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Financial Mathematics II

Exercise Sheet 3

Discussion: Thursday 07/05/2015, 16:00-17:30, He18, E60,
and Friday 15/05/2015, 08:15-09:45, He18, 120.
Handing in: Thursday 07/05/2015, beginning of the lecture.

Exercise 3.1

Let $J \in \{[0,\infty), [0,T], [0,T)\}$, $\mathbb{F} = (\mathcal{F}_t)_{t\in J}$ a filtration on $(\Omega, \mathcal{F})$ and $\tau\colon \Omega \to J \cup \{\infty\}$ an $\mathbb{F}$-stopping time. Show:

(a) $\mathcal{F}_\tau$ defined by
$$\mathcal{F}_\tau := \{A \in \mathcal{F} \colon A \cap \{\tau \le t\} \in \mathcal{F}_t \;\forall\, t \in J\}$$
is indeed a σ-algebra.

(b) τ is $\mathcal{F}_\tau$-measurable.

(c) If σ is another stopping time such that $\sigma(\omega) \le \tau(\omega)\ \forall\, \omega \in \Omega$, then $\mathcal{F}_\sigma \subset \mathcal{F}_\tau$.

(d) $\mathcal{F}_\sigma \cap \mathcal{F}_\tau = \mathcal{F}_{\sigma\wedge\tau}$.

(d) $(\mathcal{F}_{\tau\wedge t})_{t\in J}$ is a filtration in $(\Omega, \mathcal{F})$.

Exercise 3.2

Let $J \in \{[0,\infty), [0,T], [0,T)\}$, $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in J}, P)$ be a filtered probability space and $X = (X_t)_{t\in J}$ a progressively measurable process. If $\tau\colon \Omega \to J \cup \{\infty\}$ is an arbitrary stopping time, then X_τ, the process evaluated at τ, is $\mathcal{F}_\tau$-measurable. To do so, show:

(a) For any product measurable process $(X_t(\omega))_{t\in J}$ and any random time τ, i.e. a mapping $\tau\colon \Omega \to J \cup \{\infty\}$ which is measurable with respect to $\mathcal{F}$, the process evaluated at τ, i.e. $X_{\tau(\omega)}(\omega)$ is $\mathcal{F}$-measurable.

  (i) Show (a) for $X_t(\omega) = \mathbf{1}_{B\times C}(t,\omega)$ for $B \in \mathcal{B}(\overline{\mathbb{R}}_+)$ and $C \in \mathcal{F}$.

  (ii) Show for
  $$\mathcal{D} := \{D \in \mathcal{B}(\overline{\mathbb{R}}_+) \otimes \mathcal{F} \colon X_{\tau(\omega)}(\omega) \text{ is } \mathcal{F}\text{-measurable, where } X_t(\omega) = \mathbf{1}_D(t,\omega)\}$$
  that $\mathcal{D} = \mathcal{B}(\overline{\mathbb{R}}_+) \otimes \mathcal{F}$.

  (iii) Show (a) for general product measurable X.

(b) Conclude that the assertion of (a) holds true for every progressively measurable process X and every stopping time τ.

(c) If X is progressively measurable and τ a stopping time, then the stopped process X^τ is also progressively measurable.

(d) Conclude that, if X is a progressively measurable process and τ a stopping time, X_τ is $\mathcal{F}_\tau$-measurable.

Exercise 3.3

Let $J \in \{[0,\infty), [0,T], [0,T)\}$, $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in J}, P)$ be a filtered probability space and $\tau \colon \Omega \to J \cup \{\infty\}$ a stopping time. If

$$\mathcal{G} := \sigma(X_\tau \colon X \text{ is strict càdlàg and adapted})$$

then $\mathcal{G} = \mathcal{F}_\tau$.