Some Dimensionless Estimates

In order to use the generalized energ inequality effectively, we must bound the terms on the right

$$\iint (|u|^2 + 2p)\, u \cdot \nabla \phi$$

in terms of those on the left $\int |u|^2 \phi$ and $\iint |\nabla u|^2 \phi$.

Bounds of this type play a fundamental role in the proofs of both propositions 1 and 2.

Let

$$Q_\rho = Q_\rho(0,0) = \{(x,t) : |x| < \rho,\ -\rho^2 < t < 0\},$$

and take a pair of measurable functions u and p defined on Q_ρ. We consider the following quantities for $r < \rho$:

$$A(r) = \sup_{-r^2<t<0} r^{-1} \int_{B_r \times \{t\}} |u|^2, \qquad (1)$$

$$\delta(r) = r^{-1} \iint_{Q_r} |\nabla u|^2, \qquad (2)$$

$$G(r) = r^{-2} \iint_{Q_r} |u|^3, \qquad (3)$$

$$L(r) = r^{-2} \iint_{Q_r} |u||p - \bar{p}_r|, \qquad (4)$$

$$K(r) = r^{-\frac{13}{4}} \int_{-r^2}^{0} \left(\int_{B_r} |p| \right)^{5/4} d\tau \qquad (5)$$

Here $Q_r = Q_r(0,0)$, $B_r = \{x : |x| < r\}$, and

$$\bar{p}_r = \bar{p}_r(t) = \fint_{B_r} p(y,t)\, dy.$$

Our estimates will be applied to a SWS of the NSEs However, we shall only use the fact that the above quantities are finite and

$$\Delta p = -\sum_{i,j=1}^{3} \frac{\partial^2}{\partial x^i \partial x^j} u_i u_j = -\sum_{i,j=1}^{3} \partial_i \partial_j u_i u_j$$

$$\nabla \cdot u = 0$$

on $B_\rho \times \{t\}$ for a.e. t, $-\rho^2 < t < 0$. Our goal is to derive estimates for $G(r)$ and $L(r)$ in terms of A, δ and K. (Note: each of (1)-(5) has been scaled to have dimension zero). (scale-invariant)

<u>Lemma 1</u>

$$G(r) \leq C A^{3/4}(r) \left(A^{3/4}(r) + \delta^{3/4}(r) \right)$$

<u>Proof.</u> We use the Galiaro-Nirenberg inequality with $q=3$; $a = \frac{3}{4}(3-2) = \frac{3}{4}$

$$\int_{B_r} |u|^3 \leq C \left(\int_{B_r} |\nabla u|^2 \right)^{3/4} \left(\int_{B_r} |u|^2 \right)^{\frac{3}{4}} + \frac{C}{r^{3/2}} \left(\int_{B_r} |u|^2 \right)^{\frac{3}{2}}$$

Integrating in time and using Hölder inequality, we obtain

$$\iint_{Q_r} |u|^3 \leq C \int_{-r^2}^{0} \left(\int_{B_r} |\nabla u|^2 \right)^{3/4} \left(\int_{B_r} |u|^2 \right)^{\frac{3}{4}} dt + \frac{C}{r^{3/2}} \int_{-r^2}^{0} \left(\int_{B_r} |u|^2 \right)^{\frac{3}{2}} dt$$

$$\overset{\frac{3}{4}+\frac{1}{4}=1,\ \text{Hölder}}{\leq} C \left(\iint_{Q_r} |\nabla u|^2 \right)^{\frac{3}{4}} \left[\int_{-r^2}^{0} \left(\int_{B_r} |u|^2 \right)^3 dt \right]^{\frac{1}{4}} + \frac{C}{r^{3/2}} \int_{-r^2}^{0} \left(\int_{B_r} |u|^2 \right)^{3/2} dt \leq$$

$$\leq C \left(\iint_{Q_r} |\nabla u|^2 \right)^{\frac{3}{4}} \left[r^5 A^3(r) \right]^{\frac{1}{4}} + C r^2 A^{\frac{3}{2}}(r) =$$

$$= C A^{\frac{3}{4}}(r) \left(\delta^{\frac{3}{4}}(r) + A^{\frac{3}{4}}(r) \right) r^2$$ which gives the claimed inequality. ∎

Lemma 2. Let $r \leq \frac{1}{2}\rho$ then

$$L(r) \leq C \left(\frac{r}{\rho} \right)^{7/5} A^{\frac{1}{5}}(r) G^{1/5}(r) K^{\frac{4}{5}}(\rho) +$$