Lecture Notes: Navier-Stokes Equations

Emil Wiedemann, Edited: Jack Skipper
Universität Ulm, Winter 2018/19

Preface

These are lecture notes for an advanced master's course on the 3D incompressible Navier-Stokes equations at Universität Ulm in winter term 2018/19. Except for the first and the last chapter, the notes follow the excellent recent textbook [**4**]. Students are encouraged to consult further literature, such as the classical books [**1**, **3**, **5**].

Except for the very end of the course, I chose to work exclusively on the three-dimensional torus such as to simplify the presentation. However all mentioned results from the first four chapters have a straightforward extension to the whole space $\mathbb{R}^3$, or to (sufficiently regular) bounded domains, which certainly represent the physically most relevant case.

I would like to thank Dr. Jack Skipper for typing considerable parts of these notes.

Corrections and suggestions should be sent to `emil.wiedemann@uni-ulm.de`.

Contents

CHAPTER 1

Introduction

The Navier-Stokes equations are

$$\begin{aligned} \partial_t u(x,t) + (u\cdot\nabla)u(x,t) + \nabla p(x,t) &= \nu\Delta u(x,t) \\ \operatorname{div} u(x,t) &= 0. \end{aligned}$$

Here, $(x,t)\in\Omega\times[0,T]$, where $\Omega\subset\mathbb{R}^d$ some domain, and we have the unknown velocity field

$$u\colon\quad \Omega\times[0,T]\to\mathbb{R}^d;$$

the unknown pressure field

$$p\colon\quad \Omega\times[0,T]\to\mathbb{R};$$

and the given constant viscosity $\nu>0$. It can be written in components, for $i=1,\dots,d$:

$$\begin{aligned} \partial_t u_i + \sum_{j=1}^d u_j\partial_j u_i + \partial_i p &= \nu\sum_{j=1}^d \partial_{x_j}^2 u_i \\ \sum_{j=1}^d \partial_j u_j &= 0. \end{aligned}$$

The Navier-Stokes Equations (NSE) describe the time evolution of the velocity and pressure of a viscous incompressible fluid (e.g. water) without external forces.

1.1. Physical derivation (sketch)

Conservation of mass: At every time a volume element $\tilde\Omega\subset\subset\Omega$ should conserve the mass of fluid (incompressibility). This means that inflow and outflow of u into $\tilde\Omega$ have to balance:

$$\int_{\partial\tilde\Omega} u(x,t)\cdot n(x)\,\mathrm{d}S(x) = 0,$$

where $n(x)$ is the outer unit normal of the surface $\partial\tilde\Omega$ at the point x. For regular enough boundary $\partial\tilde\Omega$ and u by the Gauss-Green theorem, the surface integral is equal to

$$\int_{\tilde\Omega} \operatorname{div} u(x,t)\,\mathrm{d}x,$$

and since this should equal zero for every $\tilde\Omega$, we conclude $\operatorname{div} u = 0$ everywhere in Ω.

Conservation of momentum/Newton's 2nd law: Consider a fluid particle initially located at $x\in\Omega$ and denote its location at time t by $X(x,t)$ ("Lagrangian description").

Newton's 2nd law for this particle (point) reads "$F = ma$", and by assuming constant density ("$m = 1$") we obtain

$$\ddot X(x,t) = F(X(x,t),t).$$

The particle trajectory map is determined by the ODE

$$\begin{aligned} \dot X(x,t) &= u(X(x,t),t), \\ X(x,0) &= x, \end{aligned}$$

because the particle moves according to the flow of u. Therefore, by the chain rule,

$$\begin{aligned}\ddot{X}(x,t) &= \partial_t u(X(x,t),t) + (\dot{X}(x,t)\cdot\nabla)u(X(x,t),t)\\ &= \partial_t u(X,t) + (u(X,t)\cdot\nabla)u(X,t)\end{aligned}$$

where the second term represents the phenomenon known as *advection*.

Even without external forces (like gravity), there are two kinds of "internal" forces: The pressure: the fluid "pushes" itself due to incompressibility, and a force results called the pressure gradient $-\nabla p$. Example: rotating fluid in a disk, where the pressure gradient is precisely the centrifugal force so $-\nabla p$ is the centripetal force, orthogonal to u.

The friction due to viscosity: In a discrete setting, the velocity differences between neighbouring fluid particles would cause a friction force proportional to

$$u(x+he_j,t) - u(x,t).$$

Summing over all "neighbours" of x, we obtain

$$\sum_{j=1}^{d} \frac{u(x+he_j,t) - 2u(x,t) + u(x-he_j,t)}{h^2}$$

where $\frac{1}{h^2}$ is the appropriate scaling; indeed, then this expression is precisely the discrete Laplacian, which converges, as $h \to 0$, to $\Delta u(x,t)$.

In total, we obtain

$$\partial_t u(X,t) + (u\cdot\nabla)u(X,t) + \nabla p(X,t) = \nu\Delta u(X,t),$$

i.e. the NSE.

1.2. Elementary mathematical properties

The incompressible NSE

$$\begin{aligned}\partial_t u(x,t) + (u\cdot\nabla)u(x,t) + \nabla p(x,t) &= \nu\Delta u(x,t)\\ \operatorname{div} u(x,t) &= 0,\end{aligned}$$

have a "good part" of parabolic nature (i.e. the heat equation $\partial_t u = \nu\Delta u$). The "bad parts" non-linear advection term $(u\cdot\nabla)u$ and non-local terms ∇p and $\operatorname{div} u$. Note there is no evolution law for the pressure.

1.2.1. Energy balance. If u is smooth we can multiply the (NSE) by u and integrate over (space) Ω to obtain

$$\int_\Omega \partial_t u\cdot u \,\mathrm{d}x + \int_\Omega (u\cdot\nabla)u\cdot u\,\mathrm{d}x + \int_\Omega \nabla p\cdot u\,\mathrm{d}x = \nu\int_\Omega \Delta u\cdot u\,\mathrm{d}x. \tag{1.1}$$

The first term of (1.1) becomes

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u|^2\,\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(\cdot,t)\|^2_{L^2(\Omega)}.$$

Further, we note, by integrating by parts, that

$$\int_\Omega (u\cdot\nabla)u\cdot u\,\mathrm{d}x = \int_\Omega \sum_{i,j} u_i u_j \partial_j u_i\,\mathrm{d}x = -\int_\Omega \sum_{i,j} \partial_j u_i u_j u_i\,\mathrm{d}x - \int_\Omega \sum_{i,j} u_i \partial_j u_j u_i\,\mathrm{d}x.$$

Thanks to incompressibility ($\operatorname{div} u = 0$ or $\sum_i \partial_i u_i = 0$) we see that the last term vanishes, whereas the remaining one is precisely the negative of the left hand side, hence

$$\int_\Omega (u\cdot\nabla)u\cdot u\,\mathrm{d}x = 0.$$

For the term involving the pressure in (1.1) we can also integrate by parts to obtain

$$\int_\Omega \nabla p \cdot u \,\mathrm{d}x = -\int_\Omega p \operatorname{div} u \,\mathrm{d}x = 0$$

again by incompressibility. For the last term on the RHS of (1.1) we integrate by parts and see that

$$\nu \int_\Omega \Delta u \cdot u \,\mathrm{d}x = -\nu \int_\Omega |\nabla u|^2 \,\mathrm{d}x.$$

In total, after also integrating in time, we obtain

$$\frac{1}{2}\int_\Omega |u(x,t)|^2 \,\mathrm{d}x + \nu \int_0^t \int_\Omega |\nabla u(x,s)|^2 \,\mathrm{d}x \,\mathrm{d}s = \frac{1}{2}\int_\Omega |u(x,0)|^2 \,\mathrm{d}x.$$

This suggests that $u \in L_t^2 H_x^1(\Omega) \cap L_t^\infty L_x^2(\Omega)$ is a suitable function space for NSE (the so-called *energy space*).

1.2.2. Elimination of pressure. Note that, by virtue of incompressibility ($\operatorname{div} u = 0$), the nonlinearity can be written in divergence form (using Einstein's summation convention):

$$[(u \cdot \nabla) u]_i = u_j \partial_j u_i = \partial_j (u_j u_i) = (\operatorname{div} u \otimes u)_i,$$

were we wrote $(u \otimes u)_{ij} = u_i u_j$ and the divergence of a matrix field is taken row-wise: Let $A\colon \Omega \times [0,T] \to \mathbb{R}^{d\times d}$ then $\operatorname{div} A$ is a vector field given by $(\operatorname{div} A)_i = \sum_j \partial_j A_{ij}$.

Hence the NSE can be written in divergence form as

$$\begin{aligned} \partial_t u + \operatorname{div}(u \otimes u) + \nabla p &= \nu \Delta u, \\ \operatorname{div} u &= 0. \end{aligned}$$

Take the divergence of the NSE and we obtain

$$\operatorname{div} \partial_t u + \operatorname{div} \operatorname{div}(u \otimes u) + \operatorname{div} \nabla p = \mu \Delta \operatorname{div} u$$

and as $\operatorname{div} u = 0$ both the first term and the last term vanish. Further, we note that $\operatorname{div} \nabla p = \Delta p$ and so we obtain

$$-\Delta p = \operatorname{div} \operatorname{div}(u \otimes u),$$

which is a Poisson equation for the pressure. (In the case of a bounded domain this would be supplemented by a Neumann boundary condition.)

If $u \in L^2$ then this can be solved by some distribution p, and we can write this (symbolically) as

$$p = -\Delta^{-1} \operatorname{div} \operatorname{div}(u \otimes u)$$

and hence

$$\partial_t u + \operatorname{div}(u \otimes u) - \Delta^{-1} \operatorname{div} \operatorname{div}(u \otimes u) = \nu \Delta u.$$

The operator Δ^{-1} is given as a singular integral operator: E.g. in $\mathbb{R}^3$ we have

$$\Delta^{-1} f = C \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} \,\mathrm{d}y.$$

This is a non-local operator: Even if f is compactly supported, $\Delta^{-1} f$ will, in general, not be. For the NSE this means that fluid particles may interact, through the pressure, even when they are far away from each other.

	Existence of weak solutions	**Uniqueness**	**Regularity**
$d = 2$	Yes	Yes	Yes
$d = 3$	Yes (We will show)	Unknown/no	Unknown (Millennium Problem!)

TABLE 1. State of the Art for incompressible NSE

1.3. Related Models

1.3.1. Ideal fluids, Euler. We can set $\nu = 0$ and thus model "Ideal fluids" without friction. This gives the Euler equations

$$\begin{aligned} \partial_t u + \operatorname{div}(u \otimes u) + \nabla p &= 0 \\ \operatorname{div} u &= 0. \end{aligned}$$

Here without the parabolic term from the Laplacian, the mathematical theory is very different.

1.3.2. Compressible fluids. We can study compressible fluids (like air) where we have an extra non-negative scalar field ρ modelling the density:

$$\begin{aligned} \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) &= \operatorname{div} \mathbb{S}(\nabla u) \\ \partial_t \rho + \operatorname{div}(\rho u) &= 0, \end{aligned}$$

the (isentropic) compressible Navier-Stokes equations. Here, $\mathbb{S}$ denotes the Newtonian stress tensor, and the pressure is now a constitutively given function of the density (e.g. the polytropic pressure law $p(\rho) = \rho^\gamma$, $\gamma > 1$ the adiabatic exponent).

1.3.3. Non-Newtonian fluids. To study non-Newtonian fluids (like blood), we replace the Δu with $\operatorname{div} \mathbb{S}(\nabla u)$, where $\mathbb{S}$ is non-linear, e.g. the p-Laplacian:

$$\mathbb{S}(\nabla u) = |\nabla u|^{p-2} \nabla u$$

and we recover the standard NSE for $p = 2$. (To be precise, one usually uses only the *symmetric* part of ∇u.)

The NSE are widely used by physicists, engineers, geo-scientists etc. for atmospheric and ocean dynamics, weather forecasting, turbulence theory, etc.

CHAPTER 2

Function Spaces and Weak Solutions

We choose as a domain the three dimensional torus $\mathbb{T}^3 = \mathbb{R}^3/2\pi\mathbb{Z}^3$; it has the advantages of being compact and having no physical boundaries at the same time. In other words, we look for space periodic solutions:

$$u(x + 2\pi k, t) = u(x, t) \quad \forall k \in \mathbb{Z}^3.$$

The analysis of functions on $\mathbb{T}^3$ is simplified by the Fourier series: For $u \in L^1(\mathbb{T}^3)$, meaning

$$\int_{\mathbb{T}^3} |u(x)| \, \mathrm{d}x = \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} |u(x_1, x_2, x_3)| \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3 < \infty,$$

we can define the Fourier coefficients

$$\hat{u}_k = \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} \mathrm{e}^{-\mathrm{i}k\cdot x} u(x) \, \mathrm{d}x \in \mathbb{C}_k, \quad k \in \mathbb{Z}^3.$$

If $\sum_{k\in\mathbb{Z}^3} |u_k| < \infty$, then the Fourier inversion formula says that

$$u(x) = \sum_{k\in\mathbb{Z}^3} \hat{u}_k \mathrm{e}^{\mathrm{i}k\cdot x}.$$

We only work with real-valued functions u, which implies that $\hat{u}_k = \overline{\hat{u}_{-k}}$, for $k \in \mathbb{Z}^3$.

By Plancherel's Theorem, for $u \in L^2(\mathbb{T}^3)$ we have that

$$\int_{\mathbb{T}^3} |u(x)|^2 \, \mathrm{d}x = (2\pi)^3 \sum_{k\in\mathbb{Z}^3} |\hat{u}_k|^2,$$

and in particular $u \in L^2$ if and only if $\hat{u} \in l^2$, i.e.

$$\sum_{k\in\mathbb{Z}^3} |\hat{u}_k|^2 < \infty.$$

2.1. Fourier Characterisation of Sobolev Spaces

Let $s \in \mathbb{N}$, then one usually defines, with the multi-index $\alpha \in \mathbb{N}_0^d$, the Sobolev norm

$$\|u\|^2_{H^s(\mathbb{T}^3)} := \|u\|^2_{L^2(\mathbb{T}^3)} + \sum_{|\alpha|\le s} \|\partial^\alpha u\|^2_{L^2(\mathbb{T}^3)} = (2\pi)^3 \left[\sum_{k\in\mathbb{Z}^3} |\hat{u}_k|^2 + \sum_{|\alpha|\le s} \sum_{k\in\mathbb{Z}^3} |\widehat{\partial^\alpha u_k}|^2 \right],$$

where we used Plancherel's Theorem in the last equality. Note that the derivatives are taken in the weak (distributional) sense.

Further, we can integrate by parts to see that

$$\widehat{\partial_j u_k} = \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} \mathrm{e}^{-\mathrm{i}k\cdot x} \partial_j u(x) \, \mathrm{d}x = \mathrm{i}k_j \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} \mathrm{e}^{-\mathrm{i}k\cdot x} u(x) \, \mathrm{d}x = \mathrm{i}k_j \hat{u}_k$$

and hence

$$|\widehat{\partial^\alpha u_k}|^2 = |k|^{2|\alpha|} |\hat{u}_k|^2$$

and thus

$$\|u\|^2_{H^s(\mathbb{T}^3)} = (2\pi)^3 \sum_{k\in\mathbb{Z}^3} \sum_{|\alpha|\le s} |\hat{u}||k|^{2|\alpha|}.$$

It turns out (exercise!) that this is equivalent to the norm

$$\|u\|^2_{H^s(\mathbb{T}^3)} = (2\pi)^3 \sum_{k\in\mathbb{Z}^3} (1+|k|^{2s})|\hat{u}_k|^2.$$

Note that this is even well-defined when $s \notin \mathbb{N}$!

DEFINITION 2.1. Let $s \geq 0$, then $H^s(\mathbb{T}^3)$ contains all functions $u \in L^1(\mathbb{T}^3)$ such that

$$\|u\|^2_{H^s} = (2\pi)^3 \sum_{k\in\mathbb{Z}^3} (1+|k|^{2s})|\hat{u}_k|^2 < \infty.$$

When $s \in \mathbb{N}$, this definition coincides with the definition by weak derivatives. It will be useful to consider homogeneous Sobolev spaces, where the zero-th Fourier mode is zero.

DEFINITION 2.2. (1) The homogeneous space $\dot{L}^2(\mathbb{T}^3)$ consists of all $u \in L^2(\mathbb{T}^3)$ such that

$$\int_{\mathbb{T}^3} u(x)dx = 0 \qquad (\text{i.e.} \quad \hat{u}_0 = 0).$$

(2) The homogeneous Sobolev space $\dot{H}^s(\mathbb{T}^3)$ is defined as $H^s(\mathbb{T}^3) \cap \dot{L}^2(\mathbb{T}^3)$, with the norm

$$\|u\|^2_{\dot{H}^s} := (2\pi)^3 \sum_{k\in\mathbb{Z}^3\setminus\{0\}} |k|^{2s}|\hat{u}_k|^2.$$

(3) For $s < 0$, we define $H^{-s}(\mathbb{T}^3)$ as the dual space of $\dot{H}^s(\mathbb{T}^3)$.

An element $v \in H^{-s}(\mathbb{T}^3)$ can be represented as

$$v(x) = \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \hat{v}_k \mathrm{e}^{\mathrm{i}kx}$$

with

$$\|v\|^2_{H^{-s}(\mathbb{T}^3)} := \sum_{k\neq 0} |k|^{-2s}|\hat{v}_k|^2 < \infty.$$

Indeed, as the dual pairing is given by

$$(v,u) = \sum_{k\neq 0} \hat{v}_k \overline{\hat{u}_k} = \sum_{k\neq 0} \hat{v}_k \hat{u}_{-k},$$

using Cauchy-Schwarz we obtain

$$|(v,u)| \leq \sum_{k\neq 0} |\hat{v}_k||\hat{u}_{-k}| = \sum_{k\neq 0} \frac{|\hat{v}_k|}{|k|^s}|\hat{u}_{-k}||k|^s \leq \left(\sum_{k\neq 0} |k|^{-2s}|\hat{v}_k|^2\right)^{1/2} \left(\sum_{k\neq 0} |k|^{2s}|\hat{u}_k|^2\right)^{1/2} < \infty.$$

2.2. Helmholtz Decomposition

Consider now vector-valued maps $u \in \dot{L}^2(\mathbb{T}^3;\mathbb{R}^3)$, i.e. $u = (u_1,u_2,u_3)$ with $u_1,u_2,u_3 \in \dot{L}^2(\mathbb{T}^3;\mathbb{R})$. Recall the incompressibility (divergence-free) condition $\operatorname{div} u = 0$, which in terms of Fourier coefficients reads

$$0 = \widehat{\operatorname{div} u}_k = \sum_{j=1}^3 \widehat{\partial_j u_j} = \mathrm{i}\sum_{j=1}^3 k_j(\hat{u}_j)_k = \mathrm{i}k\cdot\hat{u}_k.$$

(Note that now $\hat{u}_k \in \mathbb{C}^3$.) This motivates the following definition of "solenoidal" (i.e. divergence free or incompressible) vector fields.

DEFINITION 2.3. We define the space $H = H(\mathbb{T}^3)$ as

$$\left\{u \in \dot{L}^2(\mathbb{T}^3;\mathbb{R}^3): \quad k\cdot u_k = 0 \quad \forall k \neq 0\right\}.$$

Note that H may contain vector fields that are not in H^1 and hence div is not well defined simply by taking derivatives. H is equipped with the L^2 norm.

Lemma 2.4. *Every $u \in H(\mathbb{T}^3)$ is weakly divergence free in the sense that*

$$\int_{\mathbb{T}^3} u(x) \cdot \nabla\phi(x)\,\mathrm{d}x = 0$$

for all $\phi \in \dot{H}^1(\mathbb{T}^3)$.

Proof. We can write u as a Fourier series as $u(x) = \sum_{j\neq 0} \hat{u}_j \mathrm{e}^{\mathrm{i}j\cdot x}$, then using orthogonality in the form $\int \mathrm{e}^{\mathrm{i}j\cdot x}\mathrm{e}^{-\mathrm{i}k\cdot x}\,\mathrm{d}x = \delta_{jk}$, we obtain

$$\int_{\mathbb{T}^3} u(x)\cdot\nabla(\mathrm{e}^{-\mathrm{i}k\cdot x})\,\mathrm{d}x = \int_{\mathbb{T}^3}\sum_{j\neq 0}\hat{u}_j\mathrm{e}^{\mathrm{i}j\cdot x}\cdot(-\mathrm{i}k)\mathrm{e}^{-\mathrm{i}k\cdot x}\,\mathrm{d}x = \int_{\mathbb{T}^3} -\mathrm{i}k\cdot\hat{u}_k\,\mathrm{d}x = 0$$

since $u \in H(\mathbb{T}^3)$. Further, since $\{\mathrm{e}^{-\mathrm{i}k\cdot x}\}_{k\in\mathbb{Z}^3\setminus\{0\}}$ form an orthonormal basis (ONB) of $\dot{H}^1(\mathbb{T}^d)$, the computation extends to any $\phi \in \dot{H}^1(\mathbb{T}^3)$. □

Definition 2.5. The space $G = G(\mathbb{T}^3)$ is defined as

$$G = \left\{g \in \dot{L}^2(\mathbb{T}^3;\mathbb{R}^3):\quad g = \nabla\phi \quad\text{for some}\quad \phi\in\dot{H}^1(\mathbb{T}^3)\right\}.$$

Hence Lemma 2.4 says that G and H are orthogonal subspaces of $\dot{L}^2$.

Theorem 2.6 (Helmholtz decomposition). *$\dot{L}^2 = G\oplus H$, i.e. for all $u\in\dot{L}^2(\mathbb{T}^3;\mathbb{R}^3)$ there exist unique $g\in G$ and $h\in H$ such that*

$$u = g + h \quad\text{and}\quad \int_{\mathbb{T}^3} g\cdot h\,\mathrm{d}x = 0.$$

Moreover, if $u\in\dot{H}^s(\mathbb{T}^3)$ then $g = \nabla\phi$ for $\phi\in\dot{H}^{s+1}(\mathbb{T}^3)$ and $h\in\dot{H}^s(\mathbb{T}^3)$.

Proof. We can write u as a Fourier series and see that $(\hat{u}_k)_{k\in\mathbb{Z}^3}\in l^2$. We can then write each $\hat{u}_k$ as a linear combination of k and a vector w_k perpendicular to k. Thus for all $k\neq 0$, let $\hat{u}_k = \alpha_k k + w_k$ with $k\cdot w_k = 0$ and $\alpha_k\in\mathbb{C}$. Note that by orthogonality

$$|\hat{u}_k|^2 = |\alpha_k|^2|k|^2 + |w_k|^2 \tag{2.1}$$

and hence

$$u(x) = \sum_{k\neq 0}\hat{u}_k\mathrm{e}^{\mathrm{i}k\cdot x} = \sum_{k\neq 0}(\alpha_k k + w_k)\mathrm{e}^{\mathrm{i}k\cdot x} = \sum_{k\neq 0} -\mathrm{i}\alpha_k\nabla\mathrm{e}^{\mathrm{i}k\cdot x} + \sum_{k\neq 0} w_k\mathrm{e}^{\mathrm{i}k\cdot x} = \nabla\phi(x) + h(x),$$

where

$$\phi(x) \coloneqq \sum_{k\neq 0} -\mathrm{i}\alpha_k\mathrm{e}^{\mathrm{i}k\cdot x} \quad\text{and}\quad h(x) \coloneqq \sum_{k\neq 0} w_k\mathrm{e}^{\mathrm{i}k\cdot x}.$$

Note that

$$\|\phi\|^2_{\dot{H}^1} = \sum_{k\neq 0}|\alpha_k|^2|k|^2 \quad\text{and}\quad \|h\|_{\dot{L}^2} = \sum_{k\neq 0}|w_k|^2$$

and so, as

$$\sum_{k\neq 0}|\alpha_k|^2|k|^2 + |w_k|^2 = \sum_{k\neq 0}|\hat{u}_k|^2 < \infty,$$

hence $\phi\in\dot{H}^1$ and $h\in\dot{L}^2$ and thus $g\in G$ and $h\in H$.

Further, suppose that $u\in\dot{H}^s(\mathbb{T}^3)$ and multiply (2.1) by $|k|^{2s}$ to conclude that $\phi\in\dot{H}^{s+1}$ and $h\in\dot{H}^s$.

Finally, we must show uniqueness. Suppose that $u = h_1 + \nabla\phi_1 = h_2 + \nabla\phi_2$ for $h_1, h_2\in H$ and $\nabla\phi_1,\nabla\phi_2\in G$, then

$$(h_1 - h_2) + \nabla(\phi_1 - \phi_2) = 0 \quad\text{implies that}\quad \|h_1 - h_2 + \nabla\phi_1 - \nabla\phi_2\|^2 = 0$$

and then we can apply Lemma 2.4 to see that

$$\|h_1 - h_2\|^2 + \|\nabla\phi_1 - \nabla\phi_2\|^2 = 0$$

and so $h_1 = h_2$ and $\nabla\phi_1 = \nabla\phi_2$. □

DEFINITION 2.7. The orthogonal projection from $\dot{L}^2(\mathbb{T}^3)$ onto H is called the *Leray projection*: If $u = h + g$ with $h \in H$ and $g \in G$, then $\mathbb{P}u = h$.

In Fourier series (exercise!)

$$\mathbb{P}u(x) = \sum_{k\neq 0} \left(\hat{u}_k - \frac{\hat{u}_k \cdot k}{|k|^2} k \right) \mathrm{e}^{\mathrm{i}k\cdot x}.$$

Note that the following useful lemma only holds true when the spatial domain has no physical boundaries.

LEMMA 2.8. *$\mathbb{P}$ commutes with derivatives, i.e. $\mathbb{P}\partial_{x_j} = \partial_{x_j}\mathbb{P}$.*

PROOF. In Fourier series we see that

$$\widehat{\mathbb{P}\partial_j u}_k = \widehat{\partial_j u}_k - \frac{\widehat{\partial_j u}_k \cdot k}{|k|^2} k = \mathrm{i}k_j \hat{u}_k - \mathrm{i}k_j \frac{\hat{u}_k \cdot k}{|k|^2} k = \mathrm{i}k_j \left(\hat{u}_k - \frac{\hat{u}_k \cdot k}{|k|^2} k \right) = \widehat{\partial_j \mathbb{P}u}_k.$$

□

2.3. The Stokes Operator

DEFINITION 2.9. The space $V = V(\mathbb{T}^3)$ is given by $V = H \cap H^1(\mathbb{T}^3; \mathbb{R}^3)$, with the $\dot{H}^1$-norm.

That is, V consists of weakly divergence-free vector fields with "extra regularity" H^1.

DEFINITION 2.10 (Stokes operator). The Stokes operator is defined as $-\mathbb{P}\Delta$, in the domain $V \cap H^2(\mathbb{T}^3; \mathbb{R}^3)$.

We notice that from Lemma 2.8 that if $u \in V \cap H^2$, then

$$-\mathbb{P}\Delta u = -\Delta \mathbb{P}u = -\Delta u$$

since $u \in H$. Hence the Stokes operator is simply $-\Delta$. However, on bounded domains this is no longer true in general – we cannot necessarily commute derivatives with the Leray projector. (On a bounded domain one includes information on the boundary condition in the definition of the space H; this amounts to a weak formulation of the slip condition $u \cdot n = 0$ on $\partial\Omega$.)

THEOREM 2.11. *There exists a family $\{w_k\}_{k\in\mathbb{N}}$ of smooth vector fields on $\mathbb{T}^3$ such that*

(1) *$\{w_k\}$ is an orthonormal basis of H,*
(2) *w_k are eigenfunctions of the Stokes operator with eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \nearrow \infty$,*
(3) *$\{w_k\}$ form an orthogonal basis of V.*

PROOF. For each $k \in \mathbb{Z}^3 \setminus \{0\}$ choose vectors $m_k, m_{-k} \in \mathbb{R}^3$ such that

- $m_k \perp k$, $m_{-k} \perp k$, $m_k \perp m_{-k}$,
- $\|m_k \cos(k\cdot x)\|_{L^2(\mathbb{T}^3)} = \|m_k \sin(k\cdot x)\|_{L^2(\mathbb{T}^3)} = 1$,
- $m_{(-k)} = m_{-k}$.

Then $\{m_k \cos(k\cdot x)\} \cup \{m_k \sin(k\cdot x)\} \subset H$: Indeed,

$$\int_{\mathbb{T}^3} m_k \cos(k\cdot x)\,\mathrm{d}x = 0, \qquad \int_{\mathbb{T}^3} m_k \sin(k\cdot x)\,\mathrm{d}x = 0,$$

and $k \cdot m_k \cos(k\cdot x), k\cdot m_k \sin(k\cdot x) = 0$ by choice of m_k. Hence, we have an orthonormal family in H whose members are also in the domain of the Stokes operator (because they are smooth). Moreover,

$$-\Delta m_k \cos(k\cdot x) = -m_k \sum \partial_j^2 \cos(k\cdot x) = +m_k|k|^2 \cos(k\cdot x) = |k|^2 m_k \cos(k\cdot x),$$

so we can define $|k|^2 := \lambda_k$, and similar for $\sin(k\cdot x)$, whence (2) is proved after re-labelling indices from $\mathbb{Z}^3 \setminus \{0\}$ to $\mathbb{N}$. To show that these functions form a Hilbert basis of H, let $u \in H$ and write

$$\begin{aligned} u(x) &= \sum_{k\neq 0} \hat{u}_k \mathrm{e}^{\mathrm{i}k\cdot x} = \frac{1}{2}\sum_{k\neq 0} \hat{u}_k \mathrm{e}^{\mathrm{i}k\cdot x} + \frac{1}{2}\sum_{k\neq 0} \hat{u}_{-k} \mathrm{e}^{-\mathrm{i}k\cdot x} \\ &= \frac{1}{2}\sum_{k\neq 0} (\hat{u}_k + \hat{u}_{-k})\cos(k\cdot x) + \frac{1}{2}\sum_{k\neq 0} \mathrm{i}(\hat{u}_k - \hat{u}_{-k})\sin(k\cdot x), \end{aligned}$$

and we see that both $(\hat{u}_k + \hat{u}_{-k})$ and $\mathrm{i}(\hat{u}_k - \hat{u}_{-k})$ are in $\mathbb{R}^3$ and perpendicular to k. This becomes, for some $a_k, b_k, c_k, d_k, \alpha_k, \beta_k \in \mathbb{R}$,

$$\begin{aligned} &\sum_{k\neq 0} (a_k m_k + b_k m_{-k})\cos(k\cdot x) + \sum_{k\neq 0} (c_k m_k + d_k m_{-k})\sin(k\cdot x) \\ &= \sum_{k\neq 0} (a_k m_k + b_{-k} m_k)\cos(k\cdot x) + \sum_{k\neq 0} (c_k m_k - d_{-k} m_k)\sin(k\cdot x) \\ &= \sum_{k\neq 0} \alpha_k m_k \cos(k\cdot x) + \sum_{k\neq 0} \beta_k m_k \sin(k\cdot x). \end{aligned}$$

Finally, show orthogonality in V: Indeed, if w_k, w_l are two of the given eigenfunctions of the Stokes operator with eigenvectors λ_k, λ_l $(k \neq l)$, then using integration by parts,

$$(\nabla w_k, \nabla w_l) = (w_k, -\Delta w_l) = \lambda_l (w_k, w_l) = 0$$

by orthogonality in H. □

REMARK 2.12. The $\dot{H}^1$-norm of w_k is $\sqrt{\lambda_k}$, because

$$\begin{aligned} \|w_k\|_{\dot{H}^1}^2 = \int_{\mathbb{T}^3} |\nabla w_k|^2 \,\mathrm{d}x = \int_{\mathbb{T}^3} \nabla w_k : \nabla w_k \,\mathrm{d}x = -\int_{\mathbb{T}^3} w_k \cdot \Delta w_k \,\mathrm{d}x \\ = \lambda_k \int_{\mathbb{T}^3} |w_k|^2 \,\mathrm{d}x = \lambda_k \|w_k\|_H^2 = \lambda_k. \end{aligned}$$

2.4. Weak Solutions

Suppose that (u, p) is a smooth solution of the NSE. Then

$$\partial_t u + (u\cdot\nabla)u - \nu\Delta u = -\nabla p \in G.$$

The requirement of $v \in G$ is equivalent, by the Helmholtz decomposition, to

$$\int_{\mathbb{T}^3} v\cdot\phi \,\mathrm{d}x = 0$$

for all $\phi \in H$. It will be convenient to choose ϕ from the smooth class of functions

$$\mathcal{D}_\sigma := \{\phi \in C_c^\infty(\mathbb{T}^3 \times [0,\infty)\colon \quad \operatorname{div}\phi(t) = 0 \quad \forall t \geq 0\}.$$

So if $\phi \in \mathcal{D}_\sigma$, then the NSE imply

$$\int_0^\infty \int_{\mathbb{T}^3} \partial_t u\cdot\phi \,\mathrm{d}x\,\mathrm{d}t + \int_0^\infty \int_{\mathbb{T}^3} (u\cdot\nabla)u\cdot\phi \,\mathrm{d}x\,\mathrm{d}t - \nu\int_0^\infty \int_{\mathbb{T}^3} \Delta u\cdot\phi \,\mathrm{d}x\,\mathrm{d}t = 0$$

where the term involving the pressure has been "projected away" by the choice of test function. Integration by parts in the ∂_t and the Δ terms gives

$$\begin{aligned} -\int_0^\infty \int_{\mathbb{T}^3} u\cdot\partial_t\phi \,\mathrm{d}x\,\mathrm{d}t + \int_0^\infty \int_{\mathbb{T}^3} (u\cdot\nabla)u\cdot\phi \,\mathrm{d}x\,\mathrm{d}t \\ + \nu\int_0^\infty \int_{\mathbb{T}^3} \nabla u : \nabla\phi \,\mathrm{d}x\,\mathrm{d}t = \int_{\mathbb{T}^3} u^0\cdot\phi(0) \,\mathrm{d}x. \quad (2.2) \end{aligned}$$

On the other hand, we have already derived the energy equality:

$$\frac{1}{2}\int_\Omega |u(x,t)|^2 \, \mathrm{d}x + \nu \int_0^t \int_\Omega |\nabla u(x,s)|^2 \, \mathrm{d}x \, \mathrm{d}s = \frac{1}{2}\int_\Omega |u^0|^2 \, \mathrm{d}x,$$

which suggests that $u \in L^2(0,\infty;V) \cap L^\infty(0,\infty;H)$ is the appropriate function space. Note that for $\phi \in \mathcal{D}_\sigma$ and $u \in L^\infty H \cap L^2 V$, (2.2) is well-defined. Thus, we have the following:

DEFINITION 2.13 (Weak solution of the NSE). A vector field $u \in L^\infty(0,\infty;H) \cap L^2(0,\infty;V)$ is called a weak (Leray-Hopf) solution of the NSE if (2.2) holds for all $\phi \in \mathcal{D}_\sigma$.

It will be convenient to check that this definition can actually be tested on a smaller class of test functions than $\mathcal{D}_\sigma$. Therefore, set

$$\tilde{\mathcal{D}}_\sigma := \left\{ \phi = \sum_{k=1}^N d_k(t) w_k(x): \quad d_k \in C_c^\infty([0,\infty)) \right\},$$

where $\{w_k\}$ is the eigenbasis of the Stokes operator from Theorem 2.11. Clearly we have that $\tilde{\mathcal{D}}_\sigma \subset \mathcal{D}_\sigma$.

LEMMA 2.14. *If $u \in L^\infty(0,\infty;H) \cap L^2(0,\infty;V)$ satisfies* (2.2) *for all $\phi \in \tilde{\mathcal{D}}_\sigma$, then it even satisfies* (2.2) *for all $\phi \in \mathcal{D}_\sigma$, i.e. it is a weak solution.*

PROOF. Let $\phi \in \mathcal{D}_\sigma$, then for every $t \geq 0$, $\phi(t) \in H$, and we can write

$$\phi(x,t) = \sum_{k=1}^\infty d_k(t) w_k(x)$$

since $\{w_k\}$ form a Hilbert basis of H. Set

$$\phi_N := \sum_{k=1}^N d_k(t) w_k(x) \in \tilde{\mathcal{D}}_\sigma.$$

Then $\phi_N \to \phi$ in $C([0,\infty);V)$. Indeed,

$$\sup_t \|\phi(t) - \phi_N(t)\|_V^2 = \sup_t \| \sum_{k=N+1}^\infty d_k(t) w_k(\cdot)\|_V^2 = \sup_t \sum_{k=N+1}^\infty \lambda_k d_k^2(t),$$

as $\{w_k\}$ are orthogonal in V and $\|w_k\|_V = \sqrt{\lambda_k}$ (from remark after Theorem 2.11). This then becomes, as λ_k increases to ∞,

$$\begin{aligned} \sup_t \sum_{k=N+1}^\infty \frac{\lambda_k^2 d_k^2(t)}{\lambda_k} &\leq \frac{1}{\lambda_N} \sup_t \sum_{k=N+1}^\infty \lambda_k^2 d_k^2(t) \\ &= \frac{1}{\lambda_N} \sup_t \sum_{k=N+1}^\infty (-\Delta w_k(x) d_k(t), -\Delta w_k(x) d_k(t))_{L^2} \end{aligned}$$

by orthogonality. We then see that we can bound this above by

$$\frac{1}{\lambda_N} \sup_t \| -\Delta\phi\|_{L^2(\mathbb{T}^3)} \leq \frac{1}{\lambda_N} \sup_t \|\phi(t)\|_{H^2(\mathbb{T}^3)} \to 0$$

as $N \to \infty$, since $\sup_t \|\phi(t)\|_{H^2(\mathbb{T}^3)}$ is independent of N.

Furthermore, $\partial_t \phi_N \to \partial_t \phi$ in $L^2(0,T;L^2(\mathbb{T}^3))$ because

$$\begin{aligned} \int_0^\infty \|\partial_t\phi - \partial_t\phi_N\|_{L^2(\mathbb{T}^3)}^2 \, \mathrm{d}t &= \int_0^\infty \| \sum_{k=N+1}^\infty d_k'(t) w_k(x)\|_{L^2(\mathbb{T}^3)}^2 \, \mathrm{d}t \\ &= \int_0^\infty \sum_{k=N+1}^\infty (d_k'(t))^2 \, \mathrm{d}t \to 0 \end{aligned}$$

as $N \to \infty$, where the latter convergence follows from

$$\partial_t \phi \in C_c^\infty(\mathbb{T}^3 \times [0,\infty)) \subset C_c^\infty([0,\infty);H)$$

and hence

$$\sup_t \sum_{k=1}^{\infty} (d_k'(t))^2 = \sup_t \|\partial_t \phi\|^2_{L^2(\mathbb{T}^3)} < \infty.$$

It follows that

$$\int_0^\infty \int \partial_t \phi_N \cdot u \, \mathrm{d}x \, \mathrm{d}t \to \int_0^\infty \int \partial_t \phi \cdot u \, \mathrm{d}x \, \mathrm{d}t,$$
$$\int_0^\infty \int \nabla \phi_N : \nabla u \, \mathrm{d}x \, \mathrm{d}t \to \int_0^\infty \int \nabla \phi : \nabla u \, \mathrm{d}x \, \mathrm{d}t,$$
$$\int_0^\infty \int \phi_N(0) \cdot u^0(x) \nabla u \, \mathrm{d}x \, \mathrm{d}t \to \int_0^\infty \int \phi(0) \cdot u^0(x) \, \mathrm{d}x \, \mathrm{d}t.$$

For the remaining term $\int\int (u\cdot\nabla u)\cdot\phi_N \, \mathrm{d}x \, \mathrm{d}t$, we will use the Sobolev embedding $H^1(\mathbb{T}^3) \subset L^6(\mathbb{T}^3)$, so it follows that

$$\sup_t \|\phi_N - \phi\|_{L^6(\mathbb{T}^3)} \le C \sup_t \|\phi_N - \phi\|_V \to 0$$

as $N \to \infty$. Thus, by Hölders's inequality,

$$\begin{aligned} \left| \int\int (u \cdot \nabla u) \cdot (\phi_N - \phi) \, \mathrm{d}x \, \mathrm{d}t \right| &\le \int\int |u||\nabla u||\phi_N - \phi| \, \mathrm{d}x \, \mathrm{d}t \\ &\le \|u\|_{L^2_t L^3_x} \|\nabla u\|_{L^2_t L^2_x} \|\phi_N - \phi\|_{L^\infty_t L^6_x} \to 0 \end{aligned}$$

as $N \to \infty$. (We note that using again the same embedding theorem we have $\|u\|_{L^2_t L^3_x} \le C\|u\|_{L^2_t L^6_x} \le C\|u\|_{L^2_t H^1_x}$ and so $\|u\|_{L^2_t L^3_x} < \infty$.) So if we consider the equation for a weak solution to the NSE (2.2) with ϕ_N used as a test function, then we see that every term will converge to the corresponding one with $\phi \in \mathcal{D}_\sigma$ and so (2.2) follows for $\phi \in \mathcal{D}_\sigma$. ☐

For later reference, we prove another lemma which allows us to test a weak solution with functions of the form $\chi_{[t_1,t_2]}\phi$ for $\phi \in \mathcal{D}_\sigma$, for almost every $t_1 < t_2$, where χ denotes the indicator function of a set. This is a consequence of the Lebesgue differentiation theorem, which we recall without proof:

Theorem 2.15 (Lebesgue differentiation theorem). *Let $\Omega \subset \mathbb{R}^n$ be measurable and $f \in L^1_{loc}(\Omega)$, then for almost every $x \in \Omega$ we have*

$$\lim_{\epsilon \to 0} \frac{1}{|B_\epsilon(x)|} \int_{B_\epsilon(x)} |f(y) - f(x)| \, \mathrm{d}y = 0.$$

A point x for which the statement of the differentiation theorem is true is called a *Lebesgue point* of f; the theorem thus says that, given a locally integrable function on a domain, almost every point in that domain is a Lebesgue point.

Lemma 2.16. *Let u be a weak (Leray-Hopf) solution of NSE. Then*

$$\begin{aligned} -\int_{t_1}^{t_2} \int_{\mathbb{T}^3} u \cdot \partial_t \phi \, \mathrm{d}x \, \mathrm{d}t + \int_{t_1}^{t_2} \int_{\mathbb{T}^3} (u \cdot \nabla) u \cdot \phi \, \mathrm{d}x \, \mathrm{d}t \\ + \nu \int_{t_1}^{t_2} \int_{\mathbb{T}^3} \nabla u : \nabla \phi \, \mathrm{d}x \, \mathrm{d}t = \int_{\mathbb{T}^3} u(t_1) \cdot \phi(t_1) \, \mathrm{d}x - \int_{\mathbb{T}^3} u(t_2) \cdot \phi(t_2) \, \mathrm{d}x \end{aligned} \tag{2.3}$$

for every $\phi \in \mathcal{D}_\sigma$ and almost all $0 \le t_1 < t_2$, including $t_1 = 0$.

Remark 2.17. Later we will see that this is even true for *all* (and not just almost all) times.

Proof. We prove only the case $t_1 = 0$ in detail. So let $t_2 > 0$ and consider a smooth (cut-off) function $\zeta : \mathbb{R} \to \mathbb{R}$ such that

- $\zeta \ge 0$,

- $\zeta(t) = 1$ for $t \le -1$ and $\zeta(t) = 0$ for $t \ge 1$,
- ζ is monotone decreasing.

Then, for every $\epsilon > 0$, set

$$\zeta_\epsilon(t) := \zeta\left(\frac{t - t_2}{\epsilon}\right).$$

Thus, ζ_ϵ (restricted to $t \ge 0$) is a smooth approximation of the indicator function $\chi_{[0;t_2]}$.

If $\phi \in \mathcal{D}_\sigma$, then the product $\zeta\phi$ is still in $\mathcal{D}_\sigma$, so using it as a test function in the weak formulation of NSE gives

$$-\int_0^\infty \int_{\mathbb{T}^3} u \cdot \partial_t(\zeta_\epsilon \phi)\,\mathrm{d}x\,\mathrm{d}t + \int_0^\infty \int_{\mathbb{T}^3} (u\cdot\nabla)u \cdot (\zeta_\epsilon\phi)\,\mathrm{d}x\,\mathrm{d}t \\ + \nu\int_0^\infty \int_{\mathbb{T}^3} \nabla u : \nabla(\zeta_\epsilon\phi)\,\mathrm{d}x\,\mathrm{d}t = \int_{\mathbb{T}^3} u^0 \cdot \phi(0)\,\mathrm{d}x$$

(for the term involving u^0, note that $\zeta_\epsilon(0) = 1$ for sufficiently small ϵ). The two integrals including space derivatives are easily seen to converge as $\epsilon \to 0$: Indeed, ζ_ϵ converges almost everywhere to $\chi_{[0,t_2]}$, and the integrand $(u\cdot\nabla)u\cdot(\zeta_\epsilon\phi)$ is bounded pointwise by $|(u\cdot\nabla)u||\phi|$, uniformly in ϵ, which is of course integrable. Hence, by the dominated convergence theorem,

$$\int_0^\infty \int_{\mathbb{T}^3} (u\cdot\nabla)u\cdot(\zeta_\epsilon\phi)\,\mathrm{d}x\,\mathrm{d}t \to \int_0^\infty \int_{\mathbb{T}^3} (u\cdot\nabla)u\cdot(\chi_{[0,t_2]}\phi)\,\mathrm{d}x\,\mathrm{d}t = \int_0^{t_2} \int_{\mathbb{T}^3} (u\cdot\nabla)u\cdot\phi\,\mathrm{d}x\,\mathrm{d}t$$

as $\epsilon \to 0$, and likewise

$$\int_0^\infty \int_{\mathbb{T}^3} \nabla u : \nabla(\zeta_\epsilon\phi)\,\mathrm{d}x\,\mathrm{d}t \to \int_0^{t_2} \int_{\mathbb{T}^3} \nabla u : \nabla\phi\,\mathrm{d}x\,\mathrm{d}t$$

(of course the space derivative does not hit ζ_ϵ, which depends only on time).

The first integral, which contains the time derivative, is a bit more delicate. We compute

$$\int_0^\infty \int_{\mathbb{T}^3} u \cdot \partial_t(\zeta_\epsilon\phi)\,\mathrm{d}x\,\mathrm{d}t = \int_0^\infty \int_{\mathbb{T}^3} u \cdot \zeta_\epsilon'\phi\,\mathrm{d}x\,\mathrm{d}t + \int_0^\infty \int_{\mathbb{T}^3} u \cdot \zeta_\epsilon\partial_t\phi\,\mathrm{d}x\,\mathrm{d}t,$$

and the latter integral is seen, as before, to converge to

$$\int_0^{t_2} \int_{\mathbb{T}^3} u \cdot \partial_t\phi\,\mathrm{d}x\,\mathrm{d}t.$$

For the integral involving ζ_ϵ', observe that by definition,

$$\zeta_\epsilon'(t) = \frac{1}{\epsilon}\zeta'\left(\frac{t-t_2}{\epsilon}\right),$$

which also implies $\int_0^\infty \zeta_\epsilon'(t)\,\mathrm{d}t = -1$ for every $\epsilon > 0$. Note also that ζ_ϵ' is supported in $B_\epsilon(t_2)$. Therefore,

$$\begin{aligned} &\left|\int_0^\infty \int_{\mathbb{T}^3} u \cdot \zeta_\epsilon'\phi\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{T}^3} u(t_2)\cdot\phi(t_2)\,\mathrm{d}x\right| \\ &\le \int_{t_2-\epsilon}^{t_2+\epsilon} |\zeta_\epsilon'(t)| \left|\int_{\mathbb{T}^3} u(t)\cdot\phi(t) - u(t_2)\cdot\phi(t_2)\,\mathrm{d}x\right|\,\mathrm{d}t \\ &\le \|\zeta\|_{C^1}\frac{1}{\epsilon}\int_{t_2-\epsilon}^{t_2+\epsilon} \left|\int_{\mathbb{T}^3} u(t)\cdot\phi(t) - u(t_2)\cdot\phi(t_2)\,\mathrm{d}x\right|\,\mathrm{d}t \to 0 \end{aligned}$$

as $\epsilon \to 0$, provided t_2 is a Lebesgue point of the map

$$t \mapsto \int_{\mathbb{T}^3} u(t)\cdot\phi(t)\,\mathrm{d}x.$$

Since, by Lebesgue's theorem, this is the case for almost every $t_2 > 0$, by collecting all terms we finally arrive at (2.3) in the case $t_1 = 0$.

In the general case, we would use the test function $\zeta_\epsilon(t)\xi_\epsilon(t)\phi(x,t)$, where ζ_ϵ is as before and

$$\xi_\epsilon(t) := \zeta\left(\frac{t_1 - t}{\epsilon}\right).$$

The passage to the limit $\epsilon \to 0$ can then be achieved in exactly the same way as above. $\square$

CHAPTER 3

Existence of Weak Solutions

3.1. Galerkin Approximation

3.1.1. A toy example: the heat equation. To illustrate the Galerkin method in the simplest possible setting, let us consider the Cauchy problem for the heat equation on the torus:

$$\begin{aligned} \partial_t u &= \Delta u \quad \text{on } \mathbb{T}^3, \\ u(t=0) &= u^0. \end{aligned} \tag{3.1}$$

Let $\{w_k\}_{k\in\mathbb{N}}$ be an eigenbasis of $-\Delta$ and "project" the problem to the finite dimensional subspace $P_N H := \operatorname{span}\{w_1,\dots,w_N\}$: If $u(t)$ is in this space for every t, then so is $\partial_t u(t)$, and thanks to the eigenfunction property also $-\Delta u(t) \in P_N H$. The projected version of the heat equation therefore simply reads

$$\begin{aligned} \partial_t u_N &= \Delta u_N \quad \text{on } \mathbb{T}^3, \\ u_N(t=0) &= P_N u^0. \end{aligned} \tag{3.2}$$

This equation is known as the *Galerkin equation* of order N, and we want to solve it in $P_N H$. To this end, take the ansatz $u_N(x,t) = \sum_{l=1}^N d_l^N(t) w_l(x)$, insert it into (3.2), multiply by w_k $(k=1,\dots N)$, and integrate in space:

$$\begin{aligned} (d_k^N)'(t) + \lambda_k d_k^N(t) &= 0, \\ d_k^N(0) &= (u^0, w_k)_{L^2}, \end{aligned}$$

where we used orthonormality and the eigenfunction property of the w_l. This is a system (actually a decoupled one in this simple case) of linear *ordinary* differential equations, which has a global smooth solution by standard ODE theory.

We wish to let $N \to \infty$ and hope to obtain a solution to the original problem in the limit. To this end, observe that multiplication of (3.2) with its solution u_N and integration in space yields (in analogy to NSE) the energy equality

$$\frac{1}{2}\int_{\mathbb{T}^3} |u_N(x,t)|^2 \,\mathrm{d}x + \int_0^t \int_{\mathbb{T}^3} |\nabla u_N(x,s)|^2 \,\mathrm{d}x\,\mathrm{d}s = \frac{1}{2}\int_{\mathbb{T}^3} |P_N u^0(x)| \,\mathrm{d}x \le \frac{1}{2}\int_{\mathbb{T}^3} |u^0(x)| \,\mathrm{d}x,$$

and thus a uniform (in N) bound of the Galerkin sequence in $L^\infty L^2 \cap L^2 H^1$. By the Banach-Alaoglu Theorem, we may therefore a weakly*-convergent subsequence (not relabelled), so that $u_N \overset{*}{\rightharpoonup} u \in L^\infty L^2$. Hence for every $\phi \in C_c^\infty(\mathbb{T}^3 \times [0,\infty))$, we have

$$\int_0^\infty \int_{\mathbb{T}^3} u_N \partial_t \phi \,\mathrm{d}x\,\mathrm{d}t \to \int_0^\infty \int_{\mathbb{T}^3} u \partial_t \phi \,\mathrm{d}x\,\mathrm{d}t,$$

$$\int_0^\infty \int_{\mathbb{T}^3} u_N \Delta \phi \,\mathrm{d}x\,\mathrm{d}t \to \int_0^\infty \int_{\mathbb{T}^3} u \Delta \phi \,\mathrm{d}x\,\mathrm{d}t,$$

and finally

$$\int_{\mathbb{T}^3} P_N u^0 \phi(t=0) dx \to \int_{\mathbb{T}^3} u^0 \phi(t=0) dx,$$

because $P_N u^0 \to u^0$ in L^2. It thus follows that u is a weak solution of (3.1). Note that we did not make any use of the L^2H^1-bound.

3.1.2. Galerkin for NSE. Recall the basis of eigenfunctions of $-\Delta$ (now viewed as the Stokes Operator) from Theorem 2.11. Let $P_N H := \operatorname{span}\{w_1, \dots, w_N\}$ and consider the projection operator $P_N : \dot{L}^2 \to P_N H$ given by

$$P_N(u) = \sum_{j=1}^{N} (u, w_j) w_j$$

using the L^2 inner product. Clearly, for all $u \in H$ we have $P_N u \to u$ in H (i.e. in the L^2-norm.) Indeed,

$$\|P_N u - u\|_{L^2}^2 = \| \sum_{j=N+1}^{\infty} (u, w_j) w_j \|^2 = \sum_{j=N+1}^{\infty} |(u, w_j)|^2 \to 0$$

as $N \to \infty$.

Definition 3.1. The N-th order Galerkin approximation of the NSE with initial data $u^0 \in H$ is the solution of the equation

$$\begin{aligned} \partial_t u_N + P_N[(u_N \cdot \nabla) u_N] &= \nu \Delta u_N, \\ u_N(0) &= P_N u_0. \end{aligned} \tag{3.3}$$

Note we have not yet proved that such u_N exists! Again we have "projected away" the pressure. Like for the heat equation, we want to take $N \to \infty$.

From the energy estimate, we expect a uniform bound of the form

$$\sup_t \int_\Omega |u_N(x,t)|^2 \, \mathrm{d}x + 2\nu \int_0^t \int_\Omega |\nabla u_N(x,s)|^2 \, \mathrm{d}x \, \mathrm{d}s \le C < \infty, \tag{3.4}$$

so by the Banach-Alaoglu Theorem we will be able to pass to weak limits:

$$u_N \rightharpoonup u, \quad \nabla u_N \rightharpoonup \nabla u$$

weakly in L^2.

So far, the general strategy seems similar as for the heat equation. However, for NSE, there are two main issues to solve:

First, the Galerkin approximation (3.3) and thus the resulting system of ODEs now feature a *quadratic* term, so that the ODE solution is *prima facie* only obtained up to a possibly finite blow-up time; indeed, the simplest quadratic ODE, $\dot{x} = x^2$, does exhibit finite-time blow-up. Even worse, the existence interval $[0, T_N)$ might depend on N and could therefore, in the worst case, converge to zero as $N \to \infty$, so that in the limit, we would be left with nothing. It turns out, luckily, that such blow-up scenarios can rather easily be ruled out by virtue of the finite dimensional energy equality (3.4).

Once we have globally existing Galerkin approximants $\{u_N\}_{N \in \mathbb{N}}$ which satisfy the uniform bound (3.4), we need to establish that the weak limit u is a weak solution. For the heat equation (and more generally for linear equations), this was trivial. However, for the NSE, in the weak formulation we have the nonlinear term

$$\int_0^\infty \int_{\mathbb{T}^3} (u_N \cdot \nabla) u_N \cdot \phi \, \mathrm{d}x \, \mathrm{d}t$$

and it is not clear whether

$$(u_N \cdot \nabla) u_N \rightharpoonup (u \cdot \nabla) u!$$

For example in 1D we see that

$$u_N(x) = \sin(Nx) \rightharpoonup 0 \quad \text{but} \quad u_N^2(x) = \sin^2(Nx) \not\rightharpoonup 0,$$

meaning that weak limits and nonlinearities do, in general, not converge. However, if we knew $u_N \rightharpoonup u$ strongly and $\nabla u_N \rightharpoonup \nabla u$ weakly, then it would follow $(u_N \cdot \nabla) u_N \rightharpoonup (u \cdot \nabla) u$ weakly. The strong convergence of u_N will be obtained by means of the following compactness result, which can be seen as a time-dependent version of the classical Rellich compactness theorem:

LEMMA 3.2 (Aubin-Lions). *Let $0 < T < \infty$ and assume for some $1 < p, q \leq \infty$ that*

$$\|u_N\|_{L^q(0,T;V)} + \|\partial_t u_N\|_{L^p(0,T;V')} \leq C,$$

for a constant $C \neq C(N)$. Here, V' is the dual space of V. Then there exists a subsequence $\{u_{N_j}\}_{j\in\mathbb{N}}$ such that $u_{N_j} \to u$ strongly in $L^q(0,T;H)$, for some $u \in L^q(0,T;H)$.

PROOF. Consider for $k \in \mathbb{N}$ the map $t \mapsto (u_N(t), w_k)_{L^2}$. It is not difficult to see (exercise!) that this map is weakly differentiable with weak derivative $(\partial_t u_N(t), w_k)$, which is a well-defined L^p function since $\partial_t u_N \in L^p_t V'$ and $w_k \in V$, and so (cf. exercise) $s \mapsto (u_N(s), w_k)$ is (absolutely) continuous and

$$(u_N(s), w_k) = (u_N(s^*), w_k) + \int_{s^*}^{s} (\partial_t u_N, w_k) \, \mathrm{d}t$$

for all $s, s^* \in [0,T]$. Again by continuity we may invoke the mean value theorem for integrals to conclude there exists an $s^* \in [0,T]$ such that

$$(u_N(s^*), w_k) = \frac{1}{T} \int_0^T (u_N, w_k) \, \mathrm{d}t.$$

Hence,

$$\begin{aligned} \sup_{0 \leq s \leq T} |(u_N(s), w_k)| &\leq |(u_N(s^*), w_k)| + \left| \int_{s^*}^{s} (\partial_t u_N, w_k) \, \mathrm{d}t \right| \\ &\leq \frac{1}{T} \int_0^T \|u_N\|_{L^2(\mathbb{T}^3)} \|w_k\|_{L^2(\mathbb{T}^3)} \, \mathrm{d}t + \int_0^T \|\partial_t u_N\|_{V'} \|w_k\|_V \, \mathrm{d}t. \end{aligned}$$

We can now use Hölder's inequality combined with the facts that $\|w_k\|_{L^2(\mathbb{T}^3)} = 1$ and $\|w_k\|_V = \sqrt{\lambda_k}$ to obtain the bound

$$\leq \frac{1}{T} T^{1-\frac{1}{q}} \|u_N\|_{L^q(0,T;H)} + T^{1-\frac{1}{p}} \|\partial_t u_N\|_{L^p(0,T;V')} \sqrt{\lambda_k} \leq C_1 + \sqrt{\lambda_k} C_2$$

where we have used that $\|\partial_t u_N\|_{L^p(0,T;V')}$ is uniformly bounded and by Poincaré's inequality we have $\|u\|_H \leq C \|u\|_V$, which gives a uniform bound on $\|u_N\|_{L^q(0,T;H)}$. It follows that $P_k u_N := \sum_{j=1}^{k} (u_N, w_j) w_j$ is in $C([0,T];H)$ and

$$\sup_{s \in (0,T)} \|P_k u_N(s)\|_H \leq \sum_{j=1}^{k} (C_1 + C_2 \sqrt{\lambda_j}) \leq k(C_1 + C_2 \sqrt{\lambda_k})$$

as λ_j is increasing.

Claim 1: For every k, $\{P_k u_N\}_{N\in\mathbb{N}}$ has a subsequence converging in $C([0,T];H)$. For this, we will use the Arzelà-Ascoli Theorem. As $P_k H$ is finite-dimensional, it suffices to check that the sequence is uniformly bounded and equicontinuous. From the previous estimate we already have uniform boundedness, so all we need to check is equicontinuity.

Since $P_k H$ is finite-dimensional, $\|\cdot\|_H$ and $\|\cdot\|_{V'}$ are equivalent in $P_k H$, and so

$$\begin{aligned}\|P_k u_N(t_2) - P_k u_N(t_1)\|_H &= \|\int_{t_1}^{t_2} \partial_t P_k u_N(s)\,\mathrm{d}s\|_H \\ &\le \int_{t_1}^{t_2} \|\partial_t P_k u_N(s)\|_H\,\mathrm{d}s \\ &\le C_k \int_{t_1}^{t_2} \|\partial_t P_k u_N(s)\|_{V'}\,\mathrm{d}s \\ &\le C_k \left(\int_{t_1}^{t_2} \|\partial_t P_k u_N(s)\|_{V'}^p\,\mathrm{d}s\right)^{\frac{1}{p}} \left(\int_{t_1}^{t_2} \mathrm{d}s\right)^{1-\frac{1}{p}} \\ &\le \tilde{C}_k |t_2 - t_1|^{1-\frac{1}{p}}\end{aligned}$$

where we used Hölder's inequality and that $\|\partial_t P_k u_N\|_{L^p(t_1,t_2;V')}$ is uniformly bounded in N. Thus we may apply the Arzelà-Ascoli theorem, which proves the claim.

Claim 2: $\{u_N\}$ has a subsequence that is Cauchy in $L^q(0,T;H)$. Clearly Claim 2 implies the theorem. By a diagonal argument (exercise), we can select a subsequence (still denoted $\{u_N\}$) so that $\{P_k u_N\}$ is convergent in $L^q(0,T;H)$ for all $k \in \mathbb{N}$ (see Claim 1). We will show this sequence is Cauchy in $L^q(0,T;H)$.

Claim 2*a*: For every $\delta > 0$ there exists $k \in \mathbb{N}$ such that

$$\int_0^T \|P_k u_N(s) - u_N(s)\|_H^q\,\mathrm{d}s < \delta$$

for all $N \ge k$. Indeed, we know that $C \ge \|u_N\|_{L^q(0,T;V)}$ and as $(\nabla w_j, \nabla w_k) = (-\Delta w_j, w_k) = \lambda_j (w_j, w_k) = \lambda_j \delta_{jk}$, then

$$\begin{aligned}C \ge \int_0^T \|\nabla u_N(s)\|_{L^2}^q\,\mathrm{d}s &= \int_0^T \left(\sum_{j=1}^\infty \lambda_j |(u_N(s), w_j)_{L^2}|^2\right)^{\frac{q}{2}}\,\mathrm{d}s \\ &\ge \int_0^T \left(\sum_{j=k+1}^\infty \lambda_j |(u_N(s), w_j)_{L^2}|^2\right)^{\frac{q}{2}}\,\mathrm{d}s,\end{aligned}$$

so as λ_j are increasing we obtain

$$\ge \lambda_{k+1}^{\frac{q}{2}} \int_0^T \left(\sum_{j=k+1}^\infty |(u_N(s), w_j)_{L^2}|^2\right)^{\frac{q}{2}}\,\mathrm{d}s = \lambda_{k+1}^{\frac{q}{2}} \int_0^T \|P_k u_N(s) - u_N(s)\|_{L^2}^q\,\mathrm{d}s,$$

and now Claim 2*a* follows from the fact that $\lambda \nearrow \infty$.

Next, recall that (for the k from Claim 2*a*) $P_k u_N$ is Cauchy in $L^q(0,T;H)$, so there is an $N_0 \in \mathbb{N}$ such that

$$\int_0^T \|P_k u_N(s) - P_k u_M(s)\|^q\,\mathrm{d}s < \delta$$

for all $N, M > N_0$. By the triangle inequality,

$$\begin{aligned}\|u_N - u_M\|_{L^q(0,T;H)} \le \|u_N - P_k u_N\|_{L^q(0,T;H)} &+ \|P_k u_N - P_k u_M\|_{L^q(0,T;H)} \\ &+ \|u_M - P_k u_M\|_{L^q(0,T;H)} \le 3\delta^{\frac{1}{q}}\end{aligned}$$

and since $\delta > 0$ was arbitrary, Claim 2 follows and we are done. □

Only one ingredient is missing before we can embark on the existence proof for weak solutions:

LEMMA 3.3 (L^p interpolation). *Let Ω be a measure space and $u \in L^p(\Omega) \cap L^q(\Omega)$ for some $1 \le p \le q \le \infty$. Then $u \in L^r(\Omega)$ for all $p \le r \le q$, and $\|u\|_{L^r} \le \|u\|_{L^p}^{\alpha} \|u\|_{L^q}^{1-\alpha}$, where $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$.*

PROOF. Using Hölder's inequality with $\frac{p}{\alpha r}, \frac{q}{(1-\alpha)r}$ we see that

$$\int_\Omega |u|^r \, \mathrm{d}x = \int_\Omega |u|^{\alpha r} |u|^{(1-\alpha)r} \, \mathrm{d}x \le \left(\int_\Omega |u|^p \, \mathrm{d}x \right)^{\frac{\alpha r}{p}} \left(\int_\Omega |u|^q \, \mathrm{d}x \right)^{\frac{(1-\alpha)r}{p}}$$
$$= \left[\|u\|_{L^p}^{\alpha} \|u\|_{L^q}^{1-\alpha} \right]^r$$

and so we are done. □

3.2. The Existence Proof

THEOREM 3.4 (Existence of weak solutions). *For every $u^0 \in H$ there exists a weak solution of the NSE with initial data u^0. Moreover, this solution satisfies $\partial_t u \in L^{4/3}_{loc}(0, T; V')$.*

PROOF. Step 1: Existence of Galerkin approximations, locally in time. Recall the N-th order Galerkin equation:

$$\partial_t u_N + P_N[(u_N \cdot \nabla) u_N] = \nu \Delta u_N,$$
$$u_N(0) = P_N u^0.$$

We take the ansatz

$$u_N(x,t) = \sum_{j=1}^N d_j^N(t) w_j(x)$$

and multiply the Galerkin equation by w_k, and integrate:

$$\partial_t \int_{\mathbb{T}^3} \sum_{j=1}^N d_j^N(t) w_j(x) w_k(x) \, \mathrm{d}x + \int_{\mathbb{T}^3} P_N\Big(\sum_{j,l=1}^N d_j^N(t) d_l^N(t) w_j(x) \cdot \nabla\Big) w_l(x) \cdot w_k(x) \, \mathrm{d}x$$
$$= \nu \sum_{j=1}^N \int_{\mathbb{T}^3} d_j^N(t) \Delta w_j(x) \cdot w_k(x) \, \mathrm{d}x.$$

By orthogonality of $\{w_k\}$ in L^2 and the eigenfunction property, this gives

$$(d_k^N)'(t) + \nu \lambda_k d_j^N(t) + \sum_{j,l=1}^N d_j^N(t) d_l^N(t) B_{kjl} = 0,$$

with $k = 1, \dots, N$ and

$$B_{kjl} := \int_{\mathbb{T}^3} (w_j(x) \cdot \nabla) w_l(x) \cdot w_k(x) \, \mathrm{d}x.$$

Indeed, $(P_N v, w_k)_{L^2} = (v, w_k)_{L^2}$ for all $v \in L^2$ by a simple linear algebra argument.

This is a system of N ODEs for the N unknown functions d_k^N ($k = 1, \dots, N$), with initial condition

$$d_k^N(0) = \int_{T^3} u_0(x) w_k(x) \, \mathrm{d}x.$$

(The latter is obtained by multiplying

$$u_N(x,0) = \sum_{j=1}^N d_j^N(0) w_j(x)$$

by w_k and employing the initial condition $u_N(x,0) = P_N u^0(x)$. Note again $(P_N u^0(x), w_k(x)) = (u^0, w_k(x))$.)

By classical ODE theory (Picard-Lindelöf/Cauchy-Lipschitz) there exists a time $T_N > 0$ and a solution $\{d_k^N\}_{k=1,\dots,N} \in C^1((0,T_N))$ of this system.

Step 2: Show $T_N = \infty$ for all N, via energy estimates. Let $s \in (0,T_N)$. Multiply the Galerkin equation at time s with $u_N(s)$ and integrate in x to get

$$\int_{\mathbb{T}^3} \partial_t u_N(s)\cdot u_N(s)\,\mathrm{d}x + \int_{\mathbb{T}^3} P_N(u_N(s)\cdot\nabla)u_N(s)\cdot u_N(s)\,\mathrm{d}x = \nu\int_{\mathbb{T}^3} \Delta u_N(s)\cdot u_N(s)\,\mathrm{d}x.$$

Observe each integral in order: for the first we see that

$$\int_{\mathbb{T}^3} \partial_t u_N(s)\cdot u_N(s)\,\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^3} |u_N(s)|^2\,\mathrm{d}x,$$

for the second,

$$\begin{aligned}\int_{\mathbb{T}^3} P_N(u_N(s)\cdot\nabla)u_N(s)\cdot u_N(s)\,\mathrm{d}x &= \int_{\mathbb{T}^3} (u_N(s)\cdot\nabla)u_N(s)\cdot u_N(s)\,\mathrm{d}x\\ &= \sum_{j,l=1}^{3}\int_{\mathbb{T}^3} u_N^j(s)\partial_{x_j}u_N^l(s)u_N^l(s)\,\mathrm{d}x\\ &= -\sum_{j,l=1}^{3}\int_{\mathbb{T}^3} \partial_{x_j}u_N^j(s)(u_N^l(s))^2\,\mathrm{d}x\\ &\quad -\sum_{j,l=1}^{3}\int_{\mathbb{T}^3} \partial_{x_j}u_N^j(s)u_N^l(s)\partial_{x_j}u_N^l(s)\,\mathrm{d}x = 0\end{aligned}$$

using incompressibility, and for the last term we see that

$$\nu\int_{\mathbb{T}^3} \Delta u_N(s)\cdot u_N(s)\,\mathrm{d}x = -\nu\int_{\mathbb{T}^3} |\nabla u_N(s)|^2\,\mathrm{d}x.$$

Hence for all $s \in (0,T_N)$ we obtain the (finite-dimensional) energy equality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_N(s)\|_{L^2(\mathbb{T}^3)}^2 + \nu\|\nabla u_N(s)\|_{L^2(\mathbb{T}^3)}^2 = 0.$$

We note that this equality implies (after integration in s) that

$$\sup_t \frac{1}{2}\|u_N(t)\|_{L^2}^2 \le \frac{1}{2}\|u_0\|_{L^2}^2 \quad\text{and}\quad \int_0^\infty \|\nabla u_N(s)\|_{L^2}^2\,\mathrm{d}s \le \frac{1}{2\nu}\|u^0\|_{L^2}^2,$$

thus $\{u_N\}$ are uniformly bounded in $L^\infty(0,\infty;H)\cap L^2(0,\infty;V)$.

In particular, since $u_N(x,s) = \sum_{k=1}^N d_k^N(s)w_k(x)$ and since $\{w_k\}$ is an ONB in L^2,

$$\|u_N(s)\|_{L^2}^2 = \sum_{k=1}^{d} |d_k^N(s)|^2$$

and this is bounded in s. It follows that $\{d_k^N(s)\}_{k=1,\dots,N}$ is uniformly bounded in s and hence $T_N = \infty$.

Step 3: Bound $\partial_t u_N$ in an appropriate norm (in order to apply Aubin-Lions).

Let $\phi \in V$ and take the L^2-inner product with the Galerkin equation:

$$\begin{aligned}(\partial_t u_N,\phi) &= \nu(\Delta u_N,\phi) - (P_N(u_N\cdot\nabla)u_N,\phi)\\ &= \nu(\Delta u_N,\phi) - ((u_N\cdot\nabla)u_N, P_N\phi),\end{aligned}$$

where for the last equality we used the self-adjointness of the projection P_N. For the first term

$$|\nu(\Delta u_N,\phi)| = \nu|(\nabla u_N,\nabla\phi)| \le \nu\|\nabla u_N\|_{L^2}\|\phi\|_V,$$

and for the second

$$|((u_N\cdot\nabla)u_N, P_N\phi)| \le \|u_N\|_{L^3}\|\nabla u_N\|_{L^2}\|P_N\phi\|_{L^6} \le \|u_N\|_{L^2}^{\frac{1}{2}}\|u_N\|_{L^6}^{\frac{1}{2}}\|\nabla u_N\|_{L^2}\|P_N\phi\|_{L^6}$$

using interpolation (Lemma 3.3). Now we can use the Sobolev embedding $\|v\|_{L^6} \le C\|v\|_{\dot{H}^1} = C\|v\|_V = C\|\nabla v\|_{L^2}$ and the projection property $\|P_N\phi\|_V \le \|\phi\|_V$ to obtain

$$|((u_N\cdot\nabla)u_N, P_N\phi)| \le C\|u_N\|_{L^2}^{\frac{1}{2}}\|\nabla u_N\|_{L^2}^{\frac{3}{2}}\|\phi\|_V.$$

It follows from the definition of the dual/operator norm

$$\|\partial_t u_N\|_{V'} \le \nu\|\nabla u_N\|_{L^2} + C\|u_N\|_{L^2}^{\frac{1}{2}}\|\nabla u_N\|_{L^2}^{\frac{3}{2}}$$

and thus for any $0 < T < \infty$

$$\int_0^T \|\partial_t u_N\|_{V'}^{\frac{4}{3}}\,\mathrm{d}s \le C\nu\int_0^T \|\nabla u_N(s)\|^{\frac{4}{3}}\,\mathrm{d}s + C\int_0^T \|u_N\|_{L^2}^{\frac{2}{3}}\|\nabla u_N(s)\|_{L^2}^2\,\mathrm{d}s.$$

Using Hölder's inequality in time on both terms (first L^3(on 1), $L^{3/2}$ second L^∞, L^1) we obtain

$$\le C\nu T^{\frac{1}{3}}\left[\left(\int_0^T \|\nabla u_N(s)\|^2\,\mathrm{d}s\right)^{\frac{1}{2}}\right]^{\frac{4}{3}} + C\|u_N\|_{L^\infty(0,T;H)}^{\frac{2}{3}}\|u_N(s)\|_{L^2(0,T;V)}^2$$

which becomes

$$\le C\nu T^{\frac{1}{3}}\|u_N(s)\|_{L^2(0,T;V)}^{\frac{4}{3}} + C\|u_N\|_{L^\infty(0,T;H)}^{\frac{2}{3}}\|u_N(s)\|_{L^2(0,T;V)}^2$$

which is bounded uniformly in N (for fixed ν and T!) owing to the energy estimates from Step 2.

Step 4: Extract a convergent subsequence.

By Banach-Alaoglu, there is a subsequence $\{u_{N_j}\}_j$ such that

$$u_{N_j} \overset{*}{\rightharpoonup} u$$

weak-$*$ in $L^\infty(0,\infty;H)$, and another subsequence $\{u_{N_{j,l}}\}_l$ such that

$$\nabla u_{N_{j,l}} \rightharpoonup w \quad \text{in } L^2(0,\infty;L^2(\mathbb{T}^3)).$$

It is easy to show that (exercise!) $w = \nabla u$. For any fixed $0 < T < \infty$, extracting yet another subsequence if necessary (not relabelled), the uniform bound on $\partial_t u_N$ gives

$$\partial_t u_N \overset{*}{\rightharpoonup} \partial_t u \quad \text{in } L^{4/3}(0,T;V').$$

Even better, by a diagonal argument we obtain a subsequence such that

$$\partial_t u_N \overset{*}{\rightharpoonup} \partial_t u \quad \text{in } L^{4/3}_{loc}(0,\infty;V').$$

Choosing yet another subsequence and applying again a diagonal argument, we obtain by Lemma 3.2 (Aubin-Lions)

$$u_N \to u$$

strongly in $L^2_{loc}(0,\infty;H)$.

Step 5: Show the limit is a solution of NSE.

By Lemma 2.14, it suffices to choose a test function of the form

$$\phi(x,t) = \sum_{k=1}^m d_k(t)w_k(x) \in \tilde{\mathcal{D}}_\sigma.$$

Let T be so large that $\operatorname{supp}(d_k) \subset [0,T)$ for all $k = 1,\dots,m$. By the Galerkin equation, for any $N \geq m$ we have

$$-\int_0^\infty \int_{\mathbb{T}^3} u_N \cdot \partial_t \phi \,\mathrm{d}x\,\mathrm{d}t + \nu \int_0^\infty \int_{\mathbb{T}^3} \nabla u_N : \nabla \phi \,\mathrm{d}x\,\mathrm{d}t \\ + \int_0^\infty \int_{\mathbb{T}^3} (u_N \cdot \nabla) u_N \cdot \phi \,\mathrm{d}x\,\mathrm{d}t = \int_{\mathbb{T}^3} u_0 \cdot \phi(0)\,\mathrm{d}x.$$

Weak convergence $u_{N_j} \overset{*}{\rightharpoonup} u$ in $L^\infty(0,T;L^2(\mathbb{T}^3))$ gives

$$-\int_0^\infty \int_{\mathbb{T}^3} u_N \cdot \partial_t \phi \,\mathrm{d}x\,\mathrm{d}t \to -\int_0^\infty \int_{\mathbb{T}^3} u \cdot \partial_t \phi \,\mathrm{d}x\,\mathrm{d}t$$

and $\nabla u_N \rightharpoonup \nabla u$ in $L^2(0,T;L^2(\mathbb{T}^3))$ gives

$$\int_0^\infty \int_{\mathbb{T}^3} \nabla u_N : \nabla \phi \,\mathrm{d}x\,\mathrm{d}t \to \int_0^\infty \int_{\mathbb{T}^3} \nabla u : \nabla \phi \,\mathrm{d}x\,\mathrm{d}t.$$

Only the non-linear term needs some more attention, we see that

$$(u_N \cdot \nabla) u_N - (u \cdot \nabla) u = ((u_N - u) \cdot \nabla) u_N - (u \cdot \nabla)(u_N - u).$$

Firstly,

$$\left| \int_0^\infty \int_{\mathbb{T}^3} ((u_N - u) \cdot \nabla) u_N \cdot \phi \,\mathrm{d}x\,\mathrm{d}t \right| \leq C_\phi \int_0^T \|u_N - u\|_{L^2} \|\nabla u_N\|_{L^2}\,\mathrm{d}t \\ \leq C_\phi \|u_N - u\|_{L^2(0,T;L^2)} \|\nabla u_N\|_{L^2(0,T;L^2)}$$

and we see that this converges to zero as $\|\nabla u_N\|_{L^2(0,T;L^2)}$ is bounded and $\|u_N - u\|_{L^2(0,T;L^2)}$ converges to zero as $N \to \infty$. Secondly,

$$\int_0^\infty \int_{\mathbb{T}^3} (u \cdot \nabla)(u_N - u) \cdot \phi \,\mathrm{d}x\,\mathrm{d}t \to 0$$

as $N \to \infty$ as $u \in L^2_{t,x}$ and $\nabla(u_N - u)$ converges to zero weakly in $L^2_{t,x}$. It follows that

$$\int_0^\infty \int_{\mathbb{T}^3} (u_N \cdot \nabla) u_N \cdot \phi \,\mathrm{d}x\,\mathrm{d}t \to \int_0^\infty \int_{T^3} (u \cdot \nabla) u \cdot \phi \,\mathrm{d}x\,\mathrm{d}t$$

and so

$$-\int_0^\infty \int_{\mathbb{T}^3} u \cdot \partial_t \phi + \nu \nabla u : \nabla \phi + (u \cdot \nabla) u \cdot \phi \,\mathrm{d}x\,\mathrm{d}t = \int_{\mathbb{T}^3} u_0 \cdot \phi(0)\,\mathrm{d}x.$$

Hence u is a weak solution. □

The bound on the time derivative allows us to obtain a useful continuity property in time, typical for balance equations in continuum mechanics:

PROPOSITION 3.5. *The solution constructed in Theorem 3.4 is (after alteration on a set of times of measure zero, if necessary) contained in the space $C([0,\infty);V')$, and it satisfies the statement of Lemma 2.16 even for* all *(and not just almost all)* $0 \leq t_1 < t_2$.

PROOF. This will be proved in the exercises, based on the property $\partial_t u \in L^{4/3}_{loc}(0,\infty;V')$. □

In fact, one can show that this proposition is true for *every* Leray-Hopf solution, not just the (possibly particular) one constructed in Theorem 3.4. Of course, if Leray-Hopf solutions are unique, this distinction is unnecessary, but uniqueness is still unknown in three dimensions.

CHAPTER 4

Strong Solutions

4.1. Some More on Bochner Spaces

For this entire chapter, let $0 < T < \infty$ be arbitrary but fixed. We collect a few technical results to be used later.

Proposition 4.1. *Let X Banach and suppose $u, w \in L^1(0,T;X)$. Then the following are equivalent:*

(1) *$\partial_t u = w$ in the weak sense;*
(2) *There exists $\xi \in X$ such that, for a.e. $t \in (0,T)$,*
$$u(t) = \xi + \int_0^t w(s)\,\mathrm{d}s;$$
(3) *For every $v \in X'$, in the weak sense it holds that*
$$\frac{\mathrm{d}}{\mathrm{d}t}(u,v) = (w,v).$$

Moreover, if one (and thus all) of these conditions holds, then u can be altered on a nullset of times so that it belongs to $C([0,T];X)$.

Proof. Exercise. $\square$

Proposition 4.2. *Let $u \in L^2(0,T;\dot{H}^1(\mathbb{T}^3))$ and $\partial_t u \in L^2(0,T;H^{-1}(\mathbb{T}^3))$. Then,*

(1) *$u \in C([0,T];\dot{L}^2(\mathbb{T}^3))$;*
(2) *the map $t \mapsto \frac{1}{2}\|u(t)\|_{\dot{L}^2}^2$ is weakly differentiable with*
$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|u(t)\|_{\dot{L}^2}^2 = (u(t), \partial_t u(t)) \quad \textit{for a.e. } t \in (0,T);$$
(3)
$$\max_{t\in[0,T]} \|u(t)\|_{\dot{L}^2} \le C\left(\|u\|_{L^2\dot{H}^1} + \|\partial_t u\|_{L^2H^{-1}}\right).$$

Proof. Let $\eta \in C_c^\infty(\mathbb{R})$ be a standard mollifier (in the time variable), that is, $\eta \ge 0$, $\int_{\mathbb{R}} \eta(\theta)\,\mathrm{d}\theta = 1$, $\operatorname{supp}\eta \subset B_1(0)$, and $\eta = \eta(|\theta|)$. Set $\eta_\epsilon(t) := \frac{1}{\epsilon}\left(\frac{t}{\epsilon}\right)$ and, for any $f \in L^1_{loc}(\mathbb{R})$, $f_\epsilon := f * \eta_\epsilon$, i.e.
$$f_\epsilon(t) = \int_{-\epsilon}^{\epsilon} f(t-\tau)\eta_\epsilon(\tau)\,\mathrm{d}\tau.$$
Consider now u as given in the statement and extend it to $t \in \mathbb{R}$ by zero, so that its mollification is well-defined on all of $[0,T]$. For $\epsilon, \delta > 0$, u_δ and u_ϵ are smooth in time, and we may thus use the standard Leibniz rule to compute
$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_\epsilon(t) - u_\delta(t)\|_{L^2}^2 = 2\int_{\mathbb{T}^3} (u_\epsilon(t) - u_\delta(t)) \cdot (\partial_t u_\epsilon(t) - \partial_t u_\delta(t))\,\mathrm{d}x. \tag{4.1}$$
Observe that for a.e. $s \in (0,T)$, $u_\epsilon(s) \to u(s)$ in $H^1(\mathbb{T}^3)$ and $\partial_t u_\epsilon(s) \overset{*}{\rightharpoonup} \partial_t u(s)$ in $H^{-1}(\mathbb{T}^3)$ (exercise). Pick such an s and integrate (4.1) from s to t to obtain
$$\|u_\epsilon(t) - u_\delta(t)\|_{L^2}^2 \le \|u_\epsilon(s) - u_\delta(s)\|_{L^2}^2 + \int_0^T |(u_\epsilon(\tau) - u_\delta(\tau), \partial_t u_\epsilon(t) - \partial_t u_\delta(t))|\,\mathrm{d}\tau$$

By choice of s, the first expression on the right hand side converges, as $\epsilon, \delta \to 0$, to zero, and so does the dual pairing under the integral, for almost every τ. But then the integral itself converges by dominated convergence (exercise).

It follows that $\{u_\epsilon\}_{\epsilon>0}$ is Cauchy in $C([0,T];L^2(\mathbb{T}^3))$, and since this space is Banach, it follows $u_\epsilon \to u \in C([0,T];L^2(\mathbb{T}^3))$, whence (1) is established.

For (2), again by the classical Leibniz rule,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_\epsilon(t)\|_{L^2}^2 = 2\int_{\mathbb{T}^3} u_\epsilon(t)\cdot\partial_t u_\epsilon(t)\,\mathrm{d}x$$

and hence, after integration from s to t,

$$\|u_\epsilon(t)\|_{L^2}^2 = \|u_\epsilon(s)\|_{L^2}^2 + 2\int_s^t\int_{\mathbb{T}^3} u_\epsilon(\tau)\cdot\partial_t u_\epsilon(\tau)\,\mathrm{d}\tau\,\mathrm{d}x, \tag{4.2}$$

and the same equality follows for u instead of u_ϵ by similar convergence arguments as before. Application of Proposition 4.1 (2) then gives the desired characterisation of the weak time derivative.

Finally, for (3), integrate (4.2) in s over $(0,T)$ to arrive at

$$T\|u_\epsilon(t)\|_{L^2}^2 \le \int_0^T \|u_\epsilon(s)\|_{L^2}^2\,\mathrm{d}s + T(\|u\|_{L^2\dot{H}^1}^2 + \|\partial_t u\|_{L^2H^{-1}}^2),$$

and thus (3) follows because t is arbitrary and the right hand side is independent of t. □

Corollary 4.3. *Let $u,v \in L^2(0,T;\dot{H}^1(\mathbb{T}^3))$ and $\partial_t u, \partial_t v \in L^2(0,T;H^{-1}(\mathbb{T}^3))$. Then the map $t \mapsto (u,v)$ is absolutely continuous with weak derivative*

$$\frac{\mathrm{d}}{\mathrm{d}t}(u,v) = (\partial_t u, v) + (u, \partial_t v).$$

Proof. This follows from the polarisation identity

$$(u,v) = \frac{1}{2}(\|u+v\|^2 - \|u\|^2 - \|v\|^2)$$

and the preceding proposition. □

The following can be seen as an extension of Proposition 4.2 to higher order Sobolev spaces:

Proposition 4.4. *Let $n \in \mathbb{N}_0$ and suppose $u \in L^2(0,T;\dot{H}^{n+2}(\mathbb{T}^3))$ and $\partial_t u \in L^2(0,T;\dot{H}^n(\mathbb{T}^3))$. Then, $u \in C([0,T];\dot{H}^{n+1}(\mathbb{T}^3))$, and*

$$\max_{t\in[0,T]}\|u(t)\|_{\dot{H}^{n+1}} \le C\left(\|u\|_{L^2\dot{H}^{n+2}} + \|\partial_t u\|_{L^2\dot{H}^n}\right).$$

Proof. We indicate only the formal argument and remark that the rigorous proof proceeds exactly as in Proposition 4.2 by time mollification.

So assume u is smooth and take for simplicity $n = 0$, then we compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{\dot{H}^1}^2 &= \frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u(t)\|_{L^2}^2\\ &= 2\int_{\mathbb{T}^3}\nabla u : \partial_t\nabla u\,\mathrm{d}x\\ &= -2\int_{\mathbb{T}^3}\Delta u\cdot\partial_t u\,\mathrm{d}x \le \|u(t)\|_{H^2}^2 + \|\partial_t u(t)\|_{L^2}^2,\end{aligned}$$

and the estimate

$$\sup_{t\in[0,T]}\|u(t)\|_{\dot{H}^{n+1}} \le C\left(\|u\|_{L^2\dot{H}^{n+2}} + \|\partial_t u\|_{L^2\dot{H}^n}\right)$$

follows, as before, by integrating first from s to t and then by integrating in s from 0 to T. □

4.2. Properties of Strong Solutions

DEFINITION 4.5 (Strong solutions). A *strong solution* of NSE is a weak solution with additional regularity

$$u \in L^\infty(0,T;H^1(\mathbb{T}^3)) \cap L^2(0,T;H^2(\mathbb{T}^3)).$$

Strong solutions have very nice properties, like energy conservation, uniqueness, and smoothness; however, given initial data $u^0 \in V$, a strong solution is known to exist only on a possibly finite time interval (whether or not the existence time can actually be finite is precisely the Navier-Stokes Millennium Problem).

LEMMA 4.6. *Let u be a strong solution, then $\partial_t u, (u\cdot\nabla)u, \Delta u \in L^2(0,T;L^2(\mathbb{T}^3))$.*

PROOF. From the assumption $u \in L^2(0,T;H^2)$ it follows immediately that $\Delta u \in L^2(0,T;L^2)$. For the nonlinear term, note that H^2 embeds continuously into L^∞, so that

$$\int_0^T \int_{\mathbb{T}^3} |u|^2 |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}t \le \int_0^T \|u(t)\|_{L^\infty}^2 \int_{\mathbb{T}^3} |\nabla u(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t \le C\|u\|_{L^\infty H^1}^2 \|u\|_{L^2 H^2}^2.$$

It remains to estimate the time derivative. In view of Lemma 2.16 and the remark after Proposition 3.5, for every smooth divergence-free vector field $\phi \in C^\infty(\mathbb{T}^3)$ and every $t \in [0,T]$ we have

$$\begin{aligned}\int_{\mathbb{T}^3} u(t)\cdot\phi \,\mathrm{d}x &= \int_{\mathbb{T}^3} u^0\cdot\phi\,\mathrm{d}x - \int_0^t \int_{\mathbb{T}^3} \nabla u : \nabla\phi + (u\cdot\nabla)u\cdot\phi \,\mathrm{d}x\,\mathrm{d}s \\ &= \int_{\mathbb{T}^3} u^0\cdot\phi\,\mathrm{d}x + \int_0^t \int_{\mathbb{T}^3} \Delta u\cdot\phi - (u\cdot\nabla)u\cdot\phi \,\mathrm{d}x\,\mathrm{d}s,\end{aligned}$$

as u has weak second space derivatives. Proposition 4.1 allows us to take the time derivative of this equality to obtain, for every $t \in [0,T]$,

$$\int_{\mathbb{T}^3} \partial_t u\cdot\phi\,\mathrm{d}x = \int_{\mathbb{T}^3} \nu\Delta u\cdot\phi - (u\cdot\nabla)u\cdot\phi\,\mathrm{d}x.$$

Let $\psi \in C^\infty(\mathbb{T}^3)$ be any smooth vector field and $\psi = \phi + \nabla\pi$ its Helmholtz decomposition, so that $\operatorname{div}\phi = 0$. Then, as $u(t) \in H$,

$$\begin{aligned}\int_{\mathbb{T}^3} \partial_t u\cdot\psi\,\mathrm{d}x &= \int_{\mathbb{T}^3} \partial_t u\cdot\psi(t)\,\mathrm{d}x \\ &= \int_{\mathbb{T}^3} \nu\Delta u\cdot\mathbb{P}\phi - (u\cdot\nabla)u\cdot\mathbb{P}\psi\,\mathrm{d}x \\ &= \int_{\mathbb{T}^3} \mathbb{P}(\nu\Delta u - (u\cdot\nabla)u)\cdot\psi\,\mathrm{d}x,\end{aligned}$$

and since ψ was arbitrary, it follows that $\partial_t u = \mathbb{P}(\Delta u - (u\cdot\nabla)u)$. By the previous estimates, this is indeed in $L^2(0,T,L^2)$.

$\square$

LEMMA 4.7. *Let u be a strong solution. Then, for any $w \in L^2(0,T;H)$,*

$$\int_0^T \int_{\mathbb{T}^3} (\partial_t u + (u\cdot\nabla)u - \nu\Delta u)\cdot w\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Note that the integral in fact is well-defined by the previous lemma.

PROOF. By similar arguments as in the proof of Lemma 2.14, the space $\tilde{\mathcal{D}}_\sigma$ is dense in $L^2(0,T;H)$, and therefore it suffices to consider $w \in \tilde{\mathcal{D}}_\sigma$. But such w may be used as a test function in the weak formulation, so that

$$\int_{\mathbb{T}^3} u(T)\cdot w(T)\,\mathrm{d}x - \int_{\mathbb{T}^3} u^0\cdot w(0)\,\mathrm{d}x = \int_0^T\int_{\mathbb{T}^3} u\cdot\partial_t w - \nu\nabla u : \nabla w - (u\cdot\nabla)u\cdot w\,\mathrm{d}x\,\mathrm{d}t.$$

But clearly, since u is a strong solution,

$$\int_0^T\int_{\mathbb{T}^3}\nabla u:\nabla w\,\mathrm{d}x\,\mathrm{d}t = -\int_0^T\int_{\mathbb{T}^3}\Delta u\cdot w\,\mathrm{d}x\,\mathrm{d}t,$$

and by Proposition 4.1 and Corollary 4.3 also

$$\int_{\mathbb{T}^3} u(T)\cdot w(T)\,\mathrm{d}x - \int_{\mathbb{T}^3} u^0\cdot w(0)\,\mathrm{d}x - \int_0^T\int_{\mathbb{T}^3} u\cdot\partial_t w\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_{\mathbb{T}^3}\partial_t u\cdot w\,\mathrm{d}x\,\mathrm{d}t.$$

Putting everything together, we arrive at the conclusion. □

LEMMA 4.8. *For any $u\in V$, we have*

$$\int_{\mathbb{T}^3}(u\cdot\nabla)u\cdot u\,\mathrm{d}x = 0.$$

PROOF. Write $b(u,u,u) := \int_{\mathbb{T}^3}(u\cdot\nabla)u\cdot u\,\mathrm{d}x = 0$, then this is a trilinear form. For smooth vector fields, we have already established $b(u,u,u)=0$ (in the formal derivation of the energy equality). Therefore, if $u_\epsilon = u * \eta_\epsilon$ denotes a standard mollification, we have

$$|b(u,u,u)| \le |b(u,u,u)-b(u_\epsilon,u,u)|+|b(u_\epsilon,u,u)-b(u_\epsilon,u_\epsilon,u)|+|b(u_\epsilon,u_\epsilon,u)-b(u_\epsilon,u_\epsilon,u_\epsilon)|.$$

But the first term is estimated as

$$\int_{\mathbb{T}^3}|u-u_\epsilon||\nabla u||u|\,\mathrm{d}x \to 0$$

as $\epsilon\to 0$: This follows from the embedding $H^1\subset L^6\subset L^3$ and Hölder's inequality with $\frac{1}{6},\frac{1}{2},\frac{1}{3}$. The other two terms are estimated in the same way. □

THEOREM 4.9 (Energy equality). *A strong solution satisfies the energy equality, i.e. for every $s<t$ we have*

$$\frac{1}{2}\int_{\mathbb{T}^3}|u(t)|^2\,\mathrm{d}x + \nu\int_s^t\int_{\mathbb{T}^3}|\nabla u(x,\tau)|^2\,\mathrm{d}\tau = \frac{1}{2}\int_{\mathbb{T}^3}|u(s)|^2\,\mathrm{d}s.$$

PROOF. In Lemma 4.7, we may take $w := \chi_{[s,t]}u$ and thus obtain

$$0 = \int_0^T\int_{\mathbb{T}^3}(\partial_t u+(u\cdot\nabla)u-\nu\Delta u)\cdot u\chi_{[s,t]}\,\mathrm{d}x\,\mathrm{d}\tau = \int_s^t\int_{\mathbb{T}^3}(\partial_t u+(u\cdot\nabla)u-\nu\Delta u)\cdot u\,\mathrm{d}x\,\mathrm{d}\tau.$$

By Proposition 4.2, $u\cdot\partial_t u = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^2}^2$, and

$$\int_s^t\int_{\mathbb{T}^3}u\cdot\partial_t u\,\mathrm{d}x\,\mathrm{d}\tau = \frac{1}{2}\int_{\mathbb{T}^3}|u(t)|^2\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{T}^3}|u(s)|^2\,\mathrm{d}s,$$

whereas (by a very simple approximation argument)

$$\int_{\mathbb{T}^3}\Delta u\cdot u\,\mathrm{d}x = -\int_{\mathbb{T}^3}|\nabla u|^2\,\mathrm{d}x,$$

and finally

$$\int_{\mathbb{T}^3}(u\cdot\nabla)u\cdot u\,\mathrm{d}x = 0$$

thanks to Lemma 4.8. This completes the proof. □

LEMMA 4.10. *Let u be a weak solution of NSE and $U \in L^2(0,T;H^2 \cap V)$ a vector field with $\partial_t U \in L^2(0,T;L^2)$. Then U is a valid test function in the definition of weak solution for u, that is,*

$$\int_{\mathbb{T}^3} u(t) \cdot U(t)\,\mathrm{d}x - \int_{\mathbb{T}^3} u^0 \cdot U(0)\,\mathrm{d}x = \int_0^t \int_{\mathbb{T}^3} u \cdot \partial_t U - \nu \nabla u : \nabla U - (u \cdot \nabla) u \cdot U\,\mathrm{d}x\,\mathrm{d}s \tag{4.3}$$

for every $t \in [0,T]$.

PROOF. We only give a sketch. As before (e.g. in the proof of Lemma 2.14), we approximate U by fields in $\tilde{\mathcal{D}}_\sigma$ by means of the projection operator P_N onto the span of the first N Stokes eigenfunctions. For every thus obtained U_N, (4.3) is valid. One then takes the limit $N \to \infty$. The only term requiring some care is $\int_{\mathbb{T}^3} u(t) \cdot U(t)\,\mathrm{d}x$, as we need to converge pointwise in t. But by Proposition 4.4 (with $n = 0$), we have

$$\sup_{t \in [0,T]} \|(U_N - U_M)(t)\|_{\dot{H}^1} \le C\left(\|U_N - U_M\|_{L^2\dot{H}^2} + \|\partial_t (U_N - U_M)\|_{L^2\dot{L}^2}\right),$$

meaning that $\{U_N\}$ is Cauchy, and thus convergent, in $C([0,T];H^1)$. The proof now proceeds as in Proposition 4.4. □

THEOREM 4.11 (Weak-strong uniqueness). *Let U be a strong solution and u a weak solution that satisfies the energy inequality, and assume both solutions share the same initial data $u^0 \in H$. Then $u \equiv U$ almost everywhere.*

PROOF. The proof relies on an estimate of the *relative energy* between u and U, defined as

$$E_{rel}(t) = \frac{1}{2} \int_{\mathbb{T}^3} |u(x,t) - U(x,t)|^2\,\mathrm{d}x.$$

In the course of the computation, we use three ingredients:

(1) The weak formulation for u, tested with U, as justified by Lemma 4.10:

$$\int_{\mathbb{T}^3} u(t) \cdot U(t)\,\mathrm{d}x - \int_{\mathbb{T}^3} u^0 \cdot U(0)\,\mathrm{d}x = \int_0^t \int_{\mathbb{T}^3} u \cdot \partial_t U - \nu \nabla u : \nabla U - (u \cdot \nabla) u \cdot U\,\mathrm{d}x\,\mathrm{d}s;$$

(2) The pointwise solution property of U, integrated against u, as justified by Lemma 4.7:

$$\int_0^T \int_{\mathbb{T}^3} (\partial_t U + (U \cdot \nabla) U - \nu \Delta U) \cdot u\,\mathrm{d}x\,\mathrm{d}t = 0;$$

(3) The energy (in)equalities for u and U, as justified by assumption and by Theorem 4.9, respectively:

$$\frac{1}{2} \int_{\mathbb{T}^3} |u(t)|^2\,\mathrm{d}x + \nu \int_s^t \int_{\mathbb{T}^3} |\nabla u(x,\tau)|^2\,\mathrm{d}\tau \le \frac{1}{2} \int_{\mathbb{T}^3} |u(s)|^2\,\mathrm{d}s$$

and similar for U.

Using (1), we obtain

$$\begin{aligned}
E_{rel}(t) &= \frac{1}{2} \int_{\mathbb{T}^3} |u(x,t) - U(x,t)|^2\,\mathrm{d}x \\
&= \frac{1}{2} \int_{\mathbb{T}^3} |u(x,t)|^2\,\mathrm{d}x + \frac{1}{2} \int_{\mathbb{T}^3} |U(x,t)|^2\,\mathrm{d}x - \int_{\mathbb{T}^3} u(t) \cdot U(t)\,\mathrm{d}x \\
&= \frac{1}{2} \int_{\mathbb{T}^3} |u(x,t)|^2\,\mathrm{d}x + \frac{1}{2} \int_{\mathbb{T}^3} |U(x,t)|^2\,\mathrm{d}x \\
&\quad - \int_0^t \int_{\mathbb{T}^3} u \cdot \partial_t U - \nu \nabla u : \nabla U - (u \cdot \nabla) u \cdot U\,\mathrm{d}x\,\mathrm{d}s - \int_{\mathbb{T}^3} u^0 \cdot U^0\,\mathrm{d}x.
\end{aligned}$$

By (3) and the assumption $u^0 = U^0$, the sum of the inital terms is non-positive and can thus be neglected in the estimate, so that

$$E_{rel}(t) \le -\int_0^t \int_{\mathbb{T}^3} u\cdot\partial_t U - \nu \nabla u : \nabla U - (u\cdot\nabla)u\cdot U \,\mathrm{d}x\,\mathrm{d}s - \nu \int_0^t \int_{\mathbb{T}^3} |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}s - \nu \int_0^t \int_{\mathbb{T}^3} |\nabla U|^2 \,\mathrm{d}x\,\mathrm{d}s.$$

Integrating by parts and using (2), we can write this as

$$\begin{aligned} E_{rel}(t) &\le -\int_0^t \int_{\mathbb{T}^3} u\cdot\partial_t U - \nu u \cdot \Delta U + (U\cdot\nabla)U\cdot u \,\mathrm{d}x\,\mathrm{d}s - \nu \int_0^t \int_{\mathbb{T}^3} |\nabla u - \nabla U|^2 \,\mathrm{d}x\,\mathrm{d}s + R \\ &= -\nu \int_0^t \int_{\mathbb{T}^3} |\nabla u - \nabla U|^2 \,\mathrm{d}x\,\mathrm{d}s + R, \end{aligned}$$

where

$$R = -\int_0^t \int_{\mathbb{T}^3} (U\cdot\nabla)U\cdot u + (u\cdot\nabla)u\cdot U \,\mathrm{d}x\,\mathrm{d}s.$$

Similar arguments as in Lemma 4.8 yield

$$\int_{\mathbb{T}^3} (U\cdot\nabla)u\cdot u \,\mathrm{d}x = 0, \qquad \int_{\mathbb{T}^3} (u\cdot\nabla)U\cdot U \,\mathrm{d}x = 0,$$

whence

$$R = -\int_0^t \int_{\mathbb{T}^3} (U\cdot\nabla)(U-u)\cdot u - (u\cdot\nabla)(U-u)\cdot U \,\mathrm{d}x\,\mathrm{d}s,$$

and finally an application of the formula $\int_{\mathbb{T}^3} (U\cdot\nabla)(U-u)\cdot(U-u)\,\mathrm{d}x = 0$ gives

$$R = -\int_0^t \int_{\mathbb{T}^3} ((U-u)\cdot\nabla)(U-u)\cdot U \,\mathrm{d}x\,\mathrm{d}s.$$

We thus obtain the estimate

$$\begin{aligned} |R| &\le \int_0^t \int_{\mathbb{T}^3} |U-u||\nabla U - \nabla u||U| \,\mathrm{d}x\,\mathrm{d}s \\ &\le \nu \int_0^t \int_{\mathbb{T}^3} |\nabla U - \nabla u|^2 \,\mathrm{d}x\,\mathrm{d}s + C(\nu) \int_0^t \|U(s)\|_{L^\infty}^2 \int_{\mathbb{T}^3} |U-u|^2 \,\mathrm{d}x\,\mathrm{d}s, \end{aligned}$$

where we used the inequality $ab \le \frac{a^2}{2\delta^2} + \delta^2\frac{b^2}{2}$ for suitable δ, depending on ν.

In total we obtain

$$E_{rel}(t) \le \int_0^t \|U(s)\|_{L^\infty}^2 E_{rel}(s)\,\mathrm{d}s,$$

and since U is a strong solution, $\|U\|_{L^\infty}^2 \in L^1(0,T)$, and we may then use Grönwall's inequality to conclude $E_{rel} \equiv 0$, which implies the theorem. □

Theorem 4.12 (Weak-strong stability). *Let U be a strong solution and u a weak solution that satisfies the energy inequality, with initial data $U^0 \in H$ and $u^0 \in H$, respectively. Then there exists a constant, depending only on the norm of U in $L^\infty H^1 \cap L^2 H^2$ and on the viscosity ν, such that for all $t \in [0,T]$*

$$\|u(t) - U(t)\|_{L^2(\mathbb{T}^3)} \le e^{Ct}\|u^0 - U^0\|_{L^2(\mathbb{T}^3)}.$$

Proof. The proof is almost identical to the one of the preceding theorem, only that the initial terms do not cancel. It is left as an exercise. The reader might also want to give an explicit formula for the constant C in terms of U and ν. □

4.3. Local Existence of Strong Solutions

LEMMA 4.13 (Agmon's inequality). *Let $u \in \dot{H}^2(\mathbb{T}^3)$, then, for some constant C,*

$$\|u\|_{L^\infty} \le C\|u\|_{H^1}^{1/2}\|u\|_{H^2}^{1/2}.$$

PROOF. We split the Fourier series of u into low and high frequencies, with $M > 0$ to be chosen later:

$$\begin{aligned} u(x) &= \sum_{k\neq 0} \hat{u}_k \mathrm{e}^{\mathrm{i}k\cdot x} \\ &= \sum_{|k|\le M} \hat{u}_k \mathrm{e}^{\mathrm{i}k\cdot x} + \sum_{|k|>M} \hat{u}_k \mathrm{e}^{\mathrm{i}k\cdot x} \\ &= \sum_{|k|\le M} \hat{u}_k \mathrm{e}^{\mathrm{i}k\cdot x}|k||k|^{-1} + \sum_{|k|>M} \hat{u}_k \mathrm{e}^{\mathrm{i}k\cdot x}|k|^2|k|^{-2} \\ &\le \|u\|_{\dot{H}^1}\left(\sum_{|k|\le M}\frac{1}{|k|^2}\right)^{1/2} + \|u\|_{\dot{H}^2}\left(\sum_{|k|>M}\frac{1}{|k|^4}\right)^{1/2} \end{aligned}$$

It is easy to see that

$$\begin{aligned} \left(\sum_{|k|\le M}\frac{1}{|k|^2}\right)^{1/2} &\le C\left(\int_{B_M(0)\setminus B_1(0)}\frac{1}{|x|^2}\,\mathrm{d}x\right)^{1/2} \\ &= C\left(\int_1^M \frac{r^2}{r^2}\,\mathrm{d}r\right)^{1/2} \le CM^{1/2}, \end{aligned}$$

and similarly

$$\begin{aligned} \left(\sum_{|k|>M}\frac{1}{|k|^4}\right)^{1/2} &\le C\left(\int_{\mathbb{R}^3\setminus B_M(0)}\frac{1}{|x|^4}\,\mathrm{d}x\right)^{1/2} \\ &= C\left(\int_M^\infty \frac{r^2}{r^4}\,\mathrm{d}r\right)^{1/2} \le CM^{-1/2}, \end{aligned}$$

so that in total

$$\|u\|_{L^\infty} \le C\left(M^{1/2}\|u\|_{\dot{H}^1} + M^{-1/2}\|u\|_{\dot{H}^2}\right).$$

The choice $M = \frac{\|u\|_{\dot{H}^2}}{\|u\|_{\dot{H}^1}}$ now yields the result. □

THEOREM 4.14 (Existence of strong solutions). *Let $u^0 \in V$. There exists a constant $C > 0$, depending only on the viscosity ν, such that there exists a strong solution at least on the interval $[0, T]$, where $T = C\|\nabla u^0\|^{-4}$.*

PROOF. Recall the Galerkin equation (3.3),

$$\begin{aligned} \partial_t u_N + P_N[(u_N\cdot\nabla)u_N] &= \nu\Delta u_N, \\ u_N(0) &= P_N u_0, \end{aligned}$$

which we have seen to have global smooth solutions. Multiply this equation by $-\Delta u_N$ and integrate in space to obtain

$$-\int_{\mathbb{T}^3}\partial_t u_N\cdot\Delta u_N\,\mathrm{d}x + \nu\int_{\mathbb{T}^3}|\Delta u_N|^2\,\mathrm{d}x - \int_{\mathbb{T}^3}(u_N\cdot\nabla)u_N\cdot\Delta u_N\,\mathrm{d}x = 0.$$

This means

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_N\|_{L^2}^2 + \nu\|\Delta u_N\|_{L^2}^2\,\mathrm{d}x = \int_{\mathbb{T}^3}(u_N\cdot\nabla)u_N\cdot\Delta u_N\,\mathrm{d}x, \tag{4.4}$$

and we wish to estimate the right-hand side. For this, we use Agmon's inequality and the estimate $\|u_N\|_{H^2} \le C\|\Delta u_N\|_{L^2}$ (which is trivial to show in Fourier):

$$\begin{aligned}\left|\int_{\mathbb{T}^3}(u_N\cdot\nabla)u_N\cdot\Delta u_N\,\mathrm{d}x\right| &\le \|u_N\|_{L^\infty}\|\nabla u_N\|_{L^2}\|\Delta u_N\|_{L^2}\\ &\le C\|u\|_{H^1}^{1/2}\|u\|_{H^2}^{1/2}\|\nabla u_N\|_{L^2}\|\Delta u_N\|_{L^2}\\ &\le C\|\nabla u_N\|_{L^2}^{3/2}\|\Delta u_N\|_{L^2}^{3/2}.\end{aligned}$$

Young's inequality $ab \le \frac{a^p}{p}+\frac{b^q}{q}$ for $\frac{1}{p}+\frac{1}{q}=1$, applied here with $p=4$ and $q=\frac{4}{3}$, gives

$$\left|\int_{\mathbb{T}^3}(u_N\cdot\nabla)u_N\cdot\Delta u_N\,\mathrm{d}x\right| \le C(\nu)\|\nabla u_N\|_{L^2}^6+\frac{1}{2}\nu\|\Delta u_N\|_{L^2}^2,$$

so that (4.4) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_N\|_{L^2}^2+\nu\|\Delta u_N\|_{L^2}^2 \le C(\nu)\|\nabla u_N\|_{L^2}^6. \tag{4.5}$$

Setting aside the Laplacian term for the moment, we see that $Y := \|\nabla u_N\|_{L^2}^2$ satisfies the ordinary differential inequality

$$Y'(t)\le CY(t)^3,\quad Y(0)=\|P_N\nabla u^0\|_{L^2}^2$$

and hence $Y(t)\le X(t)$ for the solution of the corresponding equation $X'=CX^3$, $X(0)=\|\nabla u^0\|_{L^2}^2 \ge Y(0)$. It is not difficult to compute $X(t)$ explicitly as

$$X(t)=\frac{\|\nabla u^0\|_{L^2}^2}{\sqrt{1-2Ct\|\nabla u^0\|_{L^2}^4}}.$$

If we set $T=\frac{3}{8C\|\nabla u^0\|_{L^2}^4}$, we obtain $X(T)=2\|\nabla u^0\|_{L^2}^2$ and therefore $Y(t)$ is uniformly bounded in $[0,T]$, which in turn means that the u_N are bounded in $L^\infty(0,T;V)$, uniformly in N.

Coming back to (4.5) and integrating from 0 to T, we observe (recalling that $\|\nabla u_N\|_{L^2}^2$ is bounded, on $[0,T]$, by $2\|\nabla u^0\|_{L^2}^2$)

$$\begin{aligned}\nu\int_0^T\|\Delta u_N\|_{L^2}^2\,\mathrm{d}t &\le \|\nabla P_N u^0\|_{L^2}^2-\|\nabla u_N(T)\|_{L^2}^2+C\int_0^T\|\nabla u_N\|_{L^2}^6\,\mathrm{d}t\\ &\le \|\nabla u^0\|_{L^2}^2+C\int_0^T\|\nabla u_N\|_{L^2}^6\,\mathrm{d}t\\ &\le \|\nabla u^0\|_{L^2}^2+2CT\|\nabla u^0\|_{L^2}^6\\ &=\frac{7}{4}\|\nabla u^0\|_{L^2}^2\end{aligned}$$

by choice ot T. It follows that the u_N are also bounded, uniformly in N, in the space $L^2(0,T;H^2(\mathbb{T}^3))$.

We know already that a subsequence of $\{u_N\}_{N\in\mathbb{N}}$ converges to a weak solution of NSE. Selecting from this sequence another subsequence that converges additionally in $L^\infty(0,T;V)\cap L^2(0,T;H^2(\mathbb{T}^3))$ (this exists by the Banach-Alaoglu theorem, and the bounds just derived), we see that the weak solution is in fact in $L^\infty(0,T;V)\cap L^2(0,T;H^2(\mathbb{T}^3))$, and is thus a strong solution up to time T. □

4.4. Regularity of Strong Solutions

It turns out that even when only $u^0\in V$, the strong solution of NSE will automatically be C^∞ smooth on the (open!) time interval $(0,T)$, for any T up to which the solution exists in the strong sense. This is an effect of parabolic regularisation.

First we need a crucial Banach algebra property:

THEOREM 4.15. *Let $s > \frac{3}{2}$, then $H^s(\mathbb{T}^3)$ embeds continuously into $L^\infty(\mathbb{T}^3)$ and forms a Banach algebra, that is,*

$$\|uv\|_{H^s} \le C\|u\|_{H^s}\|v\|_{H^s},$$

where the constant depends only on s.

PROOF. We give the proof only for $u, v \in \dot{H}^s$, as this is the situation we shall need. However the general statement follows very similarly (replacing $|k|^s$ by $(1+|k|)^s$). So let $x \in \mathbb{T}^3$, then

$$|u(x)| = \left|\sum_{k\neq 0} \hat{u}_k e^{\mathrm{i}k\cdot x}\right| \le \sum_{k\neq 0} |\hat{u}_k| = \sum_{k\neq 0} |\hat{u}_k||k|^s|k|^{-s} \le \left(\sum_{k\neq 0} |\hat{u}_k|^2|k|^{2s}\right)^{1/2} \left(\sum_{k\neq 0} |k|^{-2s}\right)^{1/2}.$$

But the first factor is precisely the homogeneous H^s-norm of u, and for the second factor we compute

$$\sum_{k\neq 0} |k|^{-2s} \le C \int_{\mathbb{R}^3\setminus B_0(1)} \frac{1}{|x|^{2s}}\,\mathrm{d}x = C\int_1^\infty r^{2-2s}\,\mathrm{d}r,$$

which is finite if and only if $2-2s < -1$, i.e. $s > \frac{3}{2}$. Thus we obtain the embedding assertion as required.

Let now $u, v \in \dot{H}^s(\mathbb{T}^3)$, then by the above computation the Fourier series of u and v are absolutely convergent, and thus we may form the Cauchy product to calculate

$$u(x)v(x) = \left(\sum_{k\neq 0} \hat{u}_k e^{\mathrm{i}k\cdot x}\right)\left(\sum_{j\neq 0} \hat{v}_j e^{\mathrm{i}j\cdot x}\right) = \sum_{k\in\mathbb{Z}^3}\left(\sum_{l\in\mathbb{Z}^3} \hat{u}_{k-l}\hat{v}_l\right) e^{\mathrm{i}k\cdot x}, \tag{4.6}$$

so that $\widehat{uv}_k = \sum_{l\in\mathbb{Z}^3} \hat{u}_{k-l}\hat{v}_l$.

As another ingredient we recall the inequality $|k|^s \le C(|k-l|^s + |l|^s)$, where C depends only on $s > 0$.

We can now estimate

$$\begin{aligned}
\|uv\|_{H^s}^2 &= \sum_{k\neq 0}(1+|k|^{2s})|\widehat{uv}_k|^2\\
&= \sum_{k\in\mathbb{Z}^3}(1+|k|^{2s})\left|\sum_{l\in\mathbb{Z}^3} \hat{u}_{k-l}\hat{v}_l\right|^2\\
&\le C\sum_{k\in\mathbb{Z}^3}\left|\sum_{l\in\mathbb{Z}^3}((1+|k-l|^s) + (1+|l|^s))\hat{u}_{k-l}\hat{v}_l\right|^2\\
&\le C\sum_{k\in\mathbb{Z}^3}\left|\sum_{l\in\mathbb{Z}^3}(1+|k-l|^s)\hat{u}_{k-l}\hat{v}_l\right|^2 + C\sum_{k\in\mathbb{Z}^3}\left|\sum_{l\in\mathbb{Z}^3}(1+|l|^s)\hat{u}_{k-l}\hat{v}_l\right|^2.
\end{aligned}$$

Let us define the function $(1+|\nabla|^s)u(x) := \sum_{k\neq 0}(1+|k|^s)\hat{u}_k e^{\mathrm{i}k\cdot x}$, then obviously

$$\|(1+|\nabla|^s)u\|_{L^2} \le C\|u\|_{H^s},$$

and similarly for v. By (4.6), then, we see that

$$\sum_{l\in\mathbb{Z}^3}(1+|k-l|^s)\hat{u}_{k-l}\hat{v}_l = [((1+|\nabla|^s)u)v]^\wedge_k \quad\text{and}\quad \sum_{l\in\mathbb{Z}^3}(1+|l|^s)\hat{u}_{k-l}\hat{v}_l = [u((1+|\nabla|^s)v)]^\wedge_k,$$

so that (using Plancherel's Theorem)

$$\begin{aligned}
\|uv\|_{H^s}^2 &\le C\sum_{k\neq 0}|[((1+|\nabla|^s)u)v]^\wedge_k|^2 + C\sum_{k\neq 0}|[u((1+|\nabla|^s)v)]^\wedge_k|^2\\
&= C\left(\|((1+|\nabla|^s)u)v\|_{\dot{L}^2}^2 + \|u((1+|\nabla|^s)v)\|_{\dot{L}^2}^2\right)\\
&\le C\left(\|v\|_{L^\infty}^2\|u\|_{\dot{H}^s}^2 + \|u\|_{L^\infty}^2\|v\|_{\dot{H}^s}^2\right)\\
&\le C\left(\|v\|_{\dot{H}^s}^2\|u\|_{\dot{H}^s}^2\right),
\end{aligned}$$

where in the final step we used the embedding $H^s \subset L^\infty$. □

THEOREM 4.16 (Higher regularity). *Let $m \geq 2$ and $u^0 \in V \cap H^m(\mathbb{T}^3)$. Then the strong solution of NSE with existence interval $[0,T]$ even satisfies*

$$u \in L^\infty(0,T;H^m) \cap L^2(0,T;H^{m+1}).$$

PROOF. Let (u_N) denote the smooth Galerkin approximations, as usual. The existence proof for strong solutions yielded uniform (in N) bounds for

$$\|u_N\|_{L^\infty H^1} + \|u_N\|_{L^2H^2}.$$

By induction, we will deduce such an estimate for m instead of 1. So let us assume the induction hypothesis

$$\sup_N \left(\|u_N\|_{L^\infty H^{m-1}} + \|u_N\|_{L^2H^m}\right).$$

Taking the H^m inner product of the Galerkin equation with u_N and then using Theorem 4.15, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_N\|_{H^m}^2 + \nu\|\nabla u_N\|_{H^m}^2 &= -((u_N\cdot\nabla)u_N, u_N)_{H^m}\\ &\leq \|(u_N\cdot\nabla)u_N\|_{H^m}\|u_N\|_{H^m}\\ &\leq \|u_N\|_{H^m}\|\nabla u_N\|_{H^m}\|u_N\|_{H^m}\\ &\leq \frac{1}{2}\nu\|\nabla u_N\|_{H^m}^2 + C(\nu)\|u_N\|_{H^m}^4.\end{aligned}$$

Note that in the last step, we made use of the Cauchy-Schwarz inequality with ϵ ($ab \leq \epsilon a^2 + C(\epsilon)b^2$).

This results in

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_N\|_{H^m}^2 + \nu\|\nabla u_N\|_{H^m}^2 \leq (C(\nu)\|u_N\|_{H^m}^2)\|u_N\|_{H^m}^2 = g(t)\|u_N\|_{H^m}^2, \tag{4.7}$$

where $g := C(\nu)\|u_N\|_{H^m}^2 \in L^1(0,T)$ uniformly in N, by virtue of the induction hypothesis. Grönwall's inequality therefore yields

$$\|u_N(t)\|_{H^m}^2 \leq \|u^0\|_{H^m}^2 \exp\left(\int_0^t g(s)\,\mathrm{d}s\right).$$

Since the right hand side is finite and independent of N, we obtain the bound

$$\sup_N \|u_N\|_{L^\infty H^m} < \infty.$$

Going back to estimate (4.7) and integrating in time, we get (using the bound just derived)

$$\|u_N(T)\|_{H^m}^2 - \|u^0\|_{H^m}^2 + \nu\|\nabla u_N\|_{H^m}^2 \leq \int_0^T g(t)\,\mathrm{d}t \cdot T\|u_N\|_{L^\infty H^m},$$

which entails

$$\|\nabla u_N\|_{H^m}^2 \leq \frac{1}{\nu}\left(\|u^0\|_{H^m}^2 + \int_0^T g(t)\,\mathrm{d}t\cdot T\|u_N\|_{L^\infty H^m}\right).$$

As the right hand side is finite and independent of N, we obtain the desired bound

$$\sup_N \|u_N\|_{L^2H^{m+1}} < \infty.$$

The sequence $\{u_N\}$ converges to the strong solution u of NSE, but by the bounds just obtained and the Banach-Alaoglu Theorem, it also converges weakly* in $L^\infty H^m$ and weakly in L^2H^{m+1}. It follows that u is contained in these spaces, as claimed. □

THEOREM 4.17 (Space regularity). *Let u be a strong solution of NSE on $[0,T]$. Then, for every $0 < \epsilon < T$ and every $m \in \mathbb{N}$, $u \in C([0,T];H^m(\mathbb{T}^3))$.*

PROOF. By definition, $u \in L^2(0,T;H^2)$, and therefore for almost every $s_1 \in (0,\epsilon)$, $u(s_1) \in H^2$. Choosing such s_1 and using $u(s_1)$ as initial data, and keeping in mind the uniqueness of strong solutions, we obtain from Theorem 4.16 $u \in L^2(s_1,T;H^3)$, and so we may choose an $s_2 \in (s_1,\epsilon)$ such that $u(s_2) \in H^3$. In this way we obtain a sequence $0 < s_1 < s_2 < s_3 < \ldots < \epsilon$ such that $u \in L^\infty(s_m,T;H^{m+1}) \cap L^2(s_m,T;H^{m+2})$.

Observe that this implies $\partial_t u \in L^2(\epsilon,T;H^{m-1})$, because

$$\partial_t u = -\mathbb{P}((u\cdot\nabla)u) + \nu\Delta u.$$

Clearly, the last term is in $L^2(\epsilon,T;H^{m-1})$, but also the nonlinear one: Since $u \in L^\infty(\epsilon,T;H^{m+1})$ and $\nabla u \in L^2(\epsilon,T;H^{m+1})$, by the Banach Algebra property also $(u\cdot\nabla)u \in L^2(\epsilon,T;H^{m+1})$ (so this is actually better than required).

But by virtue of Proposition 4.4, this implies

$$u \in C([\epsilon,T];H^m).$$

□

THEOREM 4.18 (Time regularity). *Let u be a strong solution, $\epsilon > 0$, and $j,k \in \mathbb{N}$. Then $\partial_t^j u \in L^\infty(\epsilon,T;H^k(\mathbb{T}^3))$.*

PROOF. We proceed by induction over j.

We use once more

$$\partial_t u = -\mathbb{P}((u\cdot\nabla)u) + \nu\Delta u. \tag{4.8}$$

Similarly as in the previous proof, we bound $\|\mathbb{P}((u\cdot\nabla)u)\|_{H^k}$ at each time by $\|u\|_{H^k}\|u\|_{H^{k+1}}$, and $\|\Delta u\|_{H^k} \le \|u\|_{H^{k+2}}$. But both these are bounded, uniformly in t, in the respective Sobolev norms by virtue of Theorem 4.17.

For the induction step, differentiate (4.8) $j-1$ times with respect to t to obtain

$$\partial_t^j u = -\sum_{i=0}^{j-1}\binom{j-1}{i}\mathbb{P}((\partial_t^i u\cdot\nabla)\partial_t^{j-1-i}u) + \nu\Delta\partial_t^{j-1}u.$$

But by induction hypothesis, the first $j-1$ time derivatives are in $L^\infty(0,T;H^m)$ for every m, and so we can apply similar arguments as in the induction base to conclude. □

Recalling that a function which is contained in Sobolev spaces of arbitrary order is in fact smooth, we obtain:

COROLLARY 4.19. *Let u be a strong solution of NSE on $[0,T]$. Then, for every $\epsilon > 0$, $u \in C^\infty(\mathbb{T}^3 \times [\epsilon,T])$.*

4.5. Blowup and the Beale-Kato-Majda Criterion

It can not be excluded that a strong solution ceases to be strong after finite time. There is a rich theory of possible blow-up scenarios, although it might turn out that blowup can actually not happen (this is the Millennium Problem).

4.5.1. Vorticity. Let u be a strong solution on $[0,T]$, then it is smooth (in space) for any $t \in (0,T]$. Define the curl operator, which acts on smooth vectorfields $u \in C^\infty(\mathbb{T}^3;\mathbb{R}^3)$ and yields another such vectorfield, by

$$(\operatorname{curl} u)_i := \epsilon_{ijk}\partial_j u_k,$$

where ϵ_{ijk} is the Levi-Civita symbol, i.e. $\epsilon_{ijk} = 1$ if (i,j,k) is an even permutation of $(1,2,3)$, $\epsilon_{ijk} = -1$ if (i,j,k) is an odd permutation, and $\epsilon_{ijk} = 0$ otherwise. Note we applied the summation convention.

It is easy to see that $\operatorname{div}\operatorname{curl} u = 0$ for any choice of u, and also $\operatorname{curl}\nabla p = 0$ for any scalar field p. Taking the curl of NSE and denoting $\omega = \operatorname{curl} u$ (which is known as the *vorticity* of the flow), we obtain

$$\partial_t\omega + \operatorname{curl}((u\cdot\nabla)u) = \nu\Delta\omega.$$

A short computation shows $\operatorname{curl}((u\cdot\nabla)u) = (u\cdot\nabla)\omega - (\omega\cdot\nabla)u$, so we arrive at the *vorticity equation*

$$\partial_t\omega + (u\cdot\nabla)\omega - \nu\Delta\omega = (\omega\cdot\nabla)u.$$

The right hand side is called the *vortex stretching term*; in 2D it is not present, and the vorticity equation is simply a linear transport-diffusion equation[1] that satisfies a maximum principle in any L^p norm (including L^∞). This is another very important way to see why 2D NSE are so much better behaved than 3D NSE.

LEMMA 4.20. *Let $u \in C^\infty(\mathbb{T}^3;\mathbb{T}^3)$ be divergence-free, then*

$$\|\omega\|_{L^2} = \|\nabla u\|_{L^2}.$$

PROOF. The computation goes like this:

$$\begin{aligned}\int |\omega|^2 &= \int (\epsilon_{ijk}\partial_j u_k)(\epsilon_{imn}\partial_m u_n)\\ &= \int (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})\partial_j u_k \partial_m u_n\\ &= \int \partial_j u_k\partial_j u_k - \partial_j u_k \partial_k u_j = \int \partial_j u_k \partial_j u_k = \int |\nabla u|^2,\end{aligned}$$

where we used $\int \partial_j u_k \partial_k u_j = 0$ owing to an integration by parts and the divergence-free property. □

4.5.2. Blowup.

DEFINITION 4.21. Let $u^0 \in V$ and u be a corresponding Leray-Hopf solution of NSE. A time $T^* > 0$ is called the *blowup time* for the solution if u is a strong solution on $[0, T^* - \epsilon]$ for any $\epsilon > 0$, but it is not a strong solution on $[0, T^*]$.

A few remarks are in order. First, it is possible that no blowup occurs and hence no (finite) blowup time exists – this is trivially so e.g. for the zero solution. Secondly, if there is a blowup, then the blowup time is uniquely determined by u^0: Indeed, as long as the strong solution exists, it is unique in the class of Leray-Hopf solutions (Theorem 4.11).

A more substantial remark is that T^* is the smallest time at which $\|\nabla u\|_{L^2}(T^*) = \infty$. Indeed, the proof of Theorem 4.14 shows that $u \in L^2(0,T;H^2)$ as long as $u \in L^\infty(0,T;H^1)$, so that a solution that exits the former space will thereby also exit the latter.

THEOREM 4.22 (Beale-Kato-Majda). *Let $u^0 \in V$ and u be a corresponding Leray-Hopf solution of NSE. If $T > 0$ is such that*

$$\int_0^T \|\omega\|_{L^\infty}\,\mathrm{d}t < \infty,$$

then u is a strong solution on $[0,T]$.

[1] Of course to solve this equation, the nonlocal coupling between u and ω needs to be taken into account. The coupling law is known by the name of Biot-Savart and is a Fourier multiplier operator of order -1.

PROOF. We multiply the vorticity equation by ω and integrate to obtain

$$\int \partial_t\omega\cdot\omega + (u\cdot\nabla)\omega\cdot\omega + \nu|\omega|^2 \,\mathrm{d}x = \int (\omega\cdot\nabla)u\cdot\omega \,\mathrm{d}x.$$

The first term equals $\frac{\mathrm{d}}{\mathrm{d}t}\int|\omega|^2\,\mathrm{d}x$, the second one is zero by the usual computation involving the divergence-free property, and the third is non-negative and can thus be dropped. Hence, using Lemma 4.20,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|^2 \le \int |\omega||\nabla u||\omega| \,\mathrm{d}x \le \|\omega\|_{L^\infty}\|\nabla u\|_{L^2}\|\omega\|_{L^2} \le \|\omega\|_{L^\infty}\|\omega\|_{L^2}^2,$$

and so by Grönwall's inequality,

$$\|\nabla u(T)\|_{L^2}^2 \le \|u^0\|_V^2 \exp\left(\int_0^T \|\omega(t)\|_{L^\infty}\,\mathrm{d}t\right).$$

Since, by assumption, the right hand side is finite, then so is $\|\nabla u(T)\|_{L^2}$, and following the remark after Definition 4.21, we conclude. □

CHAPTER 5

The Vanishing Viscosity Limit

So far we kept $\nu > 0$ constant. It has become clear that virtually all estimates during this course have crucially relied on $\nu > 0$, and blow up as $\nu \searrow 0$. In fact, except for the Beale-Kato-Majda criterion, all the results presented so far are false or, in case of existence of weak solutions, completely unknown for $\nu = 0$, in which case the resulting system is known as the *Euler equations*.

An obvious question that arises in the study of turbulent flows (for which a dimensionless number proportional to ν^{-1}, the *Reynolds number*, is typically very large) is whether the (Leray-Hopf) solutions of NSE converge, as $\nu \searrow 0$, to a solution of Euler, if the latter exists. This is known as the viscosity limit problem. It turns out there is a crucial difference between the cases with and without physical boundaries.

5.1. The Periodic Case

We give here a particularly elegant way to handle the viscosity limit, due to P.-L. Lions [**3**, Chapter 4.4].

5.1.1. Dissipative Solutions of the Euler Equations. We consider now the Euler equations,

$$\begin{aligned} \partial_t u + (u \cdot \nabla) u + \nabla p &= 0 \\ \operatorname{div} u &= 0, \end{aligned}$$

whose energy $\frac{1}{2}\int_{\mathbb{T}^3} |u|^2 \, \mathrm{d}x$ is formally conserved by a similar computation as for NSE. Therefore the function space $L^\infty(0,T;H)$ appears suitable for the study of solutions.

For the following formal computation, suppose u is a smooth solution with data u^0, and let $U \in C^\infty(\mathbb{T}^3 \times [0,T]; \mathbb{R}^3)$ be any divergence-free field. Denote

$$E(U) = -\partial_t U - \mathbb{P}((U \cdot \nabla) U),$$

which in a sense measures how far U is from being a solution of Euler.

Then, subtracting the equations for u and U, we get

$$(\partial_t + u \cdot \nabla)(u - U) + (u - U) \cdot \nabla U + \nabla \pi = E(U)$$

for some scalar field π. Next, multiply this by $u - U$ and integrate to obtain

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^3} |u-U|^2 \, \mathrm{d}x + \int_{\mathbb{T}^3} (u \cdot \nabla)(u-U) \cdot (u-U) \, \mathrm{d}x + \int_{\mathbb{T}^3} (u-U) \cdot \nabla_{sym} U (u-U) \, \mathrm{d}x \\ = \int_{\mathbb{T}^3} E(U) \cdot (u-U) \, \mathrm{d}x, \end{aligned}$$

where $\nabla_{sym} = \frac{1}{2}(\nabla + \nabla^t)$ denotes the symmetric gradient. Note that the second integral vanishes by the usual integration by parts argument, so we can estimate

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^3} |u-U|^2 \, \mathrm{d}x \le \|\nabla_{sym} U\|_{L^\infty} \int_{\mathbb{T}^3} |u-U|^2 \, \mathrm{d}x + \int_{\mathbb{T}^3} E(U) \cdot (u-U) \, \mathrm{d}x.$$

Grönwall's inequality then leads to

$$\begin{aligned}\frac{1}{2}\int |u-U|(t)^2 \,\mathrm{d}x \le \exp\left(\int_0^t \|\nabla_{sym}U\|_{L^\infty}\,\mathrm{d}s\right)\int |u^0-U(0)|^2\,\mathrm{d}x \\ + \int_0^t \int \exp\left(\int_s^t \|\nabla_{sym}U\|_{L^\infty}\,\mathrm{d}\tau\right) E(U)\cdot(u-U)\,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{5.1}$$

Recall from the exercises that $C([0,T];L^2_w)$ is the space of functions in $L^\infty(0,T;L^2)$ that are weakly continuous in the sense that, for every $t\in[0,T]$, $u(s) \rightharpoonup u(t)$ weakly in L^2, as $s\to t$.

Our formal computation motivates the following definition:

DEFINITION 5.1 (dissipative solutions). Let $u \in L^\infty(0,T;H)\cap C([0,T];L^2_w)$, with $u(0)=u^0$. Then u is a dissipative solution of Euler if (5.1) holds for every $U\in C([0,T];H)$ such that $E(U)\in L^1(0,T;L^2)$ and $\nabla_{sym}U\in L^1(0,T;L^\infty)$.

REMARK 5.2. (1) The function spaces in this definition are chosen precisely such that each term in (5.1) is well-defined.

(2) Choosing $U\equiv 0$, we obtain simply

$$\frac{1}{2}\int |u(t)|^2\,\mathrm{d}x \le \frac{1}{2}\int |u^0|^2\,\mathrm{d}x \quad \forall t\ge 0, \tag{5.2}$$

meaning that energy is not produced (but preserved or dissipated). This explains the terminology.

(3) It can be shown that every solution in the sense of distributions that satisfies the weak energy inequality (5.2) is also a dissipative solution in the sense of the given definition. Conversely, there exist dissipative solutions that are *not* solutions in the sense of distributions. As we shall see, however, dissipative solutions are useful regardless of any ontological debates as to whether dissipative solutions are "really" solutions of the Euler equations.

PROPOSITION 5.3 (Weak-strong uniqueness). *Suppose $U\in C([0,T];H)$ is such that $\nabla_{sym}U\in L^1(0,T;L^\infty)$, and is a solution of Euler in the sense that $E(U)=0$ almost everywhere. Then any dissipative solution with $u(0)=U(0)$ is equal to U almost everywhere.*

PROOF. This is a direct consequence of (5.1). □

Note that dissipative solutions coincide, in particular, with the smooth solution as long as the latter exists.

LEMMA 5.4. *Let $u\in L^\infty(0,T;H)\cap C([0,T];L^2_w)$, with $u(0)=u^0$. Then u is a dissipative solution of Euler if (5.1) holds for every smooth divergence-free $U\in C^\infty(\mathbb{T}^3\times[0,T])$.*

PROOF. Let U be as in Definition 5.1. First we observe that then $\partial_t U\in L^1(0,T;L^2)$. Indeed,

$$\partial_t U = -\mathbb{P}((U\cdot\nabla)U) - E(U).$$

By assumption, $E(U)$ has the required regularity.

Moreover, a simple calculation yields

$$(U\cdot\nabla)U = 2(\nabla_{sym}U)U - \frac{1}{2}\nabla|U|^2,$$

so that

$$\mathbb{P}((U\cdot\nabla)U) = 2\mathbb{P}((\nabla_{sym}U)U).$$

But since the latter is the product of a matrix field in L^1L^∞ and a vector field in $L^\infty L^2$, we see that $\mathbb{P}((U\cdot\nabla)U)\in L^1L^2$.

Next, let η a standard mollifier in x and set, as usual, $\eta_\epsilon(x) = \frac{1}{\epsilon^3}\eta\left(\frac{x}{\epsilon}\right)$, so that $U_\epsilon := U * \eta_\epsilon$ is a smooth function in the space variables. Moreover, by the integrability of $\partial_t U$ just shown, $\partial_t U \in L^1(0,T;C^k)$ for every $k \in \mathbb{N}$, and by Proposition 4.1 this entails $U \in C([0,T];C^k)$ for every $k \in \mathbb{N}$.

Thus, we may compute

$$\begin{aligned} E(U_\epsilon) &= -\partial_t U_\epsilon - \mathbb{P}((U_\epsilon \cdot \nabla)U_\epsilon) \\ &= [-\partial_t U - \mathbb{P}((U\cdot\nabla)U)]_\epsilon + [\mathbb{P}((U\cdot\nabla)U)_\epsilon - \mathbb{P}((U_\epsilon\cdot\nabla)U_\epsilon)] \\ &= E(U)_\epsilon + 2\mathbb{P}((\nabla_{sym}U)U)_\epsilon - 2\mathbb{P}((\nabla_{sym}U_\epsilon)U_\epsilon) \\ &= E(U)_\epsilon + 2\mathbb{P}[((\nabla_{sym}U)U)_\epsilon - (\nabla_{sym}U_\epsilon)U_\epsilon]. \end{aligned} \tag{5.3}$$

On the one hand, $E(U)_\epsilon \to E(U)$ in L^1L^2 as $\epsilon \to 0$. On the other hand, since $\nabla_{sym}U \in L^1L^\infty$ and $U \in L^\infty L^2$, both $((\nabla_{sym}U)U)_\epsilon$ and $(\nabla_{sym}U_\epsilon)U_\epsilon$ converge to $(\nabla_{sym}U)U$ in L^1L^2, so the second expression in the last line of (5.3) converges to zero in L^1L^2. Hence, $E(U_\epsilon)$ converges to $E(U)$ in L^1L^2.

Therefore, all the integrals in (5.1) converge appropriately as $\epsilon \to 0$, so that (5.1) is valid for U if it was valid for each U_ϵ.

Time regularity can be guaranteed by regularising also in t, which however poses little problem since there is no nonlinearity of $\partial_t U$. □

5.1.2. The Viscosity Limit.

THEOREM 5.5. *Suppose $U \in C([0,T];H)$ is such that $\nabla_{sym}U \in L^1(0,T;L^\infty)$, and is a solution of Euler in the sense that $E(U) = 0$ almost everywhere. Let $(u_\nu)_{\nu>0}$ be a family of Leray-Hopf solutions, satisfying the weak energy inequality, with $u_\nu(0) = U(0)$ for every $\nu > 0$. Then,*

$$\lim_{\nu\to 0} u_\nu = U \quad \textit{strongly in } L^\infty(0,T;H).$$

PROOF. First we show that a subsequence of $\{u_\nu\}$ converges weakly to a dissipative solution of Euler. It suffices to use smooth test fields, as shown in Lemma 5.3. So let $v \in C_c^\infty(\mathbb{T}^3 \times [0,T];\mathbb{R}^3)$ be divergence-free. Then, using v as a test field in the definition of weak solution of NSE, for every $t \in [0,T]$ we have

$$\begin{aligned} -\int_0^t\int_{\mathbb{T}^3} u_\nu\cdot\partial_t v \,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\mathbb{T}^3}(u_\nu\cdot\nabla)u_\nu\cdot v\,\mathrm{d}x\,\mathrm{d}s \\ + \nu\int_0^t\int_{\mathbb{T}^3}\nabla u_\nu : \nabla v\,\mathrm{d}x\,\mathrm{d}s = \int_{\mathbb{T}^3} U(0)\cdot v(0)\,\mathrm{d}x - \int_{\mathbb{T}^3} u_\nu(t)\cdot v(t)\,\mathrm{d}x. \end{aligned}$$

By definition of $E(v)$ and the divergence-free property of both fields (see Lemma 4.8), we thence get

$$\begin{aligned} \int_0^t\int_{\mathbb{T}^3} u_\nu\cdot E(v)\,\mathrm{d}x\,\mathrm{d}s - \int_0^t\int_{\mathbb{T}^3}((u_\nu - v)\cdot\nabla)v\cdot(u_\nu - v)\,\mathrm{d}x\,\mathrm{d}s \\ + \nu\int_0^t\int_{\mathbb{T}^3}\nabla u_\nu : \nabla v\,\mathrm{d}x\,\mathrm{d}s = \int_{\mathbb{T}^3} U(0)\cdot v(0)\,\mathrm{d}x - \int_{\mathbb{T}^3} u_\nu(t)\cdot v(t)\,\mathrm{d}x. \end{aligned}$$

We recall the energy inequality for u_ν,

$$\frac{1}{2}\int_{\mathbb{T}^3}|u_\nu(t)|^2\,\mathrm{d}x + \nu\int_0^t\int_{\mathbb{T}^3}|\nabla u_\nu(x,\tau)|^2\,\mathrm{d}\tau \le \frac{1}{2}\int_{\mathbb{T}^3}|U(0)|^2\,\mathrm{d}s,$$

and further observe that

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{T}^3}|v(t)|^2\,\mathrm{d}x &= \frac{1}{2}\int_{\mathbb{T}^3}|v(0)|^2\,\mathrm{d}x + \int_0^t\int_{\mathbb{T}^3} v\cdot\partial_t v\,\mathrm{d}x\,\mathrm{d}s\\ &= \frac{1}{2}\int_{\mathbb{T}^3}|v(0)|^2\,\mathrm{d}x - \int_0^t\int_{\mathbb{T}^3} v\cdot(E(v)+(v\cdot\nabla)v)\,\mathrm{d}x\,\mathrm{d}s\\ &= \frac{1}{2}\int_{\mathbb{T}^3}|v(0)|^2\,\mathrm{d}x - \int_0^t\int_{\mathbb{T}^3} v\cdot E(v)\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

Using these ingredients (and simply dropping the H^1-term in the energy inequality for u_ν), we can estimate

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{T}^3}|u_\nu(t)-v(t)|^2\,\mathrm{d}x \leq &\frac{1}{2}\int_{\mathbb{T}^3}|U(0)-v(0)|^2\,\mathrm{d}x - \int_0^t\int_{\mathbb{T}^3}((u_\nu-v)\cdot\nabla)v\cdot(u_\nu-v)\,\mathrm{d}x\,\mathrm{d}s\\ &+\nu\int_0^t\int_{\mathbb{T}^3}\nabla u_\nu:\nabla v\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\mathbb{T}^3}E(v)\cdot(u_\nu-v)\,\mathrm{d}x\,\mathrm{d}s\end{aligned}$$

and further

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{T}^3}|u_\nu(t)-v(t)|^2\,\mathrm{d}x \leq &\frac{1}{2}\int_{\mathbb{T}^3}|U(0)-v(0)|^2\,\mathrm{d}x + \int_0^t\|\nabla_{sym}v\|\int_{\mathbb{T}^3}|u_\nu-v|^2\,\mathrm{d}x\,\mathrm{d}s\\ &+C\nu\int_0^t\|\nabla u_\nu\|_{L^2}\,\mathrm{d}s + \int_0^t\int_{\mathbb{T}^3}E(v)\cdot(u_\nu-v)\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

But note that, by virtue of the energy inequality,

$$\|\nabla u_\nu\|_{L^1L^2} \leq C(T)\|\nabla u_\nu\|_{L^2L^2} \leq C(T)\nu^{-1/2}\|U(0)\|_{L^2},$$

and hence Grönwall's inequality yields

$$\begin{aligned}\frac{1}{2}\int|u_\nu-v|^2(t)\,\mathrm{d}x \leq &\exp\left(\int_0^t\|\nabla_{sym}v\|_{L^\infty}\,\mathrm{d}s\right)\int|U(0)-v(0)|^2\,\mathrm{d}x\\ &+\int_0^t\int\exp\left(\int_s^t\|\nabla_{sym}v\|_{L^\infty}\,\mathrm{d}\tau\right)E(v)\cdot(u_\nu-v)+C(T)\nu^{1/2}\|U(0)\|_{L^2}\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

As $\nu\to 0$, the last term vanishes, and the uniform bounds on u_ν in $L^\infty H$ give weak*-convergence in that space, so that on the right hand side, u_ν can be replaced by the weak* limit u. For the left hand side, one can show that u_ν even converges in the space $C([0,T];L^2_w)$, and since the functional $u\mapsto\int|u-v(t)|^2\,\mathrm{d}x$ is weakly lower semicontinuous, the left hand side can only decrease in the limit. □

5.2. Bounded Domains

As soon as physical boundaries are involved, the viscosity limit gets much more difficult. While the theory of NSE as presented in these notes carries over to smooth bounded domains in a rather straightforward way, the limit $\nu\to 0$ behaves very differently. The reason is as follows: NSE are usually equipped with Dirichlet boundary conditions ($u=0$ on $\partial\Omega$), which in the context of fluid mechanics are called "no-slip boundary conditions". Since the passage $\nu\to 0$ formally turns a second-order system into one of first order, we cannot impose the same conditions on Euler. The most common choice are the "slip boundary conditions" $u\cdot n=0$ on $\partial\Omega$, where n denotes the outer unit normal. This change of boundary conditions causes the formation of a *boundary layer*.

As an analogy, consider the 1D heat equation $\partial_t u=\nu\Delta u$ on $[0,1]\times[0,T]$ with Dirichlet boundary condition and initial data $u^0\equiv 1$. The boundary condition will instantaneously lead the (smooth) solution to attain $u(0)=u(1)=0$, and for small times the solution will be approximately 1 in the interior and decay to zero steeply in a neighbourhood of the boundary points of size $\sim\sqrt{\nu t}$. The same is expected for NSE.

DEFINITION 5.6. Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain, then $u \in C([0,T]; L^2_w(\Omega)) \cap L^2(0,T; H^1_0(\Omega))$ is said to be a weak solution of NSE if it is weakly divergence-free and satisfies

$$-\int_0^\infty \int_\Omega u \cdot \partial_t \phi \,\mathrm{d}x\,\mathrm{d}t + \int_0^\infty \int_\Omega (u \cdot \nabla) u \cdot \phi \,\mathrm{d}x\,\mathrm{d}t \\ + \nu \int_0^\infty \int_\Omega \nabla u : \nabla \phi \,\mathrm{d}x\,\mathrm{d}t = \int_\Omega u^0 \cdot \phi(0)\,\mathrm{d}x - \int_\Omega u(T) \cdot \phi(T)\,\mathrm{d}x$$

for every divergence-free $\phi \in C^1(\Omega \times [0,T])$ such that $\phi = 0$ on $\partial\Omega$.

The last condition – that the test function has to vanish on the boundary – is decisive.

THEOREM 5.7 (Kato 1984 [**2**]). *Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain, and $\{u_\nu\}_{\nu>0}$ a family of weak solutions of NSE satisfying the energy inequality, with initial u^0. Suppose there exists a smooth solution u of Euler with initial datum u^0. Then, $u_\nu \to u$ strongly in $L^\infty(0,T; L^2(\Omega))$ if and only if*

$$\lim_{\nu \to 0} \nu \int_0^T \int_{\Omega_\nu} |\nabla u_\nu|^2 \,\mathrm{d}x\,\mathrm{d}t = 0, \tag{5.4}$$

where $\Omega_\nu := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < \nu\}$.

PROOF. We only give a proof sketch. The difficult direction is to show that (5.4) implies the desired convergence. We have

$$\begin{aligned}\int_\Omega |u_\nu - u|^2 \,\mathrm{d}x &= \int_\Omega |u_\nu|^2 \,\mathrm{d}x + \int_\Omega |u|^2 \,\mathrm{d}x - 2\int_\Omega u_\nu \cdot u \,\mathrm{d}x \\ &\le 2\int_\Omega |u^0|^2 \,\mathrm{d}x - 2\int_\Omega u_\nu \cdot u \,\mathrm{d}x.\end{aligned} \tag{5.5}$$

For the last integral, we would like to use u as a test function in the weak formulation of NSE. However, this is not allowed since only $u \cdot n = 0$ at the boundary, and it may well be that the tangential component of u is non-zero.

Kato's idea is now to cutoff u near the boundary. To this end, let $\eta \in C^\infty(0,\infty)$ such that $\eta(0) = 0$, $\eta(x) = 1$ for every $x \ge 1$, and η is monotone non-decreasing. Set $d(x) := \mathrm{dist}(x, \partial\Omega)$ and let

$$\eta_\nu(x) := \eta\left(\frac{d(x)}{\nu}\right).$$

It is now tempting to use $\eta_\nu u$ as a test function, as this now satisfies the correct boundary condition. However this function will, in general, not be divergence-free!

By Poincaré's Lemma, however, since $\operatorname{div} u = 0$, there exists a smooth potential $\phi : \Omega \to \mathbb{R}^3$ such that $\operatorname{curl} \phi = u$ and $\phi = 0$ at $\partial\Omega$. Set

$$v_\nu := \operatorname{curl}(\eta_\nu \phi),$$

then v_ν is divergence-free, is zero on the boundary, and agrees with u except on Ω_ν. Moreover one can show various estimates such as

$$\|\nabla v_\nu\|_{L^2} \le C\nu^{-1/2}. \tag{5.6}$$

In particular, $\|u - v_\nu\|_{L^2} = o(1)$ as $\nu \to 0$, and therefore, (5.5) turns into

$$\int_\Omega |u_\nu - u|^2 \,\mathrm{d}x \le 2\int_\Omega |u^0|^2 \,\mathrm{d}x - 2\int_\Omega u_\nu \cdot v_\nu \,\mathrm{d}x + 0(1).$$

Using v_ν as a test function and following a computation similar to the proof of Theorem 4.11, we arrive at

$$\|u_\nu(t) - u(t)\|^2_{L^2} \le o(1) + \int_0^t (C\|u_\nu - u\|^2 + R(s))\,\mathrm{d}s,$$

where

$$R_\nu(t) = \int_\Omega (u_\nu \cdot \nabla)(u_\nu - v_\nu) \cdot u_\nu + \nu \nabla u_\nu : \nabla v_\nu \, \mathrm{d}x.$$

It remains to show $\int_0^t R_\nu \, \mathrm{d}s \to 0$.

We consider only the second term, using (5.6):

$$\begin{aligned} \left| \int_\Omega \nu \nabla u_\nu : \nabla v_\nu \, \mathrm{d}x \right| &\leq \nu \|\nabla u_\nu\|_{L^2} \|\nabla u\|_{L^2} + \nu \|\nabla u_\nu\|_{L^2(\Omega_\nu)} \|\nabla u - \nabla v_\nu\|_{L^2(\Omega_\nu)} \\ &\leq C\nu \|\nabla u_\nu\|_{L^2} + C\nu\nu^{-1/2} \|\nabla u_\nu\|_{L^2(\Omega_\nu)}. \end{aligned}$$

The time integral of the second term goes to zero by assumption, and so does the integral of the first term by virtue of the energy inequality:

$$\nu \int_0^t \|\nabla u_\nu\|_{L^2} \, \mathrm{d}s \leq C_T \nu^{1/2} \left(\nu \int_0^t \|\nabla u_\nu\|_{L^2}^2 \, \mathrm{d}s \right)^{1/2} \leq C_T v^{1/2} \to 0.$$

□

Bibliography

[1] P. Constantin and C. Foias. Navier-Stokes equations. *University of Chicago Press*, 1988.

[2] T. Kato. Remarks on zero viscosity limit for nonstationary Navier-Stokes flows with boundary. Seminar on nonlinear partial differential equations. *Springer, New York, NY*, 1984.

[3] P.-L. Lions. Mathematical topics in fluid mechanics. Vol. 1. Incompressible models. Oxford Lecture Series in Mathematics and its Applications, 3. *Oxford Science Publications. The Clarendon Press, Oxford University Press, New York*, 1996.

[4] J. C. Robinson, J. L. Rodrigo, and W. Sadowski. The Three-Dimensional Navier–Stokes equations: Classical theory. Cambridge Studies in Advanced Mathematics, Vol. 157. *Cambridge University Press*, 2016.

[5] R. Temam. Navier-Stokes equations. Theory and numerical analysis. Revised edition. Studies in Mathematics and its Applications, 2. *North-Holland Publishing Co., Amsterdam-New York*, 1979.