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Statistics of Financial Data

Exercise Sheet 7

To be discussed on Monday, December 1, 2014

Let us recall the following:

Definition: Let $I \neq \emptyset$ and $d \in \mathbb{N}$. A family $(P_J)_{J\subset I, |J|<\infty}$ of probability measures on $(\times_{j\in J} \mathbb{R}^d, \bigotimes_{j\in J} \mathbb{R}_d)$ is called a *projective family of probability measures (based on* $(\mathbb{R}^d, \mathcal{B}_d)$*)*, if

$$P_L = \pi_L^J(P_J) \quad \text{for all finite subsets } L \subset J \text{ of } I,$$

where π_J^L is the projection from $\times_{j\in J} \mathbb{R}^d \to \times_{j\in L} \mathbb{R}^d$, given by $(x(j))_{j\in J} \mapsto (x(j))_{j\in L}$.

In probability theory or in stochastic processes we have learnt:

Extension theorem of Kolmogorov:
Let $I \neq \emptyset$ and $(P_J)_{J\subset I, |J|<\infty}$ a projective family of probability measures, based on $(\mathbb{R}^d, \mathcal{B}_d)$. Then there exists an $\mathbb{R}^d$-valued stochastic process $(X_t)_{t\in I}$ whose finite dimensional distributions are exactly the P_J with $J \subset I$ finite.

Problem 7.1 (Extension of strictly stationary stochastic processes)
Let $(Y_n)_{n\in\mathbb{N}_0}$ be an $\mathbb{R}^d$-valued strictly stationary stochastic process. Show that a two-sided strictly stationary stochastic process $(X_n)_{n\in\mathbb{Z}}$ exists such that $(X_0, \dots, X_n) \stackrel{d}{=} (Y_0, \dots, Y_n)$ for all $n \in \mathbb{N}_0$.

Definition: Let $p \in \mathbb{N}$, $q \in \mathbb{N}_0$, $\alpha_0 > 0$, $\alpha_1, \dots, \alpha_{p-1} > 0$, $\alpha_p > 0$, $\beta_1, \dots, \beta_{q-1} \geq 0$, $\beta_q > 0$ and $(\varepsilon_t)_{\in\mathbb{N}_0}$ an i.i.d. sequence of random variables. Then a GARCH(p,q)-process on $\mathbb{N}_0$ with volatility process $(\sigma_t)_{t\in\mathbb{N}_0}$ is a stochastic process $(X_t)_{t\in\mathbb{N}_0}$ such that

$$X_t = \sigma_t \varepsilon_t \quad \forall\, t \in \mathbb{N}_0$$

and

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i X_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2, \quad t \geq \max\{p,q\}. \tag{1}$$

The process is called *causal* if addidionally σ_t^2 is independent of $(\varepsilon_{t+h})_{h\in\mathbb{N}_0}$ for $t = 0,\ldots,\max\{p,q\}$. By (1), the latter independence property then easily extends to hold for all $t \in \mathbb{N}_0$.

Problem 7.2
(a) Show that the following are equivalent:

(i) There exists a GARCH(p,q)-process on $\mathbb{N}_0$ with the given parameters such that $(X_t,\sigma_t)_{t\in\mathbb{N}_0}$ is strictly stationary.

(ii) There exists a solution to the GARCH(p,q) equations (3.1) and (3.2) on $\mathbb{Z}$ with the given parameters and a suitable i.i.d. noise $(\widetilde{\varepsilon}_n)_{n\in\mathbb{Z}}$, where $\widetilde{\varepsilon}_0 \stackrel{d}{=} \varepsilon_0$ and such that $(X_t,\sigma_t)_{t\in\mathbb{Z}}$ is strictly stationary.

(b) Give a necessary and sufficient condition for the existence of strictly stationary GARCH(1,1) processes on $\mathbb{N}_0$ (meaning that $(X_t,\sigma_t)_{t\in\mathbb{N}_0}$ is strictly stationary), and show that every such process must be automatically causal.