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Summer Term 2015

INSTITUTE OF MATHEMATICAL FINANCE

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Financial Mathematics II

Exercise Sheet 4

Discussion: Thursday 21/05/2015, 16:00-17:30, He18, E60,
and Friday 22/05/2015, 08:15-09:45, He18, 120.
Handing in: Thursday 21/05/2015, beginning of the lecture.

Exercise 4.1

Given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, P)$, let $X = (X_t)_{t\ \geq 0}$ be a martingale and assume that X is square integrable for all $t \geq 0$ that is

$$X_t \in \mathrm{L}^2(\Omega) := \left\{ f\colon \Omega \to \mathbb{R}\colon\ f \text{ measurable and } \int_\Omega |f(\omega)|^2 \,\mathrm{d}P < \infty \right\} \quad \forall t \geq 0\,.$$

Show if $s < t$ then

$$\mathbf{E}\left((X_t - X_s)^2\right) = \mathbf{E}\left(X_t^2\right) - \mathbf{E}\left(X_s^2\right)\,. \tag{4.1}$$

Exercise 4.2

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space. Let $X = (X_t)_{t\geq 0}$ be a càdlàg square integrable martingale such that

$$\left\|\sup_t |X_t|\right\|_2 < \infty\,,$$

i.e. X is bounded in L^2. Here

$$\|X\|_2 = \left(\int_\Omega |X(\omega)|^2 \,\mathrm{d}P\right)^{1/2} = \left(\mathbf{E}\left(|X|^2\right)\right)^{1/2}\,.$$

Show that then there exists a random variable X_∞ such that $X_\infty \in \mathrm{L}^2(\Omega, \mathcal{F}, P)$ and

$$X_t \overset{\text{a.s.}}{=} \mathbf{E}\left(X_\infty \,|\mathcal{F}_t\right)$$

for every $t \geq 0$. Further

$$X_t \xrightarrow{L_2} X_\infty\,, \quad n \to \infty\,.$$