ulm university universität uulm

Summer Term 2015

INSTITUTE OF MATHEMATICAL FINANCE

Prof. Dr. Alexander Lindner
Dirk Brandes

Financial Mathematics II

Exercise Sheet 5

Discussion: Friday 05/06/2015, 08:15-09:45, He18, 120.
Handing in: Thursday 28/05/2015, beginning of the lecture.

Exercise 5.1

Given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, P)$, let $B = (B_t)_{t\ \geq 0}$ be a standard Brownian motion with respect to $\mathbb{F}$. Show that

(a) The geometric Brownian motion $X_t := \mathrm{e}^{B_t - \frac{t}{2}}$, $t \geq 0$, is a continuous L^2-integrable $\mathbb{F}$-martingale with $\mathbf{E}X_t^2 < \infty$ for all $t \geq 0$.

(b) $Y_t := B_t^3 - 3tB_t$, $t \geq 0$, is an $\mathbb{F}$-martingale.

Definition 5.2: Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space. A stochastic Process $X = (X_t)_{t\geq 0}$ with $X_0 = 0$ almost surely is called a *local martingale* if there exists a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ with $\tau_n \nearrow \infty$ almost surely such that the stopped process

$$X_t^{\tau_n} = X_{t\wedge\tau_n} = X_t \mathbf{1}_{\{t<\tau_n\}} + X_{\tau_n}\mathbf{1}_{\{t\geq\tau_n\}}\,, \quad t \geq 0\,,$$

is a uniformly integrable martingale for all $n \in \mathbb{N}$.

Exercise 5.3

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space and $X = (X_t)_{t\geq 0}$ a local martingale. Assume further that there exists a random variable Z with $\mathbf{E}|Z| < \infty$ such that $|X_t| \leq Z$ for all $t \geq 0$. Show that then X is a martingale.

Exercise 5.4

Given a continuous time financial market $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, P, (S_t)_{t\in[0,T]}, \mathcal{A}^{si})$ with time horizon T as defined in Definition 1.5 with only bounded self-financing simple trading strategies. Assume that $S^0, S^1, \ldots, S^d$ are strictly continuous and uniformly bounded. Let Q denote an equivalent martingale measure. Show that then the market satisfies the NFLVR-condition as defined in Definition 3.9.

Exercise 5.5

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X \in \mathrm{L}^1(\Omega, \mathcal{F}, P)$. Then the family of random variables

$$\{\,\mathbf{E}\,[\,X\,|\,\mathcal{G}\,] : \mathcal{G} \text{ a sub-}\sigma\text{-algebra of } \mathcal{F}\}$$

is uniformly integrable.