Summer Term 2015

INSTITUTE OF MATHEMATICAL FINANCE

Prof. Dr. Alexander Lindner
Dirk Brandes

Financial Mathematics II

Exercise Sheet 2

Discussion: Thursday 30/04/2015, 16:00-17:30, He18, E60,
and Friday 08/05/2015, 8:15-10:00, He18, 120.
Handing in: Thursday 30/04/2015, beginning of the lecture.

Exercise 2.1

Given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, P)$. Show:

(a) Let τ and σ be stopping times. Then also

$$\tau \wedge \sigma \quad \text{and} \quad \tau \vee \sigma$$

are stopping times. Here $\tau \wedge \sigma := \min\{\tau, \sigma\}$ and $\tau \vee \sigma := \max\{\tau, \sigma\}$.

(b) If $(\tau_n)_{n\in\mathbb{N}}$ is an increasing sequence of stopping times, then

$$\tau := \lim_{n\to\infty} \tau_n$$

is a stopping time.

(c) If additionally the filtration $\mathbb{F}$ is right-continuous and $(\tau_n)_{n\in\mathbb{N}}$ is a decreasing sequence of stopping times, then

$$\tau := \lim_{n\to\infty} \tau_n$$

is a stopping time.

Exercise 2.2

Given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, P)$, let $\mathbb{F}$ be right-continuous and $(\tau_n)_{n\in\mathbb{N}}$ a sequence of stopping times, then also

$$\sup_{n\in\mathbb{N}} \tau_n\,, \quad \inf_{n\in\mathbb{N}} \tau_n\,, \quad \limsup_{n\to\infty} \tau_n\,, \quad \text{and} \quad \liminf_{n\to\infty} \tau_n$$

are stopping times.

Definition 2.3: Let $(\Omega, \mathcal{F}, P, \mathbb{F})$ be a stochastic basis and I be subinterval of $\overline{\mathbb{R}}_+$, which contains 0. A set $A \subset I \times \Omega$ is *progressively measurable* if for all $t \in I$

$$A \cap ([0,t] \times \Omega) \in \mathcal{R}_t := \mathcal{B}([0,t]) \otimes \mathcal{F}_t\,,$$

that is for all t the restriction of A to $[0,t] \times \Omega$ is measurable with respect to the product-σ-algebra

$$\mathcal{R}_t := \mathcal{B}([0,t]) \otimes \mathcal{F}_t\,.$$

The progressively measurable sets form a σ-algebra $\mathcal{R}$. We say that a stochastic process $X = (X_t)_{t\in I}$ is

(a) *progressively measurable* if it is measurable with respect to $\mathcal{R}$, that is, if for every $t \geq 0$, the mapping $[0,t] \times \Omega \to \mathbb{R}^d\,,\ (s,\omega) \mapsto X_s(\omega)$ is $\mathcal{R}_t - \mathcal{B}(\mathbb{R}^d)$-measurable.

(b) *product measurable* if it is measurable with respect to $\mathcal{B}(I) \otimes \mathcal{F}$, that is the mapping $I \times \Omega \to \mathbb{R}^d\,,\ (t,\omega) \mapsto X_t(\omega)$ is $\mathcal{B}(I) \otimes \mathcal{F} - \mathcal{B}(\mathbb{R}^d)$-measurable.

Exercise 2.4

Let $(\Omega, \mathcal{F}, P, \mathbb{F})$ be a stochastic basis and I a subinterval of $\overline{\mathbb{R}}_+$, which contains 0. Show that every adapted, left-continuous process $Y = (Y_t)_{t \in I}$ and every adapted, right-continuous process $X = (X_t)_{t \in I}$ is progressively measurable.

Example 2.5:

Construct an adapted process which is not progressively measurable.