Summer Term 2015

INSTITUTE OF MATHEMATICAL FINANCE

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Financial Mathematics II

Exercise Sheet 8

Discussion: Thursday 25/06/2015, 16:00-17:30, He18, E60,
and Friday 26/06/2015, 08:15-09:45, He18, 120.
Handing in: Thursday 18/06/2015, beginning of the lecture.

Exercise 8.1

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a stochastic basis satisfying Convention 5.9. For $X, Y \in \mathbb{D}$ or $X, Y \in \mathbb{L}$,

$$(X-Y)^* := \sup_{t\in[0,T]} |X_t - Y_t|, \quad \text{and}$$

$$d_{\mathrm{ucp}}(X,Y) := \mathbf{E}[\min\{1, (X-Y)^*\}]$$

show the following statements:

(a) d_{ucp} is a semi-metric on $\mathbb{D}$ or $\mathbb{L}$, i.e. satisfies

(i) $d_{\mathrm{ucp}}(X,Y) \in [0,\infty)$

(ii) $d_{\mathrm{ucp}}(X,Z) \leq d_{\mathrm{ucp}}(X,Y) + d_{\mathrm{ucp}}(Y,Z)$

(iii) $d_{\mathrm{ucp}}(X,Y) = d_{\mathrm{ucp}}(Y,X)$

(iv) $d_{\mathrm{ucp}}(X,X) = 0$.

(b) $d_{\mathrm{ucp}}(X,Y) = 0 \iff X$ and Y are indistinguishable.

(c) For a sequence of processes $X^{(n)}$ in $\mathbb{D}$ or $\mathbb{L}$ and X in $\mathbb{D}$ or $\mathbb{L}$, it holds $X^{(n)} \to X$, $n \to \infty$, in d_{ucp} if and only if $(X^{(n)} - X)^* \xrightarrow{P} 0$, $n \to \infty$.

(d) The spaces $\mathbb{D}$ or $\mathbb{L}$ associated with the semi-metric d_{ucp} are complete, i.e. if $(X^{(n)})_{n\in\mathbb{N}}$ is a Cauchy-sequence with respect to d_{ucp} in $\mathbb{D}$ or $\mathbb{L}$, then there exists a X in $\mathbb{D}$ or $\mathbb{L}$ with $X^{(n)} \to X$, $n \to \infty$, in d_{ucp}.

Exercise 8.2

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a stochastic basis satisfying Convention 5.9 and $X = (X_t)_{t\in[0,T]}$ be a semimartingale with respect to the filtration $\mathbb{F}$. Let $\mathbb{G}$ be a sub-filtration of $\mathbb{F}$ such that X is also adapted to $\mathbb{G}$. Show that then X is a semimartingale with respect to $\mathbb{G}$.

Exercise 8.3

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a stochastic basis satisfying Convention 5.9 and $X = (X_t)_{t\in[0,T]}$ be a continuous martingale. Let $(\tau_n)_{n\in\mathbb{N}}$ defined by $\tau_n := \inf\{t \in [0,T] \colon |X_t| \geq n\} \wedge T$ be a sequence of positive random variables increasing to T almost surely. Then X is a semimartingale. To prove this, show first that X^{τ_n} is a square integrable martingale.

Exercise 8.4

Let $(P_k)_{k\geq 1}$ be a sequence of probability measures on the measurable space $(\Omega, \mathcal{F})$ and $(\lambda_k)_{k\geq 1}$ a sequence of real numbers such that $\lambda_k \geq 0$ and $\sum_{k=1}^{\infty} \lambda_k = 1$. Define $P = \sum_{k=1}^{\infty} \lambda_k P_k$ and let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables which converges in P_k-probability for all $k \geq 1$ to a random variable X. Show that then $X_n \xrightarrow{P} X$. What does the above result imply if you assume that $X = (X_t)_{t\in[0,T]}$ is a semimartingale for each P_k, $k \geq 1$.