)}\ \mathrm{d}t\right)^{1/5}\left(\int_0^T\|p(t)\|^{5/4}_{L^{5/3}(\Omega_1)}\ \mathrm{d}t\right)^{4/5}\\ &= C\|u\|_{L^5(0,T;L^{5/2}(\Omega_1))}\|p\|_{L^{5/4}(0,T;L^{5/3}(\Omega_1))},\end{aligned}$$

by Hölder's inequality and since $\frac{3}{5}+\frac{2}{5}=\frac{4}{5}+\frac{1}{5}=1$. Thus, we have shown that for any suitable weak solution of (1.1), the right hand side of (4.9) exists.

4.3.2. Weak continuity. It can be shown, that any suitable weak solution u of (1.1) is weakly continuous in time with values in $L^2(\Omega)$, i.e. for any $w\in L^2(\Omega)$ we have

$$\int_\Omega u(x,t)w(x)\ \mathrm{d}x\to\int_\Omega u(x,t_0)w(x)\ \mathrm{d}x \text{ as } t\to t_0.$$

For a proof of this property we refer to [**Tem79**, p. 281-282]. This has some important consequences.

(i) We can evaluate u at times t and it makes sense to impose the initial condition $u(0)=u_0$ in the sense that $u(t)\rightharpoonup u_0$ in $L^2(\Omega)$ as $t\to 0$, i.e. u extends weakly continously to $[0,T)$.
(ii) The integrability condition (4.11) holds for every $t\in(0,T)$. If $t_0\in(0,T)$, then there exist $t_n\to t_0$ with $\int_\Omega|u(t_n)|^2\ \mathrm{d}x\le E_0$, otherwise (4.11) would not hold almost everywhere. But since the $L^2(\Omega)$-norm is weakly lower semicontinuous and as $u(t_n)\to u(t_0)$ as $n\to\infty$, we conclude $\int_\Omega|u(t_0)|^2\ \mathrm{d}x\le E_0$.
(iii) If (u,p) is a suitable weak solution of (1.1) on $\Omega\times(a,b)$, then for each $a<t_0<b$ and $\phi\in\mathcal{C}_0^\infty(\Omega\times(a,b))$ with $\phi\ge 0$ we have

$$\begin{aligned}\int_\Omega|u(t_0)|^2\phi(t_0)\ \mathrm{d}x+2\int_a^{t_0}\int_\Omega|\nabla u|^2\phi\ \mathrm{d}x\ \mathrm{d}t&\\ \le\int_a^{t_0}\int_\Omega\left[|u|^2(\partial_t\phi+\Delta\phi)+(|u|^2+2p)u\cdot\nabla\phi+2u\cdot f\phi\right]\ \mathrm{d}x\ \mathrm{d}t.&\end{aligned}\tag{4.29}$$

This follows from (4.10), by choosing the positive test function $\phi(x,t)\chi\left((t_0-t)/\varepsilon\right)$, where $\varepsilon>0$ and χ is a smooth function with $0\le\chi\le 1$, $\chi(s)\equiv 0$ for $s\le 0$ and $\chi(s)\equiv 1$ for $s\ge 1$. Then (4.10) yields

$$\begin{aligned}2\int_a^{t_0}\int_\Omega|\nabla u|^2\phi\chi\left((t_0-t)/\varepsilon\right)\ \mathrm{d}x\ \mathrm{d}t\le\int_a^{t_0}\int_\Omega\Big[&|u|^2\left(\partial_t\left(\phi\chi\left((t_0-t)/\varepsilon\right)\right)\right.\\ &\left.+\Delta\phi\chi\left((t_0-t)/\varepsilon\right)\right)+(|u|^2+2p)u\cdot\nabla\phi\chi\left((t_0-t)/\varepsilon\right)\\ &+2u\cdot f\phi\chi\left((t_0-t)/\varepsilon\right)\Big]\ \mathrm{d}x\ \mathrm{d}t.\end{aligned}\tag{4.30}$$

Note that for $t \le t_0$, $\chi\left({(t_0-t)}/{\varepsilon}\right) \to 1$ as $\varepsilon \to 0$. Since $0 \le \chi \le 1$, the dominated convergence theorem yields that as $\varepsilon \to 0$ in (4.30)

$$2\int_a^{t_0}\int_\Omega |\nabla u|^2 \phi \,\mathrm{d}x\,\mathrm{d}t \le \int_a^{t_0}\int_\Omega \Big[|u|^2(\partial_t\phi + \Delta\phi + (|u|^2+2p)u\cdot\nabla\phi + 2u\cdot f\phi\Big]\,\mathrm{d}x\,\mathrm{d}t \tag{4.31}$$

$$+\lim_{\varepsilon\to 0}\int_a^{t_0}\int_\Omega |u|^2\phi\partial_t\left(\chi\left({(t_0-t)}/{\varepsilon}\right)\right)\,\mathrm{d}x\,\mathrm{d}t,$$

since all terms in u and p are integrable. Taking a closer look at the last term, we observe that for u smooth enough

$$\begin{aligned}\int_a^{t_0}\int_\Omega |u|^2\phi\partial_t\left(\chi\left({(t_0-t)}/{\varepsilon}\right)\right)\,\mathrm{d}x\,\mathrm{d}t &= \int_\Omega\int_a^{t_0} |u|^2\phi\partial_t\left(\chi\left({(t_0-t)}/{\varepsilon}\right)\right)\,\mathrm{d}t\,\mathrm{d}x\\
&= \int_\Omega |u(t_0)|^2\phi(t_0)\chi(0)\,\mathrm{d}x - \int_\Omega |u(a)|^2\phi(a)\chi\left({(t_0-a)}/{\varepsilon}\right)\,\mathrm{d}x\\
&\quad - \int_a^{t_0}\int_\Omega \partial_t|u|^2\phi\chi\left({(t_0-t)}/{\varepsilon}\right)\,\mathrm{d}x\,\mathrm{d} - \int_a^{t_0}\int_\Omega |u|^2\partial_t\phi\chi\left({(t_0-t)}/{\varepsilon}\right)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

If we let $\varepsilon \to 0$ we obtain

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_a^{t_0}\int_\Omega |u|^2\phi\partial_t\left(\chi\left({(t_0-t)}/{\varepsilon}\right)\right)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad= -\int_\Omega |u(a)|^2\phi(a)\,\mathrm{d}x - \int_a^{t_0}\int_\Omega \partial_t|u|^2\phi\,\mathrm{d}x\,\mathrm{d}t - \int_a^{t_0}\int_\Omega |u|^2\partial_t\phi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad= -\int_\Omega |u(a)|^2\phi(a)\,\mathrm{d}x - \int_a^{t_0}\int_\Omega \partial_t\left(|u|^2\phi\right)\,\mathrm{d}x\,\mathrm{d}t = -\int_\Omega |u(t_0)|^2\phi(t_0)\,\mathrm{d}x,\end{aligned}$$

which together with (4.31) proves (4.29). If u is not smooth in time, we can approximate, so (4.29) holds for a.e. t_0 and any suitable weak solution (u,p). But by weak continuity this implies that (4.29) has to hold for all t_0. Like in (ii), for any $t_0 \in (a,b)$ we may find t_n such that (4.29) holds along t_n. By dominated convergence, all double integrals in (4.29) will then converge in the correct way as $t_n \to t_0$ since the involved functions are integrable on $\Omega \times (a,b)$ as (u,p) is a suitable weak solution. Moreover, for the single integral, we have using weak continuity and the Cauchy-Schwarz inequality

$$\begin{aligned}\int_\Omega |u(t_0)|^2\phi(t_0)\,\mathrm{d}x &= \lim_{n\to\infty}\int_\Omega u(t_n)\sqrt{\phi(t_n)}\cdot u(t_0)\sqrt{\phi(t_0)}\,\mathrm{d}x\\
&\le \liminf_{n\to\infty}\left(\int_\Omega |u(t_n)|^2\phi(t_n)\,\mathrm{d}x\right)^{1/2}\left(\int_\Omega |u(t_0)|^2\phi(t_0)\,\mathrm{d}x\right)^{1/2},\end{aligned}$$

hence $\int_\Omega |u(t_0)|^2\phi(t_0)\,\mathrm{d}x \le \liminf_{n\to\infty}\int_\Omega |u(t_n)|^2\phi(t_n)\,\mathrm{d}x$. Here we used that for any $v \in L^2(\Omega)$

$$\begin{aligned}&\int_\Omega \left(u(t_n)\sqrt{\phi(t_n)} - u(t_0)\sqrt{\phi(t_0)}\right)v\,\mathrm{d}x\\
&\quad= \int_\Omega u(t_n)\left(\sqrt{\phi(t_n)} - \sqrt{\phi(t_0)}\right)v\,\mathrm{d}x + \int_\Omega (u(t_n)-u(t_0))\sqrt{\phi(t_0)}v\,\mathrm{d}x \to 0,\end{aligned}$$

as $n \to \infty$ since $\|u(t_n)\|_{L^2(\Omega)}$ is bounded. This proves (4.29) for all $t_0 \in (a,b)$.

4.3.3. The measures $\mathscr{H}^k$ and $\mathscr{P}^k$. Recall that the *k-dimensional Hausdorff measure in $\mathbb{R}^d$* of a set $X \subset \mathbb{R}^d$ is given by

$$\mathscr{H}^k(X) := \lim_{\delta \to 0^+} \mathscr{H}^k_\delta(X) = \sup_{\delta > 0} \mathscr{H}^k_\delta(X),$$

where

$$\mathscr{H}^k_\delta(X) := \inf \left\{ \sum_{\ell=1}^\infty \alpha(k)(\operatorname{diam} U_\ell)^k \,\middle|\, U_\ell \subset \mathbb{R}^d \text{ closed, } X \subset \bigcup_{\ell=1}^\infty U_\ell, \operatorname{diam} U_\ell < \delta \right\},$$

where $\alpha(k)$ is chosen such that $\mathscr{H}^k([0,1]^k \times \{0\}^{d-k}) = 1$. In a completely analogous manner, we define a *"parabolic" Hausdorff meausre* via

$$\mathscr{P}^k(X) := \lim_{\delta \to 0^+} \mathscr{P}^k_\delta(X) = \sup_{\delta > 0} \mathscr{P}^k_\delta(X),$$

with

$$\mathscr{P}^k_\delta(X) := \inf \left\{ \sum_{\ell=1}^\infty r_\ell^k \,\middle|\, Q_{r_\ell} \subset \mathbb{R}^3 \times \mathbb{R}, X \subset \bigcup_{\ell=1}^\infty Q_{r_\ell}, r_\ell < \delta \right\},$$

where the supremum is taken over any parabolic cylinders, i.e. any sets

$$Q_{r,x_0,t} := \{(y,\tau) \in \mathbb{R}^3 \times \mathbb{R} \mid |y - x_0| \le r, t - r^2 \le \tau \le t\}.$$

Like for $\mathscr{H}^k$, one can show that $\mathscr{P}^k$ is an outer measure for which all Borel sets are measurable and a Borel regular measure on the σ-algebra of measurable sets.

LEMMA 4.3. *There exists $C(k) > 0$ such that $\mathscr{H}^k \le C(k)\mathscr{P}^k$.*

PROOF. Let $0 < \delta < 1$ and let $Q_\ell = Q_{r_\ell, x_\ell, t_\ell}$ be parabolic cylinders with $r_\ell < \delta$. Let $d_\ell := \operatorname{diam} Q_\ell$. Then, clearly $r_\ell \le d_\ell$. Moreover, by the Pythagorean theorem $d_\ell \le \sqrt{r_\ell + r_\ell^2} \le \sqrt{2} r_\ell$, since $r_\ell < \delta < 1$. Thus, for $X \subset \mathbb{R}^3 \times \mathbb{R}$, we have

$$\begin{aligned}
\mathscr{H}^k_\delta(X) &\le \inf \left\{ \sum_{\ell=1}^\infty \alpha(k)(d_\ell)^k \,\middle|\, Q_\ell \subset \mathbb{R}^3 \times \mathbb{R} \text{ parabolic cylinders }, X \subset \bigcup_{\ell=1}^\infty Q_\ell, d_\ell < \delta \right\} \\
&\le \alpha(k)\sqrt{2}^k \inf \left\{ \sum_{\ell=1}^\infty (r_\ell)^k \,\middle|\, Q_\ell \subset \mathbb{R}^3 \times \mathbb{R} \text{ parabolic cylinders, } X \subset \bigcup_{\ell=1}^\infty Q_\ell, r_\ell < \frac{\delta}{\sqrt{2}} \right\} \\
&= \alpha(k)\sqrt{2}^k \mathscr{P}^k_{\delta/\sqrt{2}}(X).
\end{aligned}$$

Taking $\delta \to 0$ finishes the proof. $\square$

CHAPTER 5

Talk 7: The Blow-up estimate part 1

By Lukas Niebel

The aim of this talk is to provide a partial proof of the following Proposition 5.1. This proposition gives a criterion for the regularity of certain points of suitable weak solutions by means of control of the parabolic mean of the gradient of u in cylinders shrinking to that point. Let us recall some notation first. Given any point (t,x) and a radius $r>0$ we introduce the cylinders

$$Q_r(t,x) = \left\{(s,y) \in \mathbb{R}^4 \mid t-r^2 < s < t, |x-y| < r\right\}$$
$$Q_r^*(t,x) = \left\{(s,y) \in \mathbb{R}^4 \;\middle|\; t - \frac{7}{8}r^2 < s < t + \frac{1}{8}r^2, |x-y| < r\right\}.$$

The cylinders $Q_r^*(t,x)$ are useful in the sense that $(t,x) \in Q^*_{\frac{r}{2}}(t,x)$, while $(t,x) \notin Q_r(t,x)$. Therefore we may apply Corollary 5.3 to the cylinders $Q_r^*(t,x)$ to show that the point $(t,x) \in Q^*_{\frac{r}{2}}(t,x)$ is regular.

PROPOSITION 5.1 (Proposition 2 in **[CKN82]**). *There is an aabsolute constant $\epsilon_3 > 0$ such that for all suitable weak solutions (u,p) of the Navier-Stokes in a neighborhood of a given point (t,x) satisfying*

$$\limsup_{r\to 0} \frac{1}{r}\int_{Q_r^*(t,x)} |\nabla u|^2 \, \mathrm{d}(t,x) \le \frac{1}{2}\epsilon_3$$

are regular in (t,x).

This theorem is going to be used to show Theorem B in **[CKN82]**, namely that the singular set S satisfies $\mathscr{P}^1(S) = 0$.

The proof of Proposition 5.1 is based on a rather technical decay estimate for a quantity $M_*(r)$ in terms of M_*, δ_* and F_*. These quantities are analogues to the quantities introduced in section 3. However they are defined on the translated cylinders $Q_r^*(t,x)$ instead of on the cylinder $Q_r(t,x)$. The estimate and its proof are going to be subject of the next talk. We are going to use it to prove Proposition 5.1 for now. To provide a shorthand way of writing it down we introduce several dimension-less quantities depending on u, p and f. Without loss of generality we may restrict to the case $(t,x) = (0,0)$ by translation in space and time. Writing $Q_r^* = Q_r^*(0,0)$, we define

$$G_*(r) = r^{-2}\int_{Q_r^*} |u|^3 \, \mathrm{d}(t,x) \qquad K_*(r) = r^{-\frac{13}{4}}\int_{-\frac{7}{8}r^2}^{\frac{1}{8}r^2}\left(\int_{B_r(0)} |p| \, \mathrm{d}x\right)^{\frac{5}{4}} \mathrm{d}t$$
$$J_*(r) = r^{-2}\int_{Q_r^*} |u|\,|p| \, \mathrm{d}(t,x) \qquad H_*(r) = r^{-2}\int_{Q_r^*} |u| \left||u|^2 - \overline{|u|_r^2}\right| \mathrm{d}(t,x)$$
$$\delta_*(r) = r^{-1}\int_{Q_r^*} |\nabla u|^2 \, \mathrm{d}(t,x) \qquad F_*(r) = r^{-\frac{1}{2}}\int_{Q_r^*} |f|^{\frac{3}{2}} \, \mathrm{d}(t,x),$$

where

$$\overline{|u|_r^2} = \fint_{B_r(0)} |u|^2 \, \mathrm{d}x.$$

Let us compare these quantities to their analogues from section 3. Clearly δ, G, K are exactly the same integral, with the only difference that they are now defined on the translated cylinder $Q_r^*(0,0)$. The quantity $F_*(r)$ corresponds to the quantity $F(r)$ with $q = \frac{3}{2}$ fixed and again $Q_r(0,0)$ swapped by $Q_r^*(0,0)$. The function $\delta_*(r)$ is used to provide a shorthand way of writing down the regularity condition in Proposition 5.1, i.e. $\limsup_{r\to 0} \delta_*(r) \le \frac{1}{2}\epsilon_3$. We define the function

$$M_*(r) = G_*^{\frac{2}{3}}(r) + H_*(r) + J_*(r) + K_*^{\frac{8}{5}}(r),$$

which satisfies the following decay estimate.

PROPOSITION 5.2. *Let $\rho > 0$ and let (u,p) be a suitable weak solution of the Navier-Stokes System with force f on the cylinder $Q_\rho^*(0,0)$. If it holds $\delta_*(\rho) \le 1$ and $F_*(\rho) \le 1$, then the following decay estimate holds*

$$M_*(r) \le C\left[\left(\frac{r}{\rho}\right)^{\frac{1}{5}} M_*(\rho) + \left(\frac{\rho}{r}\right)^2 \left(M_*^{\frac{1}{2}}(\rho)\delta_*^{\frac{1}{2}}(\rho) + M_*(\rho)\delta_*(\rho) + F_*(\rho) + \delta_*(\rho)\right)\right]$$

for some constant $C > 0$ and all $0 < r \le \frac{1}{4}\rho$. Moreover $M_*(r)$ is finite for all $r \le \frac{1}{4}\rho$.

COROLLARY 5.3. *There exists absolute constants $\epsilon_1, \epsilon_2 > 0$ such that the following holds. We consider a cylinder $Q_r(t,x)$ and any suitable weak solution of the Navier Stokes system in the given cylinders with a force term $f \in L^q$ for $q > \frac{5}{2}$. Suppose that*

$$r^{-2}\int_{Q_r(t,x)} |u|^3 + |u|\,|p| \;\mathrm{d}(s,y) + r^{-\frac{13}{4}}\int_{t-r^2}^{t}\left(\int_{B_r(x)} |p| \;\mathrm{d}y\right)^{\frac{5}{4}} \mathrm{d}s \le \epsilon_1$$

and

$$F_q(r) = r^{3q-5}\int_{Q_r(t,x)} |f|^q \;\mathrm{d}(s,y) \le \epsilon_2,$$

then it must hold $|u| \le Cr^{-1}$ Lebesgue almost everywhere in the smaller cylinder $Q_{\frac{r}{2}}(t,x)$. In particular u is regular on $Q_{\frac{r}{2}}(t,x)$.

PROOF OF PROPOSITION 5.1. By translation of (u,p) we may assume that $(t,x) = (0,0)$. Let (u,p) be a suitable weak solution of the Navier Stokes System in a neighborhood D of $(0,0)$. We want to apply Corollary 5.3 and verify its assumptions to prove that $(0,0)$ is a regular point. It holds $Q_r^* = Q_r(\frac{1}{8}r^2, 0)$ which suggest that we can use Corollary 5.3 applied to the point $(\frac{1}{8}r^2, 0)$. Let $r \le 1$ such that $Q_r^* \subset D$, then it holds

$$F_q(r) = r^{3q-5}\int_{Q_r} |f|^q \;\mathrm{d}(t,x) \le r^{\frac{5}{2}}\int_D |f|^q \;\mathrm{d}(t,x),$$

whence $\lim_{r\to 0} F_q(r) = 0$ due to the fact that $f \in L^1(D)$. This shows that, by Corollary 5.3, the point $(0,0) \in Q_{\frac{r}{2}}(\frac{1}{8}r^2, 0)$ is regular if for example it holds

$$\liminf_{r\to 0} r^{-2}\int_{Q_r(0,\frac{1}{8}r^2)} |u|^3 + |u|\,|p| \;\mathrm{d}(t,x) + r^{-\frac{13}{4}}\int_{-\frac{7}{8}r^2}^{\frac{1}{8}r^2}\left(\int_{B_r(0)} |p| \;\mathrm{d}y\right)^{\frac{5}{4}} \mathrm{d}s \le \epsilon_1$$

which can be written as

$$\liminf_{r\to 0} G_*(r) + J_*(r) + K_*(r) \le \epsilon_1.$$

Due to the nonnegativity of the involved terms the latter condition is clearly verified if it holds

$$\liminf_{r\to 0} M_*(r) \le \tilde{\epsilon}_1 := \min\left\{\frac{\epsilon_1}{3}, \left(\frac{\epsilon_1}{3}\right)^{\frac{2}{3}}, \left(\frac{\epsilon_1}{3}\right)^{\frac{8}{5}}\right\}.$$

We claim that there are constants $\epsilon_3 \in (0,1]$ and $\gamma \in (0,\frac{1}{4}]$ such that whenever it holds

$$M_*(\rho) > \tilde{\varepsilon}_1, F_*(\rho) \leq \epsilon_3 \text{ and } \delta_*(\rho) \leq \epsilon_3$$

for some $\rho > 0$ with $Q^*_\rho \subset D$ it follows that $M_*(\gamma\rho) \leq \frac{1}{2}M_*(\rho)$. To show the existence of such constants we choose

$$\gamma < \min\left\{\frac{1}{(C6)^5}, \frac{1}{4}\right\}$$

and then $\epsilon_3 > 0$ such that

$$\epsilon_3 < \min\left\{\frac{1}{12C}\gamma^2\tilde{\epsilon}_1, 1\right\} \text{ and } \epsilon_3 + \left(\frac{\epsilon_3}{\tilde{\epsilon}_1}\right)^{\frac{1}{2}} \leq \frac{\gamma^2}{6C}.$$

Let us suppose that $M_*(\rho) > \tilde{\epsilon}_1$, that $F_*(\rho) \leq \epsilon_3$ and that $\delta_*(\rho) \leq \epsilon_3$. In this case it holds

$$M_*^{\frac{1}{2}}(\rho) < \tilde{\epsilon}_1^{-\frac{1}{2}} M_*(\rho).$$

Using the decay estimate from Proposition 5.2 we deduce

$$M_*(r) \leq C\left\{\left(\frac{r}{\rho}\right)^{\frac{1}{5}} M_*(\rho) + \left(\frac{\rho}{r}\right)^2\left[\varepsilon_3 + \left(\frac{\varepsilon_3}{\varepsilon_1}\right)^{\frac{1}{2}}\right] M_*(\rho) + 2\left(\frac{\rho}{r}\right)^2 \varepsilon_3\right\}$$

for all $r \leq \frac{1}{4}\rho$. Choosing $r = \gamma\rho \leq \frac{1}{4}\rho$ and using the assumptions on γ and ϵ_3 we deduce

$$\begin{aligned} M_*(\gamma\rho) &\leq C\left\{\gamma^{\frac{1}{5}} M_*(\rho) + \left(\frac{1}{\gamma}\right)^2\left[\varepsilon_3 + \left(\frac{\varepsilon_3}{\varepsilon_1}\right)^{\frac{1}{2}}\right] M_*(\rho) + 2\left(\frac{1}{\gamma}\right)^2 \varepsilon_3\right\} \\ &\leq \frac{1}{6}M_*(\rho) + \frac{1}{6}M_*(\rho) + \frac{1}{6}\tilde{\epsilon}_1 \leq \frac{1}{2}M_*(\rho). \end{aligned}$$

Now let us show that

$$\liminf_{r\to 0} M_*(r) \leq \tilde{\epsilon}_1.$$

We first note that due to $q > \frac{5}{4}$ it holds

$$\begin{aligned} F_*(r) = r^{-\frac{1}{2}} \int_{Q^*_r} |f|^{\frac{3}{2}} \, \mathrm{d}(t,x) &\leq C\left(\int_{Q^*_r} |f|^q \, \mathrm{d}(t,x)\right)^{\frac{3}{2q}} r^{\frac{9}{2}-\frac{15}{2}q} \\ &\leq C r^{\frac{3}{2}} \left(\int_D |f|^q \, \mathrm{d}(t,x)\right)^{\frac{3}{2q}} \end{aligned}$$

for all $r \leq 1$ such that $Q^*_r \subset D$ by Hölder's inequality. This shows $\lim_{r\to 0} F_*(r) = 0$, which together with the assumption yields a radius $r_0 > 0$ such that $F_*(r) \leq \epsilon_3$ and $\delta_*(r) \leq \epsilon_3$ for all $r < r_0$. This is due to the assumption that $\limsup_{r\to 0} \delta_*(r) \leq \frac{1}{2}\epsilon_3 < \epsilon_3$. Let us now suppose that $\liminf_{r\to 0} M_*(r) > \tilde{\epsilon}_1$. We claim that there is $N \in \mathbb{N}$ such that $M_*(\gamma^N r_0) \leq \tilde{\epsilon}_1$. Assuming the opposite would be true it must hold that $M_*(\gamma^n r_0) > \tilde{\epsilon}_1$ for all $n \in \mathbb{N}$. Consequently as we have proven before it follows that

$$M_*(\gamma^n r_0) \leq \left(\frac{1}{2}\right)^n M_*(r_0)$$

for all $n \in \mathbb{N}$, which is a contradiction to $\liminf_{r\to 0} M_*(r) > \tilde{\epsilon}_1$. This is only due to the fact that $M_*(r_0)$ is finite. Hence, we may assume that $M_*(\gamma^N r_0) \leq \tilde{\epsilon}_1$ for some $N \in \mathbb{N}$. Now if it were true that $M_*(\gamma^{N+1} r_0) > \tilde{\epsilon}_1$ we could conclude that $\tilde{\epsilon}_1 < M_*(\gamma^{N+1} r_0) \leq \frac{1}{2}M_*(\gamma^N r_0) \leq \frac{1}{2}\tilde{\epsilon}_1$ which is a contradiction. By induction it follows that $M_*(\gamma^{N+k} r_0) \leq \tilde{\epsilon}_1$ for all $k \in \mathbb{N}$, whence $\liminf_{r\to 0} M_*(r) \leq \tilde{\epsilon}_1$. This shows that $(0,0)$ is a regular point. □

In preparation of the proof of the decay estimate we are going to start with an bound of H_* in terms of $G_*(r)$, $\delta_*(r)$ and in terms of $A_*(r)$, which is given by

$$A_*(r) = \sup_{-\frac{7}{8}r^2<t<\frac{1}{8}r^2} r^{-1}\int_{\{t\}\times B_r(0)} |u|^2(t,\cdot)\,\mathrm{d}x.$$

Let us fix a suitable weak solution (u,p) of the Navier Stokes system in a neighborhood D of $(0,0)$. Let $r>0$ such that $Q_r^*\subset D$. Clearly it holds that $A_*(r)\le r^{-1}E_0<\infty$.

LEMMA 9.4. *For any r such that $Q_r^*\subset D$ it holds*

$$H_*(r)\le C(G_*^{\frac{2}{3}}(r)+A_*(r)\delta_*(r))$$

for some constant $C>0$.

PROOF. At almost every time t it holds

$$\begin{aligned}
&\int_{B_r(0)} |u(t,x)|\left||u|^2(t,x)-\overline{|u|_r^2}(t)\right|\,\mathrm{d}x\\
&\le\left(\int_{B_r(0)}|u|^3(t)\,\mathrm{d}x\right)^{\frac{1}{3}}\left(\int_{B_r(0)}\left||u|^2(t)-\overline{|u|_r^2}(t)\right|^{\frac{3}{2}}\,\mathrm{d}x\right)^{\frac{2}{3}}\\
&\le C\left(\int_{B_r(0)}|u|^3(t)\,\mathrm{d}x\right)^{\frac{1}{3}}\int_{B_r(0)}\left|\nabla|u|^2\right|(t)\,\mathrm{d}x\\
&\le C\left(\int_{B_r(0)}|u|^3(t)\,\mathrm{d}x\right)^{\frac{1}{3}}\int_{B_r(0)}|\nabla u|(t)\,|u|(t)\,\mathrm{d}x\\
&\le C\left(\int_{B_r(0)}|u|^3(t)\,\mathrm{d}x\right)^{\frac{1}{3}}\left(\int_{B_r(0)}|\nabla u|^2(t)\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{B_r(0)}|u|^2(t)\,\mathrm{d}x\right)^{\frac{1}{2}}\\
&\le C\left(\int_{B_r(0)}|u|^3(t)\,\mathrm{d}x\right)^{\frac{1}{3}}(rA_*(r))^{\frac{1}{2}}\left(\int_{B_r(0)}|u|^2(t)\,\mathrm{d}x\right)^{\frac{1}{2}},
\end{aligned}$$

where we have used Hölder's inequality, the Poincaré inequality on the ball $B_r(0)$, the Cauchy-Schwarz inequality and the definition of $A_*(r)$. Integration in time from $-\frac{7}{8}r^2$ to $\frac{1}{8}r^2$ yields

$$\begin{aligned}
r^2H_*(r)&\le C\,(rA_*(r))^{\frac{1}{2}}\int_{-\frac{7}{8}r^2}^{\frac{1}{8}r^2}\left(\int_{B_r(0)}|u|(t)^3\,\mathrm{d}x\right)^{\frac{1}{3}}\left(\int_{B_r(0)}|u|^2(t)\,\mathrm{d}x\right)^{\frac{1}{2}}\,\mathrm{d}t\\
&\le(rA_*(r))^{\frac{1}{2}}\left(\int_{Q_r^*}|u|^3\,\mathrm{d}(t,x)\right)^{\frac{1}{3}}\left(\int_{Q_r^*}|\nabla u|^2\,\mathrm{d}(t,x)\right)^{\frac{1}{2}}r^{\frac{1}{3}}\\
&=r^2A_*^{\frac{1}{2}}(r)G_*^{\frac{1}{3}}(r)\delta_*(r)^{\frac{1}{2}}
\end{aligned}$$

by Hölders inequality. Applying Young's inequality we conclude

$$H_*(r)\le Cr^{-\frac{1}{6}}A_*^{\frac{1}{2}}(r)G_*^{\frac{1}{3}}(r)\delta_*(r)^{\frac{1}{2}}\le C(G_*^{\frac{2}{3}}(r)+A_*(r)\delta_*(r)).$$

□

REMARK 9.5. Let $r>0$ such that $Q_r^*\subset D$, then $A_*(r)\le r^{-1}E_0$ and $\delta_*(r)\le r^{-1}E_1$. It can be shown similar to Lemma 3.1 that $G_*(r)$ can be bounded by $A_*(r)$ and $\delta_*(r)$ and thus must be finite. This is going to be proven in the next talk given by Marius. By Lemma 9.4 we deduce that $H_*(r)$ is finite as well. Furthermore, due to the fact that $p\in L^{\frac{5}{4}}(D)$ it holds that $K_*(r)$ is finite. Finally $J_*(r)$ can be bounded by $A_*(r),\delta_*(r),G_*(r)$ and

$K_*(r)$, whence $M_*(r)$ must be finite. The latter claim is going to be shown in the talk given by Marius, too.

CHAPTER 10

Talk 8: The Blow-Up Estimate, Part 2

By Marius Müller

10.1. Introduction

In this talk, (u, p) will always denote a suitable weak solution in the sense of **[CKN82]** 2.1, cf. Talk 1 and Talk 2.

Our goal is to finish up the proof of Proposition 2 of **[CKN82]** (cf. Talk 7). Proposition 2 roughly states that a certain L^2-control of the gradient $|\nabla u|$ in a neighborhood of a point (x, t) is sufficient for the regularity of (x, t). For the precise statement we refer to Talk 7.

In our computations, we will assume without loss of generality that $(x, t) = (0, 0)$. Another assumption that we make for the sake of simiplicity is that the force vanishes, i.e. $f \equiv 0$. This is less general than the situation in **[CKN82]**, but it makes computations less lengthy and important concepts more obvious.

For the entire talk we set $Q_r^* := B_r(0) \times \left(-\frac{7}{8}r^2, \frac{1}{8}r^2\right)$, where the ball $B_r := B_r(0) \subset \mathbb{R}^3$ denotes a ball formed only in the x−variables. We leave out the integration measures in each integral as the integration set will always indicate clearly, whether the integral is over x or over t or even in both.

In Talk 7, Proposition 2 is shown once we accept Proposition 3, which will be proved in this talk as Proposition 10.7. We first recall some important quantities from Talk 7, which also have analogues in Section 3, cf. Talk 5.

Definition 10.1 (Some quantities, cf. Talk 7). Let u be a suitable weak solution of the Navier Stokes equations with $f \equiv 0$. With our fixed notation from above, we can define the following quantities

$$
\begin{aligned}
G_*(r) &:= \frac{1}{r^2} \int_{Q_r^*} |u|^3, \\
H_*(r) &:= \frac{1}{r^2} \int_{Q_r^*} |u| \left| |u|^2 - \fint_{B_r} |u|^2 \right|, \\
J_*(r) &:= \frac{1}{r^2} \int_{Q_r^*} |u||p|, \\
K_*(r) &:= \frac{1}{r^{\frac{13}{4}}} \int_{-\frac{7r^2}{8}}^{\frac{r^2}{8}} \left(\int_{B_r} |p| \right)^{\frac{5}{4}}, \\
M_*(r) &:= G_*^{\frac{2}{3}}(r) + H_*(r) + J_*(r) + K_*^{\frac{8}{5}}(r), \\
\delta_*(r) &:= \frac{1}{r} \int_{Q_r*} |\nabla u|^2 \\
A_*(r) &:= \sup_{t \in [-\frac{7r^2}{8}, \frac{r^2}{8}]} \int_{B_r} |u|^2.
\end{aligned}
$$

REMARK 10.2. Proposition 2 in **[CKN82]** states that there is some $\epsilon_3 > 0$ such that the condition that $\limsup_{r\to 0} \delta_*(r) \le \epsilon_3$ is sufficient for regularity of $(0,0)$. If we imagine the condition to be satisfied we can think of δ_* as a small quantity. The estimates to come seem more natural once one keeps this in mind.

REMARK 10.3. To begin with, it may be unclear whether these quantities are finite. We refer to Remark 1 in Talk 7, where arguments for the finiteness of all quantities except for J_* and M_* are given. Finiteness of J_* will follow from Lemma 10.19, which will be proved in the sequel. Given this, one can easily infer that M_* is finite as a sum of finite quantities. One has to say that another method to deduce the finiteness of J_* is to use the integrability results on the top of page 783 in **[CKN82]** and Hölder's inequality. This computation is recommended as an exercise but not very insightful for our talk, since we cannot obtain appropriate control of J_* this way.

REMARK 10.4. Smallness of $M_*(r)$ implies regularity of u in $Q^*_{\frac{r}{2}}$ by Proposition 1 of **[CKN82]**, cf. Talk 6. The strategy in the proof of Proposition 2 is therefore to show smallness of $M_*(r)$ for some sufficiently small $r > 0$ and apply Proposition 1.

Proposition 1 however requires actually a little bit less than the smallness of $M_*(r)$: It is enough if $G_*(r), J_*(r)$ and $K_*(r)$ are small, so no condition on $H_*(r)$ needs to be imposed.

This raises the question, whether H_* is actually needed in the definition of M_*, as no control of it is required for the regularity of $(0,0)$. We will justify its appearance during the proof.

REMARK 10.5. The quantity $A_*(r)$ seems to be unimportant for the proof of Proposition 2, since $M_*(r)$ does not contain it explicitly. It will however turn out to be of paramount importance since it behaves comparably to M_*. We can profit from this comparision since A_* is a quantity which is easier to handle than M_*.

We have seen part of this comparision result already in Talk 7, where Lukas prensented the inequality

$$H_*(r) \le C(G_*(r)^{\frac{2}{3}} + A_*(r)\delta_*(r)).$$

In a similar way, more quantities will be controllable by A_* and δ_*. We have already discussed in Remark 10.2 that control by δ_* is desirable. That control of quantities by A_* is also desirable will become clear when we observe an "interaction" between A_* and M_* in Lemma 10.9 and afterwards.

REMARK 10.6. The inequality we intend to prove is useful because it enables us to compare the values of M_* for different radii r and ρ. During this comparision process we will often use some obvious estimates for $r \le \rho$, for example

$$\delta_*(r) \le \frac{\rho}{r}\delta_*(\rho). \tag{10.1}$$

Indeed, this is easy to prove:

$$r\delta_*(r) = \int_{Q^*_r} |\nabla u|^2 \le \int_{Q^*_\rho} |\nabla u|^2 \le \rho\delta_*(\rho). \tag{10.2}$$

Later, the comparision with the half radius will be of particular importance, i.e. $\delta_*(\frac{\rho}{2}) \le 2\delta_*(\rho)$, which follows immediately from (10.1). Similar inequalities can be proved following the lines of (10.2) for other quantities. Let us point out one more such estimate:

$$K_*(r) \le \left(\frac{\rho}{r}\right)^{\frac{13}{4}} K_*(\rho),$$

i.e. $K_*(\frac{\rho}{2}) \le 2^{\frac{13}{4}} K_*(\rho)$. Deriving such inqualities for all given quantities we can infer that $M_*(\frac{\rho}{2}) \le CM_*(\rho)$ for some $C > 0$ independent of ρ. This will become important later.

10.2. Statement of Proposition 3

Just like in Talk 7, we state the version of Proposition 3 that we are going to prove:

MAIN PROPOSITION 10.7 (Proposition 3 in [**CKN82**] with vanishing force). Let $\rho > 0$ and let (u,p) be a suitable weak solution of the Navier Stokes System on Q^*_ρ with vanishing force $f \equiv 0$. If $\delta_*(\rho) \le 1$ then there exists a constant $C > 0$ such that

$$M_*(r) \le C\left[\left(\frac{r}{\rho}\right)^{\frac{1}{5}} M_*(\rho) + \left(\frac{\rho}{r}\right)^2 \left(M_*^{\frac{1}{2}}(\rho)\delta_*^{\frac{1}{2}}(\rho) + M_*(\rho)\delta_*(\rho) + \delta_*(\rho)\right)\right], \tag{10.3}$$

for all $r \in \left(0, \frac{\rho}{4}\right)$.

The strategy of the proof will be the following: The structure of the equation can be used to relate the growth of the quantity M_* to the quantity A_*, which controls again all the quantities that contribute to M_*, possibly for a different radius.

REMARK 10.8. In the case of $f \equiv 0$ (which is the only case we consider), we can acutally show an easier inequality, namely

$$M_*(r) \le C\left[\left(\frac{r}{\rho}\right)^{\frac{1}{5}} M_*(\rho) + \left(\frac{\rho}{r}\right)^2 \left(M_*^{\frac{1}{2}}(\rho)\delta_*^{\frac{1}{2}}(\rho) + M_*(\rho)\delta_*(\rho)\right)\right], \quad \forall r \in \left(0, \tfrac{\rho}{4}\right).$$

10.3. M_* and A_* interact because of the energy inequality

The energy inequality gives us an important relation between A_* and M_*, which we will prove now:

LEMMA 10.9 (cf. Lemma 5.5 in [**CKN82**]). *There exists a constant $C_1 > 0$ such that for all $r \in (0, \frac{1}{2}\rho]$ one has*

$$A_*(r) \le C_1 \left(\frac{\rho}{r}\right)\left(G_*^{\frac{2}{3}}(\rho) + H_*(\rho) + J_*(\rho)\right).$$

In particular,

$$A_*(r) \le C_1 \left(\frac{\rho}{r}\right) M_*(\rho) \quad \forall r \in (0, \tfrac{1}{2}\rho]. \tag{10.4}$$

PROOF. In this proof, C denotes a generic constant that can be chosen such that all the estimates are true. We use equation (2.17) in [**CKN82**] substantially, which is a slight improvment on the energy inequality. The equation reads as follows: If (u,p) is a suitable weak solution on a domain $\Omega \times (a,b)$ then each nonnegative $\phi \in C_0^\infty(\Omega \times (a,b))$ satisfies

$$\int_{\Omega\times\{t\}} |u|^2\phi + 2\int_{\Omega\times(a,t)} |\nabla u|^2 \phi \le \int_{\Omega\times(a,t)} |u|^2(\phi_t + \Delta\phi) + (|u|^2 + 2p)u\cdot\nabla\phi \quad \forall t \in (a,b).$$

In our case we will choose $\Omega = B_\rho(0)$ as well as $a = -\frac{7}{8}\rho^2$ and $b = \frac{1}{8}\rho^2$. Note that $\Omega \times (a,b) = Q^*_\rho$. Choose as well $\phi \in C_0^\infty(Q^*_\rho)$ such that $0 \le \phi \le 1$ and $\phi \equiv 1$ on Q^*_r. Moreover we can require that $|\nabla\phi| \le \frac{C}{\rho}$ and $|\phi_t| + |\Delta\phi| \le \frac{C}{\rho^2}$, see (3.8) in [**CKN82**]. Now observe for arbitrary but fixed $t \in (-\frac{7}{8}r^2, \frac{1}{8}r^2)$ that

$$\begin{aligned}\int_{B_r\times\{t\}} |u|^2 \le \int_{B_\rho\times\{t\}} |u|^2\phi &\le \int_{B_\rho\times\{t\}} |u|^2\phi + 2\int_{B_\rho\times(-\frac{7}{8}\rho^2,t)} |\nabla u|^2\phi \\ &\le \int_{B_\rho\times(-\frac{7}{8}\rho^2,t)} |u|^2(\phi_t + \Delta\phi) + \int_{B_\rho\times(-\frac{7}{8}\rho^2,t)} (|u|^2 + 2p)u\cdot\nabla\phi.\end{aligned}$$

Now note that by (weak) divergence-freeness of u one has

$$\int_{B_\rho\times(-\frac{7}{8}\rho^2,t)} \left(⨍_{B_\rho} |u|^2\right) u\cdot\nabla\phi = \int_{-\frac{7}{8}\rho^2}^{t}\int_{B_\rho}\left(⨍_{B_\rho}|u|^2\right) u\cdot\nabla\phi \tag{10.5}$$

$$= \int_{-\frac{7}{8}\rho^2}^{t} \left(\fint_{B_\rho} |u|^2 \right) \underbrace{\int_{B_\rho} u \cdot \nabla\phi}_{=0\; a.e.} = 0.$$

Hence we can insert this term into the equation above to find

$$\begin{aligned}
\int_{B_r\times\{t\}} |u|^2 &\le \int_{B_\rho\times(-\frac{7}{8}\rho^2,t)} |u|^2(\phi_t + \Delta\phi) + \int_{B_\rho\times(-\frac{7}{8}\rho^2,t)} \left(|u|^2 - \fint_{B_\rho} |u|^2\right) u\cdot\nabla\phi \\
&\quad + 2\int_{B_\rho\times(-\frac{7}{8}\rho^2,t)} pu\cdot\nabla\phi \\
&\le \int_{B_\rho\times(-\frac{7}{8}\rho^2,t)} |u|^2(|\phi_t| + |\Delta\phi|) + \int_{B_\rho\times(-\frac{7}{8}\rho^2,t)} \left||u|^2 - \fint_{B_\rho}|u|^2\right| |u|\,|\nabla\phi| \\
&\quad + 2\int_{B_\rho\times(-\frac{7}{8}\rho^2,t)} |p|\,|u|\,|\nabla\phi| \\
&\le \int_{Q_\rho^*} |u|^2(|\phi_t| + |\Delta\phi|) + \int_{Q_\rho^*} \left||u|^2 - \fint_{B_\rho}|u|^2\right| |u|\,|\nabla\phi| + 2\int_{Q_\rho^*} |p|\,|u|\,|\nabla\phi|.
\end{aligned}$$

Using the estimates for ϕ and its derivatives we obtain

$$\begin{aligned}
\int_{B_r\times\{t\}} |u|^2 &\le \frac{C}{\rho^2}\int_{Q_\rho^*} |u|^2 + \frac{C}{\rho}\int_{Q_\rho^*} \left(|u|^2 - \fint_{B_\rho}|u|^2\right)|u| + 2\frac{C}{\rho}\int_{Q_\rho} |p|\,|u| \\
&= \frac{C}{\rho^2}\int_{Q_\rho^*} |u|^2 + \rho H_*(\rho) + \rho J_*(\rho).
\end{aligned} \tag{10.6}$$

Using Hölder's inequality with $q = \frac{3}{2}, q' = 3$ in the first term, we can estimate

$$\begin{aligned}
\frac{1}{\rho^2}\int_{Q_\rho^*} |u|^2 &\le C\frac{1}{\rho^2}\rho^{\frac{5}{3}}\left(\int_{Q_\rho^*}|u|^3\right)^{\frac{2}{3}} = C\frac{1}{\rho^{\frac{1}{3}}}\left(\int_{Q_\rho^*}|u|^3\right)^{\frac{2}{3}} = C\frac{1}{\rho^{\frac{1}{3}}}\rho^{\frac{4}{3}}\left(\frac{1}{\rho^2}\int_{Q_\rho^*}|u|^3\right)^{\frac{2}{3}} \\
&= C\rho G_*^{\frac{2}{3}}(\rho).
\end{aligned}$$

Plugging this into (10.6) we find

$$\int_{B_r\times\{t\}} |u|^2 \le C\rho(G_*^{\frac{2}{3}}(\rho) + J_*(\rho) + H_*(\rho)) \quad \forall t \in (-\tfrac{7}{8}r^2, \tfrac{1}{8}r^2).$$

Dividing by r and taking the supremum over all $t \in (-\frac{7}{8}r^2, \frac{1}{8}r^2)$ we obtain the claim. □

REMARK 10.10. The above computation reveals the reason why a summand in M_* is $G_*^{2/3}$ and not G_* to any other power. This differs from section 3. Moreover the power of $\frac{2}{3}$ is really needed, since higher powers of G_* in this estimate would lead to higher powers of $M_*(\rho)$ in the right hand side of (10.3) - at least if we can only use (10.8) to estimate G_*.

REMARK 10.11. Without the trick in (10.5) we would not be able to bring H_* into play and therefore there would be no hope to control the third-power-of-u term with $G_*^{2/3}$ or with J_* (look at the scaling properties!). Hence H_* is really needed for the inequality we just proved.

REMARK 10.12. An important special case is again $r = \frac{\rho}{2}$ for which one can deduce that there exists $C > 0$ such that

$$A_*(\tfrac{\rho}{2}) \le CM_*(\rho). \tag{10.7}$$

The main task for the rest of this section is to bound the quantities G_*, J_*, H_* and K_* in terms of M_* so that we get a converse inequality that bounds $M_*(r)$ in terms of $A_*(\rho)$, $\delta_*(\rho)$. H_* has already been bounded in Talk 7, see Remark 10.5. Before we bound the other quantities we state a general and recurrent proposition, which is a refinement of Sobolev's inequality, stating that for one certain exponent, the constant in the Sobolev inequality does not depend on the domain.

PROPOSITION 10.13 (Essentially Section 5.6.1. in [**EG92**]). *Let $\Omega \subset \mathbb{R}^n$ be a C^1-smooth domain or $\Omega = \mathbb{R}^n$. Then there exists a constant $D = D(n) > 0$ such that for each $f \in W^{1,1}(\Omega)$ one has*

$$\left(\int_{B_r(x)} \left| f - \fint_{B_r(x)} f \right|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leq D \int_{B_r(x)} |\nabla f| \quad \forall x \in \Omega, r > 0 : B_r(x) \subset \Omega.$$

REMARK 10.14. The fact that the constant D in the previous Proposition does not depend on Ω becomes clear once one proves the inequality for $f \in W^{1,1}(\mathbb{R}^n)$ and argues with the extension operator.

LEMMA 10.15 (Bounding G_*, cf. Lemma 5.2 in [**CKN82**]). *Suppose that $r \leq \rho$. Then we have*

$$G_*(r) \leq C_2 \left\{ \left(\frac{r}{\rho}\right)^3 A_*^{\frac{3}{2}}(\rho) + \left(\frac{\rho}{r}\right)^3 A_*^{\frac{3}{4}}(\rho) \delta_*^{\frac{3}{4}}(\rho) \right\}. \tag{10.8}$$

PROOF. In this proof, C will again be used as a generic constant that can be determined such that all estimates below are true. First of all recall equation (2.9) of [**CKN82**], which is a Sobolev inequality with explicit embeding constants in three dimensions. It reads:

$$\int_{B_r} |u|^q \leq C \left(\int_{B_r} |\nabla u|^2\right)^a \left(\int_{B_r} |u|^2\right)^{\frac{q}{2}-a} + \frac{C}{r^{2a}} \left(\int_{B_r} |u|^2\right)^{\frac{q}{2}}, \tag{10.9}$$

for each $q \in [2, 6]$ and $a = \frac{3}{4}(q-2)$. This connects for example the L^3-norms of u (which are relevant for G_*) to δ_* and the L^2-norms of u (which are crucial to compute A_*). This explains why $G_*(r)$ can be bounded by $A_*(r)$ and $\delta_*(r)$ and gives the desired inequality in the special case $r = \rho$. We however want to make a transition between different radii. For this we can use the following insightful estimate, employing the average integral and the Sobolev inequality in Proposition 10.13. For a fixed time t we can compute

$$\begin{aligned}
\int_{B_r} |u|^2 &= \int_{B_r} \left(|u|^2 - \fint_{B_\rho} |u|^2\right) + \int_{B_r} \fint_{B_\rho} |u|^2 \leq \int_{B_r} \left| |u|^2 - \fint_{B_\rho} |u|^2 \right| + C \left(\frac{r}{\rho}\right)^3 \int_{B_\rho} |u|^2 \\
&\leq \int_{B_\rho} \left| |u|^2 - \fint_{B_\rho} |u|^2 \right| + C \left(\frac{r}{\rho}\right)^3 \int_{B_\rho} |u|^2 \\
&\underset{\text{Hölder}}{\leq} C(\rho^3)^{\frac{1}{3}} \left(\int_{B_\rho} \left| |u|^2 - \fint_{B_\rho} |u|^2 \right|^{\frac{3}{2}}\right)^{\frac{2}{3}} + C \left(\frac{r}{\rho}\right)^3 \int_{B_\rho} |u|^2 \\
&\underset{\text{Prop.10.13}}{\leq} C\rho \int_{B_\rho} |\nabla |u|^2| + C \left(\frac{r}{\rho}\right)^3 \int_{B_\rho} |u|^2 \\
&\leq C\rho \int_{B_\rho} |u||\nabla u| + C \left(\frac{r}{\rho}\right)^3 \int_{B_\rho} |u|^2 \\
&\leq C\rho \left(\int_{B_\rho} |u|^2\right)^{\frac{1}{2}} \left(\int_{B_\rho} |\nabla u|^2\right)^{\frac{1}{2}} + C \left(\frac{r}{\rho}\right)^3 \int_{B_\rho} |u|^2.
\end{aligned}$$

Further, we estimate some terms with A_* to obtain

$$\int_{B_r} |u|^2 \leq C\rho^{\frac{3}{2}} A_*(\rho)^{\frac{1}{2}} \left(\int_{B_\rho} |\nabla u|^2 \right)^{\frac{1}{2}} + C\frac{r^3}{\rho^2} A_*(\rho). \tag{10.10}$$

This gives us an estimate for the L^2-norm of u on B_r. Using (10.9) with $q = 3$ (which implies $a = \frac{3}{4}$) we find

$$\begin{aligned}
\int_{B_r} |u|^3 &\leq C \left(\int_{B_r} |\nabla u|^2 \right)^{\frac{3}{4}} \left(\int_{B_r} |u|^2 \right)^{\frac{3}{4}} + \frac{C}{r^{\frac{3}{2}}} \left(\int_{B_r} |u|^2 \right)^{\frac{3}{2}} \\
&\leq C \left(\int_{B_r} |\nabla u|^2 \right)^{\frac{3}{4}} \left(\int_{B_\rho} |u|^2 \right)^{\frac{3}{4}} + \frac{C}{r^{\frac{3}{2}}} \left(\int_{B_r} |u|^2 \right)^{\frac{3}{2}} \\
&\leq C\rho^{\frac{3}{4}} A_*(\rho)^{\frac{3}{4}} \left(\int_{B_r} |\nabla u|^2 \right)^{\frac{3}{4}} + \frac{C}{r^{\frac{3}{2}}} \left(\int_{B_r} |u|^2 \right)^{\frac{3}{2}} \\
&\leq C\rho^{\frac{3}{4}} A_*(\rho)^{\frac{3}{4}} \left(\int_{B_\rho} |\nabla u|^2 \right)^{\frac{3}{4}} + \frac{C}{r^{\frac{3}{2}}} \left(\int_{B_\rho} |u|^2 \right)^{\frac{3}{2}}
\end{aligned}$$

We can use (10.10) and the fact that for nonnegative a, b the expression $(a+b)^{\frac{3}{2}}$ is bounded by a constant multiple of $a^{\frac{3}{2}} + b^{\frac{3}{2}}$ to estimate

$$\begin{aligned}
\int_{B_r} |u|^3 &\leq C\rho^{\frac{3}{4}} A_*(\rho)^{\frac{3}{4}} \left(\int_{B_r} |\nabla u|^2 \right)^{\frac{3}{4}} + \frac{C}{r^{\frac{3}{2}}} \left(C\rho^{\frac{3}{2}} A_*(\rho)^{\frac{1}{2}} \left(\int_{B_\rho} |\nabla u|^2 \right)^{\frac{1}{2}} + C\frac{r^3}{\rho^2} A_*(\rho) \right)^{\frac{3}{2}} \\
&\leq C\rho^{\frac{3}{4}} A_*(\rho)^{\frac{3}{4}} \left(\int_{B_\rho} |\nabla u|^2 \right)^{\frac{3}{4}} + C\frac{\rho^{\frac{9}{4}}}{r^{\frac{3}{2}}} A_*(\rho)^{\frac{3}{4}} \left(\int_{B_\rho} |\nabla u|^2 \right)^{\frac{3}{4}} + C\left(\frac{r}{\rho}\right)^3 A_*(\rho)^{\frac{3}{2}}.
\end{aligned}$$

Integrating over $t \in (-\frac{7}{8}r^2, \frac{1}{8}r^2)$ and using Hölder's inequality with $q = \frac{3}{4}, q' = \frac{1}{4}$ we find that

$$\begin{aligned}
\int_{Q_r^*} |u|^3 &\leq C\left(\rho^{\frac{3}{4}} + \frac{\rho^{\frac{9}{4}}}{r^{\frac{3}{2}}}\right) A_*(\rho)^{\frac{3}{4}} \int_{-\frac{7}{8}r^2}^{\frac{1}{8}r^2} \left(\int_{B_\rho} |\nabla u|^2 \right)^{\frac{3}{4}} + Cr^2 \left(\frac{r}{\rho}\right)^3 A_*(\rho)^{\frac{3}{2}} \\
&\leq C\left(\rho^{\frac{3}{4}} + \frac{\rho^{\frac{9}{4}}}{r^{\frac{3}{2}}}\right) A_*(\rho)^{\frac{3}{4}} (r^2)^{\frac{1}{4}} \left(\int_{-\frac{7}{8}r^2}^{\frac{1}{8}r^2} \int_{B_\rho} |\nabla u|^2 \right)^{\frac{3}{4}} + Cr^2 \left(\frac{r}{\rho}\right)^3 A_*(\rho)^{\frac{3}{2}} \\
&\leq C\left(\rho^{\frac{3}{4}} + \frac{\rho^{\frac{9}{4}}}{r^{\frac{3}{2}}}\right) A_*(\rho)^{\frac{3}{4}} r^{\frac{1}{2}} \left(\int_{-\frac{7}{8}\rho^2}^{\frac{1}{8}\rho^2} \int_{B_\rho} |\nabla u|^2 \right)^{\frac{3}{4}} + Cr^2 \left(\frac{r}{\rho}\right)^3 A_*(\rho)^{\frac{3}{2}} \\
&\leq C\left(\rho^{\frac{3}{4}} + \frac{\rho^{\frac{9}{4}}}{r^{\frac{3}{2}}}\right) A_*(\rho)^{\frac{3}{4}} r^{\frac{1}{2}} \left(\int_{Q_\rho^*} |\nabla u|^2 \right)^{\frac{3}{4}} + Cr^2 \left(\frac{r}{\rho}\right)^3 A_*(\rho)^{\frac{3}{2}} \\
&\leq C\left(\rho^{\frac{3}{2}} + \frac{\rho^3}{r^{\frac{3}{2}}}\right) r^{\frac{1}{2}} A_*(\rho)^{\frac{3}{4}} \delta_*(\rho)^{\frac{3}{4}} + Cr^2 \left(\frac{r}{\rho}\right)^3 A_*(\rho)^{\frac{3}{2}}.
\end{aligned}$$

Dividing by r^2 we finally obtain

$$G_*(r) \leq C\left(\left(\frac{\rho}{r}\right)^{\frac{3}{2}} + \left(\frac{\rho}{r}\right)^3 \right) A_*(\rho)^{\frac{3}{4}} \delta_*(\rho)^{\frac{3}{4}} + C\left(\frac{r}{\rho}\right)^3 A_*(\rho)^{\frac{3}{2}}.$$

Due to the fact that $r \leq \rho$ we can estimate $\left(\frac{\rho}{r}\right)^{\frac{3}{2}} \leq \left(\frac{\rho}{r}\right)^3$ and conclude the claim. $\square$

Before we can bound J_* we prove some useful estimates on the pressure, which can be deduced with the following splitting technique.

PROPOSITION 10.16 (Splitting Technique for the pressure, cf. p. 801 in [**CKN82**]). *Suppose that $\rho > 0$ and $\phi \in C_0^\infty(B_\rho)$ is such that $0 \le \phi \le 1$ and $\phi \equiv 1$ in $B_{\frac{3}{4}\rho}$ as well as $|\nabla\phi| \le \frac{C}{\rho}$ and $|\Delta\phi| \le \frac{C}{\rho^2}$ for some $C > 0$. Then for all $x \in B_{\frac{3}{4}\rho}$ and $t \in (0,T)$ one has*

$$p(x,t) = p_4(x,t) + p_5(x,t)$$

where

$$p_4(x,t) = \frac{3}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|x-y|} p(y,t)\Delta\phi(y)\,\mathrm{d}y + \frac{3}{2\pi}\int_{\mathbb{R}^3} \sum_{i=1}^3 \frac{x_i - y_i}{|x-y|^3}\partial_i\phi(y)p(y,t)\,\mathrm{d}y$$

and

$$p_5(x,t) = \frac{3}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|x-y|}\phi(y)\sum_{i,j=1}^3 \partial_{y_i}u^j(y,t)\partial_{y_j}u^i(y,t)\,\mathrm{d}y.$$

Moreover there exists a constant $C_3 > 0$ such that

$$|p_4(x,t)| \le C_3 \fint_{B_\rho} |p| \quad \forall x \in B_{\frac{\rho}{2}} \tag{10.11}$$

and

$$\int_{B_r} |p_5|^2 \le C_3\rho\left(\int_{B_\rho} |\nabla u|^2\right)^2 \quad \forall r \in (0, \tfrac{\rho}{2}]. \tag{10.12}$$

PROOF. Let ϕ be as in the statement. Recall that the fundamental solution of the Poisson equation is given by $k(z) := -\frac{3}{4\pi}\frac{1}{|z|}$. In the following we will leave out the t-argument. Moreover, integrals without a specified set are always over $\mathbb{R}^3$. In the following we will make extensive use of equation (2.12) in [**CKN82**] which reads

$$\Delta p = -\sum_{i,j=1}^3 \partial_i u^j \partial_j u^i, \tag{10.13}$$

in the sense of distributions. For the first we will assume, that p is a smooth function on B_ρ and (10.13) holds pointwise. We have to get rid of this assumption later. This assumption is restrictive but can be gotten rid of, as wee shall discuss in Proposition 10.17. With the fundamental solution property we infer for $x \in B_{\frac{\rho}{2}}$

$$\begin{aligned} p(x)\phi(x) &= -\frac{3}{4\pi}\int \frac{1}{|x-y|}\Delta_y(\phi p)\,\mathrm{d}y \\ &= -\frac{3}{4\pi}\int \frac{1}{|x-y|}(p\Delta\phi + 2(\nabla\phi, \nabla p) + \phi\Delta p)\,\mathrm{d}y. \end{aligned} \tag{10.14}$$

Now we split the integral into three summands and integrate by parts in the second one, more precisely we compute

$$\begin{aligned} -\frac{6}{4\pi}\int \frac{1}{|x-y|}(\nabla\phi, \nabla p)\,\mathrm{d}y &= -\sum_{i=1}^3 \frac{6}{4\pi}\int \frac{1}{|x-y|}\partial_{y_i}\phi\partial_{y_i}p\,\mathrm{d}y \\ &= \sum_{i=1}^3 \frac{6}{4\pi}\int \partial_{y_i}\left(\frac{1}{|x-y|}\partial_{y_i}\phi\right)p\,\mathrm{d}y \\ &= \frac{3}{2\pi}\int p\sum_{i=1}^3 \frac{x_i - y_i}{|x-y|^3}\partial_{y_i}\phi\,\mathrm{d}y + \frac{3}{2\pi}\int p\sum_{i=1}^3 \frac{1}{|x-y|}\partial^2_{y_iy_i}\phi\,\mathrm{d}y \\ &= \frac{3}{2\pi}\int p\sum_{i=1}^3 \frac{x_i - y_i}{|x-y|^3}\partial_{y_i}\phi\,\mathrm{d}y + \frac{3}{2\pi}\int \frac{1}{|x-y|}p\Delta\phi\,\mathrm{d}y. \end{aligned}$$

Plugging this back into (10.14) and using (2.12) in [**CKN82**] we obtain

$$\begin{aligned}
p(x)\phi(x) &= \Big(-\frac{3}{4\pi}+\frac{3}{2\pi}\Big)\int \frac{1}{|x-y|}p\Delta\phi \,\mathrm{d}y + \frac{3}{2\pi}\int p\sum_{i=1}^{3}\frac{x_i-y_i}{|x-y|^3}\partial_{y_i}\phi \,\mathrm{d}y \\
&\quad -\frac{3}{4\pi}\int \frac{1}{|x-y|}\phi\Delta p \mathrm{d}y \\
&= \frac{3}{4\pi}\int \frac{1}{|x-y|}p\Delta\phi \,\mathrm{d}y + \frac{3}{2\pi}\int p\sum_{i=1}^{3}\frac{x_i-y_i}{|x-y|^3}\partial_{y_i}\phi \,\mathrm{d}y \\
&\quad +\frac{3}{4\pi}\int \frac{1}{|x-y|}\phi\sum_{i,j=1}^{3}\partial_i u^j\partial_j u^i \,\mathrm{d}y.
\end{aligned}$$

If $x\in B_{\frac{3}{4}\rho}$ then $\phi(x)=1$ and therefore we can infer the first sentence of the claim. For the pointwise estimate on p_4 in $B_{\frac{\rho}{2}}$ let $x\in B_{\frac{\rho}{2}}$ be arbitrary but fixed. We can estimate with the triangle inequality

$$|p_4(x)|\le \left|\frac{3}{4\pi}\int \frac{1}{|x-y|}p(y)\Delta\phi(y)\,\mathrm{d}y\right| + \left|\frac{3}{2\pi}\int \frac{x_i-y_i}{|x-y|^3}p(y)\partial_i\phi(y)\,\mathrm{d}y\right|. \tag{10.15}$$

Notice that $\nabla\phi\equiv 0, \Delta\phi\equiv 0$ on $B_{\frac{3}{4}\rho}$ since $\phi\equiv 1$ on $B_{\frac{3}{4}\rho}$. For the first summand we can estimate, using the properties of ϕ mentioned in the statement as well as the inverse triangle inequality

$$\begin{aligned}
\left|\frac{3}{4\pi}\int \frac{1}{|x-y|}p(y)\Delta\phi(y)\,\mathrm{d}y\right| &\le \left|\frac{3}{4\pi}\int_{B_\rho\setminus B_{\frac{3}{4}\rho}} \frac{1}{|x-y|}p(y)\Delta\phi(y)\,\mathrm{d}y\right| \\
&\le \int_{B_\rho\setminus B_{\frac{3}{4}\rho}} \frac{1}{|x-y|}|p(y)||\Delta\phi(y)|\,\mathrm{d}y \\
&\le \frac{C}{\rho^2}\frac{3}{4\pi}\int_{B_\rho\setminus B_{\frac{3}{4}\rho}} \frac{1}{|x-y|}|p(y)|\,\mathrm{d}y \\
&\le \frac{C}{\rho^2}\frac{3}{4\pi}\int_{B_\rho\setminus B_{\frac{3}{4}\rho}} \frac{1}{|y|-|x|}|p(y)|\,\mathrm{d}y \\
&\le \frac{C}{\rho^2}\frac{3}{4\pi}\int_{B_\rho\setminus B_{\frac{3}{4}\rho}} \frac{1}{\frac{3}{4}\rho-\frac{1}{2}\rho}|p(y)|\,\mathrm{d}y \\
&\le \frac{3C}{\pi\rho^3}\int_{B_\rho}|p(y)|\,\mathrm{d}y \le \frac{C_3}{2}⨍_{B_\rho}|p(y)|\,\mathrm{d}y,
\end{aligned}$$

for an appropriate choice of C_3. To estimate the second summand we use that $|x_i-y_i|\le|x-y|$ and otherwise the same techniques as above.

$$\begin{aligned}
\left|\frac{3}{4\pi}\int \frac{x_i-y_i}{|x-y|^3}p(y)\partial_i\phi(y)\,\mathrm{d}y\right| &\le \frac{3}{4\pi}\int_{B_\rho\setminus B_{\frac{3}{4}\rho}} \frac{|x_i-y_i|}{|x-y|^3}|p(y)||\partial_i\phi(y)|\,\mathrm{d}y \\
&\le \frac{3C}{4\pi\rho}\int_{B_\rho\setminus B_{\frac{3}{4}\rho}} \frac{1}{|x-y|^2}|p(y)|\,\mathrm{d}y \\
&\le \frac{3C}{4\pi\rho}\int_{B_\rho\setminus B_{\frac{3}{4}\rho}} \frac{1}{|x-y|^2}|p(y)|\,\mathrm{d}y \\
&\le \frac{3C}{4\pi\rho}\int_{B_\rho\setminus B_{\frac{3}{4}\rho}} \frac{1}{(|y|-|x|)^2}|p(y)|\,\mathrm{d}y
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{3C}{4\pi\rho}\int_{B_\rho\smallsetminus B_{\frac{3}{4}\rho}} \frac{1}{(\frac{3}{4}\rho-\frac{1}{2}\rho)^2}|p(y)|\,\mathrm{d}y\\
&\le \frac{12C}{\pi\rho^3}\int_{B_\rho\smallsetminus B_{\frac{3}{4}\rho}} |p(y)|\,\mathrm{d}y\\
&\le \frac{C_3}{2}\fint_{B_\rho} |p(y)|\,\mathrm{d}y,
\end{aligned}
$$

by possibly increasing C_3. The two previous computations imply the pointwise estimate of p_4 together with (10.15). For the L^2-estimate on p_5 fix $r \le \frac{\rho}{2}$. Estimating all derivatives of u by $|\nabla u|$ we get

$$
\begin{aligned}
\int_{B_r}|p_5|^2 &= \int_{B_r}\left|\frac{3}{4\pi}\int\frac{1}{|x-y|}\phi(y)\sum_{i,j=1}^{3}\partial_i u^j\partial_j u^i\,\mathrm{d}y\right|^2\,\mathrm{d}x\\
&\le \frac{3}{4\pi}\int_{B_r}\left(\int\frac{1}{|x-y|}|\phi(y)|\sum_{i,j=1}^{3}|\partial_i u^j||\partial_j u^i|\,\mathrm{d}y\right)^2\,\mathrm{d}x\\
&\le \frac{243}{4\pi}\int_{B_r}\left(\int\frac{1}{|x-y|}|\phi(y)||\nabla u(y)|^2\,\mathrm{d}y\right)^2\,\mathrm{d}x\\
&= \frac{243}{4\pi}\int_{B_r}\left(\int\frac{1}{|x-y|}\sqrt{|\phi(y)|}|\nabla u(y)|\sqrt{|\phi(y)|}|\nabla u(y)|\,\mathrm{d}y\right)^2\,\mathrm{d}x\\
&\underset{\text{Hölder}}{\le} \frac{243}{4\pi}\int_{B_r}\left(\int\frac{1}{|x-y|^2}|\phi(y)||\nabla u(y)|^2\,\mathrm{d}y\right)\left(\int|\phi(z)||\nabla u(z)|^2\,\mathrm{d}z\right)\,\mathrm{d}x\\
&\le \frac{243}{4\pi}\left(\int|\phi(z)||\nabla u(z)|^2\,\mathrm{d}z\right)\int_{B_r(0)}\int_{B_\rho(0)}\frac{1}{|x-y|^2}|\phi(y)||\nabla u(y)|^2\,\mathrm{d}y\,\mathrm{d}x\\
&\underset{\text{Fubini}}{\le} \frac{243}{4\pi}\left(\int|\phi(z)||\nabla u(z)|^2\,\mathrm{d}z\right)\int_{B_\rho(0)}\left(\int_{B_r(0)}\frac{1}{|x-y|^2}\,\mathrm{d}x\right)|\phi(y)||\nabla u(y)|^2\,\mathrm{d}y.
\end{aligned}
$$

Now note that for $y\in B_\rho$ one has $B_r(0)\subset B_{r+\rho}(y)\subset B_{2\rho}(y)$ and therefore

$$
\begin{aligned}
\int_{B_r}|p_5|^2 &\le \frac{243}{4\pi}\left(\int|\phi(z)||\nabla u(z)|^2\,\mathrm{d}z\right)\int_{B_\rho(0)}\left(\int_{B_{2\rho}(y)}\frac{1}{|x-y|^2}\,\mathrm{d}x\right)|\phi(y)||\nabla u(y)|^2\,\mathrm{d}y\\
&\underset{\text{Subst.}w=x-y}{\le} \frac{243}{4\pi}\left(\int|\phi(z)||\nabla u(z)|^2\,\mathrm{d}z\right)\int_{B_\rho(0)}\left(\int_{B_{2\rho}(0)}\frac{1}{|w|^2}\,\mathrm{d}w\right)|\phi(y)||\nabla u(y)|^2\,\mathrm{d}y\\
&\le \frac{243}{4\pi}\left(\int|\phi(z)||\nabla u(z)|^2\,\mathrm{d}z\right)^2\int_{B_{2\rho}(0)}\frac{1}{|w|^2}\,\mathrm{d}w.
\end{aligned}
$$

By radial integration one has

$$
\int_{B_{2\rho}(0)}\frac{1}{|w|^2}\,\mathrm{d}w = \int_0^{2\rho}(4\pi s^2)\frac{1}{s^2}ds = 8\pi\rho
$$

and hence we can conclude

$$
\int_{B_r}|p_5|^2 \le C\rho\left(\int|\phi(z)||\nabla u(z)|^2\,\mathrm{d}z\right)^2 \le C\rho\left(\int_{B_\rho}|\nabla u(z)|^2\,\mathrm{d}z\right)^2.
$$

Remark 10.17. In the fundamental solution argument in (10.14) we have used the additional assumption that p is smooth in B_ρ, which is not satisfied in general. If p is not smooth on B_ρ we follow the lines of the proof after (10.14), replacing p with $p*\phi_\epsilon$ for fixed

$\epsilon > 0$, where $(\phi_\epsilon)_{\epsilon>0}$ denotes the standard mollifier family. Note that by (10.13)

$$\Delta(p * \phi_\epsilon) = p * \Delta\phi_\epsilon = -\sum_{i,j=1}^{3} (\partial_i u^j \partial_j u^i) * \phi_\epsilon.$$

Using this and the fact that $p * \phi_\epsilon \to p$ almost everywhere by [**EG92**, Theorem 1 (iv), Section 4.2] one can possibly repeat the above computations and pass to the limit as $\epsilon \to 0$.

REMARK 10.18. Possibly one can circumvent adjustments in the previous Remark with a maximal regularity argument for (10.13). For this however, more a-priori regularity of p and higher integrability of derivatives of u have to be shown first (in case that these are actually true).

LEMMA 10.19 (Bounds for J_*, cf. Lemma 5.3 in [**CKN82**]). *For each $r \le \frac{\rho}{2}$ one has*

$$J_*(r) \le C_4 \left\{ \left(\frac{r}{\rho}\right)^{\frac{1}{5}} A_*^{\frac{1}{5}}(\rho) G_*^{\frac{1}{5}}(r) K_*^{\frac{4}{5}}(\rho) + \left(\frac{\rho}{r}\right)^2 A_*^{\frac{1}{2}}(\rho)\delta_*(\rho) \right\}.$$

PROOF. We start using the splitting of p to get

$$J_*(r) = \frac{1}{r^2}\int_{Q_r^*} |u||p| \le \frac{1}{r^2}\int_{Q_r^*} |u||p_4| + \frac{1}{r^2}\int_{B_r} |u||p_5|. \tag{10.16}$$

We estimate both summands seperately, starting with the first one. As usual, we compute for a fixed time $t \in (-\frac{7}{8}r^2, \frac{1}{8}r^2)$ using (10.11)

$$\begin{aligned}
\int_{B_r} |u||p_4| &\le C\left(\int_{B_r}|u|\right)\left(\fint_{B_\rho}|p|\right) \le C\left(\int_{B_r}|u|\right)^{\frac{2}{5}}\left(\int_{B_r}|u|\right)^{\frac{3}{5}}\left(\fint_{B_\rho}|p|\right)\\
&\underset{\text{Hölder}}{\le} Cr^{\frac{3}{5}}\left(\int_{B_r}|u|^2\right)^{\frac{1}{5}}\left(\int_{B_r}|u|\right)^{\frac{3}{5}}\left(\fint_{B_\rho}|p|\right)\\
&\underset{\text{Hölder}}{\le} Cr^{\frac{3}{5}}\left(\int_{B_r}|u|^2\right)^{\frac{1}{5}} r^{\frac{6}{5}}\left(\int_{B_r}|u|^3\right)^{\frac{1}{5}}\left(\fint_{B_\rho}|p|\right)\\
&\le Cr^{\frac{9}{5}}\left(\int_{B_r}|u|^2\right)^{\frac{1}{5}}\left(\int_{B_r}|u|^3\right)^{\frac{1}{5}}\left(\fint_{B_\rho}|p|\right)\\
&\le Cr^{\frac{9}{5}}\left(\int_{B_\rho}|u|^2\right)^{\frac{1}{5}}\left(\int_{B_r}|u|^3\right)^{\frac{1}{5}}\left(\fint_{B_\rho}|p|\right)\\
&\le Cr^{\frac{9}{5}}(\rho A_*(\rho))^{\frac{1}{5}}\left(\int_{B_r}|u|^3\right)^{\frac{1}{5}}\left(\fint_{B_\rho}|p|\right).
\end{aligned}$$

Integrating in time we obtain

$$\begin{aligned}
\int_{Q_r^*}|u||p_4| &\le Cr^{\frac{9}{5}}\rho^{\frac{1}{5}}A_*^{\frac{1}{5}}(\rho)\int_{-\frac{7}{8}r^2}^{\frac{1}{8}r^2}\left(\int_{B_r}|u|^3\right)^{\frac{1}{5}}\left(\fint_{B_\rho}|p|\right)\\
&\le Cr^{\frac{9}{5}}\rho^{\frac{1}{5}}A_*^{\frac{1}{5}}(\rho)\left(\int_{-\frac{7}{8}r^2}^{\frac{1}{8}r^2}\int_{B_r}|u|^3\right)^{\frac{1}{5}}\left(\int_{-\frac{7}{8}r^2}^{\frac{1}{8}r^2}\left(\fint_{B_\rho}|p|\right)^{\frac{5}{4}}\right)^{\frac{4}{5}}\\
&\le Cr^{\frac{9}{5}}\rho^{\frac{1}{5}}A_*^{\frac{1}{5}}(\rho)\left(\int_{Q_r^*}|u|^3\right)^{\frac{1}{5}}\frac{1}{\rho^3}\left(\int_{-\frac{7}{8}r^2}^{\frac{1}{8}r^2}\left(\int_{B_\rho}|p|\right)^{\frac{5}{4}}\right)^{\frac{4}{5}}\\
&\le Cr^{\frac{9}{5}}\rho^{\frac{1}{5}}A_*^{\frac{1}{5}}(\rho)\left(\int_{Q_r^*}|u|^3\right)^{\frac{1}{5}}\frac{1}{\rho^3}\left(\int_{-\frac{7}{8}\rho^2}^{\frac{1}{8}\rho^2}\left(\int_{B_\rho}|p|\right)^{\frac{5}{4}}\right)^{\frac{4}{5}}
\end{aligned}$$

$$
\begin{aligned}
&\le Cr^{\frac{9}{5}}\rho^{\frac{1}{5}}A_*^{\frac{1}{5}}(\rho)\left(\int_{Q_r^*}|u|^3\right)^{\frac{1}{5}}\frac{1}{\rho^3}\left(\rho^{\frac{13}{4}}K_*(\rho)\right)^{\frac{4}{5}}\\
&\le Cr^{\frac{9}{5}}\rho^{\frac{1}{5}}A_*^{\frac{1}{5}}(\rho)\left(\int_{Q_r^*}|u|^3\right)^{\frac{1}{5}}\frac{1}{\rho^{\frac{2}{5}}}K_*(\rho)^{\frac{4}{5}}\\
&\le Cr^{\frac{9}{5}}A_*^{\frac{1}{5}}(\rho)\frac{1}{\rho^{\frac{1}{5}}}(r^2G_*(r))^{\frac{1}{5}}K_*(\rho)^{\frac{4}{5}} = Cr^{\frac{11}{5}}\frac{1}{\rho^{\frac{1}{5}}}A_*^{\frac{1}{5}}(\rho)G_*(r)^{\frac{1}{5}}K_*(\rho)^{\frac{4}{5}}.
\end{aligned}
$$

Dividing by r^2 we conclude

$$
\frac{1}{r^2}\int_{Q_r^*}|u||p_4| \le C\left(\frac{r}{\rho}\right)^{\frac{1}{5}}A_*^{\frac{1}{5}}(\rho)G_*^{\frac{1}{5}}(\rho)K_*^{\frac{4}{5}}(\rho). \tag{10.17}
$$

To estimate the second summand in (10.16) we compute with (10.12)

$$
\begin{aligned}
\int_{B_r}|u||p_5| &\le \left(\int_{B_r}|u|^2\right)^{\frac{1}{2}}\left(\int_{B_r}|p_5|^2\right)^{\frac{1}{2}} \le \left(\int_{B_\rho}|u|^2\right)^{\frac{1}{2}}\rho^{\frac{1}{2}}\int_{B_\rho}|\nabla u|^2\\
&\le C(\rho A_*(\rho))^{\frac{1}{2}}\rho^{\frac{1}{2}}\int_{B_\rho}|\nabla u|^2 = C\rho A_*(\rho)^{\frac{1}{2}}\int_{B_\rho}|\nabla u|^2.
\end{aligned}
$$

Integrating with respect to t we obtain

$$
\int_{Q_r^*}|u||p_5| \le C\rho A_*(\rho)^{\frac{1}{2}}\int_{Q_\rho^*}|\nabla u|^2 = C\rho^2A_*(\rho)^{\frac{1}{2}}\delta_*(\rho).
$$

Dividing by r^2 we obtain

$$
\frac{1}{r^2}\int_{Q_r^*}|u||p_5| \le C\left(\frac{\rho}{r}\right)^2A_*(\rho)^{\frac{1}{2}}\delta_*(\rho).
$$

This and (10.17) yield the claim. □

The proof of the following lemma will most likely be omitted in the talk, since it is somewhat technical. Nevertheless it is highly recommendable to read, since it presents useful refinements of the pressure estimate.

LEMMA 10.20. *[An estimate for K_*, see Lemma 5.4 in [**CKN82**]] If $r \le \frac{1}{2}\rho$ then*

$$
K_*(r) \le C\left\{\left(\frac{r}{\rho}\right)^{\frac{1}{2}}K_*(\rho) + \left(\frac{\rho}{r}\right)^{\frac{5}{4}}A_*^{\frac{5}{8}}(\rho)\delta_*^{\frac{5}{8}}(\rho)\right\}.
$$

Before we can prove this lemma we have to prove another splitting property of the pressure

PROPOSITION 10.21 (Refinement of the pressure splitting and L^1-control of p_5, cf. p. 803 in [**CKN82**]). *Let ρ, ϕ, p_4, p_5 be as in Proposition 10.16. Then p_5 can be split as follows*

$$
p_5(x,t) = p_6(x,t) + p_7(x,t),
$$

where

$$
p_6(x,t) = -\frac{3}{4\pi}\int\sum_{i=1}^{3}\frac{x_i-y_i}{|x-y|^3}\phi(y)(u\cdot\nabla u^i)(y,t)\,\mathrm{d}y,
$$

$$
p_7(x,t) = -\frac{3}{4\pi}\int\frac{1}{|x-y|}\sum_{i=1}^{3}\partial_{y_i}\phi(y)(u\cdot\nabla u^i)(y).
$$

Moreover, for each $r \leq \frac{\rho}{2}$ one has

$$\int_{B_r} |p_5| \leq Cr\rho^{\frac{1}{2}} A_*(\rho)^{\frac{1}{2}} \left(\int_{B_\rho} |\nabla u|^2 \right)^{\frac{1}{2}}. \tag{10.18}$$

PROOF. Just like in the proof of Proposition 10.16, we leave out the t-argument. To simplify the following computation, we assume first that u is smooth in B_r, an assumption which is not at all justified but can be gotten rid of, as we will discuss after the computation. We integrate by parts in the expression for p_5 that we obtained in Proposition 10.16 to obtain

$$\begin{aligned}
p_5(x) &= \frac{3}{4\pi} \int \sum_{i,j=1}^{3} \frac{1}{|x-y|} \phi \partial_{y_j} u^i \partial_{y_i} u^j \, \mathrm{d}y \\
&= -\sum_{i,j=1}^{3} \frac{3}{4\pi} \int \partial_{y_i} \left(\frac{1}{|x-y|} \phi \partial_{y_j} u^i \right) u^j \, \mathrm{d}y. \\
&= -\sum_{i,j=1}^{3} \frac{3}{4\pi} \int \partial_{y_i} \left(\frac{1}{|x-y|} \phi \partial_{y_j} u^i \right) u^j \, \mathrm{d}y \\
&= -\sum_{i,j=1}^{3} \frac{3}{4\pi} \Bigg(\int \frac{x_i - y_i}{|x-y|^3} \phi \partial_{y_j} u^i u^j \, \mathrm{d}y + \int \frac{1}{|x-y|} \partial_{y_i} \phi u^j \partial_{y_j} u^i \, \mathrm{d}y \\
&\qquad\qquad + \int \frac{1}{|x-y|} \phi \partial^2_{y_i y_j} u^i u^j \Bigg) \\
&= p_6(x) + p_7(x) + \sum_{j=1}^{3} \int \frac{1}{|x-y|} u^j \sum_{i=1}^{3} \partial^2_{y_j y_i} u^i,
\end{aligned}$$

where we have rewritten the j−sums as with the dot product in the last step. Now observe that by Schwarz's Lemma (or Clairaut's Theorem)

$$\sum_{i=1}^{3} \partial^2_{y_i y_j} u^i = \partial_{y_j} \sum_{i=1}^{3} \partial_{y_i} u^i = \partial_{y_j} \operatorname{div}(u) = 0,$$

as u was assumed to be divergence-free. This implies that $p_5 = p_6 + p_7$ as claimed. The point where we apply Schwarz's Lemma is however exactly the point where the additional regularity assumption kicks in. We will now briefly comment on how we can overcome the unjustified regularity assumption. In the first step we rewrite

$$\frac{3}{4\pi} \int \sum_{i,j=1}^{3} \frac{1}{|x-y|} \phi \partial_{y_j} u^i \partial_{y_i} u^j \, \mathrm{d}y = \lim_{\epsilon \to 0} \frac{3}{4\pi} \int \sum_{i,j=1}^{3} \frac{1}{|x-y|} \phi (\partial_j u^i * \phi_\epsilon)(y) \partial_{y_i} u^j \, \mathrm{d}y,$$

where $(\phi_\epsilon)_{\epsilon > 0}$ is the standard mollifier family. Following the lines of the proof and using that $u(t, \cdot) \in W^{1,2}(\Omega)$ (which is true at least for almost every t) we obtain

$$p_5(x) = p_6(x) + p_7(x) + \lim_{\epsilon \to 0} \frac{-3}{4\pi} \sum_{j=1}^{3} \int \frac{1}{|x-y|} \phi u^j \sum_{i=1}^{3} \partial_{y_i} (\partial_j u^i * \phi_\epsilon)(y) \, \mathrm{d}y.$$

Now observe that

$$\begin{aligned}
\partial_{y_i} (\partial_j u^i * \phi_\epsilon) &= \partial_{y_i} \int \partial_{z_j} u^i(z) \phi_\epsilon(y-z) dz = -\partial_{y_i} \int u^i(z) \partial_{z_j} \phi_\epsilon(y-z) dz \\
&= -\int u^i(z) \partial_{y_i} \partial_{z_j} [\phi_\epsilon(y-z)] dz \underset{\text{Chain Rule}}{=} \int u^i(z) \partial^2_{z_i z_j} [\phi_\epsilon(y-z)] dz \\
&\underset{\text{Schwarz's Lemma}}{=} \int u^i(z) \partial_{z_i} [\partial_{z_j} \phi_\epsilon(y-z)] dz.
\end{aligned}$$

Summing over i and use the definition of the dot product we obtain

$$\sum_{i=1}^{3} \partial_{y_i}(\partial_j u^i * \phi_\epsilon)(y) = \int u(z) \cdot \nabla \partial_{z_j} \phi_\epsilon(y-z) dz = 0 \quad \forall \epsilon > 0,$$

since u is weakly divergence-free and hence L^2-orthogonal to $\nabla \partial_j \phi_\epsilon(y - \cdot)$ for each $\epsilon > 0$. We have shown the desired decomposition. To show (10.18) we show

$$\int_{B_r} |p_6| \le C r \rho^{\frac{1}{2}} A_*(\rho)^{\frac{1}{2}} \left(\int_{B_\rho} |\nabla u|^2 \right)^{\frac{1}{2}}$$

and

$$\int_{B_r} |p_7| \le C r \rho^{\frac{1}{2}} A_*(\rho)^{\frac{1}{2}} \left(\int_{B_\rho} |\nabla u|^2 \right)^{\frac{1}{2}}.$$

Given the two previous inequalities, the desired estimate follows easily with the triangle inequality. First we obtain the L^1-control for p_6:

$$\begin{aligned}
\int_{B_r} |p_6| &= \int_{B_r} \left| \int_{B_\rho} \frac{3}{4\pi} \sum_{i=1}^{3} \frac{x_i - y_i}{|x-y|^3} \phi(y)(u \cdot \nabla u^i)(y)\, \mathrm{d}y \right| \mathrm{d}x \\
&\le C \int_{B_r} \int_{B_\rho} \frac{1}{|x-y|^2} |\phi(y)|\, |u(y)|\, |\nabla u(y)|\, \mathrm{d}y\, \mathrm{d}x \\
&\underset{\text{Fubini}}{=} C \int_{B_\rho} \left(\int_{B_r} \frac{1}{|x-y|^2} \mathrm{d}x \right) |\phi|\, |u|\, |\nabla u| \\
&= C \int_{B_{2r}} \left(\int_{B_r} \frac{1}{|x-y|^2} \mathrm{d}x \right) |\phi|\, |u|\, |\nabla u| \\
&\quad + C \int_{B_\rho \setminus B_{2r}} \left(\int_{B_r} \frac{1}{|x-y|^2} \mathrm{d}x \right) |\phi|\, |u|\, |\nabla u|.
\end{aligned}$$

We estimate both summands seperately. For the first summand we use that $y \in B_{2r}(0)$ implies $B_r(0) \subset B_{3r}(y)$ and hence

$$\begin{aligned}
\int_{B_{2r}} \left(\int_{B_r} \frac{1}{|x-y|^2} \mathrm{d}x \right) |\phi|\, |u|\, |\nabla u| &\le \int_{B_{2r}} \left(\int_{B_{3r}(y)} \frac{1}{|x-y|^2} \mathrm{d}x \right) |\phi|\, |u|\, |\nabla u| \\
&\underset{z:=x-y}{=} \left(\int_{B_{2r}} |\phi|\, |u|\, |\nabla u| \right) \left(\int_{B_{3r}(0)} \frac{1}{|z|^2}\, \mathrm{d}z \right) \\
&\underset{\text{Radial integration}}{=} \left(\int_{B_{2r}} |\phi|\, |u|\, |\nabla u| \right) \left(\int_0^{3r} (4\pi s^2) \frac{1}{s^2}\, \mathrm{d}s \right) \\
&\le 12\pi r \left(\int_{B_{2r}} |\phi|\, |u|\, |\nabla u| \right).
\end{aligned} \tag{10.19}$$

For the other integral we use the inverse triangle inequality to estimate for $x \in B_r(0)$ and $|y| \ge 2r$

$$\frac{1}{|x-y|^2} \le \frac{1}{(|y|-|x|)^2} \le \frac{1}{(2r-r)^2} \le \cdot \frac{1}{r^2}$$

Therefore

$$\begin{aligned}
\int_{B_\rho \setminus B_{2r}} \left(\int_{B_r} \frac{1}{|x-y|^2} \mathrm{d}x \right) |\phi|\, |u|\, |\nabla u| &\le \int_{B_\rho \setminus B_{2r}} \frac{1}{r^2} |B_r(0)|\, |\phi|\, |u|\, |\nabla u| \\
&= \frac{4\pi}{3} r \int_{B_\rho \setminus B_{2r}} |\phi|\, |u|\, |\nabla u|,
\end{aligned}$$

where we have used that $|B_r(0)| = \frac{4}{3}\pi r^3$ is the volume of $B_r(0)$. Plugging both considerations back into (10.19) we obtain

$$\begin{aligned}\int_{B_r} |p_6| &\le Cr \int_{B_\rho} |\phi|\,|u|\,|\nabla u| \le Cr \int_{B_\rho} |u|\,|\nabla u| \le Cr \left(\int_{B_\rho} |u|^2\right)^{\frac{1}{2}} \left(\int_{B_\rho} |\nabla u|^2\right)^{\frac{1}{2}} \\ &\le Cr\rho^{\frac{1}{2}} A_*^{\frac{1}{2}}(\rho) \left(\int_{B_\rho} |\nabla u|^2\right)^{\frac{1}{2}}, \end{aligned} \tag{10.20}$$

which is the desired estimate for p_6. Now for the estimation of p_7 fix $r \le \frac{\rho}{2}$ and $x \in B_r(0)$ to estimate with the properties of ϕ (cf. statement of Proposition 10.16)

$$\begin{aligned}|p_7(x)| &\le \frac{3}{4\pi} \int \frac{1}{|x-y|} |\nabla\phi(y)|\,|u(y)|\,|\nabla u(y)|\,\mathrm{d}y \\ &\underset{\text{Choice of } \phi}{=} \frac{3}{4\pi} \int_{B_\rho \setminus B_{\frac{3}{4}\rho}} \frac{1}{|x-y|} |\nabla\phi(y)|\,|u(y)|\,|\nabla u(y)|\,\mathrm{d}y \\ &\underset{\text{Choice of } \phi}{\le} \frac{C}{\rho} \int_{B_\rho \setminus B_{\frac{3}{4}\rho}} \frac{1}{|x-y|} |u(y)|\,|\nabla u(y)|\,\mathrm{d}y \\ &\underset{\text{Inv. triangle inequality}}{\le} \frac{C}{\rho} \int_{B_\rho \setminus B_{\frac{3}{4}\rho}} \frac{1}{|y|-|x|} |u(y)|\,|\nabla u(y)|\,\mathrm{d}y \\ &\underset{x \in B_{\frac{\rho}{2}}}{\le} \frac{C}{\rho} \int_{B_\rho \setminus B_{\frac{3}{4}\rho}} \frac{1}{\frac{3}{4}\rho - \frac{1}{2}\rho} |u(y)|\,|\nabla u(y)|\,\mathrm{d}y \\ &\le \frac{C}{\rho^2} \int_{B_\rho \setminus B_{\frac{3}{4}\rho}} |u(y)|\,|\nabla u(y)|\,\mathrm{d}y \le \frac{C}{\rho^2} \int_{B_\rho} |u(y)|\,|\nabla u(y)|\,\mathrm{d}y. \end{aligned}$$

Integrating over $x \in B_r(0)$ we obtain

$$\begin{aligned}\int_{B_r} |p_7| &\le \frac{C}{\rho^2} r^3 \int_{B_\rho} |u(y)|\,|\nabla u(y)|\,\mathrm{d}y \le C\left(\frac{r}{\rho}\right)^2 r \left(\int_{B_\rho} |u|^2\right)^{\frac{1}{2}} \left(\int_{B_\rho} |\nabla u|^2\right)^{\frac{1}{2}} \\ &\le Cr\rho^{\frac{1}{2}} A_*(\rho)^{\frac{1}{2}} \left(\int_{B_\rho} |\nabla u|^2\right)^{\frac{1}{2}}, \end{aligned} \tag{10.21}$$

where we used that $\frac{r}{\rho} < 1$ in the last step. As we discussed before, the claim follows from (10.20) and (10.21). □

Proof of Lemma 10.20. Let r, ρ be as in the statement. By (10.11) we conclude that

$$\int_{B_r} |p_4| \le C\left(\frac{r}{\rho}\right)^3 \int_{B_\rho} |p|,$$

in particular

$$\left(\int_{B_r} |p_4|\right)^{\frac{5}{4}} \le C\left(\frac{r}{\rho}\right)^{\frac{15}{4}} \left(\int_{B_\rho} |p|\right)^{\frac{5}{4}}.$$

Integrating over $t \in (-\frac{7}{8}r^2, \frac{1}{8}r^2)$ we obtain

$$\begin{aligned}\int_{-\frac{7}{8}r^2}^{\frac{r^2}{8}} \left(\int_{B_r} |p_4|\right)^{\frac{5}{4}} &\le C\left(\frac{r}{\rho}\right)^{\frac{15}{4}} \int_{-\frac{7}{8}r^2}^{\frac{r^2}{8}} \left(\int_{B_\rho} |p|\right)^{\frac{5}{4}} \le C\left(\frac{r}{\rho}\right)^{\frac{15}{4}} \int_{-\frac{7}{8}\rho^2}^{\frac{\rho^2}{8}} \left(\int_{B_\rho} |p|\right)^{\frac{5}{4}} \\ &= C\left(\frac{r}{\rho}\right)^{\frac{15}{4}} \rho^{\frac{13}{4}} A_*(\rho) = Cr^{\frac{13}{4}} \left(\frac{r}{\rho}\right)^{\frac{1}{2}} K_*(\rho). \end{aligned}$$

Dividing by $r^{\frac{13}{4}}$ yields

$$\frac{1}{r^{\frac{13}{4}}}\int_{-\frac{7}{8}r^2}^{\frac{r^2}{8}}\left(\int_{B_r}|p_4|\right)^{\frac{5}{4}} \leq C\left(\frac{r}{\rho}\right)^{\frac{1}{2}} K_*(\rho). \tag{10.22}$$

Furthermore, using (10.18) we find

$$\begin{aligned}
\int_{-\frac{7}{8}r^2}^{\frac{r^2}{8}}\left(\int_{B_r}|p_5|\right)^{\frac{5}{4}} &\leq C\int_{-\frac{7}{8}r^2}^{\frac{r^2}{8}} r^{\frac{5}{4}}\rho^{\frac{5}{8}}A_*^{\frac{5}{8}}(\rho)\left(\int_{B_\rho}|\nabla u|^2\right)^{\frac{5}{8}}\\
&\leq Cr^{\frac{5}{4}}\rho^{\frac{5}{8}}A_*^{\frac{5}{8}}(\rho)\int_{-\frac{7}{8}r^2}^{\frac{r^2}{8}}\left(\int_{B_\rho}|\nabla u|^2\right)^{\frac{5}{8}}\\
&\underset{\text{Hölder } p=\frac{8}{5},q=\frac{8}{3}}{\leq} Cr^{\frac{5}{4}}\rho^{\frac{5}{8}}A_*^{\frac{5}{8}}(\rho)(r^2)^{\frac{3}{8}}\left(\int_{-\frac{7}{8}r^2}^{\frac{r^2}{8}}\int_{B_\rho}|\nabla u|^2\right)^{\frac{5}{8}}\\
&\leq Cr^2\rho^{\frac{5}{8}}A_*^{\frac{5}{8}}(\rho)\left(\int_{-\frac{7}{8}\rho^2}^{\frac{\rho^2}{8}}\int_{B_\rho}|\nabla u|^2\right)^{\frac{5}{8}} \leq Cr^2\rho^{\frac{5}{8}}A_*^{\frac{5}{8}}(\rho)(\rho\delta_*(\rho))^{\frac{5}{8}}\\
&= Cr^2\rho^{\frac{5}{4}}A_*^{\frac{5}{8}}(\rho)\delta_*^{\frac{5}{8}}(\rho).
\end{aligned}$$

Dividing by $r^{\frac{13}{4}}$ we infer

$$\frac{1}{r^{\frac{13}{4}}}\int_{-\frac{7}{8}r^2}^{\frac{r^2}{8}}\left(\int_{B_r}|p_5|\right)^{\frac{5}{4}} \leq C\left(\frac{\rho}{r}\right)^{\frac{5}{4}} A_*^{\frac{5}{8}}(\rho)\delta_*^{\frac{5}{8}}(\rho). \tag{10.23}$$

Given (10.22) and (10.23) we conclude

$$\begin{aligned}
K_*(r) = \frac{1}{r^{\frac{13}{4}}}\int_{-\frac{7}{8}r^2}^{\frac{r^2}{8}}\left(\int_{B_r}|p|\right)^{\frac{5}{4}} &\leq \frac{1}{r^{\frac{13}{4}}}\int_{-\frac{7}{8}r^2}^{\frac{r^2}{8}}\left(\int_{B_r}|p_4|+\int_{B_r}|p_5|\right)^{\frac{5}{4}}\\
&\leq C\left(\frac{1}{r^{\frac{13}{4}}}\int_{-\frac{7}{8}r^2}^{\frac{r^2}{8}}\left(\int_{B_r}|p_4|\right)^{\frac{5}{4}} + \frac{1}{r^{\frac{13}{4}}}\int_{-\frac{7}{8}r^2}^{\frac{r^2}{8}}\left(\int_{B_r}|p_5|\right)^{\frac{5}{4}}\right)\\
&\leq C\left\{\left(\frac{r}{\rho}\right)^{\frac{1}{2}}K_*(\rho) + \left(\frac{\rho}{r}\right)^{\frac{5}{4}}A_*^{\frac{5}{8}}(\rho)\delta_*^{\frac{5}{8}}(\rho)\right\}.
\end{aligned}$$

□

10.4. Proof of the main proposition

Proof. Let $r \leq \frac{1}{4}\rho$. Recall that the statement imposes the condition $\delta_*(\rho) \leq 1$. We bound $G_*^{\frac{2}{3}}(r), H_*(r), J_*(r)$ and $K_*(r)$ seperately in terms of M_* and δ_*. Again, we use C to denote a generic constant which we possibly have to increase after each estimate.

Step 1: Estimating G_*. First we can use (10.8) with input parameters $\widetilde{r} := r$ and $\widetilde{\rho} := \frac{\rho}{2}$ to obtain

$$\begin{aligned}
G_*^{\frac{2}{3}}(r) &\underset{(10.8)}{\leq} \left\{\left(\frac{r}{\rho/2}\right)^3 A_*^{\frac{3}{2}}(\tfrac{\rho}{2}) + \left(\frac{\rho/2}{r}\right)^3 A_*^{\frac{3}{4}}(\tfrac{\rho}{2})\delta_*^{\frac{3}{4}}(\tfrac{\rho}{2})\right\}^{\frac{2}{3}}\\
&\leq C\left\{\left(\frac{r}{\rho/2}\right)^2 A_*(\tfrac{\rho}{2}) + \left(\frac{\rho/2}{r}\right)^2 A_*(\tfrac{\rho}{2})^{\frac{1}{2}}\delta_*(\tfrac{\rho}{2})^{\frac{1}{2}}\right\}
\end{aligned}$$

$$
\begin{aligned}
&\le C\left\{4\left(\frac{r}{\rho}\right)^2 A_*(\tfrac{\rho}{2}) + \frac{1}{4}\left(\frac{\rho}{r}\right)^2 A_*(\tfrac{\rho}{2})^{\frac{1}{2}}\delta_*(\tfrac{\rho}{2})^{\frac{1}{2}}\right\}\\
&\le C\left\{\left(\frac{r}{\rho}\right)^2 A_*(\tfrac{\rho}{2}) + \left(\frac{\rho}{r}\right)^2 A_*(\tfrac{\rho}{2})^{\frac{1}{2}}\delta_*(\tfrac{\rho}{2})^{\frac{1}{2}}\right\}\\
&\underset{(10.4)}{\le} \left\{\left(\frac{r}{\rho}\right)^2 M_*(\rho) + \left(\frac{\rho}{r}\right)^2 M_*(\rho)^{\frac{1}{2}}\delta_*(\tfrac{\rho}{2})^{\frac{1}{2}}\right\}\\
&\underset{\text{Remark 10.6}}{\le} \left\{\left(\frac{r}{\rho}\right)^2 M_*(\rho) + \left(\frac{\rho}{r}\right)^2 M_*(\rho)^{\frac{1}{2}}\delta_*(\rho)^{\frac{1}{2}}\right\}. \qquad (10.24)
\end{aligned}
$$

Step 2: Estimating H_*.

$$
\begin{aligned}
H_*(r) &\underset{\text{Remark 10.5}}{\le} C(G_*^{\frac{2}{3}}(r) + A_*(r)\delta_*(r)) \underset{\text{Remark 10.6}}{\le} C\left\{G_*^{\frac{2}{3}}(r) + \left(\frac{\rho}{r}\right) A_*(r)\delta_*(\rho)\right\}\\
&\underset{(10.4)}{\le} C\left\{G_*^{\frac{2}{3}}(r) + \left(\frac{\rho}{r}\right)^2 M_*(\rho)\delta_*(\rho)\right\}\\
&\underset{(10.24)}{\le} C\left\{\left(\frac{r}{\rho}\right)^2 M_*(\rho) + \left(\frac{\rho}{r}\right)^2\left[M_*(\rho)\delta_*(\rho) + M_*^{\frac{1}{2}}(\rho)\delta_*^{\frac{1}{2}}(\rho)\right]\right\}.
\end{aligned}
$$

We can merge the estimates in Step 1 and Step 2 to get

$$
\begin{aligned}
G_*^{\frac{2}{3}}(r) + H_*(r) &\le C\left\{\left(\frac{r}{\rho}\right)^2 M_*(\rho) + \left(\frac{\rho}{r}\right)^2\left[M_*(\rho)\delta_*(\rho) + M_*^{\frac{1}{2}}(\rho)\delta_*^{\frac{1}{2}}(\rho)\right]\right\}\\
&\le C\left\{\left(\frac{r}{\rho}\right)^{\frac{1}{5}} M_*(\rho) + \left(\frac{\rho}{r}\right)^2\left[M_*(\rho)\delta_*(\rho) + M_*^{\frac{1}{2}}(\rho)\delta_*^{\frac{1}{2}}(\rho)\right]\right\}, \qquad (10.25)
\end{aligned}
$$

where we used in the last estimate that $\frac{r}{\rho} \le 1$.

Step 3: Estimating J_*. By Lemma 10.19 and similar techniques as in the first three estimates of Step 1 we obtain

$$
\begin{aligned}
J_*(r) &\le C\left\{\left(\frac{r}{\rho}\right)^{\frac{1}{5}} A_*(\tfrac{\rho}{2})^{\frac{1}{5}} G_*(r)^{\frac{1}{5}} K_*(\tfrac{\rho}{2})^{\frac{4}{5}} + \left(\frac{\rho}{r}\right)^2 A_*(\tfrac{\rho}{2})^{\frac{1}{2}}\delta_*(\tfrac{\rho}{2})\right\}\\
&\underset{(10.4),\text{Remark 10.6}}{\le} C\left\{\left(\frac{r}{\rho}\right)^{\frac{1}{5}} A_*(\tfrac{\rho}{2})^{\frac{1}{5}} G_*(r)^{\frac{1}{5}} K_*(\tfrac{\rho}{2})^{\frac{4}{5}} + \left(\frac{\rho}{r}\right)^2 M_*(\rho)^{\frac{1}{2}}\delta_*(\rho)\right\}\\
&\underset{\delta_*(\rho)\le 1}{\le} C\left\{\left(\frac{r}{\rho}\right)^{\frac{1}{5}} A_*(\tfrac{\rho}{2})^{\frac{1}{5}} G_*(r)^{\frac{1}{5}} K_*(\tfrac{\rho}{2})^{\frac{4}{5}} + \left(\frac{\rho}{r}\right)^2 M_*(\rho)^{\frac{1}{2}}\delta_*(\rho)^{\frac{1}{2}}\right\}. \qquad (10.26)
\end{aligned}
$$

We can now use the generalized Young inequality $abc \le C(a^{p_1} + b^{p_2} + c^{p_3})$ whenever $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$ to estimate $A_*(\frac{\rho}{2})^{\frac{1}{5}} G_*(r)^{\frac{1}{5}} K_*(\frac{\rho}{2})^{\frac{4}{5}}$. Here the choice $p_1 = 5, p_2 = \frac{10}{3}, p_3 = 2$ yields

$$
\begin{aligned}
A_*(\tfrac{\rho}{2})^{\frac{1}{5}} G_*(r)^{\frac{1}{5}} K_*(\tfrac{\rho}{2})^{\frac{4}{5}} &\le C\left(A_*(\tfrac{\rho}{2}) + G_*^{\frac{2}{3}}(r) + K_*^{\frac{8}{5}}(\tfrac{\rho}{2})\right)\\
&\le C\left(M_*(\rho) + K_*^{\frac{8}{5}}(\rho) + G_*^{\frac{2}{3}}(r)\right)
\end{aligned}
$$

where we have used (10.4) and Remark 10.6 in the last step. Notice that one can also estimate $K_*^{\frac{8}{5}}(\rho) \le M_*(\rho)$ to simplify

$$
A_*(\tfrac{\rho}{2})^{\frac{1}{5}} G_*(r)^{\frac{1}{5}} K_*(\tfrac{\rho}{2})^{\frac{4}{5}} \le C\left(M_*(\rho) + G_*^{\frac{2}{3}}(r)\right).
$$

Plugging this into (10.26) we obtain

$$
\begin{aligned}
J_*(r) &\le C\left\{\left(\frac{r}{\rho}\right)^{\frac{1}{5}} M_*(\rho) + \left(\frac{r}{\rho}\right)^{\frac{1}{5}} G_*^{\frac{2}{3}}(r) + \left(\frac{\rho}{r}\right)^{2} M_*(\rho)^{\frac{1}{2}}\delta_*(\rho)^{\frac{1}{2}}\right\} \\
&\underset{(10.24)}{\le} C\left\{\left(\frac{r}{\rho}\right)^{\frac{1}{5}} M_*(\rho) + \left(\frac{r}{\rho}\right)^{\frac{1}{5}}\left(\left(\frac{r}{\rho}\right)^{2} M_*(\rho) + \left(\frac{\rho}{r}\right)^{2} M_*(\rho)^{\frac{1}{2}}\delta_*(\rho)^{\frac{1}{2}}\right)\right. \\
&\qquad \left. + \left(\frac{\rho}{r}\right)^{2} M_*(\rho)^{\frac{1}{2}}\delta_*(\rho)^{\frac{1}{2}}\right\} \\
&\le \left\{\left(\left(\frac{r}{\rho}\right)^{\frac{1}{5}} + \left(\frac{r}{\rho}\right)^{\frac{11}{5}}\right) M_*(\rho) + \left(\left(\frac{\rho}{r}\right)^{2} + \left(\frac{\rho}{r}\right)^{\frac{9}{5}}\right) M_*(\rho)^{\frac{1}{2}}\delta_*(\rho)^{\frac{1}{2}}\right\}.
\end{aligned}
$$

Using that $r < \rho$ we can determine which power of $\frac{r}{\rho}$ or $\frac{\rho}{r}$ respectively dominates and infer

$$
J_*(r) \le \left\{\left(\frac{r}{\rho}\right)^{\frac{1}{5}} M_*(\rho) + \left(\frac{\rho}{r}\right)^{2} M_*(\rho)^{\frac{1}{2}}\delta_*(\rho)^{\frac{1}{2}}\right\}. \tag{10.27}
$$

Step 4: Estimating $K_*^{\frac{8}{5}}$. By Lemma 10.20 and $(a+b)^{\frac{8}{5}} \le C(a^{\frac{8}{5}} + b^{\frac{8}{5}})$ we obtain that

$$
\begin{aligned}
K_*^{\frac{8}{5}}(r) &\le \left(C\left\{\left(\frac{r}{\rho}\right)^{\frac{1}{2}} K_*(\tfrac{\rho}{2}) + \left(\frac{\rho}{r}\right)^{\frac{5}{4}} A_*^{\frac{5}{8}}(\tfrac{\rho}{2})\delta_*^{\frac{5}{8}}(\tfrac{\rho}{2})\right\}\right)^{\frac{8}{5}} \\
&\le C\left\{\left(\frac{r}{\rho}\right)^{\frac{4}{5}} K_*^{\frac{8}{5}}(\tfrac{\rho}{2}) + \left(\frac{\rho}{r}\right)^{2} A_*(\tfrac{\rho}{2})\delta_*(\tfrac{\rho}{2})\right\} \\
&\le C\left\{\left(\frac{r}{\rho}\right)^{\frac{4}{5}} K_*^{\frac{8}{5}}(\rho) + \left(\frac{\rho}{r}\right)^{2} A_*(\tfrac{\rho}{2})\delta_*(\rho)\right\}
\end{aligned}
$$

We can use that by definition of M_* one has $K_*^{\frac{8}{5}}(\rho) \le M_*(\rho)$ as well as (10.7) to obtain

$$
\begin{aligned}
K_*^{\frac{8}{5}}(r) &\le C\left\{\left(\frac{r}{\rho}\right)^{\frac{4}{5}} M_*(\rho) + \left(\frac{\rho}{r}\right)^{2} M_*(\rho)\delta_*(\rho)\right\} \\
&\le C\left\{\left(\frac{r}{\rho}\right)^{\frac{1}{5}} M_*(\rho) + \left(\frac{\rho}{r}\right)^{2} M_*(\rho)\delta_*(\rho)\right\},
\end{aligned} \tag{10.28}
$$

where the last step uses again that $r < \rho$.

Step 5: The claim follows now by adding up (10.25), (10.27) and (10.28), all of which consist only of terms that appear on the right hand side of the desired inequality. ☐

CHAPTER 11

Talk 9: Estimating the Singular Set and Estimates for u and p in Weighted Norms

By Dennis Gallenmüller

This talk splits into two independent sections. On the one hand, we prove the main theorem subject to this seminar, namely Theorem B in [**CKN82**], which corresponds to section 6 in the paper. On the other hand, this talk will prepare the proof of Theorems C and D in [**CKN82**], which corresponds to section 7 of the paper. For this we provide two lemmas concerning estimates of the velocity field u and the pressure p of a suitable weak solution in some specific weighted norms.

11.1. Estimating the Singular Set

11.1.1. Completing the Proof of the Main Theorem. For convenience we recall the main theorem and Proposition 2 from [**CKN82**].

THEOREM 11.1 (Caffarelli, Kohn, Nirenberg (Theorem B)). *For any suitable weak solution of the Navier-Stokes system on an open set in space-time, the associated singular set satisfies $\mathcal{P}^1(S) = 0$.*

PROPOSITION 11.2 (Caffarelli, Kohn, Nirenberg (Proposition 2)). *There is an absolute constant $\varepsilon_3 > 0$ with the following property. If (u, p) is a suitable weak solution of the Navier-Stokes system near (x, t) and if $\limsup_{r \to 0} \frac{1}{r} \int_{Q_r^*(x,t)} |\nabla u|^2 \le \varepsilon_3$, then (x, t) is a regular point.*

The idea of the proof of Theorem 11.1 is to use Proposition 11.2 (cf. Proposition 1.6 in talk 1) and a variant of Vitali's covering lemma for parabolic cylinders (see Lemma 11.3) to estimate the one-dimensional parabolic Hausdorff measure of S.
First, let us state and prove this variant of Vitali's covering lemma for parabolic cylinders. The classical Vitali lemma considers balls, but cylinders are more convenient for our discussion due to the structure of the Navier-Stokes equations.

LEMMA 11.3. *Let $\mathcal{C} = \left(Q_{r_i}^*(x_i, t_i)\right)_{i \in I}$ be any collection of parabolic cylinders contained in a bounded subset of $\mathbb{R}^3 \times \mathbb{R}$. Then there exists a finite or countable sub-collection*

$$\mathcal{C}' = \left(Q_{r_{i_j}}^*(x_{i_j}, t_{i_j})\right)_{j \in I'}$$

i.e. $I' \subset I$, which is disjoint and has the property that for all $Q^* \in \mathcal{C}$ there is a $j \in I'$ such that $Q^* \subset Q_{5 r_{i_j}}^*(x_{i_j}, t_{i_j})$.

REMARK 11.4. As in the other talks we use the notation

$$Q_r^*(x, t) := \left\{(y, \tau) \ : \ |y - x| < r, \ t - \frac{7}{8} r^2 < \tau < t + \frac{1}{8} r^2\right\}.$$

PROOF. Set $\mathcal{C}_0 := \mathcal{C}$. Moreover, since $\left(Q_{r_i}^*\right)$ is contained in a bounded subset, we have $\sup_{i \in I} r_i < \infty$. Hence, we can choose $Q_1^* \subset \mathcal{C}_0$ such that $\frac{3}{2} r_{Q_1^*} \ge \sup_{i \in I} r_i$. Note that this is possible, since by the definition of the supremum and $\frac{1}{3} \sup_{i \in I} r_i > 0$ we find some r such that

$r \geq \sup_{i\in I} r_i - \frac{1}{3}\sup_{i\in I} r_i = \frac{2}{3}\sup_{i\in I} r_i$.
Let us choose a countable sub-collection of $\mathcal{C}$ inductively as follows:
Assume for $n \in \mathbb{N}$ we have already chosen $\left(Q_k^*\right)_{k=1}^n$. Then set

$$\mathcal{C}_n := \{Q^* \in \mathcal{C} \ : \ Q^* \cap Q_k^* = \varnothing, \ k = 1, ..., n\}.$$

Moreover, as long as $\mathcal{C}_n \neq \varnothing$ choose $Q_{n+1}^* \in \mathcal{C}_n$ such that $\sup_{Q^*\in\mathcal{C}_n} r_{Q^*} \leq \frac{3}{2} r_{Q_{n+1}^*}$. This is again possible by the definition of the supremum as above.
Thus, the subcollection $\mathcal{C}' := \left(Q_k^*\right)_k$ is disjoint and countable or finite by construction. The latter case is considered if $\mathcal{C}_n = \varnothing$ for some $n \in \mathbb{N}$.
Now, we claim that given a $Q^* \in \mathcal{C}\backslash\mathcal{C}'$ there exists a $n \in \mathbb{N}_0$ such that $Q^* \in \mathcal{C}_n$ but $Q^* \notin \mathcal{C}_{n+1}$. In the case that $\mathcal{C}'$ is finite this is obvious, since $\mathcal{C}_{n+1} = \varnothing$ for some $n \in \mathbb{N}_0$. In the case that $\mathcal{C}'$ is countably infinite, the pairwise distjointness of $\mathcal{C}'$ and the fact that $\mathcal{C}$ is contained in a bounded set imply that $r_{Q_n^*}$ tend to zero as $n \to \infty$. Now, given a $Q^* \in \mathcal{C}\backslash\mathcal{C}'$ by the same reasoning as just mentioned there are only finitely many pairwise disjoint cylinders $\tilde{Q}^* \in \mathcal{C}$ such that $\frac{3}{2} r_{\tilde{Q}^*} \geq r_{Q^*}$. Assume now that Q^* would not be deleted by intersecting one of these $\tilde{Q}^*$. Then eventually after finitely many, say $n \in \mathbb{N}$ many, selection processes holds

$$r_{Q^*} > \frac{3}{2} r_{Q'^*}$$

for all $Q'^* \in \mathcal{C}_n$. As Q^* has not yet been deleted, we have $Q^* \in \mathcal{C}_n$. Therefore, we have to make the selection $Q^* = Q_{n+1}^*$ contradicting the fact that $Q^* \notin \mathcal{C}'$. Thus, Q^* has to be deleted after finitely many steps yielding the claim.
The claim implies by definition of the selection process, that for every $Q^* \in \mathcal{C}\backslash\mathcal{C}'$ there is a $n \in \mathbb{N}_0$ such that $Q^* \cap Q_{n+1}^* \neq \varnothing$ and $r_{Q^*} \leq \frac{3}{2} r_{Q_{n+1}^*}$.
Let us write $r_{n+1} := r_{Q_{n+1}^*}$. Therefore, the diameter of Q^* in space direction is at most $3r_{n+1}$ and in time direction at most $\left(\frac{3}{2} r_{n+1}\right)^2$. Hence, the maximal distance of a point $(x,t) \in Q^*$ to the parabolic center of Q_{n+1}^* in space is $4r_{n+1}$. In time direction the maximal distance of (x,t) to the parabolic center of Q_{n+1}^* has to be considered for forewards and backwards direction seperately, since the definition of the Q^* involves different scaling forewards and backwards in time. To be precise, the maximal distance backwards in time is

$$\frac{7}{8} r_{n+1}^2 + \frac{9}{4} r_{n+1}^2 \overset{!}{\leq} \frac{7}{8} (a r_{n+1})^2. \tag{11.1}$$

Here, we introduced some $a \in \mathbb{R}$ to be chosen such that the cylinder $Q_{ar_{n+1}}^*$ contains Q^*. From (11.1) it follows that $a \geq \sqrt{\frac{25}{7}}$, where the latter is less than 2.
For the forewards time direction we have to ensure that

$$\frac{1}{8} r_{n+1}^2 + \frac{9}{4} r_{n+1}^2 \leq \frac{1}{8} (a r_{n+1})^2.$$

Thus, $a \geq \sqrt{19}$, which is less than 5. All in all, we showed that $Q^* \subset Q_{5r_{n+1}}^*$ for all $Q^* \in \mathcal{C}$ as the latter is obviously true for $Q^* \in \mathcal{C}'$, as $5 > 1$. □

We have collected all tools needed to prove Theorem 11.1.

Proof. Let (u,p) be a suitable weak solution of the Navier-Stokes system. It suffices to assume that (u,p) is only defined on an open bounded subset of $\mathbb{R}^3 \times \mathbb{R}$. Indeed, let $(D_i)_{i=0}^\infty$ be a countable open covering of the potentially unbounded domain of definition of (u,p). Then (u,p) is also a suitable weak solution on $\mathcal{S} \cap D_i$ for all i by restricting the set of testfunctions to those with support in D_i. Assume we have already shown the theorem for bounded domains, then $\mathcal{P}^1(\mathcal{S} \cap D_i) = 0$ for all i. Let $\delta > 0$ and $\varepsilon > 0$. Now, for all i choose a countable collection of parabolic cylinders $\left(Q_{r_j^i}^*\left(x_j^i, t_j^i\right)\right)$ covering $\mathcal{S} \cap D_i$ with

$r_j^i < \delta$ such that $\sum_j r_j^i < \mathcal{P}^1_\delta(\mathcal{S} \cap D_i) + \frac{\varepsilon}{2^i}$. Then $\bigcup_i \bigcup_j Q^*_{r_j^i}\left(x_j^i, t_j^i\right) = \bigcup_{i,j} Q^*_{r_j^i}\left(x_j^i, t_j^i\right)$ is a countable collection covering $\mathcal{S}$. Thus,

$$\mathcal{P}^1_\delta(\mathcal{S}) \leq \sum_{i,j} r_j^i = \sum_i \sum_j r_j^i \leq \sum_i \left(\mathcal{P}^1_\delta(\mathcal{S} \cap D_i) + \frac{\varepsilon}{2^i}\right) = \sum_i \mathcal{P}^1_\delta(\mathcal{S} \cap D_i) + \varepsilon.$$

We can let $\varepsilon \to 0$ to infer the countable subadditivity of $\mathcal{P}^1_\delta$ for all $\delta > 0$. By definition of $\mathcal{P}^1_\delta$ and the infimum, we know that $\mathcal{P}^1_{\delta'}(\mathcal{S} \cap D^i) \leq \mathcal{P}^1_\delta(\mathcal{S} \cap D^i)$ for $\delta \leq \delta'$. Thus, by monotone convergence

$$\mathcal{P}^1(\mathcal{S}) = \lim_{\delta \to 0} \mathcal{P}^1_\delta(\mathcal{S}) \leq \lim_{\delta \to 0} \sum_i \mathcal{P}^1_\delta(\mathcal{S} \cap D_i) = \sum_i \lim_{\delta \to 0} \mathcal{P}^1_\delta(\mathcal{S} \cap D_i) = \sum_i \mathcal{P}^1(\mathcal{S} \cap D_i) = \sum_i 0 = 0.$$

Therefore, we assume (u,p) to be defined on the open bounded set $D \subset \mathbb{R}^3 \times \mathbb{R}$.
By Proposition 11.2 there is a constant $\varepsilon_3 > 0$ such that for all $(x,t) \in \mathcal{S}$ holds

$$\limsup_{r \to 0} \frac{1}{r} \int_{Q_r^*(x,t)} |\nabla u|^2 > \varepsilon_3. \tag{11.2}$$

Now, let $\delta > 0$ and $V \subset D$ be a neighborhood of $\mathcal{S}$. By the strictness of the inequality in (11.2) for every $(x,t) \in \mathcal{S}$ we can choose a parabolic cylinder $Q_r^*(x,t)$ with $0 < r < \delta$ such that

$$\frac{1}{r} \int_{Q_r^*(x,t)} |\nabla u|^2 > \varepsilon_3 \tag{11.3}$$

and $Q_r^*(x,t) \subset V$.
Now the covering Lemma for parabolic cylinders (Lemma 11.3) yields a disjoint countable subcollection $\left(Q^*_{r_i}(x_i,t_i)\right)_i$ such that

$$\mathcal{S} \subset \bigcup_{(x,t) \in \mathcal{S}} Q_r^*(x,t) \subset \bigcup_i Q^*_{5r_i}(x_i,t_i).$$

Moreover, since $Q^*_{r_i} \subset V$ are disjoint and by (11.3) we obtain

$$\sum_i r_i \leq \sum_i \frac{1}{\varepsilon_3} \int_{Q^*_{r_i}(x_i,t_i)} |\nabla u|^2 \leq \frac{1}{\varepsilon_3} \int_V |\nabla u|^2, \tag{11.4}$$

where the right hand side is independend of the choice of r_i and hence independent of δ.
Therefore, we estimate the Lebesgue measure of the singular set by using $r < \delta$

$$|\mathcal{S}| \leq \left|\bigcup_i Q^*_{5r_i}(x_i,t_i)\right| \leq \sum_i \left|Q^*_{5r_i}(x_i,t_i)\right| = \sum_i C r_i^5 \leq C\delta^4 \sum_i r_i \leq C \frac{1}{\varepsilon_3} \int_V |\nabla u|^2 \cdot \delta^4,$$

where in the last inequality we used (11.4). Since $\delta > 0$ was arbitrary, we conclude that $|\mathcal{S}| = 0$.
Also (11.4) and $r_i < \delta$ imply that

$$\mathcal{P}^1_\delta(\mathcal{S}) \leq \sum_i 5 r_i \leq \frac{5}{\varepsilon_3} \int_V |\nabla u|^2$$

for all $\delta > 0$, hence also the limit for $\delta \to 0$, i.e. the parabolic Hausdorff measure $\mathcal{P}^1(\mathcal{S})$, is less or equal than $\frac{5}{\varepsilon_3} \int_V |\nabla u|^2$.
Still the neighborhood V is arbitrary. Thus, we can choose a sequence V_n for example by

$$V_n := D \cap \left\{(y,s) : \operatorname{dist}(\mathcal{S},(y,s)) < \frac{1}{n}\right\}.$$

We need to show that the indicator function of V_n tends pointwise almost everywhere in D to zero as $n \to \infty$. Indeed, $\mathcal{S}$ is closed, since its complement $\mathcal{R}$, defined as

$$\mathcal{R} := \{(x,t) : \exists \text{ neighborhood } U \text{ of } (x,t) \text{ such that } u \in L^\infty(U)\},$$

is obviously an open set, because every point in $\mathcal{R}$ has an open neighborhood that contains again only points in $\mathcal{R}$ by definition. Now, let $(x,t) \in \mathcal{R}$. Then, there is some ball $B_\varepsilon(x,t)$ around this point lying in $\mathcal{R}$. Hence, for all $n \in \mathbb{N}$ large enough such that $\frac{1}{n} < \frac{\varepsilon}{2}$ we achieve $(x,t) \notin V_n$, because else $B_\varepsilon(x,t) \cap \mathcal{S} \neq \varnothing$ would be a contradiction. Since, $\mathcal{R}^c = \mathcal{S}$ is a Lebesgue null set, we infer that the indicator of V_n tends to zero as $n \to \infty$ almost everywhere.
We also have $\int_{V_n} |\nabla u|^2 \le \int_D |\nabla u|^2 < \infty$. So, dominated convergence implies that

$$\mathcal{P}^1(\mathcal{S}) \le \frac{5}{\varepsilon_3} \int_{V_n} |\nabla u|^2 \overset{n\to\infty}{\to} 0,$$

finishing the proof. □

11.1.2. Some Corollaries. In general, Theorem 11.1 is not strong enough to imply the uniqueness or strong time-continuity for suitable weak solutions, since still $\mathcal{S}$ could be non-empty. On the other hand, there are interesting direct consequences, some of them listed in the following.

COROLLARY 11.5. *On $\mathbb{T}^3$ holds $\mathcal{H}^{\frac{1}{2}}(\mathcal{T}) = 0$, where $\mathcal{T}$ denotes the set of positive singular times.*

PROOF. From Lemma 16.3 in **[RRS16]** we know that on $\mathbb{T}^3$ holds $\mathcal{T} = \mathrm{pr}_t(\mathcal{S})$, where the latter denotes the projection of $\mathcal{S}$ onto the time coordinate. Thus, it is sufficient to prove the inequality

$$\mathcal{H}^{\frac{1}{2}}(\mathrm{pr}_t X) \le C\mathcal{P}^1(X) \tag{11.5}$$

for all $X \subset \mathbb{R}^3 \times \mathbb{R}$.
Indeed, for every covering by parabolic cylinders $(Q_{r_i}(x_i,t_i))_{i=0}^\infty$ of X holds in particular that $\mathrm{pr}_t X \subset \bigcup\limits_i \mathrm{pr}_t Q_{r_i}$. Thus, for all $\delta > 0$ holds

$$\begin{aligned}
\mathcal{P}^1_\delta(X) &= \inf\left\{\sum_i r_i \,:\, X \subset \bigcup_i Q_{r_i},\ r_i < \delta\right\}\\
&\ge \inf\left\{\sum_i r_i \,:\, \mathrm{pr}_t X \subset \bigcup_i \mathrm{pr}_t Q_{r_i},\ r_i < \delta\right\}\\
&= \inf\left\{\sum_i r_i \,:\, \mathrm{pr}_t X \subset \bigcup_i (t_i - r_i^2, t_i),\ r_i < \delta\right\}\\
&= \inf\left\{\sum_i \sqrt{s_i} \,:\, \mathrm{pr}_t X \subset \bigcup_i (t_i - s_i, t_i),\ s_i < \delta^2\right\}\\
&\ge \inf\left\{\sum_i \sqrt{\mathrm{diam}([t_i - s_i, t_i])} \,:\, \mathrm{pr}_t X \subset \bigcup_i [t_i - s_i, t_i],\ s_i < \delta^2\right\}\\
&\ge C \cdot \mathcal{H}^{\frac{1}{2}}_{\delta^2}(\mathrm{pr}_t X).
\end{aligned}$$

Passing to the limit $\delta \to 0$ on both sides yields the desired inequality (11.5), completing the proof. □

COROLLARY 11.6. *Let $\int \left(\int |\nabla u|^2 \,\mathrm{d}x\right)^2 \,\mathrm{d}t < \infty$ for a suitable weak solution (u,p) defined on $D \subset \mathbb{R}^3 \times \mathbb{R}$. Then (u,p) is regular on D.*

PROOF. Let $(x,t) \in D$ be any point. We estimate using Hölder in the time-integration

$$\limsup_{r\to 0} \frac{1}{r} \int_{Q_r^*(x,t)} |\nabla u|^2 = \limsup_{r\to 0} \frac{1}{r} \int_{t-\frac{7}{8}r^2}^{t+\frac{1}{8}r^2} 1 \cdot \int_{B_r(x)} |\nabla u|^2 \,\mathrm{d}y \,\mathrm{d}s$$

$$
\begin{aligned}
&\leq \limsup_{r\to 0} \frac{1}{r}\left(\int_{t-\frac{7}{8}r^2}^{t+\frac{1}{8}r^2}\left(\int_{B_r(x)} |\nabla u|^2 \,\mathrm{d}y\right)^2 \mathrm{d}s\right)^{\frac{1}{2}} \cdot \sqrt{r^2}\\
&\leq \limsup_{r\to 0} \left(\int_{t-\frac{7}{8}r^2}^{t+\frac{1}{8}r^2}\left(\int_{D\cap\{s=t\}} |\nabla u|^2 \,\mathrm{d}y\right)^2 \mathrm{d}s\right)^{\frac{1}{2}}\\
&= 0,
\end{aligned}
$$

where in the last step we used dominated convergence as clearly the indicator of the time interval $\chi_{[t-\frac{7}{8}r^2, t+\frac{1}{8}r^2]}(s) \to 0$ for all $s \neq t$ and

$$
\left(\int_{t-\frac{7}{8}r^2}^{t+\frac{1}{8}r^2}\left(\int_{D\cap\{s=t\}} |\nabla u|^2 \,\mathrm{d}y\right)^2 \mathrm{d}s\right) \leq \left(\int_{\mathrm{pr}_t D}\left(\int_{D\cap\{s=t\}} |\nabla u|^2 \,\mathrm{d}y\right)^2 \mathrm{d}s\right)
$$

which is bounded by assumption.
Thus, Proposition 11.2 implies that (x,t) is regular. ☐

A similar result follows by Proposition 1 in the paper.

COROLLARY 11.7. *Let $\int\left(\int |u|^s + |p|^{\frac{s}{2}} \,\mathrm{d}x\right)^{\frac{s'}{s}} \mathrm{d}t < \infty$ for a suitable weak solution (u,p) defined on $D \subset \mathbb{R}^3 \times \mathbb{R}$ and $3 < s \leq s'$ satisfying $\frac{3}{s} + \frac{2}{s'} = 1$. Then (u,p) is regular on D.*

PROOF. Let $(x,t) \in D$ be any point. We estimate using Hölder with exponents $\frac{s'}{3}$ and $\frac{1}{1-\frac{3}{s'}}$ in the time-integration, but first we Hölder in space in both summands with Hölder exponents $\frac{s}{3}$ and $\frac{1}{1-\frac{3}{s}} = \frac{s'}{2}$.

$$
\begin{aligned}
&\limsup_{r\to 0} \frac{1}{r^2}\int_{Q_r^*(x,t)} |u|^3 + |p|^{\frac{3}{2}}\\
=&\limsup_{r\to 0} \frac{1}{r^2}\int_{t-\frac{7}{8}r^2}^{t+\frac{1}{8}r^2}\int_{B_r(x)} 1\cdot|u|^3 + 1\cdot|p|^{\frac{3}{2}} \,\mathrm{d}y\,\mathrm{d}\tau\\
\leq&\limsup_{r\to 0} \frac{1}{r^2}\int_{t-\frac{7}{8}r^2}^{t+\frac{1}{8}r^2}\left(\left(\int_{B_r(x)} |u|^s \,\mathrm{d}y\right)^{\frac{3}{s}} + \left(\int_{B_r(x)} |p|^{\frac{s}{2}} \,\mathrm{d}y\right)^{\frac{3}{s}}\right)\cdot |B_r(x)|^{\frac{2}{s'}} \,\mathrm{d}\tau\\
\leq&\limsup_{r\to 0} \frac{1}{r^2}2\int_{t-\frac{7}{8}r^2}^{t+\frac{1}{8}r^2}\left(\int_{B_r(x)} |u|^s + |p|^{\frac{s}{2}} \,\mathrm{d}y\right)^{\frac{3}{s}}\cdot 1\,\mathrm{d}\tau \cdot |B_r(x)|^{\frac{2}{s'}}\\
\leq&\limsup_{r\to 0} \frac{2}{r^2}\left(\int_{t-\frac{7}{8}r^2}^{t+\frac{1}{8}r^2}\left(|u|^s + |p|^{\frac{s}{2}} \,\mathrm{d}y\right)^{\frac{s'}{s}} \mathrm{d}\tau\right)^{\frac{3}{s'}} (r^2)^{1-\frac{3}{s'}}|B_r(x)|^{\frac{2}{s'}}.
\end{aligned}
$$

Now the second factor involving the time integral over the measure of the ball $B_r(x)$ is proportional to $r^{(2-\frac{6}{s'})+\frac{6}{s'}} = r^2$. Hence, this factors cancelles the prefactor of $\frac{1}{r^2}$ and infer that the limit $r \to 0$ tends to zero on the right hand side by dominated convergence similarly as in the previous proof, which is valid due to $\int\left(\int |u|^s + |p|^{\frac{s}{2}} \,\mathrm{d}x\right)^{\frac{s'}{s}} \mathrm{d}t$ being bounded by assumption. ☐

COROLLARY 11.8. *Let (u,p) be a suitable weak solution of the Navier-Stokes system which has cylindrical symmetry about some axis. Then singularities can only occur on the symmetry axis.*

PROOF. Assume there would be an off-axis singularity. Then, due to symmetry, this would give rise to a whole circle on which the solution would be singular. But this contradicts the fact that $\mathcal{H}^1(\mathcal{S}) \leq C\mathcal{P}^1(\mathcal{S}) = 0$. So, possible singularities can only lie on the axis of symmetry. ☐

11.2. Estimates for u and p in Weighted Norms

In the following we will prove two lemmas that play an important role in the proof of Theorems C and D, which are subject to the subsequent talk.
The first lemma provides a weighted interpolation estimate.

LEMMA 11.9. *Let $\alpha, \beta, \gamma, r, s$ be such that:*
(i) $r \geq 2$, $\gamma + \frac{3}{r} > 0$, $\alpha + \frac{3}{2} > 0$, $\beta + \frac{3}{2} > 0$, *and* $s \in \left[\frac{1}{2}, 1\right]$,
(ii) $\gamma + \frac{3}{r} = s\left(\alpha + \frac{1}{2}\right) + (1-s)\left(\beta + \frac{3}{2}\right)$,
(iii) $s(\alpha - 1) + (1-s)\beta \leq \gamma \leq s\alpha + (1-s)\beta$.
Then there exists a constant $C = C(\alpha, \beta, \gamma, r, s)$ such that for all $\varepsilon \geq 0$ holds the inequality

$$\left\|(\varepsilon + |x|^2)^{\frac{\gamma}{2}} u\right\|_{L^r(\mathbb{R}^3)} \leq C \left\|(\varepsilon + |x|^2)^{\frac{\alpha}{2}} |\nabla u|\right\|^s_{L^2(\mathbb{R}^3)} \left\|(\varepsilon + |x|^2)^{\frac{\beta}{2}} u\right\|^{1-s}_{L^2(\mathbb{R}^3)} \tag{11.6}$$

for all $u \in H^1(\mathbb{R}^3)$ with $\left\|(\varepsilon + |x|^2)^{\frac{\alpha-1}{2}} u\right\|_{L^2(\mathbb{R}^3)} < \infty$.

REMARK 11.10. Note that for functions $u \in H^1(\mathbb{R}^3)$ with compact support of course we do not need to assume the weighted L^2-norm of u with exponent $\alpha - 1$ to be finite. Also in the original paper, this assumption is not stated even for non compactly supported functions u. Nevertheless, it is not clear to us how to relax this condition or even skip it. Hence, we kept this assumption for completeness of the present notes.

PROOF. Suppose we have already proven the lemma for $\varepsilon = 1$, then for $\varepsilon > 0$ we have by rescaling

$$\begin{aligned}
&\left\|(\varepsilon + |x|^2)^{\frac{\gamma}{2}} u\right\|_{L^r} \\
&= \left\|\left(1 + \frac{|x|^2}{\varepsilon}\right)^{\frac{\gamma}{2}} u(x)\right\|_{L^r} \varepsilon^{\frac{\gamma}{2}} \\
&= \left\|(1 + |y|^2)^{\frac{\gamma}{2}} u(\sqrt{\varepsilon} y)\right\|_{L^r} \varepsilon^{\frac{\gamma}{2} + \frac{3}{2r}} \\
&\leq C \left\|(1 + |y|^2)^{\frac{\alpha}{2}} |\nabla_y (u(\sqrt{\varepsilon} y))|^2\right\|^s_{L^2} \left\|(1 + |y|^2)^{\frac{\beta}{2}} u(\sqrt{\varepsilon} y)\right\|^{1-s}_{L^2} \varepsilon^{\frac{\gamma + \frac{3}{r}}{2}} \\
&= C \left\|(1 + |y|^2)^{\frac{\alpha}{2}} \left|\nabla_{\sqrt{\varepsilon} y} (u(\sqrt{\varepsilon} y)) \sqrt{\varepsilon}\right|^2\right\|^s_{L^2} \left\|(1 + |y|^2)^{\frac{\beta}{2}} u(\sqrt{\varepsilon} y)\right\|^{1-s}_{L^2} \varepsilon^{\frac{1}{2}(s(\alpha + \frac{1}{2}) + (1-s)(\beta + \frac{3}{2}))} \\
&= C \left\|(\varepsilon + |\sqrt{\varepsilon} y|^2)^{\frac{\alpha}{2}} \left|\nabla_{\sqrt{\varepsilon} y} (u(\sqrt{\varepsilon} y))\right|^2\right\|^s_{L^2} \left\|(\varepsilon + |\sqrt{\varepsilon} y|^2)^{\frac{\beta}{2}} u(\sqrt{\varepsilon} y)\right\|^{1-s}_{L^2} \varepsilon^{\frac{1}{2}(s\frac{3}{2} + (1-s)\frac{3}{2})} \\
&= C \left\|(\varepsilon + |x|^2)^{\frac{\alpha}{2}} |\nabla u|^2\right\|^s_{L^2} \left\|(\varepsilon + |x|^2)^{\frac{\beta}{2}} u\right\|^{1-s}_{L^2},
\end{aligned}$$

where we used assumption (ii) from the lemma. So, the case $\varepsilon > 0$ follows from $\varepsilon = 1$. For $\varepsilon = 0$ we let $\varepsilon \to 0$ in the inequality for $\varepsilon > 0$. To do so, we use dominated convergence, which is valid as the pointwise almost everywhere convergence $(\varepsilon + |x|^2)^{\frac{\gamma}{2} r} |u|^r \to |x|^{r\gamma} |u|^r$ is clear and for ε small enough and $\gamma \geq 0$ holds $(\varepsilon + |x|^2)^{\frac{\gamma}{2} r} |u|^r \leq (1 + |x|^2)^{\frac{\gamma}{2} r} |u|^r$. This last function is integrable because

$$\begin{aligned}
\left\|(1 + |x|^2)^{\frac{\gamma}{2}} u\right\|_{L^r} &\leq C \left\|(1 + |x|^2)^{\frac{\alpha}{2}} |\nabla u|^2\right\|^s_{L^2} \left\|(1 + |x|^2)^{\frac{\beta}{2}} u\right\|^{1-s}_{L^2} \\
&\leq M + C \left\||x|^\alpha |\nabla u|^2\right\|^s_{L^2} \left\||x|^\beta u\right\|^{1-s}_{L^2} < \infty
\end{aligned}$$

for some number $M < \infty$, and we assume the right hand side of (11.6) to be finite for $\varepsilon = 0$, since else the statement of the lemma is trivially true. For $\gamma < 0$, we simply estimate $(\varepsilon + |x|^2)^{\frac{\gamma}{2} r} |u|^r \leq |u|^r \in L^1$ and respecting (11.9). The convergence $\varepsilon \to 0$ on the right hand side of (11.6) is treated similarly.
We now want to simplify the proof in terms of which functions u need to be considered.

Again, without loss of generality, we assume that both weighted norms of u on the right hand side of (11.6) are finite, since else the inequality is trivially satisfied.
Assume further that we have shown the inequality (11.6) already for all functions in H^1 with compact support. Then for a general $u \in H^1(\mathbb{R}^3)$ we define $\varphi \in C_0^\infty(\mathbb{R}^3)$ to be a radially-symmetric function with $0 \le \varphi \le 1$, $\varphi = 1$ on $B_1(0)$ and $\varphi = 0$ outside $B_2(0)$, and $|\nabla\varphi| \le 2$. Then, the sequence of smooth functions $(\varphi_n)_n := \left(\varphi\left(\frac{\cdot}{n}\right)\right)_n$ tends to 1 a.e. on $\mathbb{R}^3$ and $(\nabla\varphi_n)_n$ tends to zero almost everywhere. So, $\varphi_n u \in H^1(\mathbb{R}^3)$ has compact support and we can estimate

$$\begin{aligned}\left\|(1+|x|^2)^{\frac{\gamma}{2}}\varphi_n u\right\|_{L^r(\mathbb{R}^3)} &\le C\left\|(1+|x|^2)^{\frac{\alpha}{2}}|\nabla(\varphi_n u)|\right\|_{L^2(\mathbb{R}^3)}^s \left\|(1+|x|^2)^{\frac{\beta}{2}}(\varphi_n u)\right\|_{L^2(\mathbb{R}^3)}^{1-s} \\ &\le C\left\|(1+|x|^2)^{\frac{\alpha}{2}}|\nabla\varphi_n| u\right\|_{L^2(\mathbb{R}^3)}^s \left\|(1+|x|^2)^{\frac{\beta}{2}}\varphi_n u\right\|_{L^2(\mathbb{R}^3)}^{1-s} \\ &\quad + C\left\|(1+|x|^2)^{\frac{\alpha}{2}}\varphi_n|\nabla u|\right\|_{L^2(\mathbb{R}^3)}^s \left\|(1+|x|^2)^{\frac{\beta}{2}}\varphi_n u\right\|_{L^2(\mathbb{R}^3)}^{1-s} \quad (11.7)\end{aligned}$$

All norms in (11.7) involving φ_n tend to the desired norm by monotone convergence, e.g.

$$\left\|(1+|x|^2)^{\frac{\gamma}{2}}\varphi_n u\right\|_{L^r(\mathbb{R}^3)} \overset{n\to\infty}{\to} \left\|(1+|x|^2)^{\frac{\gamma}{2}} u\right\|_{L^r(\mathbb{R}^3)}.$$

The only problematic term is $\left\|(1+|x|^2)^{\frac{\alpha}{2}}|\nabla\varphi_n| u\right\|_{L^2(\mathbb{R}^3)}$ as $(\nabla\varphi_n)_n$ is not a monotonically increasing sequence of functions. But here (and in fact only here) the assumption discussed in Remark 11.10 comes into play and we have

$$\begin{aligned}\left|(1+|x|^2)^{\frac{\alpha}{2}}|\nabla\varphi_n| u\right|^2 &\le \left|(1+|x|^2)^{\frac{\alpha}{2}}\frac{2}{n}u\cdot\chi_{B_{2n}(0)\setminus B_n(0)}\right|^2 \\ &= \left|(1+|x|^2)^{\frac{\alpha}{2}}\frac{4\sqrt{2}}{\sqrt{2}2n}u\cdot\chi_{B_{2n}(0)\setminus B_n(0)}\right|^2 \\ &\le \left|\frac{4\sqrt{2}(1+|x|^2)^{\frac{\alpha}{2}}}{\sqrt{1+|x|^2}}u\cdot\chi_{B_{2n}(0)\setminus B_n(0)}\right|^2 \\ &\le \left|4\sqrt{2}(1+|x|^2)^{\frac{\alpha-1}{2}}u\right|^2 \in L^1(\mathbb{R}^3),\end{aligned}$$

since $\sqrt{|x|^2+1} \le \sqrt{2|x|^2} \le \sqrt{2}2n$ for all $n \in \mathbb{N}$. By dominated convergence we infer that all terms in (11.7) converge to the desired norms.
Now it suffices to show the inequality (11.6) only for smooth compactly supported functions. Indeed, assume u is supported in $B_M(0)$, i.e. $u \in H^1(B_M(0))$. Clearly, for all $1 \le q \le \infty$ and all measurable functions f holds

$$\|f\|_{L^q(B_M(0))} \le \left\|(1+|x|^2)^{\frac{\delta}{2}}f\right\|_{L^q(\mathbb{R}^3)} \le (1+M^2)^{\frac{\delta}{2}}\|f\|_{L^q(B_M(0))}.$$

Thus, the weighted norm and the unweighted norm on $L^q(B_M(0))$ are equivalent. Note that by (11.9) (see below) and assumption (i) we have $r \in [2,6]$. So by the continuous Sobolev embedding $H^1(B_M(0)) \hookrightarrow L^6(B_M(0))$ and the fact that $u \in H_0^1(B_M(0))$ we can choose a sequence $u_n \in C_0^\infty(B_M(0))$ with $u_n \to u$ in H^1 as $n \to \infty$. Note that the convergence in H^1 implies the convergence in L^6 and hence in L^r and L^2 by Hölder. Thus,

$$\begin{aligned}\left\|(1+|x|^2)^{\frac{\gamma}{2}}u\right\|_{L^r(B_M(0))} &\leftarrow \left\|(1+|x|^2)^{\frac{\gamma}{2}}u_n\right\|_{L^r(B_M(0))} \\ &\le C\left\|(1+|x|^2)^{\frac{\alpha}{2}}|\nabla u_n|\right\|_{L^2(B_M(0))}^s \left\|(1+|x|^2)^{\frac{\beta}{2}}u_n\right\|_{L^2(B_M(0))}^{1-s} \\ &\to C\left\|(1+|x|^2)^{\frac{\alpha}{2}}|\nabla u|\right\|_{L^2(B_M(0))}^s \left\|(1+|x|^2)^{\frac{\beta}{2}}u\right\|_{L^2(B_M(0))}^{1-s},\end{aligned}$$

hence the lemma follows for u compactly supported.
To sum up, it suffices to prove the lemma for $\varepsilon = 1$ and $u \in C_0^\infty(\mathbb{R}^3)$.
We introduce the notation $\tau := (1+|x|^2)^{\frac{1}{2}}$, $A := \|\tau^\alpha|\nabla u|\|_{L^2}$, and $B := \|\tau^\beta u\|_{L^2}$.
We first consider the case $r = 2$:
Assumption (i) implies $\gamma > -\frac{3}{2}$ and assumption (ii) implies

$$\gamma = s\left(\alpha + \frac{1}{2}\right) + (1-s)\left(\beta + \frac{3}{2}\right) - \frac{3}{2} = s(\alpha - 1) + (1-s)\beta.$$

We now introduce spherical coordinates (ρ, θ) on $\mathbb{R}^3$ to obtain

$$\begin{aligned}\int_{\mathbb{R}^3} \tau^{2\gamma}|u|^2 \,\mathrm{d}x &= \int_{S^2}\int_0^\infty \tau^{2\gamma}|u|^2\rho^2 \,\mathrm{d}\rho\,\mathrm{d}\theta \\ &= \int_{S^2}\int_0^\infty \tau^{2\gamma}|u|^2\tau\rho \,\mathrm{d}\rho\,\mathrm{d}\theta + \int_{S^2}\int_0^\infty \tau^{2\gamma}|u|^2(\rho^2 - \tau\rho)\,\mathrm{d}\rho\,\mathrm{d}\theta.\end{aligned}$$

By partial integration in ρ, while recalling $u \in C_0^\infty$ and the triviality $\rho\big|_{\rho=0} = 0$, we get

$$\begin{aligned}\int_{S^2}\int_0^\infty \tau^{2\gamma+1}\rho|u|^2 \,\mathrm{d}\rho\,\mathrm{d}\theta &= -\int_{S^2}\int_0^\infty \frac{\rho^2}{2}\left(u\cdot\partial_r u\tau^{2\gamma+1} + |u|^2(2\gamma+1)\tau^{2\gamma}\frac{\rho}{\tau}\right)\mathrm{d}\rho\,\mathrm{d}\theta \\ &\leq -\int_{S^2}\int_0^\infty \rho^3\tau^{2\gamma-1}|u|^2\left(\gamma + \frac{1}{2}\right) + \rho^2\tau^{2\gamma+1}|u||\nabla u|\,\mathrm{d}\rho\,\mathrm{d}\theta.\end{aligned}$$

Thus,

$$\int_{\mathbb{R}^3} \tau^{2\gamma}|u|^2 \,\mathrm{d}x \leq \int_{S^2}\int_0^\infty \tau^{2\gamma+1}\rho^2|u||\nabla u| + |u|^2\tau^{2\gamma}\rho^2\left(1 - \frac{\tau}{\rho} - \left(\gamma + \frac{1}{2}\frac{\rho}{\tau}\right)\right)\mathrm{d}\rho\,\mathrm{d}\theta.$$

Now notice that by $\gamma > -\frac{3}{2}$ we have $\gamma \geq -\frac{3}{2} + \bar{C}$ for some constant $1 > \bar{C} > 0$. Hence,

$$\begin{aligned}\frac{\tau}{\rho} + \left(\gamma + \frac{1}{2}\right)\frac{\rho}{\tau} &\geq (1-\bar{C})\frac{\sqrt{1+|x|^2}}{|x|} - (1-\bar{C})\frac{|x|}{\sqrt{1+|x|^2}} + \bar{C}\frac{\sqrt{1+|x|^2}}{|x|} \\ &\geq \bar{C}\frac{\sqrt{1+|x|^2}}{|x|} \geq \bar{C}.\end{aligned}$$

Using the last estimate and rearranging yields

$$\begin{aligned}\int_{\mathbb{R}^3} \tau^{2\gamma}|u|^2 &\leq \frac{1}{\bar{C}}\int_{S^2}\int_0^\infty \tau^{2\gamma+1}\rho^2|u||\nabla u|\,\mathrm{d}\rho\,\mathrm{d}\theta \\ &= \frac{1}{\bar{C}}\int_{\mathbb{R}^3} \tau^{2\gamma+1}|u||\nabla u|\,\mathrm{d}x \\ &= \frac{1}{\bar{C}}\int_{\mathbb{R}^3} \tau^{1+\frac{\gamma}{2}}(\tau^\gamma|u|)^{2-\frac{1}{s}}|\nabla u||u|^{\frac{1}{s}-1}\,\mathrm{d}x \\ &= \frac{1}{\bar{C}}\int_{\mathbb{R}^3} (\tau^\gamma|u|)^{2-\frac{1}{s}}(\tau^\alpha|\nabla u|)(\tau^\beta|u|)^{\frac{1}{s}-1}\,\mathrm{d}x,\end{aligned}$$

where we used (ii), i.e. $1 + \frac{\gamma}{s} = \alpha + \beta\left(\frac{1}{s} - 1\right)$.
Note that $\frac{1}{s} \in [1,2]$ and $\frac{1}{2} + \frac{1}{2}\left(\frac{1}{s} - 1\right) + \frac{1}{2}\left(2 - \frac{1}{s}\right) = 1$. Thus, we can apply the Hölder inequality for three factors with exponents 2, $\frac{2}{\frac{1}{s}-1}$, and $\frac{2}{2-\frac{1}{s}}$ to obtain

$$\begin{aligned}\int_{\mathbb{R}^3} \tau^{2\gamma}|u|^2 &\leq \frac{1}{\bar{C}}\left(\int_{\mathbb{R}^3}(\tau^\gamma|u|)^2\,\mathrm{d}x\right)^{1-\frac{1}{2s}}\left(\int_{\mathbb{R}^3}(\tau^\beta|u|)^2\,\mathrm{d}x\right)^{\frac{1}{2s}-\frac{1}{2}}\left(\int_{\mathbb{R}^3}\tau^\alpha|\nabla u|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \\ &= \frac{1}{\bar{C}}\left(\int_{\mathbb{R}^3}(\tau^\gamma|u|)^2\,\mathrm{d}x\right)^{1-\frac{1}{2s}} B^{\frac{1}{s}-1}A.\end{aligned}$$

Rearranging and taking the whole inequality to the power s yields

$$\left(\int_{\mathbb{R}^3} \tau^{2\gamma}|u|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \leq CB^{1-s}A^s,$$

finishing the proof in the case $r = 2$.

Now consider $r > 2$:
Define $R_k := \{2^{k-1} < |x| \leq 2^k\}$. We note that $q := \frac{1}{\frac{1}{2}-\frac{s}{3}} = \frac{6}{3-2s} \in [3,6]$ as $s \in \left[\frac{1}{2},1\right]$. Therefore, by standard Lebesgue space interpolation on R_k with $\frac{1}{q} = \frac{1-s}{2} + \frac{s}{6}$ and Poincare's inequality on balls (cf. section 4.5.2 in [**EG92**]) holds

$$\begin{aligned}
\|u\|_{L^q(R_k)} &\leq \|u-\bar{u}\|_{L^q(R_k)} + \|\bar{u}\|_{L^q(R_k)} \\
&\leq \|u-\bar{u}\|^{1-s}_{L^2(R_k)}\|u-\bar{u}\|^s_{L^6(R_k)} + \|\bar{u}\|_{L^q(R_k)} \\
&\leq C\|u-\bar{u}\|^{1-s}_{L^2(R_k)}\|\nabla u\|^s_{L^2(R_k)} + Cd_k^{-3+\frac{3}{q}}\int_{R_k}|u|\,\mathrm{d}x \\
&\leq C\|u\|^{1-s}_{L^2(R_k)}\|\nabla u\|^s_{L^2(R_k)} + Cd_k^{-3+\frac{3}{q}}\sqrt{|R_k|}\cdot\|u\|_{L^2(R_k)} \\
&\leq C\|\nabla u\|^s_{L^2(R_k)}\|u\|^{1-s}_{L^2(R_k)} + \frac{C}{d_k^s}\|u\|_{L^2(R_k)},
\end{aligned} \tag{11.8}$$

where we introduced the shorthand notation $d_k := \operatorname{diam}(R_k)$ and $\bar{u} := \frac{1}{|R_k|}\int_{R_k} u \,\mathrm{d}x$. We also used the estimate

$$\begin{aligned}
\|u-\bar{u}\|_{L^2(R_k)} &= \left(\int_{R_k}\left|u - \frac{1}{|R_k|}\int_{R_k} u\,\mathrm{d}y\right|^2 \mathrm{d}x\right)^{\frac{1}{2}} \\
&\leq C\|u\|_{L^2(R_k)} + C\left(\int_{R_k}\left|\frac{1}{|R_k|}\int_{R_k} u\,\mathrm{d}y\right|^2\right)^{\frac{1}{2}} \\
&\leq C\|u\|_{L^2(R_k)} + C\left(\int_{R_k}\mathrm{d}x\frac{1}{|R_k|}\int_{R_k}|u|^2\,\mathrm{d}y\right)^{\frac{1}{2}} \\
&= C\|u\|_{L^2(R_k)},
\end{aligned}$$

which follows from Jensen's inequality. Note that C does not depend on d_k. Another remark on (11.8) is concerning the use of the Poincare inequality for balls: By Theorem 1 in section 5.4 in [**Eva10**] there is a continuous linear extension operator $E\colon H^1(R_k) \to H^1\left(B_{22^k}(0)\right)$. This means we estimate as follows

$$\begin{aligned}
\|u-\bar{u}\|_{L^6(R_k)} &= \|Eu - \overline{Eu}\|_{L^6(R_k)} \leq \|Eu-\overline{Eu}\|_{L^6\left(B_{22^k}(0)\right)} \\
&\leq C\|\nabla Eu\|_{L^2\left(B_{22^k}(0)\right)} \leq C\|\nabla u\|_{L^2(R_k)}.
\end{aligned}$$

Moreover, assumptions (ii) and (iii) yield $s\left(\alpha+\frac{1}{2}\right)+(1-s)\left(\beta+\frac{3}{2}\right) = \gamma+\frac{3}{r} \leq s\alpha+(1-s)\beta+\frac{3}{r}$. Hence, $\frac{s}{2}+\frac{3}{2}-\frac{3s}{2} \leq \frac{3}{r}$, which rearranged gives

$$\frac{1}{r} \geq \frac{1}{2} - \frac{s}{3}. \tag{11.9}$$

Or in other words, $r \leq q$. So, Hölder with exponents $\frac{q}{r}$ and $\frac{q}{q-r}$leads to

$$\|\tau^\gamma u\|_{L^r(R_k)} \leq C|R_k|^{\frac{q-r}{qr}}\|\tau^\gamma u\|_{L^q(R_k)} \leq Cd_k^{3\left(\frac{1}{r}-\frac{1}{q}\right)+\gamma}\|u\|_{L^q(R_k)}, \tag{11.10}$$

where we used that d_k is comparable to τ on R_k. Indeed, $d_k = 2\cdot 2^k$, hence

$$\tau = \sqrt{1+|x|^2} \geq |x| \geq 2^{k-1} = \frac{1}{4}d_k.$$

On the other hand,

$$\tau = \sqrt{1+|x|^2} \le \sqrt{1+2^{2k}} \le \sqrt{2\cdot 2^{2k}} = \sqrt{2}2^k = \sqrt{2}d_k.$$

Thus, we can combine the non-weighted interpolation estimate (11.8) with the Hölder estimate (11.10) to obtain

$$\begin{aligned}\|\tau^\gamma u\|_{L^r(R_k)} &\le C d_k^{3\left(\frac{1}{r}-\frac{1}{q}\right)+\gamma}\left(\|\nabla u\|^s_{L^2(R_k)}\|u\|^{1-s}_{L^2(R_k)} + \frac{1}{d_k^s}\|u\|_{L^2(R_k)}\right)\\ &\le C\|\tau^\alpha|\nabla u|\|^s_{L^2(R_k)}\|\tau^\beta u\|^{1-s}_{L^2(R_k)} + C\|\tau^\delta u\|_{L^2(R_k)},\end{aligned} \tag{11.11}$$

where we used the definition $\delta := \gamma + \frac{3}{r} - \frac{3}{2} = \gamma + \frac{3}{r} + \frac{3}{q} - s$ and the identity

$$\begin{aligned}\frac{3}{r} - \frac{3}{q} + \gamma &= s\left(\alpha + \frac{1}{2}\right) + (1-s)\left(\beta + \frac{3}{2}\right) - \frac{3}{q} = s\alpha + (1-s)\beta + \frac{s}{2} + (1-s)\frac{3}{2} - \frac{3}{q}\\ &= s\alpha + (1-s)\beta,\end{aligned}$$

which follows from (ii).

We now take the sum of the inequalities (11.11) over k to obtain the inequality on the whole $\mathbb{R}^3$. Note that then the left hand side is estimated by $a^{\frac{1}{r}} + b^{\frac{1}{r}} \le 2^{1-\frac{1}{r}}(a+b)^{\frac{1}{r}}$ for $a, b \ge 0$ by concavity, and by using that $R_k \cap R_j = \varnothing$ for $k \ne j$ the sum over the integrals is simply the integral over $\mathbb{R}^3$. On the right hand side we use Minkowski's inequality on the summands with $\tau^\delta u$ and for the others we estimate using concavity of the square root and Hölder with exponents $\frac{1}{s}$ and $\frac{1}{1-s}$ in the sum over k to get

$$\begin{aligned}&\sum_{k=0}^\infty \left(\int_{R_k}\tau^{2\alpha}|\nabla u|^2 dx\right)^{\frac{s}{2}}\left(\int_{R_k}\tau^{2\beta}|u|^2 dx\right)^{\frac{1-s}{2}}\\ &\le \sqrt{2}\left(\sum_{k=0}^\infty\left(\int_{R_k}\tau^{2\alpha}|\nabla u|^2 dx\right)^s\cdot\left(\int_{R_k}\tau^{2\beta}|u|^2 dx\right)^{1-s}\right)^{\frac{1}{2}}\\ &\le \sqrt{2}\left(\left(\sum_{k=0}^\infty\int_{R_k}\tau^{2\alpha}|\nabla u|^2 dx\right)^s\cdot\left(\sum_{k=0}^\infty\int_{R_k}\tau^{2\beta}|u|^2 dx\right)^{1-s}\right)^{\frac{1}{2}}\\ &= \sqrt{2}A^s B^{1-s}.\end{aligned}$$

We conclude the proof by applying the case $r = 2$ to the term $C\|\tau^\delta u\|_{L^2(\mathbb{R}^3)}$ on the right hand side with δ playing the role of γ, which is valid since $\delta = \gamma + \frac{3}{r} - \frac{3}{2} > -\frac{3}{2}$ by (i). This means, we can estimate

$$\|\tau^\delta u\|_{L^2(\mathbb{R}^3)} \le CA^s B^{1-s},$$

which finishes the proof also for the case $r > 2$. □

Let us now prove a lemma concerning weighted-norm bounds of the singular integral operator $(-\Delta)^{-1}\operatorname{div}\operatorname{div}$, i.e. for later use we want to relate weighted norms of the pressure and the velocity field of a suitable weak solution.

LEMMA 11.11. *If $p \in L^3(\mathbb{R}^3)$ is the solution of the differential equation*

$$-\Delta p = \operatorname{div}\operatorname{div}(u\otimes u) \tag{11.12}$$

on $\mathbb{R}^3$ for a function $u \in H^1(\mathbb{R}^3)$. Then for r, γ satisfying $1 < r < \infty$ and $-\frac{3}{r} < \gamma < 3 - \frac{3}{r}$ there exists a constant C such that for all $\varepsilon \ge 0$ holds

$$\left\|(\varepsilon + |x|^2)^{\frac{\gamma}{2}} p\right\|_{L^r(\mathbb{R}^3)} \le C\left\|(\varepsilon+|x|^2)^{\frac{\gamma}{4}} u\right\|^2_{L^{2r}(\mathbb{R}^3)}. \tag{11.13}$$

PROOF. Without loss of generality assume the right hand side of (11.13) is finite, else the statement is trivial.
Define the operator $T_{ij}f := c\partial_i\partial_j\left(\frac{1}{x}\right)\star f$ for $f \in C_0^\infty(\mathbb{R}^3)$. By Theorem B.6 in [**RRS16**] this extends to a linear bounded operator from $L^3(\mathbb{R}^3)$ to itself. Note that $u_iu_j \le |u|^2 \in L^3(\mathbb{R}^3)$ for all i,j, since $u \in H^1$. Thus, for all i,j choose a sequence of testfunctions $\varphi_n^{ij} \overset{L^3}{\to} u_iu_j$. Moreover, define

$$p := -|u|^2 + \sum_{i\neq j} T_{ij}(u_iu_j).$$

By the above we know $p \in L^3(\mathbb{R}^3)$. This is also the unique solution to (11.12) in L^3. Indeed, let $\tilde{p} \in L^3$ be another distributional solution. We check that p is a distributional solution. For that let $\psi \in C_c^\infty(\mathbb{R}^3)$ and observe

$$\begin{aligned}
\langle -\Delta p, \psi\rangle &= \sum_i \langle -u_iu_i, -\partial_i\partial_i\psi\rangle + \sum_{i\neq j}\langle T_{ij}(u_iu_j), -\Delta\psi\rangle \\
&\leftarrow \sum_i \langle u_iu_i, \partial_i\partial_i\psi\rangle + \sum_{i\neq j}\langle T_{ij}(\varphi_n^{ij}), -\Delta\psi\rangle \\
&= \sum_i \langle \partial_i\partial_i u_iu_i, \psi\rangle + \sum_{i\neq j}\langle \varphi_n^{ij}, \partial_i\partial_j(-\Delta)^{-1}(-\Delta)\psi\rangle \\
&\to \sum_i \langle \partial_i\partial_i u_iu_i, \psi\rangle + \sum_{i\neq j}\langle u_iu_j, \partial_i\partial_j\psi\rangle \\
&= \sum_i \langle \partial_i\partial_i u_iu_i, \psi\rangle + \sum_{i\neq j}\langle \partial_i\partial_j(u_iu_j), \psi\rangle \\
&= \langle \operatorname{div}\operatorname{div}(u\otimes u), \psi\rangle.
\end{aligned}$$

Therefore, $\Delta(p-\tilde{p}) = 0$ and $p-\tilde{p} \in L^3$, hence in particular $p-\tilde{p} \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$. By Weyl's Lemma (cf. Theorem C.3 in [**RRS16**]) we obtain that $p-\tilde{p}$ is smooth. But since $p-\tilde{p}$ also lies in L^3 we get by the mean value property for all $x \in \mathbb{R}^3$ using Hölder

$$|(p-\tilde{p})(x)| \le \frac{C}{r^3}\int_{B_r(x)} |p-\tilde{p}|\,\mathrm{d}x \le \frac{C}{r^{3-\frac{3}{2}}}\|p-\tilde{p}\|_{L^3(\mathbb{R}^3)} \overset{r\to\infty}{\to} 0.$$

Thus, p defined above is the unique solution in L^3.
We show that for $i \neq j$ holds

$$\|(1+|x|^2)^{\frac{\gamma}{2}}T_{ij}(f)\|_{L^r(\mathbb{R}^3)} \le C\|(1+|x|^2)^{\frac{\gamma}{2}}f\|_{L^r(\mathbb{R}^3)} \tag{11.14}$$

for all f such that the right hand side is finite. This then proves the lemma, since by scaling we infer for $\varepsilon > 0$

$$\begin{aligned}
\left(\int_{\mathbb{R}^3}(\varepsilon+|x|^2)^{\frac{r\gamma}{2}}(T_{ij}f)^r(x)\,\mathrm{d}x\right)^{\frac{1}{r}} &= \varepsilon^{\frac{\gamma}{2}}\left(\int_{\mathbb{R}^3}\left(1+\left(\frac{|x|}{\sqrt{\varepsilon}}\right)^2\right)^{\frac{r\gamma}{2}}(T_{ij}f)^r(x)\,\mathrm{d}x\right)^{\frac{1}{r}} \\
&= \varepsilon^{\frac{\gamma}{2}+\frac{3}{2r}}\left(\int_{\mathbb{R}^3}(1+|y|^2)^{\frac{r\gamma}{2}}(T_{ij}f)^r(\sqrt{\varepsilon}y)\,\mathrm{d}y\right)^{\frac{1}{r}} \\
&= \varepsilon^{\frac{\gamma}{2}+\frac{3}{2r}}\left(\int_{\mathbb{R}^3}(1+|y|^2)^{\frac{r\gamma}{2}}(T_{ij}(f(\sqrt{\varepsilon}\cdot))(y))^r\,\mathrm{d}y\right)^{\frac{1}{r}} \\
&\le C\varepsilon^{\frac{\gamma}{2}+\frac{3}{2r}}\left(\int_{\mathbb{R}^3}(1+|y|^2)^{\frac{r\gamma}{2}}f^r(\sqrt{\varepsilon}y)\,\mathrm{d}y\right)^{\frac{1}{r}}
\end{aligned}$$

$$= C\varepsilon^{\frac{\gamma}{2}} \left(\int_{\mathbb{R}^3} \left(1 + \left(\frac{|x|}{\sqrt{\varepsilon}} \right)^2 \right)^{\frac{r\gamma}{2}} f^r(x) \,\mathrm{d}x \right)^{\frac{1}{r}}$$

$$= C \left(\int_{\mathbb{R}^3} (\varepsilon + |x|^2)^{\frac{r\gamma}{2}} f^r(x) \,\mathrm{d}x \right)^{\frac{1}{r}}.$$

In the above we used

$$\begin{aligned}
(T_{ij}f)(\sqrt{\varepsilon}y) &= \mathrm{p.v.} \int_{\mathbb{R}^3} C \left(\partial_i \partial_j \frac{1}{|\cdot|} \right) (x - \sqrt{\varepsilon}y) f(x) \,\mathrm{d}x \\
&= \mathrm{p.v.} \int_{\mathbb{R}^3} C \frac{(\sqrt{\varepsilon}y_i - x_i)(\sqrt{\varepsilon}y_j - x_j)}{|\sqrt{\varepsilon}y - x|^5} f(x) \,\mathrm{d}x \\
&= \mathrm{p.v.}\, C\varepsilon^{-\frac{3}{2}} \int_{\mathbb{R}^3} \frac{\left(y_i - \frac{x_i}{\sqrt{\varepsilon}}\right)\left(y_j - \frac{x_j}{\sqrt{\varepsilon}}\right)}{\left|y - \frac{x}{\sqrt{\varepsilon}}\right|^5} f(x) \,\mathrm{d}x \\
&= \mathrm{p.v.}\, C\varepsilon^{-\frac{3}{2}+\frac{3}{2}} \int_{\mathbb{R}^3} \frac{(y_i - z_i)(y_j - z_j)}{|y - z|^5} f(\sqrt{\varepsilon}z) \,\mathrm{d}z \\
&= T(f(\sqrt{\varepsilon}\cdot))(y)
\end{aligned}$$

Note that the case $\varepsilon = 0$, i.e.

$$\left\| |x|^\gamma T_{ij}(f) \right\|_{L^r(\mathbb{R}^3)} \le C \left\| |x|^\gamma f \right\|_{L^r(\mathbb{R}^3)},$$

is proven in **[Ste57]**, where the bounds on γ stated in the lemma are exactly chosen to fit into Stein's theorem. Indeed, using $\partial_i \partial_j \left(\frac{1}{|x|} \right) = 3 \frac{x_i x_j}{|x|^5}$ the function $H(x, x-y)$ appearing in Stein's theorem, which is defined by

$$\left(\partial_i \partial_j \frac{1}{|\cdot|} \right) (x - y) =: \frac{1}{|x-y|^3} \cdot H(x, x-y),$$

satisfies the bound

$$|H(x, x-y)| = 3 \frac{|x_i - y_i||x_j - y_j|}{|x-y|^2} \le 3.$$

Thus, all requirements for Stein's theorem are satisfied.
So, assume (11.14) holds, then for $\varepsilon \ge 0$ we get

$$\begin{aligned}
\left\| (\varepsilon + |x|^2)^{\frac{\gamma}{2}} p \right\|_{L^r(\mathbb{R}^3)} &= \left\| (\varepsilon + |x|^2)^{\frac{\gamma}{2}} \left(-|u|^2 + \sum_{i \ne j} T_{ij}(u_i u_j) \right) \right\|_{L^r(\mathbb{R}^3)} \\
&\le \left\| (\varepsilon + |x|^2)^{\frac{\gamma}{2}} u \right\|^2_{L^{2r}(\mathbb{R}^3)} + \sum_{i \ne j} C \left\| (\varepsilon + |x|^2)^{\frac{\gamma}{2}} u_i u_j \right\|_{L^r} \\
&\le \left\| (\varepsilon + |x|^2)^{\frac{\gamma}{2}} u \right\|^2_{L^{2r}(\mathbb{R}^3)} + \sum_{i \ne j} C \left\| (\varepsilon + |x|^2)^{\frac{\gamma}{2}} u \right\|^2_{L^{2r}} \\
&\le \left\| (\varepsilon + |x|^2)^{\frac{\gamma}{2}} u \right\|^2_{L^{2r}(\mathbb{R}^3)}.
\end{aligned}$$

Hence, it is left to prove (11.14). The case $\gamma = 0$ corresponds to the classical Calderon-Zygmund estimate

$$\|T_{ij}(f)\|_{L^r} \le C \|f\|_{L^r}.$$

Now decompose f into $f = f_1 + f_2$, where $f_1 = \chi_{|x| \le 1}$ and $f_2 = \chi_{|x|>1}$. Note that $(1 + |x|^2)^{\frac{\gamma}{2}} \le 1 \le 2^{\frac{\gamma}{2}}(1 + |x|^\gamma)$ for $\gamma \le 0$ and else for $|x| \le 1$ holds

$$(1 + |x|^2)^{\frac{\gamma}{2}} \le 2^{\frac{\gamma}{2}} \le 2^{\frac{\gamma}{2}}(1 + |x|^\gamma),$$

whereas for $|x| > 1$ holds

$$(1+|x|^2)^{\frac{\gamma}{2}} \le 2^{\frac{\gamma}{2}}|x|^\gamma \le 2^{\frac{\gamma}{2}}(1+|x|^\gamma).$$

Now using the estimates for $\gamma = 0$ and $\varepsilon = 0$ proven before we obtain for positive γ

$$\begin{aligned}
\left\|(1+|x|^2)^{\frac{\gamma}{2}}T_{ij}f\right\|_{L^r}^r \le& \left\|(1+|x|^2)^{\frac{\gamma}{2}}T_{ij}f_1\right\|_{L^r}^r + \left\|(1+|x|^2)^{\frac{\gamma}{2}}T_{ij}f_2\right\|_{L^r}^r \\
\le& C\left\|T_{ij}f_1\right\|_{L^r}^r + C\left\||x|^\gamma T_{ij}f_1\right\|_{L^r}^r + C\left\|T_{ij}f_2\right\|_{L^r}^r + C\left\||x|^\gamma T_{ij}f_2\right\|_{L^r}^r \\
\le& C\left\|f_1\right\|_{L^r}^r + C\left\||x|^\gamma f_1\right\|_{L^r}^r + C\left\|f_2\right\|_{L^r}^r + C\left\||x|^\gamma f_2\right\|_{L^r}^r \\
\le& C\left\|(1+|x|^2)^{\frac{\gamma}{2}}f_1\right\|_{L^r}^r + C\left\|(1+|x|^2)^{\frac{\gamma}{2}}f_1\right\|_{L^r}^r \\
&+ C\left\|(1+|x|^2)^{\frac{\gamma}{2}}f_2\right\|_{L^r}^r + C\left\|(1+|x|^2)^{\frac{\gamma}{2}}f_2\right\|_{L^r}^r \\
=& C\left\|(1+|x|^2)^{\frac{\gamma}{2}}f\right\|_{L^r}^r.
\end{aligned}$$

For $\gamma \le 0$ we estimate

$$\begin{aligned}
\left\|(1+|x|^2)^{\frac{\gamma}{2}}T_{ij}f\right\|_{L^r}^r \le& \left\|(1+|x|^2)^{\frac{\gamma}{2}}T_{ij}f_1\right\|_{L^r}^r + \left\|(1+|x|^2)^{\frac{\gamma}{2}}T_{ij}f_2\right\|_{L^r}^r \\
\le& \left\|T_{ij}f_1\right\|_{L^r}^r + \left\||x|^\gamma T_{ij}f_2\right\|_{L^r}^r \\
\le& C\left\|\frac{2}{2}\cdot f_1\right\|_{L^r}^r + C\left\|\frac{2}{2|x|^{-\gamma}}f_2\right\|_{L^r}^r \\
\le& C\left\|2(1+|x|^2)^{\frac{\gamma}{2}}f_1\right\|_{L^r}^r + C\left\|2(1+|x|^2)^{\frac{\gamma}{2}}f_2\right\|_{L^r}^r \\
=& C\left\|(1+|x|^2)^{\frac{\gamma}{2}}f\right\|_{L^r}^r.
\end{aligned}$$

This completes the proof of the lemma. □

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