Exercise 1: Calculate and draw the empirical distribution function of the following sample: $(-15.4, -8.8, 8.2, 3.4, -7.1, 4.5, -12.7, 5.2, -10.6, -11.2)$

Solution:

x	< 15.4	-15.4	-12.7	-11.2	-10.6	-8.8	-7.1	3.4	4.5	5.2	$8.2 \geq$
$F_n(x)$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1

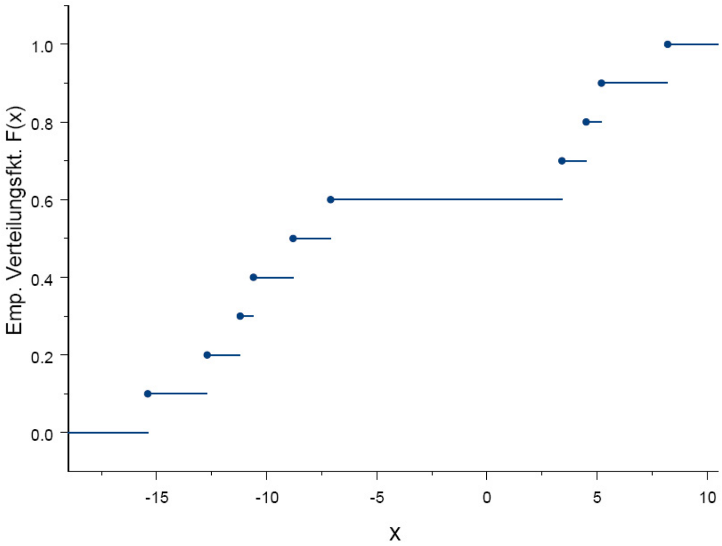

Figure 1: Empirical distribution function $F(x)$

Exercise 2: A gambling machine has the following probability distribution for the profit X in each game (in €).

x	-1	0	1
$P(X = x)$	p	p	$1 - 2p$

The manufacturer of the machine has assigned a statistician to the problem of approximating p to get an idea of whether the value of p has changed since the initiation of the machines.

a) The statistician takes a sample of size $n = 6$, i.e. he uses the machine exactly six times and remembers the corresponding profit. The sample $(X_1, X_2, X_3, X_4, X_5, X_6)$ has realisation $(-1, 1, -1, 0, 1, 1)$. Describe, in words, the realisation.

b) How would you estimate the probability of profit X per game on the basis of the sample in part (a) provided you had no information on the probability distribution of X?

c) What is the value of $P[(X_1, X_2, X_3, X_4, X_5, X_6) = (-1, 1, -1, 0, 1, 1)]$ assuming X to be distributed as in the above table.

d) Determine the Maximum-Likelihood-estimating-function for p in this problem. Here, you are asked for the general estimating function (i.e. for m games).

e) Calculate the expectation of the maximum-likelihood-estimating function.

f) Estimate p by using the sample and the maximum-likelihood-method.

g) Determine a moment-estimator for p.

h) Calculate the expectation of the moment-estimator.

i) Estimate p using the sample and the moment-estimator.

Solution:

a) Lost first game, won second game, ...

b) Empirical probabilities

x	-1	0	1
$P(X = x)$	$\frac{2}{6}$	$\frac{1}{6}$	$\frac{3}{6}$

c) We get

$$\begin{aligned} P[(X_1, X_2, X_3, X_4, X_5, X_6) = (-1, 1, -1, 0, 1, 1)] &= P[X_1 = -1] \cdot \ldots \cdot P[X_6 = 1] \\ &= p^2 p(1-2p)^3 \\ &= p^3(1-2p)^3 \end{aligned}$$

d) There are n games of which v were lost ($X = -1$), u were draws ($X = 0$) and g were won ($X = 1$). The likelihood-function is then:

$$\begin{aligned} L(p; x_1, \ldots, x_n) &= P[X_1 = x_1] \cdot \ldots \cdot P[X_n = x_n] \\ &= P[X = -1]^v P[X = 0]^u P[X = 1]^g \\ &= p^{u+v}(1-2p)^{n-u-v} \end{aligned}$$

Now we maximise (writing $x = u + v$):

$$\begin{aligned} \frac{\partial L}{\partial p} &= (u+v)p^{u+v-1}(1-2p)^{n-u-v} + p^{u+v}(n-u-v)(1-2p)^{n-u-v-1}(-2) \\ &= p^{x-1}(1-2p)^{n-x-1}(x(1-2p) - 2p(n-x)) = 0 \end{aligned}$$

But now we can ignore trivial solutions ($p = 0$ or $p = 0.5$) as they would correspond to winning always ($p = 0$) or never ($p = 0.5$). Hence $x(1-2p) = 2p(n-x)$ and then $\hat{p} = \frac{x}{2n} = \frac{u+v}{2n}$.

e) We have

$$\mathbb{E}[\hat{p}] = \frac{1}{2n}(\mathbb{E}[U] + \mathbb{E}[V]) = p$$

because $\mathbb{E}[U] = \mathbb{E}[V] = 2np$ as U and V are $B(n, p)$ distributed random variables.

f) We get

$$\hat{p} = \frac{3}{2 \cdot 6} = \frac{1}{4}$$

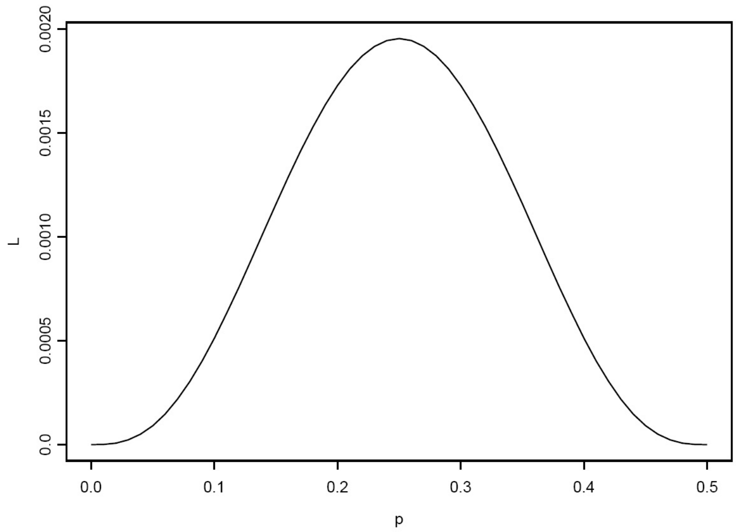

Figure 2: Likelihood function of the sample

g) Let X be the profit per game. Then

$$\mathbb{E}[X] = -p + (1-2p) = 1 - 3p$$

and we define

$$\overline{X} = \mathbb{E}[X] = 1 - 3p$$

Now we solve for p and get

$$\hat{p} = \frac{1}{3}(1 - \overline{X})$$

And with the variables defined in part (d) we obtain:

$$\overline{X} = \frac{1}{n}(g - v)$$
$$(1 - \overline{X}) = \frac{n}{n} - \frac{g}{n} + \frac{v}{n} = \frac{u + 2v}{n}$$

and hence

$$\hat{p} = \frac{1}{3}(1 - \overline{X}) = \frac{1}{3n}(u + 2v)$$

h) We get

$$\mathbb{E}[\hat{p}] = \frac{1}{3n} 3np = p$$

i) An easy calculation yields:

$$\frac{1 + 2 \cdot 2}{3 \cdot 6} = \frac{5}{18}$$

Exercise 3: Due to a malfunction a manufacturing facility fails frequently. Examine the random variable T denoting the period of time until the facility fails next (in weeks).

a) What is the distribution of T?

b) Determine the maximum-likelihood-estimating-function for the parameter of the distribution of T.

c) Determine the moment-estimator of the parameter.

d) Calculate the expectation of the maximum-likelihood-estimating-function and the expectation of the moment-estimator.

e) Determine the maximum-likelihood-estimator and the moment-estimator for the following set of data (each number denoting the period of time between two failures in weeks): $(1, 2.3, 1.7, 0.9, 1.8, 3.1)$. Assume the production facility does not change and that the failures are independent of each other.

Solution:

a) $T \sim Exp(\lambda)$

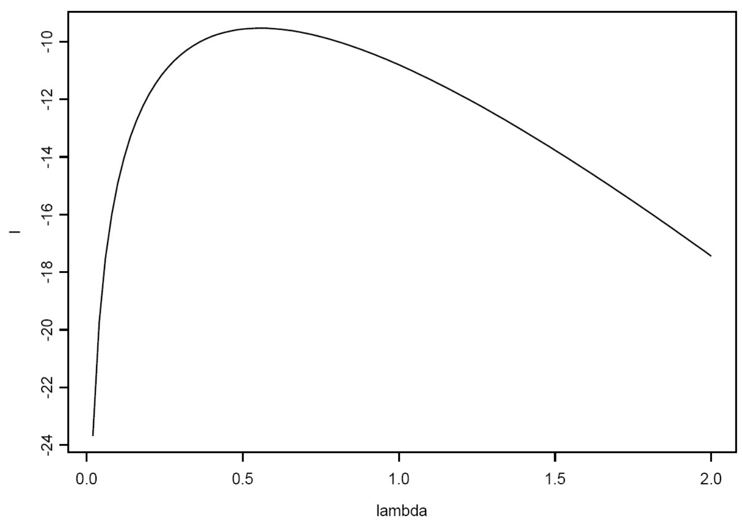

Figure 3: Log-Likelihood function for the sample given in question 2, part (e)

b) We have $T_1, \ldots, T_n =$ *observed failure times (in weeks)* and thus

$$\begin{aligned} L(\lambda; t_1, \ldots, t_n) &= f_{Exp(\lambda)}(t_1) \cdot \ldots \cdot f_{Exp(\lambda)}(t_n) \\ &= \lambda e^{-\lambda t_1} \cdot \ldots \cdot \lambda e^{-\lambda t_n} \\ &= \lambda^n \exp(-\lambda \sum_{i=1}^{n} t_i) \end{aligned}$$

And the log-likelihood function

$$l(\lambda; t_1, \ldots, t_n) = \log L(\lambda; t_1, \ldots, t_n) = n \log(\lambda) - \lambda \sum_{i=1}^{n} t_i$$

And now we maximise

$$\frac{\partial l}{\partial \lambda} = n\frac{1}{\lambda} - \sum_{i=1}^{n} t_i = 0 \Longrightarrow \hat{\lambda} = \frac{n}{\sum_{i=1}^{n} t_i} = \frac{1}{\overline{t}}$$

c) We have $\mathbb{E} = \frac{1}{\lambda}$ and hence $\overline{T} = \mathbb{E}[T] = \frac{1}{\hat{\lambda}}$ and thus $\hat{\lambda} = \frac{1}{\overline{T}}$

d) See sheet 8, question 1.

e) $\overline{t} = 10.8/7 = 1.8$ thus $\hat{\lambda} = 1/1.8 \approx 0.5555556$

Exercise 3: Let $X_1, \ldots, X_n \sim Gamma(\alpha, \lambda)$ be a simple sample. We also know that

$$\mathbb{E}[X_1] = \frac{\alpha}{\lambda}$$
$$Var(X_1) = \frac{\alpha}{\lambda^2}$$

Determine an estimator for α and λ using the moment method.

Solution: We obtain the following system of linear equations:

$$\frac{1}{n}\sum_{i=1}^{n} x_i = \hat{\mu}$$
$$\frac{1}{n}\sum_{i=1}^{n} x_i^2 = \hat{\mu}^2 + \hat{\sigma}^2$$

where $\hat{\mu} = \frac{\hat{\alpha}}{\hat{\lambda}}$ and $\hat{\sigma}^2 = \frac{\hat{\alpha}}{\hat{\lambda}^2}$. Solving this system for $\hat{\lambda}$ and $\hat{\alpha}$ we get:

$$\hat{\lambda} = \frac{\hat{\mu}}{\hat{\sigma}^2} = \frac{\frac{1}{n}\sum_{i=1}^{n} x_i}{\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2}$$
$$\hat{\alpha} = \frac{\hat{\mu}^2}{\hat{\sigma}^2} = \frac{\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2}{\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2}$$