Some Dimensionless Estimates

In order to use the generalized energ inequality effectively, we must bound the terms on the right

$$\iint (|u|^2 + 2p)\, u \cdot \nabla \phi$$

in terms of those on the left $\int |u|^2 \phi$ and $\iint |\nabla u|^2 \phi$.

Bounds of this type play a fundamental role in the proofs of both propositions 1 and 2.

Let

$$Q_\rho = Q_\rho(0,0) = \{(x,t) : |x| < \rho, \ -\rho^2 < t < 0\},$$

and take a pair of measurable functions u and p defined on Q_ρ. We consider the following quantities for $r < \rho$:

$$A(r) = \sup_{-r^2 < t < 0} r^{-1} \int_{B_r \times \{t\}} |u|^2, \qquad (1)$$

$$\delta(r) = r^{-1} \iint_{Q_r} |\nabla u|^2, \qquad (2)$$

$$G(r) = r^{-2} \iint_{Q_r} |u|^3, \qquad (3)$$

$$L(r) = r^{-2} \iint_{Q_r} |u| |p - \bar{p}_r|, \qquad (4)$$

$$K(r) = r^{-\frac{13}{4}} \int_{-r^2}^{0} \left(\int_{B_r} |p| \right)^{5/4} d\tau \qquad (5)$$

Here $Q_r = Q_r(0,0)$, $B_r = \{x : |x| < r\}$, and

$$\bar{p}_r = \bar{p}_r(t) = ⨍_{B_r} p(y,t)\, dy.$$

Our estimates will be applied to a SWS of the NSEs However, we shall only use the fact that the above quantities are finite and

$$\Delta p = -\sum_{i,j=1}^{3} \frac{\partial^2}{\partial x^i \partial x^j} u_i u_j = -\sum_{i,j=1}^{3} \partial_i \partial_j u_i u_j$$

$$\nabla \cdot u = 0$$

on $B_\rho \times \{t\}$ for a.e. t, $-\rho^2 < t < 0$. Our goal is to derive estimates for $G(r)$ and $L(r)$ in terms of A, δ and K. (Note: each of (1)-(5) has been scaled to have dimension zero). (scale-invariant)

<u>Lemma 1</u>

$$G(r) \leq C A^{3/4}(r) \left(A^{3/4}(r) + \delta^{3/4}(r) \right)$$

<u>Proof.</u> We use the Galiaro-Nirenberg inequality with $q=3$; $a = \frac{3}{4}(3-2) = \frac{3}{4}$

$$\int_{B_r} |u|^3 \leq C \left(\int_{B_r} |\nabla u|^2 \right)^{3/4} \left(\int_{B_r} |u|^2 \right)^{\frac{3}{4}} + \frac{C}{r^{3/2}} \left(\int_{B_r} |u|^2 \right)^{\frac{3}{2}}$$

Integrating in time and using Hölder inequality, we obtain

$$\iint_{Q_r} |u|^3 \leq C \int_{-r^2}^{0} \left(\int_{B_r} |\nabla u|^2 \right)^{3/4} \left(\int_{B_r} |u|^2 \right)^{\frac{3}{4}} dt + \frac{C}{r^{3/2}} \int_{-r^2}^{0} \left(\int_{B_r} |u|^2 \right)^{\frac{3}{2}} dt$$

$$\overset{\frac{3}{4}+\frac{1}{4}=1,\ \text{Hölder}}{\leq} C \left(\iint_{Q_r} |\nabla u|^2 \right)^{\frac{3}{4}} \left[\int_{-r^2}^{0} \left(\int_{B_r} |u|^2 \right)^3 dt \right]^{\frac{1}{4}} + \frac{C}{r^{3/2}} \int_{-r^2}^{0} \left(\int_{B_r} |u|^2 \right)^{3/2} dt \leq$$

$$\leq C \left(\iint_{Q_r} |\nabla u|^2 \right)^{\frac{3}{4}} \left[r^5 A^3(r) \right]^{\frac{1}{4}} + C r^2 A^{\frac{3}{2}}(r) =$$

$$= C A^{\frac{3}{4}}(r) \left(\delta^{\frac{3}{4}}(r) + A^{\frac{3}{4}}(r) \right) r^2$$ which gives the claimed inequality. $\square$

Lemma 2. Let $r \leq \frac{1}{2}\rho$ then

$$L(r) \leq C \left(\frac{r}{\rho} \right)^{7/5} A^{\frac{1}{5}}(r) G^{1/5}(r) K^{\frac{4}{5}}(\rho) +$$

$$+ C\left(\frac{r}{\rho}\right)^{\frac{5}{3}} G^{1/3}(r)\, G^{\frac{2}{3}}(\rho) + C\, G^{\frac{1}{3}}(r)\, G^{2/3}(2r) +$$

$$+ C r^3 G^{1/3}(r) \sup_{-r^2<t<0} \int_{2r<|y|<\rho} \frac{|u|^2}{|y|^4} \qquad (7)$$

<u>Proof.</u> We choose $\phi \in C_c^\infty(\mathbb{R}^3)$ s.t. $0 \le \phi \le 1$

(8) $\phi(y) = \begin{cases} 1, & \text{if } |y| \le \frac{3}{4}\rho \\ 0 & \text{if } |y| \ge \rho \end{cases}$, $|\nabla\phi| \le C\rho^{-1}$, $|\partial_i\partial_j\phi| \le C\rho^{-2}$.

$C \ne C(\rho)$.

Now, recall,

$$\phi(x)\, p(x,t) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|x-y|} \Delta(\phi p)\, dy =$$

$$= -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|x-y|}\left[p\Delta\phi + 2(\nabla\phi\cdot\nabla p) + \phi\Delta p\right] dy =$$

$$= \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|x-y|}\left[\phi\partial_i\partial_j(u_i u_j) - 2(\nabla\phi\cdot\nabla p) - p\Delta\phi\right] dy$$

We now integrate by parts to remove any derivatives from p and u to give

$$(\phi p)(x) = \frac{1}{4\pi}\int \left[\partial_i\partial_j \frac{1}{|x-y|}\right] \phi u_i u_j\, dy -$$

$$-\frac{1}{2\pi}\int \frac{x_j - y_j}{|x-y|^3}(\partial_i\phi) u_i u_j\, dy + \frac{1}{4\pi}\int \frac{1}{|x-y|}(\partial_i\partial_j\phi) u_i u_j\, dy$$

$$-\frac{1}{2\pi}\int \frac{x_i - y_i}{|x-y|^3}\, p\,\partial_i\phi\, dy + \frac{1}{4\pi}\int \frac{1}{|x-y|}\, p\,\Delta\phi\, dy$$

which we write as $\phi p = \tilde{p} + p_3 + p_4$.

We decompose $\tilde{p}$ further as $\tilde{p} = p_1 + p_2$

$$P_1 = \frac{1}{4\pi} \int_{|y|<2r} \left[\partial_i \partial_j \frac{1}{|x-y|} \right] \cdot \phi u_i u_j \, dy$$

$$P_2 = \frac{1}{4\pi} \int_{|y|>2r} \left[\partial_i \partial_j \frac{1}{|x-y|} \right] \cdot \phi u_i u_j \, dy$$

We note that $\quad (\phi p = p \text{ on } Q_{\frac{3s}{4}})$

$$|p - \bar{p}_r| = \left| \sum_{i=1}^{4} P_i - \bar{P}_i \right| \le \sum_{i=1}^{4} |P_i - \bar{P}_i|, \quad \bar{P}_i = \fint_{B_r} P_i$$

and we estimate the four terms separately.

For P_1, recall that each operator

$$T_{ij}(\psi) = \left(\partial_i \partial_j \frac{1}{4\pi |x|} \right) * \psi$$

is bounded operator on $L^q(\mathbb{R}^3)$ for $1<q<\infty$, by Calderon-Zygmund theorem. Taking $\psi = \phi u_i u_j |_{B_{2r}}$ and $q = \frac{3}{2}$, we obtain

$$\| P_1 \|_{L^{3/2}(B_r)} \le \| P_1 \|_{L^{3/2}(\mathbb{R}^3)} \overset{C\text{-}Z}{\le} C \| |u|^2 \|_{L^{3/2}(B_{2r})} = C \left(\int_{B_{2r}} |u|^3 \right)^{\frac{2}{3}}$$

and so

$$\int_{B_r} |u| |P_1 - \bar{P}_1| \le C \| u \|_{L^3(B_r)} \| P_1 - \bar{P}_1 \|_{L^{3/2}} \le C \| u \|_{L^3(B_r)} \| P_1 \|_{L^{3/2}(B_r)} \le$$

$$\le C \left(\int_{B_r} |u|^3 \right)^{\frac{1}{3}} \left(\int_{B_{2r}} |u|^3 \right)^{\frac{2}{3}}. \qquad (9)$$

We bound the remaining terms by estimating their gradients and using the Mean Value Theorem.

First, for P_2, we have

$$|\nabla P_2| \le C \int_{|y|>2r} \frac{1}{|x-y|^4} \phi |u|^2 \, dy$$ and since $|x-y| > \frac{|y|}{2}$ for

$|x|<r$ and $|y|>2r$, we can bound

$$\|\nabla P_2\|_{L^\infty(B_r)} \le 16C \int_{2r<|y|<\rho} \frac{|u|^2}{|y|^4}\, dy \qquad \left(\text{Note: } \operatorname{supp}\phi \subset B_\rho,\ |\phi|\le 1\right)$$

and so

$$(10)\quad \int_{B_r} |u|\,|P_2 - \overline{P_2}| \le \|u\|_{L^3(B_r)} \|P_i - \overline{P_i}\|_{L^{3/2}} \le C r^2 \left(\int_{B_r} |u|^3\right)^{\frac{1}{3}} \|P_i - \overline{P_i}\|_{L^\infty(B_r)}$$

$$\le C r^3 \|u\|_{L^3(B_r)} \|\nabla P_i\|_{L^\infty(B_r)} \le C r^3 \|u\|_{L^3(B_r)} \int_{2r<|y|<\rho} \frac{|u|^2}{|y|^4}\, dy$$

We estimate P_3 similarly. Dealing with

$$P_{3,1} := -\frac{1}{2\pi} \int \frac{x_j - y_j}{|x-y|^3}\, \partial_i\phi\, u_i u_j\, dy$$

we have $\qquad \partial_i \phi = 0$ for $|y| \le \frac{3}{4}\rho$, $|y| \ge \rho$

$$|\nabla P_{3,1}| \le C \sum_{j=1}^{3} \int \frac{1}{|x-y|^3} |(\partial_i\phi) u_i u_j|\, dy \le$$

$$\le C \sum_{j=1}^{3} \int_{\frac{3}{4}\rho \le |y| \le \rho} \frac{1}{|x-y|^3} \underbrace{|\partial_i \phi|}_{\le \frac{C}{\rho}} |u_i u_j|\, dy$$

$$|x| \le r \le \frac{\rho}{2} \text{ and } |y| \ge \frac{3}{4}\rho \implies |x-y| \ge \frac{\rho}{4}$$

So
$$\|\nabla P_{3,1}\|_{L^\infty(B_r)} \le C \rho^{-4} \int_{B_\rho} |u|^2\, dy$$

Similarly, we obtain
$$\|\nabla P_{3,2}\|_{L^\infty(B_r)} \le C \rho^{-4} \int_{B_\rho} |u|^2\, dy$$

$$\implies \|\nabla P_3\|_{L^\infty(B_r)} \le C \rho^{-4} \int_{B_\rho} |u|^2\, dy$$

Hence,

$$(11)\quad \int_{B_r} |u|\,|P_3 - \overline{P_3}| \le C \frac{r^3}{\rho^4} \|u\|_{L^3(B_r)} \|\nabla P_3\|_{L^\infty(B_r)}$$

$$\le C \frac{r^3}{\rho^4} \|u\|_{L^3(B_r)} \int_{B_\rho} |u|^2 \overset{\substack{\frac{2}{3}+\frac{1}{3}=1 \\ \text{Hölder}}}{\le} C \frac{r^3}{\rho^3} \|u\|_{L^3(B_r)} \|u\|^2_{L^3(B_\rho)}$$

Similar arguments show that

$$\|\nabla P_4(x)\|_{L^\infty(B_r)} \le C\rho^{-4}\int_{B_\rho}|P|.$$

Then,

$$\int_{B_r}|u||P_4-\bar{P}_4| \le C\left(\int_{B_r}|u|\right)\sup_{x\in B_r}|P_4(x)-\bar{P}_4| \le C\left(\int_{B_r}|u|\right)\|\nabla P_4\|_{L^\infty(B_r)}$$

$$\le C\frac{r}{\rho^4}\left(\int_{B_r}|u|\right)\int_{B_\rho}|P| \overset{\substack{\frac{1}{5}+\dots+\frac{1}{5}=1\\ \text{Hölder}}}{\le} C\frac{r^{14/5}}{\rho^4}\Big(\underbrace{\int_{B_r}|u|^2}_{\le rA(r)}\Big)^{\frac{1}{5}}\left(\int_{B_r}|u|^3\right)^{\frac{1}{5}}\|P\|_{L^1(B_\rho)}$$

$$\le C\frac{r^3}{\rho^4}A(r)^{\frac{1}{5}}\left(\int_{B_r}|u|^3\right)^{\frac{1}{5}}\int_{B_\rho}|P|. \qquad (12)$$

Finally, we integrate each term (9)-(12) in time, using Hölder's inequality as appropriate, to obtain

$$\iint_{Q_r}|u||P-\bar{P}_r| \le C\left(\iint_{Q_r}|u|^3\right)^{\frac{1}{3}}\left(\iint_{Q_{2r}}|u|^3\right)^{2/3} + Cr^{\frac{13}{3}}\left(\iint_{Q_r}|u|^3\right)^{\frac{1}{3}}\cdot\sup_{-r^2<t<0}\int_{2r<|y|<\rho}\frac{|u|^2}{|y|^4}dy$$

$$+ C\frac{r^3}{\rho^3}\left(\iint_{Q_r}|u|^3\right)^{\frac{1}{3}}\left(\iint_{Q_\rho}|u|^3\right)^{\frac{2}{3}} + C\frac{r^3}{\rho^4}A(r)^{\frac{1}{5}}\left(\iint_{Q_r}|u|^3\right)^{\frac{1}{5}}\left[\int_{-\rho^2}^{0}\left(\int_{B_\rho}|P|\right)^{\frac{5}{4}}\right]^{\frac{4}{5}}$$

The assertion of the lemma follows, by normalizing each term. $\square$