Summer Term 2015

INSTITUTE OF MATHEMATICAL FINANCE

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Financial Mathematics II

Exercise Sheet 1

Discussion: Thursday 23/04/2015, 16:00-17:30, He18, E60,
and Friday 24/04/2015, 8:15-10:00, He18, 120.
Handing in: Thursday 23/04/2015, beginning of the lecture.

Exercise 1.1

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Show the following three statements:

(a) Let $B = (B_t)_{t\geq 0}$ be a stochastic process on $(\Omega, \mathcal{F}, P)$, and $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ the natural filtration of B. Then the following are equivalent:

- (i) B is a Standard Brownian Motion.
- (ii) B is an $\mathbb{F}$-Standard Brownian Motion.

(b) A Brownian Motion $W = (W_t)_{t\geq 0}$ with drift μ and variance σ^2 is characterized by Definition 1.7 a) ii), iii), iv) and

- (i') $W_t \stackrel{d}{=} N(\mu t, \sigma^2 t) \quad \forall t \geq 0.$

(c) If B is an $\mathbb{F}$-Standard Brownian Motion, where $\mathbb{F}$ is a general filtration, then it also holds

- (ii") $\sigma(B_t - B_s \colon t \geq s) \perp\!\!\!\perp \mathcal{F}_s \quad \forall s \geq 0.$

Exercise 1.2

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X, X_1, X_2, \ldots : \Omega \to \overline{\mathbb{R}}$ be random variables. Let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra of $\mathcal{F}$. Show the following statements:

(a) [Monotone Convergence]
If $0 \leq X_n \leq X_{n+1}\ \forall\, n \in \mathbb{N}$ and $X := \lim_{n\to\infty} X_n$, then

$$\lim_{n\to\infty} E(X_n|\mathcal{G}) = E(X|\mathcal{G}) \quad P\text{-a.s.}$$

(b) [Fatou's Lemma]
If $X_n \geq 0\ \forall\, n \in \mathbb{N}$, then

$$E(\liminf_{n\to\infty} X_n|\mathcal{G}) \leq \liminf_{n\to\infty} E(X_n|\mathcal{G}) \quad P\text{-a.s.}$$

(c) [Dominated Convergence]
If $X_1, X_2, \ldots$ are integrable for all $n \in \mathbb{N}$ and $|X_n(\omega)| \leq W(\omega)\ \forall\, \omega \in \Omega$ with $E(W) < \infty$, and if $X_n \xrightarrow{a.s.} X$ for $n \to \infty$, then it holds

$$E(X_n|\mathcal{G}) \xrightarrow{a.s.} E(X|\mathcal{G}) \quad (n \to \infty),$$

$$E\left[|X_n - X|\middle|\mathcal{G}\right] \xrightarrow{a.s.} 0 \quad (n \to \infty)\,,$$

and also

$$E(X_n|\mathcal{G}) \xrightarrow{L_1} E(X|\mathcal{G}) \quad (n \to \infty).$$