ulm university universität uulm

Summer Term 2015

INSTITUTE OF MATHEMATICAL FINANCE

Prof. Dr. Alexander Lindner
Dirk Brandes

Financial Mathematics II

Exercise Sheet 7

Discussion: Thursday 18/06/2015, 16:00-17:30, He18, E60,
and Friday 19/06/2015, 08:15-09:45, He18, 120.
Handing in: Thursday 11/06/2015, beginning of the lecture.

Exercise 7.1

Let $[a,b] \subset \mathbb{R}$ and $f\colon [a,b] \to \mathbb{R}$. Define $\pi = (t_0, \dots, t_n)$ to be a *partition* of $[a,b]$, i.e. a finite sequence of points $a = t_0 < t_1 < \dots < t_n = b$, $n \in \mathbb{N}$, $\Pi_{[a,b]}$ the *set of all partitions of* $[a,b]$, and

$$v_{[a,b]}(f) = \sup_{\pi \in \Pi_{[a,b]}} \sum_{j=1}^{n} |f(t_j) - f(t_{j-1})|$$

the *total variation of* f *on* $[a,b]$.

(a) Let $f\colon [a,b] \to \mathbb{R}$ monotone. Show that $v_{[a,b]}(f) = |f(b) - f(a)|$.
(b) Show $v_{[a,b]}(f) = v_{[a,c]}(f) + v_{[c,b]}(f) \quad \forall c \in (a,b)$.
(c) Calculate the total variation $v_{[0,1]}(f_j)$ of f_j, $j = 1, 2$, where

$$f_1(t) = 5\cos(2\pi t)\,,$$
$$f_2(t) = \begin{cases} 7\log(t)\,, & \text{if } t \neq 0\,, \\ 0\,, & \text{if } t = 0\,, \end{cases}$$

for all $t \in [0,1]$.
(d) Let $g\colon [0,\infty) \to \mathbb{R}$ be of finite variation and right-continuous. Show that then also $v_t(g)$ and $v_t(g) - g$ are right-continuous.

Exercise 7.2

(a) Let $[0,T] \subset \mathbb{R}$, $f\colon [0,T] \to \mathbb{R}$ be Borel-measurable and bounded on compacts, and $g\colon [0,\infty) \to \mathbb{R}$ continuously differentiable. Then

$$\int_{[0,T]} f(t)\,\mathrm{d}g(t) = f(0)g(0) + \int_{(0,T]} f(t)g'(t)\,\mathrm{d}t\,.$$

(b) Let $[0,T] \subset \mathbb{R}$, $f\colon [0,T] \to \mathbb{R}$ be continuous, and $g\colon [0,T] \to \mathbb{R}$ of finite variation and right-continuous. Show that

$$\lim_{n\to\infty} \sum_{j=1}^{m(n)} f(\xi_{n,j}) \cdot (g(t_{n,j}) - g(t_{n,j-1})) = \int_{0+}^{T} f(x)\,\mathrm{d}g(x)\,,$$

where $\pi^{(n)} := (t_{n,0}, \ldots, t_{n,m(n)})$ be a sequence of partitions of the interval $[0,T]$ such that

$$\delta(\pi^{(n)}) := \max_{i \in \{0,\ldots,m(n)\}} |t_{n,i} - t_{n,i-1}| \to 0\,, \quad n \to \infty\,,$$

and, for all $j \in \{1, \ldots, m(n)-1\}$, $\xi_{n,j} \in [t_{n,j-1}, t_{n,j}]$.

Exercise 7.3

For $A \subset [0,T]$ and $f \cdot \mathbf{1}_A$ Borel-measurable and bounded on compacts define

$$\int_A f(x)\,\mathrm{d}g(x) := \int_{[0,T]} f(x)\mathbf{1}_A(x)\,\mathrm{d}g(x)\,.$$

and

$$\int_a^b f(x)\,\mathrm{d}g(x) := \int_{[a,b]} f(x)\,\mathrm{d}g(x)\,, \quad \text{for } [a,b] \subset [0,T].$$

Calculate the Lebesgue-Stieltjes-integrals I_j, $j = 1,2,3$,

$$\begin{aligned}
I_1 &= \int_1^3 \frac{1}{x}\,\mathrm{d}(\ln(x))\,,\\
I_2 &= \int_1^4 \frac{1}{x}\,\mathrm{d}((x-2)^2)\,,\\
I_3 &= \int_0^1 x\,\mathrm{d}g_5(x)\,, \quad \text{where } g_5(x) = \begin{cases} -x/2\,, & \text{if } x \in [0,1/2)\,,\\ 3(x-1/2)\,, & \text{if } x \in [1/2,1]\,. \end{cases}
\end{aligned}$$