Exercise 2.1
Given a filtered probability space \((\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, P)\). Show:

(a) Let \(\tau\) and \(\sigma\) be stopping times. Then also
\[
\tau \land \sigma \quad \text{and} \quad \tau \lor \sigma
\]
are stopping times. Here \(\tau \land \sigma := \min\{\tau, \sigma\}\) and \(\tau \lor \sigma := \max\{\tau, \sigma\}\).

(b) If \((\tau_n)_{n \in \mathbb{N}}\) is an increasing sequence of stopping times, then
\[
\tau := \lim_{n \to \infty} \tau_n
\]
is a stopping time.

(c) If additionally the filtration \(\mathbb{F}\) is right-continuous and \((\tau_n)_{n \in \mathbb{N}}\) is a decreasing sequence of stopping times, then
\[
\tau := \lim_{n \to \infty} \tau_n
\]
is a stopping time.

Exercise 2.2
Given a filtered probability space \((\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, P)\), let \(\mathbb{F}\) be right-continuous and \((\tau_n)_{n \in \mathbb{N}}\) a sequence of stopping times, then also
\[
\sup_{n \in \mathbb{N}} \tau_n, \quad \inf_{n \in \mathbb{N}} \tau_n, \quad \limsup_{n \to \infty} \tau_n, \quad \text{and} \quad \liminf_{n \to \infty} \tau_n
\]
are stopping times.

Definition 2.3: Let \((\Omega, \mathcal{F}, P, \mathbb{F})\) be a stochastic basis and \(I\) be subinterval of \(\mathbb{R}_+\), which contains 0. A set \(A \subset I \times \Omega\) is \textit{progressively measurable} if for all \(t \in I\)
\[
A \cap ([0, t] \times \Omega) \in \mathcal{R}_t := \mathcal{B}([0, t]) \otimes \mathcal{F}_t,
\]
that is for all \(t\) the restriction of \(A\) to \([0, t] \times \Omega\) is measurable with respect to the product-\(\sigma\)-algebra
\[
\mathcal{R}_t := \mathcal{B}([0, t]) \otimes \mathcal{F}_t.
\]
The progressively measurable sets form a \(\sigma\)-algebra \(\mathcal{R}\). We say that a stochastic process \(X = (X_t)_{t \in I}\) is
(a) *progressively measurable* if it is measurable with respect to \( \mathcal{R} \), that is, if for every \( t \geq 0 \), the mapping \([0, t] \times \Omega \to \mathbb{R}^d\), \((s, \omega) \mapsto X_s(\omega)\) is \( \mathcal{R}_t - \mathcal{B}(\mathbb{R}^d) \)-measurable.

(b) *product measurable* if it is measurable with respect to \( \mathcal{B}(I) \otimes \mathcal{F} \), that is the mapping \( I \times \Omega \to \mathbb{R}^d\), \((t, \omega) \mapsto X_t(\omega)\) is \( \mathcal{B}(I) \otimes \mathcal{F} - \mathcal{B}(\mathbb{R}^d) \)-measurable.

**Exercise 2.4**

Let \((\Omega, \mathcal{F}, P, \mathbb{F})\) be a stochastic basis and \( I \) a subinterval of \( \mathbb{R}_+ \), which contains 0. Show that every adapted, left-continuous process \( Y = (Y_t)_{t \in I} \) and every adapted, right-continuous process \( X = (X_t)_{t \in I} \) is progressively measurable.

**Example 2.5:**

Construct an adapted process which is not progressively measurable.