Infinite-dimensional and continuous-time moving average processes

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Abstract

This thesis consists of two parts both dealing with topics in time series analysis.

In Chapter 2 we study necessary and sufficient conditions for the existence of strictly stationary solutions of ARMA equations in a separable complex Banach space $\mathcal{B}$ of the form

$$Y_t - A_1 Y_{t-1} - \ldots - A_p Y_{t-p} = B_0 Z_t + \ldots + B_q Z_{t-q}, \quad t \in \mathbb{Z},$$

where $A_1, \ldots, A_p$ and $B_0, \ldots, B_q$ are linear continuous operators in $\mathcal{B}$ with $A_p \neq 0$ and $B_q \neq 0$. First, we obtain conditions for ARMA(1,q) equations by excluding zero and the unit circle from the spectrum of the operator of the AR part, where we use a decomposition of A_1 similar to the Jordan decomposition of matrices. We then extend this to ARMA(p,q) equations by using a state space representation of an ARMA(p,q) process as an ARMA(1,q) process. We also show that many ARMA processes in Banach spaces possess a moving average process representation of the form

$$Y_t = \sum_{k=-\infty}^{\infty} \psi_k Z_{t-k}, \quad t \in \mathbb{Z},$$

where the coefficients $(\psi_k)_{k\in\mathbb{Z}}$ can be calculated as the coefficients of a Laurent series. Finally, we discuss various examples illustrating what may happen if one drops the assumptions we made.

In Chapter 3 we study the asymptotic behaviour of the covariance estimator for a continuous-time moving average process with long memory. A continuous-time moving average is here a process $(X_t)_{t\in\mathbb{R}}$ of the form

$$X_t := \int_{-\infty}^{\infty} f(t-s)\, dL_s, \quad t \in \mathbb{R},$$

where f is a real function in $L^2(\mathbb{R})$ and $(L_t)_{t\in\mathbb{R}}$ is a two-sided Lévy process with $\mathbb{E}[L_1] = 0$ and $\mathrm{Var}(L_1) < \infty$. We choose f to be decaying polynomially slowly at infinity such that $(X_t)_{t\in\mathbb{R}}$ exhibits the long-memory property

$$\sum_{k=-\infty}^{\infty} |\gamma(k)| = \infty,$$

where $\gamma(h) := \mathrm{Cov}(X_0, X_h)$. We then show, depending on the speed of the polynomial decay of f and on the tail behaviour of L_1, that the covariance estimator is asymptotically Rosenblatt, stable or normal distributed.

Zusammenfassung in deutscher Sprache

Diese Dissertation besteht aus zwei Teilen, die sich beide mit Fragestellungen aus der Zeitreihenanalyse beschäftigen.

In Kapitel 2 studieren wir notwendige und hinreichende Bedingungen für die Existenz von strikt stationären Lösungen von ARMA-Gleichungen in einem separablen, komplexen Banachraum $\mathcal{B}$ der Form

$$Y_t - A_1 Y_{t-1} - \ldots - A_p Y_{t-p} = B_0 Z_t + \ldots + B_q Z_{t-q}, \quad t \in \mathbb{Z},$$

wobei $A_1, \ldots, A_p$ und $B_0, \ldots, B_q$ lineare stetige Operatoren in $\mathcal{B}$ mit $A_p \neq 0$ and $B_q \neq 0$ sind. Zuerst erhalten wir Bedingungen für ARMA(1,q)-Gleichungen, indem wir die Null und den Einheitskreis vom Spektrum des Operators des AR-Teils ausschließen, wobei wir eine Zerlegung von A_1 benutzen, die ähnlich zur Jordanzerlegung von Matrizen ist. Wir erweitern dies dann auf ARMA(p,q)-Gleichungen, indem wir eine Zustandsraumdarstellung eines ARMA(p,q)-Prozesses als einen ARMA(1,q)-Prozess benutzen. Wir zeigen außerdem, dass viele ARMA-Prozesse in Banachräumen eine Moving-Average-Prozess-Darstellung der Form

$$Y_t = \sum_{k=-\infty}^{\infty} \psi_k Z_{t-k}, \quad t \in \mathbb{Z}$$

besitzen, wobei die Koeffizienten $(\psi_k)_{k\in\mathbb{Z}}$ als Koeffizienten einer Laurentreihe berechnet werden können. Schließlich diskutieren wir mehrere Beispiele, die illustrieren, was passieren kann, wenn man die Annahmen weglässt, die wir getroffen haben.

In Kapitel 3 studieren wir das asymptotische Verhalten des Kovarianzschätzers für einen zeitstetigen Moving-Average-Prozess mit Long-Memory. Ein zeitstetiger Moving-Average-Prozess ist hier ein Prozess $(X_t)_{t\in\mathbb{R}}$ der Form

$$X_t := \int_{-\infty}^{\infty} f(t-s)\, dL_s, \quad t \in \mathbb{R},$$

wobei f eine reelle Funktion aus $L^2(\mathbb{R})$ und $(L_t)_{t\in\mathbb{R}}$ ein zweiseitiger Lévyprozess mit $\mathbb{E}[L_1] = 0$ und $\mathrm{Var}(L_1) < \infty$ sind. Wir wählen f polynomiell langsam bei unendlich fallend, sodass $(X_t)_{t\in\mathbb{R}}$ die Long-Memory-Eigenschaft

$$\sum_{k=-\infty}^{\infty} |\gamma(k)| = \infty$$

zeigt, wobei $\gamma(h) := \mathrm{Cov}(X_0, X_h)$ ist. Wir zeigen dann, dass der Kovarianzschätzer abhängig von der Geschwindigkeit des polynomiellen Abfalls von f und vom Tailverhalten von L_1 asymptotisch Rosenblatt-, stabil- oder normalverteilt ist.

Contents

Chapter 1

Introduction

1.1 Time series

Time series are time-dependent observed data. We encounter many time series in our daily life, such as stock prices or weather and unemployment data to name a few ubiquitous. The mathematical discipline time series analysis tries to find models for such data. It also analyses the mathematical properties of such models and develops statistical methods for them.

This thesis consists of two parts. In the first part in Chapter 2, we analyse the mathematical property of such a model, namely the existence of stationary solutions of a certain time series model, more specifically the ARMA model in function spaces. In the second part in Chapter 3, we analyse the distributional property of an estimator in a time series model with long memory.

We introduce in this section the definitions of a time series and of stationarity. In the next section, we introduce the ARMA model and motivate Chapter 2. In the third section, we explain existing central limit theorems for time series and motivate Chapter 3. In the fourth section, we summarise the main results of Chapter 2 and 3.

A stochastic process is a family of random variables $(X_t)_{t \in T}$ on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. From a mathematical point of view, a time series is a stochastic process, where we interpret the index set T as time. Usually, T is $\mathbb{R}$ in a continuous-time model or $\mathbb{Z}$ in a discrete-time model. For an introduction to time series, see [7].

One fundamental assumption that is often made on the time series is stationarity. That means that the stochastic behaviour of the time series should depend at all times only on relative time differences. To make it mathematically precise, we introduce the two notions of stationarity. The following definition of strict stationarity for discrete-time time series can be found in [7], Definition 1.3.3.

Definition 1.1.1. A time series $(X_t)_{t \in \mathbb{Z}}$ is said to be *strictly stationary* if the joint distributions of $(X_{t_1}, \dots, X_{t_k})$ and $(X_{t_1+h}, \dots, X_{t_k+h})$ are the same for all positive integers k and for all $t_1, \dots, t_k, h \in \mathbb{Z}$.

There is also the notion of weak stationarity which is defined by the covariance. The covariance function of a time series with $\mathrm{Var}(X_t) < \infty$ for all $t \in \mathbb{Z}$ is defined by

$\gamma(r,s) := \mathrm{Cov}(X_r, X_s)$ for $r,s \in \mathbb{Z}$, see [7] Definition 1.3.1. The following definition of weak stationarity can be found in [7], Definition 1.3.2.

Definition 1.1.2. A complex-valued time series $(X_t)_{t\in\mathbb{Z}}$ is said to be *weakly stationary* if

1. $\mathbb{E}[|X_t|^2] < \infty$ for all $t \in \mathbb{Z}$,
2. $\mathbb{E}[X_t] = m$ for all $t \in \mathbb{Z}$ and
3. $\gamma(r,s) = \gamma(r+t, s+t)$ for all $r,s,t \in \mathbb{Z}$.

For a weakly stationary time series, one defines the *autocovariance function* by

$$\gamma(h) := \mathrm{Cov}(X_h, X_0) \text{ for } h \in \mathbb{Z}.$$

The autocovariance is a measure for dependence and the property of long memory is defined in terms of it, see Definition 1.3.2. Note that a strictly stationary time series with finite second moments is weakly stationary as well.

1.2 ARMA processes

One fundamental model for time series is the ARMA model. ARMA is an acronym for autoregresssive moving average. We need the notion of white noise for the definition of the model. We call an i.i.d. sequence $(Z_t)_{t\in\mathbb{Z}}$ *strict white noise*. We call a weakly stationary uncorrelated sequence $(Z_t)_{t\in\mathbb{Z}}$ *weak white noise*.

Definition 1.2.1. An *ARMA(p,q) process* with $p,q \in \mathbb{N}_0$ is a complex-valued stochastic process $(Y_t)_{t\in\mathbb{Z}}$ that fulfils

$$Y_t - a_1 Y_{t-1} - \ldots - a_p Y_{t-p} = b_0 Z_t + \ldots + b_q Z_{t-q}, \quad t \in \mathbb{Z}. \tag{1.1}$$

Here $a_1, \ldots, a_p$ and $b_0, \ldots, b_q$ are complex numbers with $a_p \neq 0$, $b_q \neq 0$ and $(Z_t)_{t\in\mathbb{Z}}$ is strict or weak complex-valued white noise.

If $p = 0$, then the process is called *moving average process of order q* or MA(q) process. If $q = 0$, then the process is called *autoregressive process of order p* or AR(p) process. Note that it might look peculiar that we allow the coefficients and the processes in the model to be complex-valued. In practice one would usually only expect real numbers. It makes the mathematical analysis easier as we will soon see. For that, one defines two polynomials $a(z) := 1 - a_1 z - \cdots - a_p z^p$ and $b(z) := b_0 + b_1 z + \cdots + b_q z^q$.

We are interested when there are strictly or weakly stationary ARMA processes for strict respectively weak white noise.

The question of existence of strictly stationary ARMA processes was completely answered by Brockwell and Lindner [9], Theorem 1:

Theorem 1.2.1. *Suppose that $(Z_t)_{t\in\mathbb{Z}}$ is a non-deterministic i.i.d. sequence. Then the ARMA equation (1.1) admits a strictly stationary solution $(Y_t)_{t\in\mathbb{Z}}$ if and only if*

1. *all singularities of $b(z)/a(z)$ on the unit circle are removable and* $\mathbb{E}[\log^+ |Z_1|] < \infty$,

2. or all singularities of $b(z)/a(z)$ are removable.

If (1) or (2) holds, then a strictly stationary solution is given by

$$Y_t = \sum_{k=-\infty}^{\infty} \psi_k Z_{t-k}, \quad t \in \mathbb{Z}, \tag{1.2}$$

where

$$\sum_{k=-\infty}^{\infty} \psi_k z^k = \frac{b(z)}{a(z)}, \quad 1-\delta < |z| < 1+\delta \text{ for some } \delta \in (0,1),$$

is the Laurent expansion of $b(z)/a(z)$. The sum in (1.2) converges absolutely almost surely. If b does not have a zero on the unit circle, then (1.2) is the unique strictly stationary solution of (1.1).

Example 1.2.1. The equation $Y_t - a_1 Y_{t-1} = Z_t$ has a strictly stationary solution if and only if $|a_1| \neq 1$ and $\mathbb{E}[\log^+ |Z_1|] < \infty$ assuming that the white noise is non-deterministic and that $a_1 \neq 0$. If $a_1 = 0$, then there is always a solution, namely $Y_t = Z_t$. If $|a_1| = 1$, then the white noise has to vanish almost surely. A strictly stationary solution is then in general not unique and is not of the form (1.2), see [9], Theorem 3.

Note that it is common knowledge that there exists a weakly stationary solution of (1.1) if and only if all singularities of $b(z)/a(z)$ on the unit circle are removable, which can be shown by spectral theory, see the introduction of [9].

The (unique) solution of (1.1) is of the form $Y_t = \sum_{k=-\infty}^{\infty} \psi_k Z_{t-k}$. Processes of this form are called moving average processes of infinite order. Hence most ARMA processes are moving average processes.

Definition 1.2.2. A process $(Y_t)_{t\in\mathbb{Z}}$ of the form

$$Y_t = \sum_{k=-\infty}^{\infty} \psi_k Z_{t-k}$$

for a white noise sequence $(Z_t)_{t\in\mathbb{Z}}$ and a sequence of coefficients $(\psi_k)_{k\in\mathbb{Z}}$ is called *moving average process of infinite order* or MA(∞) process. Note that one has to consider in which sense the series is defined. If we have weak white noise and $\sum_{k=-\infty}^{\infty} |\psi_k|^2 < \infty$, then the series converges unconditionally in the L^2-sense. If we have strict white noise with finite second moments, then the series converges by the Itô-Nisio theorem almost surely as well.

By replacing the coefficients $a_1, \dots, a_p$ and $b_0, \dots, b_q$ by matrices and assuming that $(Y_t)_{t\in\mathbb{Z}}$ and $(Z_t)_{t\in\mathbb{Z}}$ are multivariate processes, one obtains the notion of a multivariate ARMA process. The existence of strictly stationary solutions of multivariate ARMA equations was completely characterised by Brockwell, Lindner and Vollenbröker [10].

For ARMA(1,q) equations, they diagonalise A_1 via the Jordan decomposition. The existence of a strictly stationary solution then depends on projections on the blocks of the Jordan matrix. For the blocks with eigenvalues λ with $|\lambda| \neq 0, 1$ there is also a $\log^+$-moment condition. For the blocks with $|\lambda| = 1$ there is a deterministic part of the white noise and for $\lambda = 0$ there is no moment condition, see Theorem 2.1 in [10]. Note that this parallels Example 1.2.1 in the one-dimensional case.

By replacing the coefficients by linear operators in a Banach space $\mathcal{B}$ and assuming that $(Y_t)_{t\in\mathbb{Z}}$ and $(Z_t)_{t\in\mathbb{Z}}$ are $\mathcal{B}$-valued, we obtain a model for functional time series. For an introduction to time series in function spaces, see [4]. The aim of the first part of our thesis is to derive necessary and sufficient conditions for the existence of strictly stationary solutions of ARMA equations in Banach spaces similar to the results in [9] and [10].

Our approach is to solve it by the spectrum. The spectrum of a bounded linear operator A on a Banach space $\mathcal{B}$ is defined by $\sigma(A) := \{\lambda \in \mathbb{C} | \lambda \operatorname{id} - A \text{ is not invertible}\}$. In the case $\mathcal{B} = \mathbb{C}^n$ the spectrum coincides with the set of eigenvalues. By functional calculus, there is also a decomposition with respect to the spectrum of an operator similar to the Jordan decomposition, hence we think this is the right vehicle to solve the problem for ARMA(1,q) equations. We obtain a partial characterisation, extend it to the ARMA(p,q) case and discuss why we think a complete characterisation is not feasible by discussing various examples. Our results extend known sufficient conditions for the existence of causal weakly and strictly stationary solutions of AR(p) equations in Hilbert spaces and AR(1) equations in Banach spaces with finite second moment white noise, see Theorem 5.1 and Theorem 6.1 in [4].

1.3 Central limit theorems

The well-known classical central limit theorem states the following:

Theorem 1.3.1. *Let $(X_t)_{t\in\mathbb{N}}$ be an i.i.d. sequence of real random variables with $\mathbb{E}[X_1] = \mu$ and $\operatorname{Var}(X_1) = \sigma^2 \in (0, \infty)$. Then*

$$\frac{\sum_{t=1}^{N} X_t - N\mu}{\sigma\sqrt{N}} \xrightarrow{d} N(0,1) \text{ as } N \to \infty,$$

where $N(0,1)$ denotes the standard normal distribution.

This theorem can be used for example for the calculation of confidence intervals for the empiral mean. There are also many central limit theorems for time series, see for example [7]. If one wants to estimate the autocovariance function $\gamma(h)$ of a weakly stationary real-valued time series, then a canonical estimator is

$$\hat{\gamma}_N(h) := \frac{1}{N}\sum_{t=1}^{N} X_t X_{t+|h|}, \quad h \in \mathbb{Z},$$

if one assumes that the time series has expectation zero.

There is a central limit theorem for this autocovariance estimator for a moving average process of infinite order as in (1.2), which can be found as Proposition 7.3.3. in [7]:

Theorem 1.3.2. *Let $(Z_t)_{t\in\mathbb{Z}}$ be an i.i.d. sequence of real random variables with $\mathbb{E}[Z_0] = 0$, $\mathbb{E}[Z_0^2] = \sigma^2 < \infty$ and $\mathbb{E}[Z_1^4] = \eta\sigma^4 < \infty$. Define a moving average process $(X_t)_{t\in\mathbb{Z}}$ by*

$$X_t := \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}, \quad t \in \mathbb{Z},$$

where we assume that $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$ *and* $\psi_j \in \mathbb{R}$ *for* $j \in \mathbb{Z}$. *Then*

$$\sqrt{N}(\hat{\gamma}_N(0) - \gamma(0), \dots, \hat{\gamma}_N(H) - \gamma(H)) \xrightarrow{d} N(0, V) \text{ as } N \to \infty,$$

where $N(0, V)$ *is a normal distribution with covariance matrix* $V = (v_{pq})_{p,q=0,\dots,H}$ *given by*

$$v_{pq} = (\eta - 3)\gamma(p)\gamma(q) + \sum_{k=-\infty}^{\infty} [\gamma(k)\gamma(k - p + q) + \gamma(k + q)\gamma(k - p)].$$

The condition $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$ is essential. Hence an interesting question is what happens if $\sum_{j=-\infty}^{\infty} |\psi_j|^2 < \infty$ but $\sum_{j=-\infty}^{\infty} |\psi_j| = \infty$. Horváth and Kokoszka [23] showed that the asymptotic distribution can be a Rosenblatt or stable distribution if this condition is not fulfilled. More precisely they showed this under the assumption that the moving average process is of the form $X_t := \sum_{j=0}^{\infty} \psi_j Z_{t-j}$ for $t \in \mathbb{Z}$ with $\psi_j = j^{d-1} l(j)$ for $d \in (0, \frac{1}{2})$ and $l(j) \to C_d > 0$ as $j \to \infty$. This process exhibits the so-called long memory property. We explain the Rosenblatt and stable distributions and the long memory property in the following subsections.

There is a continuous-time analogue to discrete-time moving average processes as in (1.2):

Definition 1.3.1. Let $(L_t)_{t\in\mathbb{R}}$ be a two-sided Lévy process, i.e. a stochastic process with independent and stationary increments, càdlàg paths and $L_0 = 0$. Let f be a real function in $L^2(\mathbb{R})$. Assume further that $\mathbb{E}[L_1] = 0$ and $\mathrm{Var}(L_1) < \infty$. Then one defines a *continuous-time moving average process* $(X_t)_{t\in\mathbb{R}}$ by

$$X_t := \int_{-\infty}^{\infty} f(t - s)\, dL_s, \quad t \in \mathbb{R},$$

where the integral is defined in the L^2-sense of stochastic integrals.

Cohen and Lindner [11] showed that for continuous-time moving average processes with further conditions on f a theorem similar to Theorem 1.3.2 holds.

Our aim is to show a central limit theorem for continuous-time moving average processes with long memory similar to the results of Horváth and Kokoszka [23].

1.3.1 Long memory property

One way to describe the dependence structure of a time series is to look at its autocovariance structure. The following definition is taken from [21], Definition 3.1.2.

Definition 1.3.2. Let $(X_k)_{k\in\mathbb{Z}}$ be a weakly stationary real-valued time series. We say that $(X_k)_{k\in\mathbb{Z}}$ exhibits *short memory*, if

$$\sum_{k=-\infty}^{\infty} |\gamma(k)| < \infty \quad \text{and} \quad \sum_{k=-\infty}^{\infty} \gamma(k) > 0.$$

We further say that it exhibits *long memory*, if

$$\sum_{k=-\infty}^{\infty} |\gamma(k)| = \infty$$

and *negative memory*, if

$$\sum_{k=-\infty}^{\infty} |\gamma(k)| < \infty \quad \text{and} \quad \sum_{k=-\infty}^{\infty} \gamma(k) = 0.$$

For an introduction to time series with long memory, see [21]. Most weakly stationary ARMA process exhibit short memory as the autocovariance decreases exponentially fast which can be seen from the Laurent expansion of the moving average representation.

If we consider a moving average process of the form

$$X_t := \sum_{j=0}^{\infty} \psi_j Z_{t-j}, \quad t \in \mathbb{Z},$$

with $\psi_j = j^{d-1}l(j)$ for $d \in (0, \frac{1}{2})$ and $l(j) \to C_d > 0$ as $j \to \infty$, then one can easily see that this process exhibits long memory.

1.3.2 Rosenblatt distribution(s)

The following example is taken from [42] and goes back to Rosenblatt. Let $(Y_k)_{k\in\mathbb{Z}}$ be a strictly stationary Gaussian sequence with $\mathbb{E}[Y_k] = 0$ and $\mathrm{Var}(Y_k) = 1$ for all $k \in \mathbb{Z}$ and autocovariance function satisfying $\gamma(k) = \mathbb{E}[Y_0 Y_k](1+k^2)^{-D/2} \sim k^{-D}$ as $k \to \infty$ and $D \in (0, \frac{1}{2})$. Consider then

$$X_k := Y_k^2 - 1.$$

Then $Z_N := \frac{\sigma}{N^{1-D}} \sum_{k=1}^N X_k$ with $\sigma = \sqrt{\frac{1}{2}(1-2D)(1-D)}$ converges in distribution to a non-Gaussian distribution with mean zero and variance one. This distribution is called the *Rosenblatt distribution* in [42]. Note that it depends on the choice of D, so there is a family of Rosenblatt distributions. Note further that $(X_k)_{k\in\mathbb{Z}}$ exhibits long memory as $\sum_{k=1}^{\infty} \mathbb{E}[X_0 X_k] = \infty$ as is pointed out in [40] on page 33. Properties of the Rosenblatt distributions can be found in [42]. They compute numerically their moments and their distribution and density functions.

If one sets $\sigma = 1$, then Z_N converges in distribution to $U_d(1)$, where $(U_d(t))_{t\in\mathbb{R}}$ is the *Rosenblatt process* defined by

$$U_d(t) := 2 \int_{x_1<x_2<t} [\int_0^t (v-x_1)_+^{d-1} (v-x_2)_+^{d-1}\, dv]\, W(dx_1) W(dx_2), \quad t \in \mathbb{R}, \tag{1.3}$$

where W is a standard Gaussian random measure on $\mathbb{R}$, i.e. standard Brownian motion and where $d = \frac{1}{2}(1-D)$, see [40] section 7. Note that we omit the normalisation constant like [23] and unlike [42]. We define this integral in the next subsection.

Horváth and Kokoszka show in Theorem 3.3 in [23] in the situation of Theorem 1.3.2 but under the assumption $\psi_j = j^{d-1}l(j)$, with $\psi_j = j^{d-1}l(j)$ for $d \in (0, \frac{1}{2})$ and $l(j) \to C_d > 0$ as $j \to \infty$ that

$$N^{1-2d}(\hat{\gamma}_N(0) - \gamma(0), \ldots, \hat{\gamma}_N(H) - \gamma(H)) \xrightarrow{d} C_d^2 \sigma^2 U_d(1)(1, \ldots, 1) \text{ as } N \to \infty,$$

if $d \in (\frac{1}{4}, \frac{1}{2})$ and asymptotic normality if $d \in (0, \frac{1}{4})$. An intuitive explanation for the fact that there are two cases is that in the case $d \in (\frac{1}{4}, \frac{1}{2})$ the process $(X_t X_{t+h})_{t \in \mathbb{Z}}$ exhibits long memory while in the case $d \in (0, \frac{1}{4})$ it exhibits short memory.

We show in Theorem 3.2.1 that for continuous-time moving average processes a similar result holds.

1.3.3 Multiple Wiener-Itô integrals

In this section we stick to the introduction to multiple Wiener-Itô integrals in section 14.3 in [21]. Let f be a function in $L^2(\mathbb{R}^2)$. We then want to define

$$I_2(f) = \int_{\mathbb{R}^2} f(x_1, x_2)\, W(dx_1) W(dx_2),$$

where W is a standard Gaussian random measure on $\mathbb{R}$, or equivalently a two-sided Brownian motion. For convenience, we stick to the case $\mathbb{R}^2$ but it its also possible to define it for general $\mathbb{R}^k$. For a simple function of the form $f = 1_{(a,b] \times (c,d]}$ such that the function vanishes on the diagonal, we define

$$I_2(1_{(a,b] \times (c,d]}) := (W(b) - W(a))(W(d) - W(c)),$$

where we interpret here W as a two-sided Brownian motion. For a function of the form $\sum_{i=1}^n K_i 1_{(a_i,b_i] \times (c_i,d_i]}$ such that the function vanishes on the diagonal, we define

$$I_2(\sum_{i=1}^n K_i 1_{(a_i,b_i] \times (c_i,d_i]}) := \sum_{i=1}^n K_i (W(b_i) - W(a_i))(W(d_i) - W(c_i)).$$

The set of functions of this form is a dense subset of $L^2(\mathbb{R}^2)$ and we denote it by $S(\mathbb{R}^2)$. I_2 is a bounded (and well-defined) mapping from $S(\mathbb{R}^2)$ to $L^2(\mathbb{P})$, since

$$\mathbb{E}[I_2(f)^2] \leq 2||f||_2^2$$

for $f \in S(\mathbb{R}^2)$ which follows from (14.3.7) in [21]. Hence it can be extended in a unique way to a bounded linear operator I_2 mapping from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{P})$. This finishes the construction of the multiple Wiener-Itô integral. Note that the fact that one excludes the diagonal in the construction suits our needs well, since these integrals represent limits that arise from sums of products of white noise where we exclude squares of the white noise.

1.3.4 Stable distributions

If we have an i.i.d. sequence of random variables without finite variance, then it is possible that the partial sums of the random variables converge asymptotically towards a stable distribution. The following definition is taken from [34], Definition 1.1.6.

Definition 1.3.3. A random variable X is said to have a *stable distribution* if there are parameters $\alpha \in (0,2]$, $\tau \geq 0$, $\beta \in [-1,1]$ and $\mu \in \mathbb{R}$ such that its characteristic function has the following form:

$$\mathbb{E}[\exp(i\theta X)] = \begin{cases} \exp\Big(-\tau^\alpha |\theta|^\alpha (1 - i\beta(\text{sign}(\theta))\tan(\frac{\pi\alpha}{2})) + i\mu\theta\Big) & \text{if } \alpha \neq 1 \\ \exp\Big(-\tau^\alpha |\theta|^\alpha (1 + i\beta\frac{2}{\pi}(\text{sign}(\theta))\ln(|\theta|)) + i\mu\theta\Big) & \text{if } \alpha = 1. \end{cases}$$

We denote its distribution with $S_\alpha(\tau,\beta,\mu)$, see (1.1.6), p. 9, in [34].

Note that in the case $\alpha = 2$ it does not depend on the choice of β and the stable distribution is then a normal distribution.

A function $l : (0,\infty) \to (0,\infty)$ is called *regularly varying with index* ρ, if $\lim_{t\to\infty} \frac{l(tx)}{l(t)} = x^\rho$ for all $x > 0$. We call a random variable X *regularly varying with index* α, if the tail function $\bar{F}(x) := \mathbb{P}[|X| > x]$ is regularly varying with index $-\alpha$. We say that X fulfils a *tail balance condition*, if there is a $p \in [0,1]$ such that

$$\lim_{x\to\infty} \frac{\mathbb{P}[X > x]}{\mathbb{P}[|X| > x]} = p. \tag{1.4}$$

The following theorem is a reformulation of Theorem 2.2.15 and Propositions 2.2.13/14 in [19]:

Theorem 1.3.3. *Let $(X_t)_{t\in\mathbb{N}}$ be an i.i.d. sequence of real random variables that are regularly varying with index $\alpha \in (0,2)$ and fulfil a tail balance condition. Define*

$$a_N := \inf\{y : \mathbb{P}[|X_1| > y] < \frac{1}{N}\} \text{ and } b_N := \mathbb{E}[X_1 1_{\{|X_1| \leq a_N\}}].$$

Then

$$\frac{\sum_{t=1}^N X_t - N b_N}{a_N} \xrightarrow{d} S_\alpha(\tau,\beta,\mu) \text{ as } N \to \infty,$$

with $\tau \geq 0$, $\beta \in [-1,1]$ and $\mu \in \mathbb{R}$.

For an introduction to stable distributions, see [34]. The name stable stems from the fact that the convolution of stable distributions is a stable distribution again, more precisely

$$S_\alpha(\tau_1,\beta_1,\mu_1) * S_\alpha(\tau_2,\beta_2,\mu_2) = S_\alpha((\tau_1^\alpha + \tau_2^\alpha)^{1/\alpha}, \frac{\beta_1\tau_1^\alpha + \beta_2\tau_2^\alpha}{\tau_1^\alpha + \tau_2^\alpha}, \mu_1 + \mu_2),$$

see Property 1.2.1 in [34]. Every non-degenerate stable distribution has a continuous Lebesgue density but only in few cases an explicit elementary form is known, see p. 10 in [34]. Stable distributions do not have finite second moments unless $\alpha = 2$.

If $(K_s)_{s\in[0,1]}$ is a Lévy process with $K_1 \overset{d}{=} S_\alpha(\tau,\beta,\mu)$ and f is a function with $f \in L^\alpha([0,1]) \cap L^1([0,1])$ where $0 < \alpha < 2$ (and additionally $\int_0^1 |f(x)\beta \ln|f(x)||\,dx < \infty$ if $\alpha = 1$), then we can define

$$I_1(f) := \int_0^1 f(s)\,dK_s.$$

see [34], Section 3.4. Note that in [34] the stochastic integral is defined in a more general setting for stable random measures. Note further that in [34] the location parameter of the stable random measure is zero, so that we have to add the additional condition $f \in L^1([0,1])$. Note finally that since the Lévy process $(K_s)_{s\in[0,1]}$ does not have a finite second moment, this integral cannot be defined in the L^2-sense. However, it can be constructed via Cauchy sequences in probability, see [34], Section 3.4.

Note that $I_1(f)$ is also stable distributed with location parameter $\mu\int_0^1 f(x)\,dx$, skewness $\beta\frac{\int_0^1 \operatorname{sign}(f(x))|f(x)|^\alpha\,dx}{\int_0^1 |f(x)|^\alpha\,dx}$ and scale parameter $\tau(\int_0^1 |f(x)|^\alpha\,dx)^{1/\alpha}$, see [34], Section 3.4.

Horváth and Kokoszka [23] show if one has a moving average process with white noise which is regularly varying with index $\alpha \in (2,4)$ and with $\psi_j = j^{d-1}l(j)$ for $d \in (0,\frac{1}{2})$ and $l(j) \to C_d > 0$ as $j \to \infty$, that the autocovariance is asymptotically stable or Rosenblatt distributed. More precisely, if $\frac{1}{\alpha} < d$, then

$$N^{1-2d}(\hat\gamma_N(0)-\gamma(0),\ldots,\hat\gamma_N(H)-\gamma(H)) \overset{d}{\to} C_d^2\sigma^2 U_d(1)(1,\ldots,1) \text{ as } N\to\infty,$$

and if $\frac{1}{\alpha} > d$, then

$$\frac{N}{a_N^2}\Big(\hat\gamma_N(0)-\gamma(0),\ldots,\hat\gamma_N(H)-\gamma(H)\Big) \overset{d}{\to} (S-\frac{\alpha}{\alpha-2})(\sum_{j=1}^\infty \psi_j^2,\ldots,\ldots\sum_{j=1}^\infty \psi_j\psi_{j+H}),$$

as $N\to\infty$ for a stable random variable S with index $\alpha/2$, see [23] Theorem 3.1.

We prove a similar result for continuous-time moving average processes, see Theorem 3.3.1.

1.4 Main results of this thesis

1.4.1 Strictly stationary solutions of ARMA equations in Banach spaces

In Chapter 2 we study conditions for the existence of solutions of ARMA equations in a separable complex Banach space $\mathcal{B}$ of the form

$$Y_t - A_1Y_{t-1} - \ldots - A_pY_{t-p} = B_0Z_t + \ldots + B_qZ_{t-q}, \quad t\in\mathbb{Z}, \tag{1.5}$$

where $A_1,\ldots,A_p$ and $B_0,\ldots,B_q$ are linear continuous operators in $\mathcal{B}$ with $A_p \neq 0$ and $B_q \neq 0$. This extends results by Brockwell and Lindner [9] on univariate ARMA processes and results on multivariate ARMA processes by Brockwell, Lindner and Vollenbröker [10]. Chapter 2 is based on [38].

The proof of the characterisation of the existence of strictly stationary multivariate ARMA(1,q) processes in [10] makes use of the Jordan canonical decomposition. A Jordan decomposition for operators in Banach spaces does not exist in full generality. Therefore, we restrict ourselves to operators in the AR part whose spectra, denoted by $\sigma(A_1)$, do not contain elements of the unit circle $\mathbb{S} = \{z \in \mathbb{C}||z| = 1\}$. Let A_1 be an operator with $\sigma(A_1)\cap\mathbb{S} = \emptyset$. We can find closed subspaces $\mathcal{B}_1, \mathcal{B}_2$ of $\mathcal{B}$ and an invertible linear continuous operator $S : \mathcal{B}_1 \times \mathcal{B}_2 \to \mathcal{B}$ such that

$$S^{-1}A_1S = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix},$$

where $\Lambda_1 : \mathcal{B}_1 \to \mathcal{B}_1$ and $\Lambda_2 : \mathcal{B}_2 \to \mathcal{B}_2$ are bounded linear operators with $\sigma(\Lambda_1) = \{z \in \mathbb{C} : |z| < 1\} \cap \sigma(A_1)$ and $\sigma(\Lambda_2) = \{z \in \mathbb{C} : |z| > 1\} \cap \sigma(A_1)$, see Theorem 6.17 in Kato [27] Chapter III.6.4.

We derive under the assumption that $\sigma(A_1) \cap \mathbb{S} = \emptyset$ and $0 \notin \sigma(A_1)$ based on this diagonalisation necessary and sufficient conditions for the existence of a strictly stationary solution of

$$Y_t - A_1Y_{t-1} = B_0Z_t + \ldots + B_qZ_{t-q}, \quad t \in \mathbb{Z}. \tag{1.6}$$

The result can be found in Chapter 2 as Theorem 2.2.1. It is a partial generalisation of the result for multivariate ARMA(1,q) processes by [10].

Based on this we then derive necessary and sufficient conditions for the existence of a a strictly stationary solution of (1.5). This is the content of Theorem 2.2.2.

We additionally derive a moving average representation for ARMA processes of the form

$$Y_t = \sum_{k=-\infty}^{\infty} \psi_k Z_{t-k}, \quad t \in \mathbb{Z},$$

where the coefficients $(\psi_k)_{k\in\mathbb{Z}}$ are the coefficients of the Laurent series of

$$(\mathrm{id} - zA_1 - z^2A_2 - \cdots z^pA_p)^{-1}(B_0 + zB_1 + \ldots + z^qB_q),$$

see Theorem 2.2.3.

We finally discuss several examples in section 3 of Chapter 2 that illustrate what can happen if we drop the assumption that $\sigma(A_1) \cap \mathbb{S} = \emptyset$ or $0 \notin \sigma(A_1)$.

1.4.2 A central limit theorem for the sample autocovariance of a continuous-time moving average process with long memory

In Chapter 3 we study the asymptotic behaviour of the autocovariance estimator defined by

$$\hat{\gamma}_N(h) := \frac{1}{N}\sum_{t=1}^{N} X_tX_{t+|h|}, \quad h \in \mathbb{Z},$$

for continuous-time moving average processes $(X_t)_{t\in\mathbb{R}}$ of the form

$$X_t := \int_{-\infty}^{\infty} f(t-s)\,dL_s, \quad t \in \mathbb{R},$$

where $(L_t)_{t\in\mathbb{R}}$ is a two-sided Lévy process with $\mathbb{E}[L_1] = 0$ and $\mathrm{Var}(L_1) < \infty$ and f is a real-valued function in $L^2(\mathbb{R})$. We assume that $f(t) = 0$ for $t \leq 0$, f is bounded and $f(t) \sim C_d t^{d-1}$ as $t \to \infty$ with $d \in (0, \frac{1}{2})$ and $C_d > 0$. $(X_t)_{t\in\mathbb{R}}$ exhibits by this assumption long memory, i.e.

$$\sum_{k=-\infty}^{\infty} |\gamma(k)| = \infty,$$

where $\gamma(h) := \mathrm{Cov}(X_0, X_h)$.

Horváth and Kokoszka [23] studied the asymptotic behaviour of the autocovariance estimator for discrete-time moving average processes under similar conditions. Cohen and Lindner [11] studied the asymptotic behaviour for continuous-time moving average processes with short memory property. Chapter 3 is based on [39].

In analogy to [23], we show under the additional assumption $\mathbb{E}[L_1^4] = \eta\sigma^4 < \infty$ that the autocovariance estimator is asymptotically Rosenblatt distributed if $d \in (\frac{1}{4}, \frac{1}{2})$ or normal distributed if $d \in (0, \frac{1}{4})$. This result can be found as Theorem 3.2.1 and as Remark 3.4.1 in Chapter 3. We show under the assumption that L_1 is regularly varying with index $\alpha \in (2, 4)$ that the autocovariance is asymptotically stable distributed if $\frac{1}{\alpha} > d$ or Rosenblatt distributed if $\frac{1}{\alpha} < d$. This result can be found as Theorem 3.3.1 in Chapter 3.

Chapter 2

Strictly stationary solutions of ARMA equations in Banach spaces

Based on [38]: F. Spangenberg Strictly stationary solutions of ARMA equations in Banach spaces *Journal of Multivariate Analysis*, Volume 121, Pages 127-138, 2013

Abstract. We obtain necessary and sufficient conditions for the existence of strictly stationary solutions of ARMA equations in Banach spaces with independent and identically distributed noise under certain assumptions. First, we obtain conditions for ARMA(1,q) equations by excluding zero and the unit circle from the spectrum of the operator of the AR part. We then extend this to ARMA(p,q) equations. Finally, we discuss various examples.

2.1 Introduction

An ARMA(p,q) process is a stochastic process $(Y_t)_{t\in\mathbb{Z}}$ that fulfils the following recursion equation (ARMA, AutoRegressive Moving Average equation)

$$Y_t - a_1Y_{t-1} - \ldots - a_pY_{t-p} = b_0Z_t + \ldots + b_qZ_{t-q}, \quad t \in \mathbb{Z}.$$

Here $a_1, \ldots, a_p$ and $b_0, \ldots, b_q$ are usually complex numbers with $a_p \neq 0$, $b_q \neq 0$ and $(Z_t)_{t\in\mathbb{Z}}$ is a sequence of random variables, which are mostly either i.i.d. or uncorrelated. Such a sequence is called white noise. Y and Z are complex-valued stochastic processes as this facilitates many technical problems. An excellent introduction to time series in general and ARMA processes in particular is the monograph by Brockwell and Davis [7].

A natural extension is to consider multivariate ARMA processes and one can take a further step by looking at ARMA processes in infinite dimensional vector spaces. For an introduction to time series in Banach spaces, see the monographs by Bosq [4] and Horváth and Kokoszka [24] and the survey articles by Hörmann and Kokoszka [22] and Mas and Pumo [30]. ARMA processes in Banach spaces can be applied for example in climate prediction and financial modelling, see e.g. [1] and [37]. For work in functional data with heavy tails, see for instance the article by Meinguet and Segers [36] which deals with extreme value theory of functional time series. One finds explicit examples for the use of functional data models in chapter 1 of the book by Horváth and Kokoszka [24], where they give the examples of the magnetic field of the Earth, stock prices, pollution levels in a city and credit card transactions. In functional data analysis, one usually considers functions on the unit interval. The functions are usually approximated by a series of functions, for example an orthonormal basis of $L^2[0,1]$. To this end, one endows the function space with a norm that renders it into a Banach space. The coefficients of infinite linear combinations of these approximation functions give rise to sequence spaces. In the case of the $L^2[0,1]$ and an orthonormal basis as approximation functions this gives a Hilbert space isomorphism between $L^2[0,1]$ and the space of square integrable sequences $\ell^2(\mathbb{N})$. Hence it is sensible to consider $\ell^2(\mathbb{N})$ instead of $L^2[0,1]$.

In this chapter we study conditions for the existence of solutions of ARMA equations in a separable complex Banach space $\mathcal{B}$ of the form

$$Y_t - A_1 Y_{t-1} - \ldots - A_p Y_{t-p} = B_0 Z_t + \ldots + B_q Z_{t-q}, \quad t \in \mathbb{Z}, \tag{2.1}$$

where $A_1, \ldots, A_p$ and $B_0, \ldots, B_q$ are linear continuous operators in $\mathcal{B}$ with $A_p \neq 0$ and $B_q \neq 0$. Any $\mathcal{B}$-valued stochastic process $(Y_t)_{t\in\mathbb{Z}}$ which satisfies this equation is called a solution of the ARMA(p,q) equation or an ARMA(p,q) process. We investigate strictly stationary solutions of ARMA equations, where the white noise $(Z_t)_{t\in\mathbb{Z}}$ is a series of Banach-space-valued i.i.d. random variables. This extends the work by Bosq [4] which deals with causal weakly and strictly stationary solutions of AR equations with finite second moment white noise.

Firstly, we generalise results by Brockwell and Lindner [9] and Brockwell, Lindner and Vollenbröker [10]. They give necessary and sufficient conditions for the existence of strictly stationary solutions for multivariate ARMA(1,q) equations in terms of the eigenvalues of the matrix of the AR part and $\log^+$-moment conditions on the white noise. Our approach is to partially generalise this by investigating the spectrum of the operator of the AR(1) part. Assuming that the spectrum of the autoregressive operator does not contain zero and has empty intersection with the unit circle, we derive necessary and sufficient conditions for the existence of a strictly stationary solution in terms of finiteness of $\log^+$-moments. Secondly, we extend our results to ARMA(p,q) processes by using their representations as $\mathcal{B}^p$-valued ARMA(1,q) processes. Thirdly, we give an additional representation of the solution as a moving average process of infinite order by employing Laurent series. Finally, we look at what can happen in the case when zero is in the spectrum or the intersection of the spectrum and the unit circle is nonempty by looking at various examples.

2.2 Results

We now state our main results. We first give necessary and sufficient conditions for the existence of strictly stationary ARMA(1,q) processes when the spectrum does not contain zero and elements of the unit circle. We extend these results to ARMA(p,q) processes. Finally, we give a representation of the solution as a moving average process of infinite order using Laurent series.

2.2.1 Strictly stationary solutions of ARMA(1,q) equations

The proof of the characterisation of the existence of strictly stationary multivariate ARMA(1,q) processes in [10] makes use of the Jordan canonical decomposition. A Jordan decomposition for operators in Banach spaces does not exist in full generality. Therefore, we restrict ourselves to operators in the AR part whose spectra, denoted by $\sigma(A_1)$, do not contain elements of the unit circle $\mathbb{S}$. Let A_1 be an operator with $\sigma(A_1) \cap \mathbb{S} = \emptyset$. We can find closed subspaces $\mathcal{B}_1, \mathcal{B}_2$ of $\mathcal{B}$ and an invertible linear continuous operator $S : \mathcal{B}_1 \times \mathcal{B}_2 \to \mathcal{B}$ such that

$$S^{-1}A_1S = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix},$$

where $\Lambda_1 : \mathcal{B}_1 \to \mathcal{B}_1$ and $\Lambda_2 : \mathcal{B}_2 \to \mathcal{B}_2$ are bounded linear operators with $\sigma(\Lambda_1) = \{z \in \mathbb{C} : |z| < 1\} \cap \sigma(A_1)$ and $\sigma(\Lambda_2) = \{z \in \mathbb{C} : |z| > 1\} \cap \sigma(A_1)$. This can be proven by using functional calculus for holomorphic functions. For a proof see Theorem 6.17 in Kato [27] Chapter III.6.4: S^{-1} is given by $S^{-1} = (P, \mathrm{id} - P)$, where $P = -\frac{1}{2\pi i}\int_\gamma R(z, A_1)\, dz$, with γ being a path following the unit circle and $R(z, A_1)$ being the resolvent operator of A_1. $\mathcal{B}_1$ and $\mathcal{B}_2$ are the images of P and $\mathrm{id} - P$. The cartesian product $\mathcal{B}_1 \times \mathcal{B}_2$ is endowed with a norm, e.g. the maximum norm, that renders it into a Banach space, and S is an isomorphism between $\mathcal{B}_1 \times \mathcal{B}_2$ and $\mathcal{B}$. $\mathcal{B}_1 \times \mathcal{B}_2$ is the Banach space direct sum of $\mathcal{B}_1$ and $\mathcal{B}_2$. It is usually denoted by $\mathcal{B}_1 \oplus \mathcal{B}_2$.

An operator A_1 that fulfils $\sigma(A_1) \cap \mathbb{S} = \emptyset$ is called *hyperbolic*. The notion of hyperbolicity is used in the theory of operator semigroups in the context of stability, see for example [17].

We use the same diagonalisation of the ARMA(1,q) equation as in [10]: the ARMA(1,q) equation

$$Y_t - A_1Y_{t-1} = B_0Z_t + \ldots + B_qZ_{t-q}, \quad t \in \mathbb{Z} \tag{2.2}$$

has a strictly stationary solution $(Y_t)_{t\in\mathbb{Z}}$ if and only if the corresponding equation for $X_t := S^{-1}Y_t$

$$X_t - S^{-1}A_1SX_{t-1} = S^{-1}B_0Z_t + \cdots + S^{-1}B_qZ_{t-q}, \quad t \in \mathbb{Z},$$

has a strictly stationary solution. We define I_i as the projection on the i-th component of $\mathcal{B}_1 \oplus \mathcal{B}_2$ and $X_t^{(i)} = I_iX_t$ for $i = 1, 2$. We will see in the proof of Theorem 2.2.1 that there is a strictly stationary solution of the original equation if and only if there are strictly stationary solutions for

$$X_t^{(i)} - \Lambda_iX_{t-1}^{(i)} = I_iS^{-1}B_0Z_t + \cdots + I_iS^{-1}B_qZ_{t-q}, \quad t \in \mathbb{Z}, \quad i = 1, 2. \tag{2.3}$$

Recall that the spectral radius $r(A_1) = \lim_{n\to\infty} \sqrt[n]{||A_1^n||}$ coincides with $\sup\{|\lambda| : \lambda \in \sigma(A_1)\}$ which justifies the name spectral radius. It is easy to see that $r(A_1) < 1$ if and only if A_1 is uniformly exponentially stable, i.e. there exist constants $a \geq 0$, $0 < b < 1$ such that $||A_1^j|| \leq ab^j$ for all $j \in \mathbb{N}$. The latter condition corresponds to condition (c_1) on p. 74 in [4]. Hence the condition $\sigma(A) \subset \{z \in \mathbb{C} : |z| < 1\}$ is equivalent to the conditions in Lemma 3.1 of [4].

We now give an extension of Theorem 1 in [10]:

Theorem 2.2.1. *Let $\mathcal{B}$ be a complex separable Banach space and let $(Z_t)_{t\in\mathbb{Z}}$ be an i.i.d. sequence of $\mathcal{B}$-valued random variables. Let A_1 and $B_0,\ldots,B_q$ be linear continuous operators in $\mathcal{B}$ and assume that A_1 is hyperbolic, i.e. $\sigma(A_1)\cap\mathbb{S}=\emptyset$. Let S be the operator as given above such that $S^{-1}A_1S=\begin{pmatrix}\Lambda_1 & 0\\ 0 & \Lambda_2\end{pmatrix}$. Then the ARMA(1,q) equation (2.2) has a strictly stationary solution $Y=(Y_t)_{t\in\mathbb{Z}}$ if*

$$\mathbb{E}\log^+\Big|\Big|\Big(\sum_{k=0}^{q}A_1^{q-k}B_k\Big)Z_0\Big|\Big|<\infty.$$

This moment condition is also necessary under the additional assumption $0\notin\sigma(A_1)$. The strictly stationary solution is unique in both cases. It is given by $Y_t=SX_t$ with

$$X_t^{(1)}=\sum_{j=0}^{q-1}(\sum_{k=0}^{j}\Lambda_1^{j-k}I_1S^{-1}B_k)Z_{t-j}+\sum_{j=q}^{\infty}\Lambda_1^{j-q}(\sum_{k=0}^{q}\Lambda_1^{q-k}I_1S^{-1}B_k)Z_{t-j}\tag{2.4}$$

and

$$X_t^{(2)}=-\sum_{j=1-q}^{\infty}\Lambda_2^{-j-q}(\sum_{k=(1-j)\vee 0}^{q}\Lambda_2^{q-k}I_2S^{-1}B_k)Z_{t+j},\tag{2.5}$$

where the series defining $X_t^{(1)}$ and $X_t^{(2)}$ converge almost surely absolutely.

Proof. The proof follows along the lines of the proofs of Theorem 1 and Corollary 1 in [10]. We first consider the case $\sigma(A_1)\subset\{z\in\mathbb{C}:|z|<1\}$, then the case $\sigma(A_1)\subset\{z\in\mathbb{C}:1<|z|\}$ and finally the general case.

Case 1: $\sigma(A_1)\subset\{z\in\mathbb{C}:|z|<1\}$ i) Sufficiency: Assume that the moment condition is fulfilled. The sufficiency of the conditions follows along the lines of Section 3.2 in [10]. Note that because $r(A_1)<1$, there are $a>0$ and $0<b<1$ such that $||A_1^j||\le ab^j$. We show that this moment condition is sufficient for the almost sure absolute convergence of

$$Y_t=\sum_{j=0}^{q-1}\Big(\sum_{k=0}^{j}A_1^{j-k}B_k\Big)Z_{t-j}+\sum_{j=q}^{\infty}A_1^{j-q}\Big(\sum_{k=0}^{q}A_1^{q-k}B_k\Big)Z_{t-j}.$$

By using the Borel-Cantelli-Lemma, we show that only finitely many summands have norm greater than b'^j for $b<b'<1$:

$$\begin{aligned}
&\sum_{j=q}^{\infty}\mathbb{P}\Big[\Big|\Big|A_1^{j-q}\Big(\sum_{k=0}^{q}A_1^{q-k}B_k\Big)Z_{t-j}\Big|\Big|>b'^j\Big]\\
\le\ &\sum_{j=q}^{\infty}\mathbb{P}\Big[ab^{-q}\Big|\Big|\Big(\sum_{k=0}^{q}A_1^{q-k}B_k\Big)Z_{t-j}\Big|\Big|>(\frac{b'}{b})^j\Big]\\
=\ &\sum_{j=q}^{\infty}\mathbb{P}\Big[\log^+\Big(ab^{-q}\Big|\Big|\Big(\sum_{k=0}^{q}A_1^{q-k}B_k\Big)Z_{t-j}\Big|\Big|\Big)>j\log\frac{b'}{b}\Big].
\end{aligned}$$

The last series is finite because of our moment assumption. Hence the series defining Y_t converges almost surely absolutely. Obviously, $(Y_t)_{t\in\mathbb{Z}}$ is a strictly stationary process and one can check that it defines a solution of (2.2).

ii) Necessity and uniqueness: Assume that there is a strictly stationary solution. By iterating the ARMA(1,q) equation (2.2) (see equation (22) in [10]), we have

$$\begin{aligned} Y_t &= \sum_{j=0}^{q-1}\Big(\sum_{k=0}^{j} A_1^{j-k}B_k\Big)Z_{t-j} + \sum_{j=q}^{n-1} A_1^{j-q}\Big(\sum_{k=0}^{q} A_1^{q-k}B_k\Big)Z_{t-j} \\ &\quad + \sum_{j=0}^{q-1} A_1^{n+j-q}\Big(\sum_{k=j+1}^{q} A_1^{q-k}B_k\Big)Z_{t-(n+j)} + A_1^n Y_{t-n}. \end{aligned} \tag{2.6}$$

By taking the limit in probability as $n \to \infty$, the last two summands converge to 0 in probability (as they converge to 0 in distribution), since $(Y_t)_{t\in\mathbb{Z}}$ and $(Z_t)_{t\in\mathbb{Z}}$ are both strictly stationary. Hence we get

$$Y_t = \sum_{j=0}^{q-1}\Big(\sum_{k=0}^{j} A_1^{j-k}B_k\Big)Z_{t-j} + \mathbb{P} - \lim_{n\to\infty}\sum_{j=q}^{n-1} A_1^{j-q}\Big(\sum_{k=0}^{q} A_1^{q-k}B_k\Big)Z_{t-j}.$$

This shows uniqueness. Now assume $0 \notin \sigma(A_1)$. The Itô-Nisio-Theorem is also valid in Banach spaces (see Theorem 6.1 in [28]), therefore we also get almost sure convergence. Hence only finitely many summands may have norm greater than 1, which gives the last inequality of the following (in)equalities by the Borel-Cantelli-Lemma:

$$\begin{aligned} &\sum_{j=q}^{\infty}\mathbb{P}\Big[\Big|\Big|\Big(\sum_{k=0}^{q} A_1^{q-k}B_k\Big)Z_0\Big|\Big| > \underbrace{||A_1^{-1}||}_{>1}{}^{j-q}\Big] \\ \leq &\sum_{j=q}^{\infty}\mathbb{P}\Big[||A_1^{q-j}||\Big|\Big|A_1^{j-q}\Big(\sum_{k=0}^{q} A_1^{-k}B_k\Big)Z_{-j}\Big|\Big| > ||A_1^{q-j}||\Big] \\ = &\sum_{j=q}^{\infty}\mathbb{P}\Big[\Big|\Big|A_1^{j-q}\Big(\sum_{k=0}^{q} A_1^{q-k}B_k\Big)Z_{-j}\Big|\Big| > 1\Big] < \infty. \end{aligned}$$

This gives $\mathbb{E}\log^+\Big|\Big|\Big(\sum_{k=0}^{q} A_1^{q-k}B_k\Big)Z_0\Big|\Big| < \infty$.

Case 2: $\sigma(A_1) \subset \{z \in \mathbb{C} : 1 < |z|\}$ i) Sufficiency: Define

$$\begin{aligned} Y_u &:= -\sum_{j=0}^{q-1} A_1^j\Big(\sum_{k=j+1}^{q} A_1^{-k}B_k\Big)Z_{u-j} - \sum_{j=-\infty}^{-1} A_1^j\Big(\sum_{k=0}^{q} A_1^{-k}B_k\Big)Z_{u-j} \\ &= -\sum_{j=-\infty}^{q-1} A_1^j\Big(\sum_{k=\max(0,j+1)}^{q} A_1^{-k}B_k\Big)Z_{u-j}. \end{aligned}$$

As in case 1, it follow from the moment condition that the defining series Y_u converges almost surely absolutely and by similar calculations it follows that $(Y_u)_{u\in\mathbb{Z}}$ is a strictly stationary solution of (2.2).

ii) Necessity and uniqueness: For obtaining the same result in the case $\sigma(A_1) \subset \{z \in \mathbb{C} : 1 < |z|\}$, one proceeds as in subsection 3.1.2. in [10]: Note that $\sigma(A_1^{-1}) \subset \{z \in \mathbb{C} :$

$0 < |z| < 1\}$ by holomorphic functional calculus. One obtains (see equation (25) in [10])

$$\begin{aligned} Y_u &= -\sum_{j=0}^{q-1} A_1^j \Big(\sum_{k=j+1}^{q} A_1^{-k} B_k \Big) Z_{u-j} - \sum_{j=1}^{n-q} A_1^{-j} \Big(\sum_{k=0}^{q} A_1^{-k} B_k \Big) Z_{u+j} \\ &\quad - \sum_{j=0}^{q-1} A_1^{-n+j} \Big(\sum_{k=0}^{j} A_1^{-k} B_k \Big) Z_{u+n-j} + A_1^{-n} Y_{u+n}. \end{aligned}$$

Again, the last two summands converge to 0 in probability as $n \to \infty$ and

$$\sum_{j=1}^{\infty} A_1^{-j} \Big(\sum_{k=0}^{q} A_1^{-k} B_k \Big) Z_{u+j}$$

has to converge almost surely. This shows uniqueness and gives the same necessary moment condition by the same argument.

Case 3: If there is a strictly stationary solution of (2.2), then (2.3) obviously admits a strictly stationary solution for $i = 1, 2$. Conversely, if there are strictly stationary solutions of (2.3) for $i = 1, 2$, then they are unique and given by (2.4) and (2.5). It is easy to see that $Y_t = S(X_t^1, X_t^2)^T$ defines a strictly stationary solution of (2.2). Hence by the cases 1 and 2, there is a strictly stationary solution, if (and only if in the case $0 \notin \sigma(A_1)$)

$$\mathbb{E} \log^+ ||(\sum_{k=0}^{q} \Lambda_i^{q-k} I_i S^{-1} B_k) Z_0|| < \infty \quad \text{for } i = 1, 2$$

which is in turn equivalent to

$$\begin{aligned} \infty &> \mathbb{E} \log^+ ||(\sum_{k=0}^{q} \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix}^{q-k} S^{-1} B_k) Z_0|| \\ &= \mathbb{E} \log^+ ||(\sum_{k=0}^{q} (S^{-1} A_1 S)^{q-k} S^{-1} B_k) Z_0|| \\ &= \mathbb{E} \log^+ ||(\sum_{k=0}^{q} S^{-1} A_1^{q-k} B_k) Z_0||. \end{aligned}$$

Finally, this is true if and only if $\mathbb{E} \log^+ ||(\sum_{k=0}^{q} A_1^{q-k} B_k) Z_0|| < \infty$ because S is invertible and continuous. The uniqueness and the specific form of the solutions are clear from the cases 1 and 2. □

One can hope that one can also use a continuous operator A_1 with the property that $\lim_{n\to\infty} A_1^n x = 0$ for all $x \in \mathcal{B}$ as we then still have $A_1^n Z \xrightarrow{d} 0$ for a random variable Z and hence $A_1^n Z_{t-n} \xrightarrow{d} 0$ for a strictly stationary process $(Z_t)_{t\in\mathbb{Z}}$. An operator with this property is called *strongly stable*, see [16], Definition 2.1. There are strongly stable operators with spectral radius one. For example, consider a multiplication operator A_1 on $\ell^2(\mathbb{N})$ defined by $A_1(x_0, x_1, \ldots) = (\lambda_0 x_0, \lambda_1 x_1, \ldots)$. Recall that $\sigma(A_1) = \overline{\{\lambda_i, i \in \mathbb{N}\}}$. If we choose a sequence $0 < \lambda_i < 1$ tending to 1, then $r(A_1) = 1$, but A_1 is strongly stable. However, if $r(A_1) = 1$, then $||A_1^n||$ does not converge exponentially fast to zero and hence sufficient $\log^+$-moment conditions cannot be derived in this way. Still, we have the following:

Corollary 2.2.1. *Let $\mathcal{B}$ be a complex separable Banach space and let $(Z_t)_{t\in\mathbb{Z}}$ be an i.i.d. sequence of $\mathcal{B}$-valued random variables. Let A_1 and $B_0, \dots, B_q$ be linear continuous operators in $\mathcal{B}$. Further assume that A_1 is strongly stable. Then the ARMA(1,q) equation (2.2) has a strictly stationary solution $Y = (Y_t)_{t\in\mathbb{Z}}$ if and only if*

$$\mathbb{P} - \lim_{n\to\infty} \sum_{j=q}^{n-1} A_1^{j-q} \Big(\sum_{k=0}^{q} A_1^{q-k} B_k \Big) Z_{t-j} \text{ exists.}$$

If there is a strictly stationary solution, it is unique and is given by

$$Y_t = \sum_{j=0}^{q-1} \Big(\sum_{k=0}^{j} A_1^{j-k} B_k \Big) Z_{t-j} + \mathbb{P} - \lim_{n\to\infty} \sum_{j=q}^{n-1} A_1^{j-q} \Big(\sum_{k=0}^{q} A_1^{q-k} B_k \Big) Z_{t-j}.$$

Proof. The necessity and uniqueness follows as in the proof of Theorem 2.2.1 case 1 since the strong stability implies that $\sum_{j=0}^{q-1} A_1^{n+j-q} \Big(\sum_{k=j+1}^{q} A_1^{q-k} B_k \Big) Z_{t-(n+j)} \xrightarrow{\mathbb{P}} 0$ and $A_1^n Y_{t-n} \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$ as observed above, so that (2.6) gives the desired convergence conditions. The sufficiency follows from the same arguments as given in the proof of Theorem 2.2.1 since convergence in probability is preserved by application of continuous operators. $\square$

An example of a strictly stationary AR(1) process with strongly stable A_1 but with $r(A_1) = 1$ is given in Proposition 2.3.2.

2.2.2 Strictly stationary solutions of ARMA(p,q) equations

We can extend our results to ARMA(p,q) processes by using the representation as an ARMA(1,q) process with new state space $\mathcal{B}^p$ endowed with a suitable norm, e.g. the maximum of the norms of the components. We follow here Section 5.1 of Bosq [4]. If we have the ARMA(p,q) equation $Y_t - A_1 Y_{t-1} - \ldots - A_p Y_{t-p} = B_0 Z_t + B_1 Z_{t-1} + \ldots + B_q Z_{t-q}$, we set

$$\begin{aligned} \widetilde{Y_t} &= (Y_t, \dots, Y_{t-p+1})^T, \\ \widetilde{B_k} &= \begin{pmatrix} B_k & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \\ \widetilde{Z_t} &= (Z_t, 0, \dots, 0)^T. \end{aligned} \tag{2.7}$$

We further define the operator A on $\mathcal{B}^p$ by

$$A = \begin{pmatrix} A_1 & A_2 & \cdots & \cdots & A_p \\ \mathrm{id} & 0 & \cdots & \cdots & 0 \\ 0 & \mathrm{id} & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \mathrm{id} & 0 \end{pmatrix}. \tag{2.8}$$

We can now regard the old ARMA(p,q) equation as the new ARMA(1,q) equation

$$\widetilde{Y_t} - A\widetilde{Y_{t-1}} = \widetilde{B_0}\widetilde{Z_t} + \widetilde{B_1}\widetilde{Z_{t-1}} + \ldots + \widetilde{B_q}\widetilde{Z_{t-q}}. \tag{2.9}$$

Note that $(\widetilde{Z_t})_{t\in\mathbb{Z}}$ is also strict white noise. We now consider the operator-valued polynomial Q defined by

$$Q(z) = z^p \,\mathrm{id} - z^{p-1}A_1 - \cdots - zA_{p-1} - A_p. \tag{2.10}$$

Lemma 2.2.1. *We have*

$$\sigma(A) := \{z \in \mathbb{C} | z\,\mathrm{id} - A \text{ is not invertible}\} = \{z \in \mathbb{C} | Q(z) \text{ is not invertible}\}.$$

Proof. This follows from the proof of Theorem 5.2 p. 130 in Bosq [4]. Bosq states only the inclusion $\subset$ in his proof but his arguments also give equality. $\square$

We now apply this representation to Theorem 2.2.1:

Theorem 2.2.2. *Let $A_1, \ldots, A_p$ and $B_0, \ldots, B_q$ be continuous linear operators in a separable complex Banach space $\mathcal{B}$. Let $(Z_t)_{t\in\mathbb{Z}}$ be an i.i.d. sequence of $\mathcal{B}$-valued random variables. Define Q as in (2.10) and assume that $Q(z)$ is invertible for all $z \in \mathbb{C}$ with $|z| = 1$. Define $\widetilde{B_k}$, $\widetilde{Z_t}$ and A as in (2.7) and (2.8). Then a strictly stationary solution of the ARMA(p,q) equation (2.1) exists if*

$$\mathbb{E}\log^+ \left|\left|\Big(\sum_{k=0}^{q} A^{q-k}\widetilde{B_k}\Big)\widetilde{Z_0}\right|\right| < \infty. \tag{2.11}$$

If $Q(0)$ is also invertible, then this moment condition is also necessary. The stationary solution is unique.

Proof. There is a solution for the ARMA(p,q) equation (2.1) if and only if there is a solution for the rewritten ARMA(1,q) equation (2.9). The spectrum $\sigma(A)$ of A is $\{z \in \mathbb{C}|Q(z)$ is not invertible$\}$. Hence by Theorem 2.2.1, a strictly stationary solution exists if (and only if in the case $Q(0)$ is invertible) (2.11) holds. If there is a solution for the rewritten ARMA(1,q) equation, it is unique and hence so is the corresponding solution of the ARMA(p,q) equation that is given by its first component. $\square$

2.2.3 Representation of stationary solutions as moving averages

The theory of holomorphic functions extends to holomorphic operator-valued functions, see e.g. [13], VII §4. We formulate a generalised Laurent series development for our convenience:

Lemma 2.2.2. *Let f be a holomorphic operator-valued function in the annulus $\{z : R_1 < |z-a| < R_2\}$ with $0 \leq R_1 < R_2 \leq \infty$ and $a \in \mathbb{C}$. Then*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n,$$

where the convergence is absolute. The coefficients are given by

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}} \, dz,$$

where γ is the circle $|z-a| = r$ for any r with $R_1 < r < R_2$. Moreover, this series is unique.

Proof. The proof follows the lines of the usual Laurent series development, see VII Lemma 6.11 in [13]. Actually, Lemma 6.11 in [13] is only formulated for a punctured disc and the resolvent, but the proof of the Laurent Series development 1.11 in [12] only makes use of Cauchy's Theorem and Cauchy's Integral Formula. These are also true for operator-valued holomorphic functions, see [13], §4.1 and §4.2. □

We now give another representation of the unique strictly stationary solution:

Theorem 2.2.3. *Let $(Z_t)_{t\in\mathbb{Z}}$ be strict white noise in a separable complex Banach space $\mathcal{B}$ satisfying $\mathbb{E}\log^+ ||Z_0|| < \infty$. Let $A_1, \ldots, A_p$ and $B_0, \ldots, B_q$ be continuous linear operators and let Q be defined by (2.10). Assume $Q(z)$ is invertible for all $z \in \mathbb{C}$ with $|z| = 1$. Then the unique strictly stationary solution $Y = (Y_t)_{t\in\mathbb{Z}}$ of the ARMA(p,q) equation (2.1) is given by*

$$Y_t = \sum_{k=-\infty}^{\infty} \psi_k Z_{t-k},$$

where the series converges almost surely absolutely and where $(\psi_k)_{k\in\mathbb{Z}}$ denote the coefficients of the Laurent series of $(\mathrm{id} - zA_1 - z^2A_2 - \cdots z^pA_p)^{-1}(B_0 + zB_1 + \ldots + z^qB_q)$ for an annulus containing the unit circle $\mathbb{S}$. The coefficients $(\psi_k)_{k\in\mathbb{Z}}$ are given by

$$\psi_n = \frac{1}{2\pi i} \int_\gamma z^{-n-1} (\mathrm{id} - zA_1 - z^2A_2 - \cdots z^pA_p)^{-1}(B_0 + zB_1 + \ldots + z^qB_q) \, dz.$$

Proof. We first show that $Q(z)^{-1}$ is a holomorphic function on the resolvent set $\rho(A) := \{z \in \mathbb{C} | z\,\mathrm{id} - A \text{ is invertible}\} = \{z \in \mathbb{C} | Q(z) \text{ is invertible}\}$. For that, observe that from the proof of Theorem 5.2 in [4] it holds

$$M(z)(z\,\mathrm{id} - A)N(z) = \begin{pmatrix} \mathrm{id} & 0 & \ldots & 0 & 0 \\ 0 & \mathrm{id} & \ldots & \vdots & \vdots \\ \vdots & & & \mathrm{id} & 0 \\ 0 & \ldots & \ldots & 0 & Q(z) \end{pmatrix} =: R(z).$$

Here $N(z)$ is a suitable upper triangular matrix with diagonal entries id and upper entries of the form z^k id. The matrix M is given by

$$M(z) = \begin{pmatrix} 0 & -\mathrm{id} & 0 & \dots & 0 \\ 0 & 0 & -\mathrm{id} & \dots & 0 \\ \vdots & & & & \\ 0 & \dots & \dots & 0 & -\mathrm{id} \\ \mathrm{id} & Q_1(z) & \dots & \dots & Q_{p-1}(z) \end{pmatrix},$$

where $Q_1(z), \dots, Q_{p-1}(z)$ are operator-valued polynomials. The inverse of M is given by

$$M(z)^{-1} = \begin{pmatrix} Q_1(z) & \dots & \dots & Q_{p-1}(z) & \mathrm{id} \\ -\mathrm{id} & 0 & 0 & \dots & 0 \\ 0 & \ddots & & & \\ 0 & \dots & -\mathrm{id} & 0 & 0 \\ 0 & \dots & \dots & -\mathrm{id} & 0 \end{pmatrix}.$$

Thus $M(z)^{-1}$ is a holomorphic function. The inverse of $N(z)$ can be easily given by Cramer's rule and is hence holomorphic as well. The inverse of $(z\,\mathrm{id} - A)$ is the resolvent of A and it is known that the resolvent is a holomorphic function. Hence $R(z)^{-1}$ is holomorphic on $\rho(A)$ and hence so is $Q(z)^{-1}$.

Now observe that $(\mathrm{id} - zA_1 - z^2A_2 - \cdots z^pA_p)^{-1} = (z^pQ(z^{-1}))^{-1}$. Hence this function is holomorphic on an annulus containing the unit circle. This shows that we can develop $(\mathrm{id} - zA_1 - z^2A_2 - \cdots z^pA_p)^{-1}(B_0 + zB_1 + \ldots + z^qB_q)$ as a Laurent series by Lemma 2.2.2.

Finally, we show that the series defining Y_t converges almost surely absolutely: the Laurent series $\sum_{k=-\infty}^{\infty} z^k\psi_k$ converges absolutely on an annulus containing the unit circle. As in the proof of Theorem 3.1.1 in [7] we find the same exponential decay of $||\psi_k||$ as in the one-dimensional case: the Laurent series is absolutely convergent, hence $\sum_{n=1}^{\infty}(1+\varepsilon)^n\psi_n$ and $\sum_{n=1}^{\infty}(1-\varepsilon)^{-n}\psi_{-n}$ are also absolutely convergent for $\varepsilon > 0$ small enough. Hence there are $a > 0$ and $0 < b < 1$ such that $||\psi_n|| < ab^{|n|}$. The proof of the almost sure absolute convergence is the same as in the proof of Theorem 2.2.1 and one can check that $(Y_t)_{t\in\mathbb{Z}}$ is a strictly stationary solution.

□

Remark 2.2.1. The assumption that $B_0, \dots, B_q$ are continuous operators is in fact not needed. In Theorem 2.2.1 the white noise could be an E-valued sequence, where E is a measure space with $B_k : E \to \mathcal{B}$ being measurable mappings. In Theorem 2.2.3, the mappings can be continuous operators from a normed space $\mathcal{A}$ to $\mathcal{B}$ or also only measurable mappings, if we assume $\mathbb{E}[\log^+ ||B_0Z_t + B_1Z_{t-1} + \ldots + B_qZ_{t-q}||] < \infty$.

If $(Z_t)_{t\in\mathbb{Z}}$ is centred weak white noise, i.e. a sequence of second order centred random variables with constant covariance operator and whose cross covariance operators vanish, then $Y_t = \sum_{k=-\infty}^{\infty}\psi_kZ_{t-k}$ is well-defined and defines a weakly stationary sequence and hence a weakly stationary solution of the ARMA equation. See Definition 2.4 and Definition 3.1 in [4] for the definitions of weak white noise and weak stationarity. Hence we get a corollary which generalises Theorem 3.1 in [4]:

Corollary 2.2.2. *Let $A_1, \ldots, A_p$ and $B_0, \ldots, B_q$ be continuous linear operators in a separable complex Banach space $\mathcal{B}$. Define Q as in (2.10) and assume that $Q(z)$ is invertible for all $z \in \mathbb{C}$ with $|z| = 1$. Let $(Z_t)_{t\in\mathbb{Z}}$ be weak white noise with $\mathbb{E}[Z_0] = 0$. The ARMA(p,q) equation (2.1) then has a unique weakly stationary solution $Y = (Y_t)_{t\in\mathbb{Z}}$ given by*

$$Y_t = \sum_{k=-\infty}^{\infty} \psi_k Z_{t-k},$$

where the series converges almost surely absolutely and in the L^2-sense.

Proof. The existence as an almost surely absolutely convergent series can be established similar to Proposition 3.1.1. in [7]. The sequence $(\sum_{k=-n}^{n} \psi_k Z_{t-k})_{n\in\mathbb{N}}$ is a Cauchy sequence in the space of square integrable random variables as

$$\mathbb{E}|| \sum_{k=-n}^{n} \psi_k Z_{t-k} - \sum_{k=-m}^{m} \psi_k Z_{t-k}||^2 \leq \Big(\sum_{k=-n}^{n} ||\psi_k|| - \sum_{k=-m}^{m} ||\psi_k||\Big)^2 \mathbb{E}||Z_0||^2,$$

which follows from the proof of Theorem 6.1 in [4]. The uniqueness follows along the lines of the proof of Lemma 3.1 in [8]. However, the last argument of the proof uses Slutsky's Lemma. This cannot be applied, but the convergence in probability follows from Tchebychev's inequality. Finally, it it obvious that $(Y_t)_{t\in\mathbb{Z}}$ is weakly stationary and one can check that it is a solution. □

2.3 Examples

2.3.1 Weaker moment conditions if $\sigma(A_1) = \{0\}$

We already know that a $\log^+$-moment is sufficient for the existence of a solution of the AR(1) equation $Y_t - A_1 Y_{t-1} = Z_t$ if $\sigma(A_1) = \{0\}$. We now give examples to show that this sufficient condition is not necessary in this case.

Example 2.3.1. The first example is that A_1 is the zero operator or more generally nilpotent, i.e. there is a power A_1^n that vanishes. The spectral radius is then $r(A_1) = 0$, hence $\sigma(A_1) = \{0\}$. The series $\sum_{n=0}^{\infty} A_1^n Z_{t-n}$ always converges. Hence there is no necessary moment condition.

In the next example, the spectral radius of the operator A_1 vanishes, i.e. $r(A_1) = 0$ and $\sigma(A_1) = \{0\}$ but A_1 is not nilpotent. Recall that such operators are called *quasinilpotent*.

Example 2.3.2. Let $c_0(\mathbb{N})$ denote the space of sequences that converge to zero endowed with the supremum norm. Consider the weighted right shift A_1 on $c_0(\mathbb{N})$ given by $A_1(x_0, x_1, x_2, \ldots) = (0, a_1 x_0, a_2 x_1, \ldots)$ for a monotone sequence $a_i \geq 0$ tending to zero. We then have $||A_1^n|| = a_1 a_2 \cdots a_n$. We set $a_n = \frac{e^{-e^n}}{e^{-e^{n-1}}}$ for $n > 1$ and $a_1 = e^{-e}$, hence $||A_1^n|| = e^{-e^n}$. We claim that the condition $\mathbb{E}[\log^+ \log^+ ||Z_0||] < \infty$ is sufficient for the existence of a solution of the corresponding AR(1) equation and that this moment condition is sharp inasmuch as there cannot be a better sufficient moment condition. What is more, the condition is indeed different from the usual $\log^+$-moment condition in that there are distributions with $\log^+ \log^+$-moments without $\log^+$-moments.

Proof. Sufficiency: We know from the proof of Theorem 2.2.1 that there is a strictly stationary solution for $Y_t - A_1 Y_{t-1} = Z_t$ if $\sum_{n=0}^{\infty} A_1^n Z_{t-n}$ converges almost surely absolutely. Hence it suffices to show that $||A_1^n Z_{t-n}|| > e^{-n}$ only finitely often. We show this by the Borel-Cantelli-Lemma. First note that there is a K such that $\log(e^n - n) = n + \log(1 - ne^{-n}) > n - \frac{1}{2}$ for $n \geq K$. Hence

$$\begin{aligned} & \sum_{n=K}^{\infty} \mathbb{P}[||A_1^n Z_{t-n}|| > e^{-n}] \\ \leq & \sum_{n=K}^{\infty} \mathbb{P}[\log^+ \log^+ ||Z_0|| > \log(e^n - n)] \\ \leq & \sum_{n=K}^{\infty} \mathbb{P}[\log^+ \log^+ ||Z_0|| > n - \frac{1}{2}]. \end{aligned}$$

The last series is finite if $\mathbb{E}[\log^+ \log^+ ||Z_0||] < \infty$. The Borel-Cantelli-Lemma then shows that a strictly stationary solution exists.

Sharpness: We denote the n-th component of Y_t and Z_t by $Y_t^{(n)}$ and $Z_t^{(n)}$. One can show that a solution has to fulfil $Y_t^{(n)} = \sum_{i=0}^{n} (\prod_{j=i+1}^{n} a_j) Z_{t+i-n}^{(i)}$. We also know that $\lim_{n\to\infty} Y_t^{(n)} = 0$ almost surely because the solution is in $c_0(\mathbb{N})$. If we assume that only $Z_t^{(0)}$ is nondeterministic and all other components vanish, then the $\log^+ \log^+$-moment condition is in fact necessary: then the components of $Y_t^{(n)}$ are independent (for fixed t) and we get by the Borel-Cantelli-Lemma, because only finitely many components have absolute value greater than 1, the following inequality. The finiteness of the first series then gives the $\log^+ \log^+$-moment:

$$\begin{aligned} & \sum_{n=1}^{\infty} \mathbb{P}[\log^+ \log^+ |Z_0^{(0)}| > n] \\ = & \sum_{n=1}^{\infty} \mathbb{P}[\log^+ |Z_0^{(0)}| > \underbrace{-\sum_{i=1}^{n} \log a_i}_{=e^n}] \\ = & \sum_{n=1}^{\infty} \mathbb{P}[|Z_0^{(0)}| > \prod_{i=1}^{n} \frac{1}{a_i}] \\ = & \sum_{n=1}^{\infty} \mathbb{P}[|(\prod_{i=1}^{n} a_i) Z_{t-n}^{(0)}| > 1] < \infty. \end{aligned}$$

Finally, let P be a real-valued random variable with Pareto distribution on $[1, \infty)$ with index $\alpha = 1$. Define $Z = xe^P$ where x is a vector in $\mathcal{B}$ with norm 1. Z has $\log^+ \log^+$-moment but no finite $\log^+$-moment, i.e. there is white noise $(Z_t)_{t\in\mathbb{Z}}$ fulfilling these moment conditions.

The spectrum of the operator in the next example contains zero but the $\log^+$ condition is necessary.

Example 2.3.3. Let A_1 be the rescaled right shift operator on $\ell^2(\mathbb{N})$ given by

$$A_1(x_0, x_1, \dots) := \frac{1}{2}(0, x_0, x_1, \dots).$$

Then A_1 is not invertible, thus we have $0 \in \sigma(A_1)$, but A_1 has a left inverse given by $A_1^{-1}(x_0, x_1, \dots) = 2(x_1, x_2, \dots)$. It is known that $\sigma(A_1) = \{z \in \mathbb{C} : |z| \le \frac{1}{2}\}$. The arguments of the proof of Theorem 2.2.1 show that the $\log^+$-condition is necessary.

We now give another example with a different sufficient condition. Let Γ be the gamma function given by $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$ for $z > 0$. We know that $\Gamma(n) = (n-1)!$. Let $K > 0$ be a constant such that Γ is strictly increasing on $[K, \infty)$. We then denote by Γ^{-1} the inverse function of Γ on the interval $[K, \infty)$.

Example 2.3.4. Let $\mathcal{B} = C[0,1]$ be endowed with the supremum norm. Let A_1 be the Volterra integral operator given by $A_1 x(s) = \int_0^s x(t)\,dt$. Then $\mathbb{E}[\Gamma^{-1}(||Z_0|| \vee K)] < \infty$ is a sufficient condition for the existence of a solution for the AR(1) equation given by A_1. This condition is sharp as well.

Proof. We will show that $Y_t := \sum_{n=0}^\infty A_1^n Z_{t-n}$ converges almost surely absolutely and we know that this then defines a solution for the AR(1) equation. We know that $||A_1^n|| = \frac{1}{n!}$, see for example [13] p.217 . It suffices to show that $||\frac{1}{n!}Z_{t-n}|| > \frac{1}{n(n-1)}$ for only finitely many n. We do so by using the Borel-Cantelli-Lemma:

$$\begin{aligned} &\sum_{n=2}^\infty \mathbb{P}\Big[||\frac{1}{n!}Z_0|| > \frac{1}{n(n-1)}\Big] \\ = &\sum_{n=2}^\infty \mathbb{P}[||Z_0|| > (n-2)!] \\ \le &\sum_{n=2}^\infty \mathbb{P}[||Z_0|| \vee K > (n-2)!] \\ \le &\sum_{n=2}^\infty \mathbb{P}[\Gamma^{-1}(||Z_0|| \vee K) > (n-1)]. \end{aligned}$$

The last series is finite because of the moment asumption. Finally, we show that this condition cannot be improved: let $(\widetilde{Z}_t)_{t\in\mathbb{Z}}$ be $\mathbb{C}$-valued i.i.d. white noise and define $Z_t = 1_{[0,1]}\widetilde{Z}_t$. Now we assume that there is a solution Y_t. We evaluate this solution as a function at 1, i.e. $\widetilde{Y}_t = Y_t(1) = \sum_{n=0}^\infty \frac{1}{n!}\widetilde{Z_{t-n}}$. This series necessarily converges if there is a solution Y_t. Hence again by the Borel-Cantelli-Lemma, we get $\sum_{n=0}^\infty \mathbb{P}[|\widetilde{Z_{t-n}}| > n!] < \infty$, thus $\sum_{n=0}^\infty \mathbb{P}[|\widetilde{Z_0}| \vee K > n!] < \infty$ and $\sum_{n=0}^\infty \mathbb{P}[\Gamma^{-1}(|\widetilde{Z_0}| \vee K) > n+1] < \infty$. This shows that under this choice of white noise, the moment condition is necessary. ☐

Now, we give an example of a distribution with Γ^{-1}-moment but without $\log^+$-moment: It can be shown that $\Gamma^{-1}(x)$ behaves asymptotically like $\frac{\log x}{\log\log x}$: by Stirling's formula, $\Gamma(y)$ behaves asymptotically like $x := \sqrt{y}(\frac{y}{e})^y$. Taking the logarithm, we obtain

$$\log\sqrt{y} + y(\log y - 1) = \log x =: z.$$

By using the ansatz $y = \frac{z}{\log z} r(z)$ and inserting it in the last equation, one can show that $\lim_{z\to\infty} r(z) = 1$, giving the desired asymptotic behaviour of $\Gamma^{-1}(x)$ as $x \to \infty$.

Now, define $f(x) := \frac{1}{x(\log x)^2(\log\log x)}$. Then, for large x_1

$$\int_{x \geq x_1} f(x) \log x \, dx = \int_{x\geq x_1} \frac{1}{x \log x \log\log x} \, dx$$

$$= \int_{y \geq \log x_1} \frac{1}{e^y y \log y} e^y \, dy = \int_{y\geq \log x_1} \frac{1}{y \log y} \, dy$$

and

$$\int_{x\geq x_1} f(x) \frac{\log x}{\log\log x} \, dx = \int_{y \geq \log x_1} \frac{1}{y(\log y)^2} \, dy.$$

Antiderivatives of the integrands are $\log\log y$ and $\frac{-1}{\log y}$. Hence by restricting f on an interval $[x_1, \infty)$ and normalising it, we obtain a density that defines a distribution with the desired properties of having finite Γ^{-1}-moment without having $\log^+$-moment.

We can also give an example of an operator on $C[0,1]$ comparable to the operator in Example 2.3.2:

Example 2.3.5. Define the operator $T : C[0,1] \to C[0,1]$ for $f \in C[0,1]$ by

$$Tf(x) := \begin{cases} 2xf(0), & \text{if } x \in [0, \frac{1}{2}] \\ f(2(x-\frac{1}{2})), & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

For a monotone sequence of positive numbers $(a_n)_{n\geq 1}$ converging to zero, we define $g \in C[0,1]$ by setting $g(0) := a_1$, $g(\frac{1}{2}) := a_2$, $g(\frac{3}{4}) := a_3$ and $g(\sum_{i=1}^n (\frac{1}{2})^i) = a_{n+1}$ in general, interpolating linearly in between and setting $g(1) := 0$. The function g is continuous in 1 because $\lim a_n = 0$. Now define $S : C[0,1] \to C[0,1]$ by setting $Sf := fg$ for $f \in C[0,1]$. Finally we define $A_1 := TS$. Then one can show that $||A_1^n|| = a_1 a_2 \cdots a_n$. Like in Example 2.3.2, one can show that $\mathbb{E}[\log^+ \log^+ ||Z_0||] < \infty$ is sufficient for the existence of a solution, if we set $a_n = \frac{e^{-e^n}}{e^{-e^{n-1}}}$ for $n > 1$ and $a_1 = e^{-e}$. Like in that example, one can show the sharpness of this condition: Let $(\widetilde{Z}_t)_{t\in\mathbb{Z}}$ be $\mathbb{C}$-valued i.i.d. white noise and define $Z_t(x) = (1 - 10x)1_{[0,\frac{1}{10}]}(x)\widetilde{Z}_t$ for $x \in [0,1]$. Let $(Y_t)_{t\in\mathbb{Z}}$ be a solution of the corresponding AR(1) equation. We know that a solution has to fulfil $Y_t = \sum_{n=0}^\infty A_1^n Z_{t-n}$. We see that $Y_t(1) = 0$ because $(A_1^n Z_t)(1) = 0$ for all $n \geq 0$ by our choice of white noise. Because of that and since we chose $C[0,1]$ as our state space, we have $\lim_{n\to\infty} Y_t(\sum_{i=1}^n (\frac{1}{2})^i) = 0$. Like in Example 2.3.2, one can show that $\mathbb{E}[\log^+ \log^+ ||Z_0||] < \infty$ is also necessary in this case: We have $Y_t(\sum_{i=1}^n (\frac{1}{2})^i) = \left(\prod_{j=1}^n a_j\right) \widetilde{Z}_{t-n}$. Hence for fixed t, $(Y_t(\sum_{i=1}^n (\frac{1}{2})^i))_{n\in\mathbb{N}}$ is an i.i.d. sequence whose absolute value is only finitely often greater than 1. By using the Borel-Cantelli-Lemma, one can show that $\mathbb{E}[\log^+ \log^+ ||Z_0||] < \infty$.

□

2.3.2 Examples when $\sigma(A_1)$ is the closed unit disc

We consider two further examples. In the first one the operator is an isometry and there is no nondeterministic solution, while in the second one the operator is a multiplication operator and there is a nondeterministic solution.

For the proof of the next proposition, we need some preparation: a family of random variables $(X_i)_{i\in I}$ is said to *bounded in probability*, if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\mathbb{P}[||X_i|| > \delta] < \varepsilon \text{ for all } i \in I.$$

Recall that if the partial sums S_n of independent random variables in a separable space are bounded in probability, then they are also almost surely bounded, see [41], Theorem V.2.2. b) p. 266. Note that [41] define boundedness in probability by boundedness with respect to a metric space that induces convergence in probability. Then they show that it coincides with this definition (Proposition III.1.2. p. 93). Recall that almost sure boundedness means that the supremum is almost surely finite, i.e. $\mathbb{P}[\sup_n ||S_n|| < \infty] = 1$. Finally, recall that a Banach space $\mathcal{B}$ does not contain subspaces being isomorphic to $c_0(\mathbb{N})$ if and only if for every sequence of independent symmetric Radon random variables in $\mathcal{B}$, the almost sure boundedness of the sequence of the partial sums S_n implies its convergence, see [28], Theorem 9.29.

Proposition 2.3.1. *Let A_1 be an isometry and $\mathcal{B}$ be a separable Banach space, that does not contain a subspace being isomorphic to $c_0(\mathbb{N})$, e.g. a Hilbert space. Then a necessary condition for the existence of a strictly stationary solution for the AR(1) equation $Y_t - A_1Y_{t-1} = Z_t$ with $(Z_t)_{t\in\mathbb{Z}}$ i.i.d. is that Z_0 is almost surely deterministic.*

Proof. We will mimic the argument of the proof of Theorem 1 in [9] c). We may symmetrise the processes by taking the differences of two independent copies of the white noise and the solution. We then show that Z_0 vanishes almost surely. We now assume that the white noise and the solution are symmetric. We again obtain

$$Y_t - A_1^n Y_{t-n} = \sum_{k=0}^{n-1} A_1^k Z_{t-k}.$$

From the stationarity and the fact that A_1^n are isometries, for each $\varepsilon > 0$ there is a $K > 0$ such that

$$\mathbb{P}[||Y_t - A_1^n Y_{t-n}|| > K] < \varepsilon$$

as we have

$$\mathbb{P}[||Y_t - A_1^n Y_{t-n}|| > K] \leq \mathbb{P}[\{||Y_t|| > \frac{K}{2}\} \cup \{||A^n Y_t|| > \frac{K}{2}\}] \leq 2\mathbb{P}[||Y_t|| > \frac{K}{2}].$$

Hence

$$\mathbb{P}[||\sum_{k=0}^{n-1} A_1^k Z_{t-k}|| > K] < \varepsilon$$

and the $\sum_{k=0}^{n-1} A_1^k Z_{t-k}$ are bounded in probability, hence almost surely bounded and thus $\sum_{k=0}^{\infty} A_1^k Z_{t-k}$ converges almost surely (by the theorems refered to above). We now have

$$\sum_{k=0}^{\infty} \mathbb{P}[||Z_0|| > r] = \sum_{k=0}^{\infty} \mathbb{P}[||A_1^k Z_{t-k}|| > r] < \infty \quad \text{for all } r > 0.$$

Thus Z_0 vanishes almost surely. □

Example 2.3.6. Let $\mathcal{B}$ be $\ell^\infty(\mathbb{N})$ and let A_1 be the unilateral right shift defined by $A_1(x_0, x_1, \ldots) = (0, x_1, x_2, \ldots)$. We set $Z_t = (Z_t^{(0)}, Z_t^{(1)}, \ldots)$ and $Y_t = (Y_t^{(0)}, Y_t^{(1)}, \ldots)$. Then $Y_t^{(n)} := \sum_{i=0}^{n} Z_{t+i-n}^{(i)}$ is a stationary solution as long as Y_t is in $\ell^\infty(\mathbb{N})$. If we assume $\sup_{\omega \in \Omega} \sup_{t \in \mathbb{Z}} |Z_t^{(i)}| \leq 2^{-i}$, then the Y_t^i are always bounded by 2, hence a strictly stationary solution exists though the shift is an isometry. This example does not contradict our proposition above as we have $c_0(\mathbb{N}) \subset \ell^\infty(\mathbb{N})$ and $\ell^\infty(\mathbb{N})$ is not separable.

We again consider the AR(1) equation in the case that A_1 is a multiplication operator given by $A_1(x_0, x_1, \ldots) = (\lambda_0 x_0, \lambda_1 x_1, \ldots)$ on $\ell^2(\mathbb{N})$. We set $Z_t = (Z_t^{(0)}, Z_t^{(1)}, \ldots)$ and $Y_t = (Y_t^{(0)}, Y_t^{(1)}, \ldots)$. Define $\sigma_i^2 = \operatorname{Var}(Z_t^{(i)})$. If we consider a component of a solution of the AR(1) equation $Y_t - A_1 Y_{t-1} = Z_t$, we have $Y_t^{(n)} = \sum_{i=0}^{\infty} \lambda_n^i Z_{t-i}^{(n)}$, hence $\operatorname{Var}(Y_t^{(n)}) = \sum_{i=0}^{\infty} (\lambda_n^i)^2 \operatorname{Var}(Z_{t-i}^{(n)}) = \frac{\sigma_n^2}{1-\lambda_n^2}$. If $\sum_{n=0}^{\infty} \operatorname{Var}(Y_t^{(n)})$ is finite, then Y_t is almost surely in $\ell^2(\mathbb{N})$ and we choose a modification of Y that is in $\ell^2(\mathbb{N})$. Hence we get:

Proposition 2.3.2. *Let A_1 be a multiplication operator in $\ell^2(\mathbb{N})$ defined by a sequence $(\lambda_i)_{i \in \mathbb{N}}$ with $|\lambda_i| < 1$ for all i. Let $(Z_t)_{t \in \mathbb{Z}}$ be i.i.d. and with the notation introduced above assume $\sum_{i=0}^{\infty} \sigma_i^2 < \infty$. Then there is a solution for the AR(1) equation given by A_1 if*

$$\sum_{i=0}^{\infty} (\frac{\sigma_i^2}{1 - \lambda_i^2}) < \infty.$$

If the white noise is Gaussian, then this condition is also necessary. In particular, $\mathbb{E}[||Z_t||^2] < \infty$ is not a sufficient condition for the existence of a strictly stationary solution.

Proof. The sufficiency has already been observed above. To show necessity when $(Z_t)_{t \in \mathbb{Z}}$ is Gaussian, consider only the first $n+1$ components of the AR(1) equation. For this multivariate equation, there is always a solution and the distribution of the solution $(Y_t^{(0)}, \ldots, Y_t^{(n)})$ is Gaussian. If Y_t is well-defined in its state space $\ell^2(\mathbb{N})$, then its distribution is Gaussian, because every linear functional of Y_t is an almost sure limit of the linear functionals of $(Y_t^{(0)}, \ldots, Y_t^{(n)})$. These are normal distributed and the normal distributions are closed under convergence in distribution. The diagonal entries of the covariance operator of the distribution of Y_t are given by $\frac{\sigma_i^2}{1-\lambda_i^2}$. Recall that the diagonal entries of a covariance operator of a Gaussian measure on $\ell^2(\mathbb{N})$ have to be summable, see [41], Theorem V.5.6 p. 334. □

It is well-known that the spectrum of the unilateral right shift R in $\ell^2(\mathbb{N})$ has the closed unit disc as spectrum. It is also well-known that the spectrum of the multiplication operator M is $\overline{\{\lambda_i : i \in \mathbb{N}\}}$. We can now choose a sequence $(\lambda_i)_{i\in\mathbb{N}}$ with $|\lambda_i| < 1$, such that $\sigma(R) = \sigma(M)$. The shift is an isometry, hence we already know that the corresponding AR(1) equation has no strictly stationary solution but we also know that in the case of the multiplication operator M we have a solution, if we choose a priori appropiate white noise.

The operator in the next example has the disc with radius 2 as spectrum.

Example 2.3.7. Let A_1 be the rescaled right shift operator on $\ell^2(\mathbb{N})$ given by

$$A_1(x_0, x_1, \ldots) := 2(0, x_0, x_1, \ldots).$$

Then A_1 has a left inverse given by $A_1^{-1}(x_0, x_1, \ldots) = \frac{1}{2}(x_1, x_2, \ldots)$. There is no solution for the corresponding AR(1) equation if Z_0 is nondeterministic.

Proof. We mimic the arguments given in [7] p. 81 for the noncausal univariate AR(1) equation. If there is a solution for the AR equation, we multiply the equation with A_1^{-1} and get $A_1^{-1}X_t - X_{t-1} = A_1^{-1}Z_t$ and rearrange it to $X_t = -A_1^{-1}Z_{t+1} + A_1^{-1}X_{t+1}$. Iteration gives $X_t = -A_1^{-1}Z_{t+1} - \cdots - A_1^{-n}Z_{t+n} + A_1^{-n}X_{t+n}$. By taking the limit in distribution and applying again the Itô-Nisio-Theorem one sees that if there is a solution for the corresponding AR(1) equation for A_1, then it is unique and fulfils $X_t = -\sum_{j=1}^{\infty} A_1^{-j} Z_{t+j}$ as an almost sure limit. Hence on the one hand, we have $X_t^{(0)} = -\sum_{j=1}^{\infty} 2^{-j} Z_{t+j}^{(j)}$. On the other hand we also have $X_t^{(0)} = Z_t^{(0)}$ but this is impossible unless Z_0 is deterministic. □

2.4 Summary

The aim of this chapter was to find necessary and sufficient conditions for the existence of strictly stationary solutions of ARMA equations in Banach spaces with i.i.d. white noise.

Firstly, we generalised a result on multivariate ARMA(1,q) processes by Brockwell, Lindner and Vollenbröker [10] by excluding the unit circle and zero from the spectrum of the operator in the autogressive part. Secondly, we extended this to ARMA(p,q) processes by using their representations as ARMA(1,q) processes. Thirdly, we gave an additional representation of the solution by using Laurent series. Finally, we provided various examples illustrating what can happen when we drop our assumptions. These examples show that a complete characterisation of the existence and uniqueness of strictly stationary solutions by the spectrum and moment conditions is not possible.

Our work extends the results in the book by Bosq [4] in that we allow for noncausal solutions, a broader class of operators, white noise without finite second moments and a moving average part.

Chapter 3

A central limit theorem for the sample autocovariance of a continuous-time moving average process with long memory

Based on [39]: F. Spangenberg A central limit theorem for the sample autocovariance of a continuous-time moving average process with long memory, Preprint 2015[1]

Abstract. We examine the asymptotic behaviour of the sample autocovariance in a continuous-time moving average model with long-range dependence. We show that it is either asymptotically Rosenblatt distributed or stable distributed. This shows that results by Horváth and Kokoszka [23] for discrete-time moving average processes with long memory also hold for continuous-time moving average processes.

3.1 Introduction

Let $(Z_t)_{t\in\mathbb{Z}}$ be an i.i.d. sequence of real random variables with $\mathbb{E}[Z_0] = 0$ and $\mathbb{E}[Z_0^2] = \sigma^2 < \infty$. Let $(\psi_j)_{j\in\mathbb{N}_0}$ be a square summable real sequence. Then a one-sided discrete-time moving average process of infinite order $(X_t)_{t\in\mathbb{Z}}$ is defined by

$$X_t := \sum_{j=0}^{\infty} \psi_j Z_{t-j}, \quad t \in \mathbb{Z},$$

where the limit exists in the L^2-sense and here as a sum of independent summands almost surely as well. The autocovariances of the process are given by

$$\gamma(h) := \operatorname{Cov}(X_t, X_{t+h}) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|h|}, \quad h \in \mathbb{Z}.$$

[1]Supported by Deutsche Forschungsgemeinschaft Grant LI 1026/4-2

A canonical estimator for the autocovariances is given by

$$\hat{\gamma}_N(h) := \frac{1}{N}\sum_{t=1}^{N} X_t X_{t+|h|}, \quad h \in \mathbb{Z}.$$

Horváth and Kokoszka [23] examined the asymptotic behaviour of this estimator under the assumption that $\psi_j = j^{d-1}l(j)$, with $d \in (0, \frac{1}{2})$, $l(j) \to C_d > 0$ as $j \to \infty$ and that either $\mathbb{E}[Z_0^4] = \eta\sigma^4 < \infty$ or that Z_0 is regularly varying with index $\alpha \in (2, 4)$. In the case $d \in (0, \frac{1}{4})$, they showed that the estimator is asymptotically normal distributed. In the case with $d \in (\frac{1}{4}, \frac{1}{2})$ and finite fourth moments, the estimator is asymptotically Rosenblatt distributed and in the case with regularly varying noise, the estimator is asymptotically Rosenblatt distributed when $d > \frac{1}{\alpha}$ and asymptotically stable distributed when $d < \frac{1}{\alpha}$.

Let $(L_t)_{t\in\mathbb{R}}$ be a two-sided Lévy process with $\mathbb{E}[L_1] = 0$ and $\mathbb{E}[L_1^2] = \sigma^2 \in (0, \infty)$. Let f be a real-valued function in $L^2(\mathbb{R})$. Then a continuous-time moving average process $(X_t)_{t\in\mathbb{R}}$ is defined by

$$X_t := \int_{-\infty}^{\infty} f(t-s)\, dL_s, \quad t \in \mathbb{R}, \tag{3.1}$$

where the integral is defined in the L^2-sense of stochastic integrals. This process is easily seen to be strictly and weakly stationary. The autocovariance can be easily seen by Itô's isometry to be

$$\gamma(h) = \sigma^2 \int_{-\infty}^{\infty} f(s)f(s+h)\, ds, \quad h \in \mathbb{R}.$$

Cohen and Lindner [11] investigated the asymptotic behaviour of the sample mean, the sample autocovariance and the sample autocorrelation for continuous-time moving average processes under certain conditions. Among other things, they showed that the sample autocovariance is asymptotically normal distributed in their case.

The aim of this chapter is to derive the asymptotic behaviour of the sample autocovariance in the continuous-time case under assumptions similar to those in the article by Horváth and Kokoszka [23]. To this end, we assume that $f(t) = 0$ for $t \leq 0$, f is bounded and $f(t) \sim C_d t^{d-1}$ as $t \to \infty$ with $d \in (0, \frac{1}{2})$ and $C_d > 0$.

A class of discrete-time processes that are of the above form is given by ARFIMA processes, see for example Chapter 7 in Giraitis et al. [21]. A class of continuous-time processes of the above form is given by FICARMA processes, which go back to Peter Brockwell, see [5] and [6].

In this chapter, we stick to the notation of Horváth and Kokoszka [23]. They define the Rosenblatt process $(U_d(t))_{t\in\mathbb{R}}$ by

$$U_d(t) := 2\int_{x_1<x_2<t} \Big[\int_0^t (v-x_1)_+^{d-1}(v-x_2)_+^{d-1}\, dv\Big]\, W(dx_1)W(dx_2), \quad t \in \mathbb{R}, \tag{3.2}$$

where W is a standard Gaussian random measure on $\mathbb{R}$, i.e. standard Brownian motion. Note that this definition depends on d. We call the distribution of $U_d(1)$ the Rosenblatt distribution. For an introduction to multiple Wiener integrals, see Chapter 14 in [21].

We show in the second section under the assumption that $\mathbb{E}[L_1^4] = \eta\sigma^4 < \infty$ and $d \in (\frac{1}{4}, \frac{1}{2})$, that the sample autocovariance is asymptotically Rosenblatt distributed. This is in contrast to the case when $\mathbb{E}[L_1^4] = \eta\sigma^4 < \infty$ and $d \in (0, \frac{1}{4})$ in which the sample autocovariance function is asymptotically normal distributed, as follows easily from results of Cohen and Lindner [11] and is shortly discussed in section 4. We show in section 3 under the assumption that L_1 is regularly varying with index $\alpha \in (2,4)$, that it is aymptotically either stable or Rosenblatt distributed if $d \in (0, \frac{1}{2}) \setminus \{\frac{1}{\alpha}\}$. In the case with regularly varying L_1, we have to restrict ourselves to symmetric Lévy processes, but we believe that this assumption is not too severe as we already assumed that its expectation vanishes. In section 4, we further discuss briefly that FICARMA processes satisfy the assumptions on the kernel function and that our results can also be applied to calculate the asymptotics of the sample autocorrelation and the asymptotics for an estimator for d for fractional Lévy noise.

We conclude the introduction with some (notational) remarks. From the assumptions on f, we can conclude that there is a constant $K > 0$ such that

$$|f(t)| \leq K \max(1, t^{d-1}), \tag{3.3}$$

which we use throughout the chapter. Further, we assume that we are given a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. By $\mathbb{E}$ and Var, we denote the expectation and variance with respect to $\mathbb{P}$. By $L^1(\mathbb{P})$ and $L^2(\mathbb{P})$, we denote the Banach spaces of integrable and square integrable random variables, by $L^2(\mathbb{R}^d)$, we denote the space of square integrable functions with respect to the Lebesgue measure. By $\lfloor x \rfloor$ for $x \in \mathbb{R}$, we denote the largest integer that is not larger than x. We define $x_+^{d-1} := x^{d-1} 1_{(0,\infty)}(x)$ for $x \in \mathbb{R}$. By g, G, u, v and w we will denote auxiliary functions. We call a series unconditionally convergent, if its limit does not depend on the order of summation. By $h \in \mathbb{N}_0$, we denote the lag of the autocovariance function. Since the autocovariance function is symmetric, it suffices to assume $h \in \mathbb{N}_0$. We set $\varepsilon := \frac{1}{m}$ with $m \in \mathbb{N}$. Finally, when we use $o(N^d)$ and $O(N^d)$, we mean the asymptotics as $N \to \infty$.

3.2 Theorem for finite fourth moments

In this section we assume that the Lévy process $(L_t)_{t\in\mathbb{R}}$ has finite fourth moments and that $d \in (\frac{1}{4}, \frac{1}{2})$. Define the sample and the actual autocovariance function of the process $(X_t)_{t\in\mathbb{R}}$ defined in (3.1) by

$$\hat{\gamma}_N(h) := \frac{1}{N} \sum_{t=1}^{N} X_t X_{t+h}, \quad h \in \mathbb{N}_0$$

and

$$\gamma(h) := \mathrm{Cov}(X_t, X_{t+h}) = \sigma^2 \int_{-\infty}^{\infty} f(t) f(t+h)\, dt, \quad h \in \mathbb{N}_0.$$

We show that the sample autocovariance is then asymptotically Rosenblatt distributed. This is the statement of the following theorem, which parallels Theorem 3.3 (b) in [23].

Theorem 3.2.1. *Let $(L_t)_{t\in\mathbb{R}}$ be a two-sided Lévy process with $\mathbb{E}[L_1] = 0$, $\mathrm{Var}(L_1) = \sigma^2 \in (0,\infty)$ and $\mathbb{E}[L_1^4] = \eta\sigma^4 < \infty$. Define a continuous-time moving average process $(X_t)_{t\in\mathbb{R}}$*

by

$$X_t := \int_{-\infty}^{\infty} f(t-s)\,dL_s, \quad t \in \mathbb{R},$$

where we assume that $f(t) = 0$ *for* $t \leq 0$, f *is bounded and* $f(t) \sim C_d t^{d-1}$ *as* $t \to \infty$ *with* $d \in (\frac{1}{4}, \frac{1}{2})$ *and* $C_d > 0$. *Then*

$$N^{1-2d}(\hat{\gamma}_N(0) - \gamma(0), \ldots, \hat{\gamma}_N(H) - \gamma(H)) \xrightarrow{d} C_d^2 \sigma^2 U_d(1)(1, \ldots, 1) \text{ as } N \to \infty,$$

where $U_d(1)$ *is the marginal distribution of the Rosenblatt process at time* 1 *defined in (3.2).*

The rest of this section is devoted to the proof of Theorem 3.2.1. Our proof consists of two parts. In the first part, we approximate f by

$$f_m := \sum_{k=0}^{\infty} f(\varepsilon k) 1_{[k\varepsilon,(k+1)\varepsilon)},$$

where $m \in \mathbb{N}$ and $\varepsilon := \frac{1}{m}$. We then show that the *non-diagonal terms* of the sample autocovariance function which we denote by $r_{N,h,\varepsilon}$ and are defined in (3.9) converge towards the Rosenblatt distribution, which is the statement of Lemma 3.2.2. We then use Slutsky's Lemma in the second part to show the asymptotics of the sample autocovariance for general kernel functions.

Define

$$Z_i := L_{\varepsilon i} - L_{\varepsilon(i-1)}, \quad i \in \mathbb{Z}. \tag{3.4}$$

Observe that $\mathbb{E}[Z_i] = 0$ and $\mathbb{E}[Z_i^2] = \varepsilon\sigma^2$. Define

$$X_t^{(m)} := \int_{-\infty}^{\infty} f_m(t-s)\,dL_s = \sum_{k=0}^{\infty} f(\varepsilon k)(L_{t-\varepsilon k} - L_{t-\varepsilon(k+1)}) = \sum_{k=0}^{\infty} f(\varepsilon k) Z_{mt-k}, \quad t \in \mathbb{Z},$$

where the two series converge as series of independent, hence orthogonal elements in the Hilbert space $L^2(\mathbb{P})$, unconditionally in $L^2(\mathbb{P})$.

We divide the summands of the sample autocovariance in diagonal terms and non-diagonal terms with respect to products of $(Z_t)_{t\in\mathbb{Z}}$. To this end, we need the following technical lemma:

Lemma 3.2.1. *Let* A, B *and* C *be random variables. Let* $(A_u)_{u\in\mathbb{Z}}$ *and* $(B_v)_{v\in\mathbb{Z}}$ *be sequences of random variables such that* $A = \sum_{u\in\mathbb{Z}} A_u$ *and* $B = \sum_{v\in\mathbb{Z}} B_v$ *as unconditional* $L^2(\mathbb{P})$ *limits. Assume further that* $C = \sum_{u,v\in\mathbb{Z}} A_u B_v$ *as an unconditional* $L^1(\mathbb{P})$ *limit. Then* $AB = C$ *almost surely.*

Proof. Let $\delta > 0$. By our assumptions, we find an $N \in \mathbb{N}$ such that $||A - \sum_{|u|\leq N} A_u||_2 < \delta$,

$||B - \sum_{|v|\le N} B_v||_2 < \delta$ and $||C - \sum_{|u|,|v|\le N} A_u B_v||_1 < \delta$. We then obtain

$$\begin{aligned}
||C - AB||_1 &\le ||C - \sum_{|u|,|v|\le N} A_u B_v||_1 \\
&\quad + ||\sum_{|u|,|v|\le N} A_u B_v - (\sum_{|u|\le N} A_u)B||_1 + ||(\sum_{|u|\le N} A_u)B - AB||_1 \\
&\le \delta + ||(\sum_{|u|\le N} A_u)(\sum_{|v|\le N} B_v - B)||_1 + ||(\sum_{|u|\le N} A_u - A)B||_1 \\
&\le \delta + ||\sum_{|u|\le N} A_u||_2 ||\sum_{|v|\le N} B_v - B||_2 + ||\sum_{|u|\le N} A_u - A||_2 ||B||_2 \\
&\le \delta + (||A||_2 + \delta)\delta + \delta ||B||_2.
\end{aligned}$$

□

Now the following rearrangement is justified by Lemma 3.2.1, once we have shown that the right-hand side converges unconditionally in $L^1(\mathbb{P})$. Let $h \in \mathbb{N}_0$. Then

$$X_t^{(m)} X_{t+h}^{(m)} = (\sum_{i=0}^{\infty} f(\varepsilon i) Z_{mt-i})(\sum_{j=0}^{\infty} f(\varepsilon j) Z_{m(t+h)-j}) \tag{3.5}$$

$$= \sum_{i=0}^{\infty} f(\varepsilon i) f(\varepsilon i + h)(Z_{mt-i})^2 + \sum_{j\ne i+mh, i,j\in\mathbb{N}_0} f(\varepsilon i) f(\varepsilon j) Z_{mt-i} Z_{m(t+h)-j}. \tag{3.6}$$

Observe that $f(x)$ vanishes for $x \le 0$, hence the sum can also be taken over $i, j \in \mathbb{Z}$. Note that $\sum_{i=0}^{\infty} f(\varepsilon i) f(\varepsilon i + h)(Z_{mt-i})^2$ converges absolutely almost surely and unconditionally in $L^1(\mathbb{P})$ because $\sum_{i=0}^{\infty} f(\varepsilon i) f(\varepsilon i + h)$ is absolutely summable and $(Z_{mt-i})^2$ has finite expectation. Setting $k = mt - i$ and $k' = m(t+h) - j$, the last summand can be rewritten as

$$\sum_{k\ne k', k,k'\in\mathbb{Z}} f(t - \varepsilon k) f(t + h - \varepsilon k') Z_k Z_{k'}.$$

We decompose this series as

$$\sum_{k\ne k', k,k'\in\mathbb{Z}} f(t - \varepsilon k) f(t + h - \varepsilon k') Z_k Z_{k'} \tag{3.7}$$

$$= \sum_{k<k'} f(t - \varepsilon k) f(t + h - \varepsilon k') Z_k Z_{k'} + \sum_{k>k'} f(t - \varepsilon k) f(t + h - \varepsilon k') Z_k Z_{k'}. \tag{3.8}$$

Since $(Z_k)_{k\in\mathbb{Z}}$ is i.i.d. with expectation zero and finite variance, both $(Z_k Z_{k'})_{k<k'}$ and $(Z_k Z_{k'})_{k>k'}$ are families of orthogonal elements in $L^2(\mathbb{P})$ with constant variance and since $(f(t - \varepsilon k) f(t + h - \varepsilon k'))_{k\ne k'}$ is square summable by assumption, the two series in (3.8) converge unconditionally in $L^2(\mathbb{P})$ and thus in $L^1(\mathbb{P})$ as well, hence the series in (3.7) can be seen to converge unconditionally as well. Hence (3.6) converges unconditionally in $L^1(\mathbb{P})$ and hence by Lemma 3.2.1 (3.5) and (3.6) are equal. Now define

$$r_{N,h,\varepsilon} := \frac{1}{N}\sum_{t=1}^{N}\sum_{k\neq k'} f(t-\varepsilon k)f(t+h-\varepsilon k')Z_k Z_{k'}, \quad h \in \mathbb{N}_0. \tag{3.9}$$

Thus $r_{N,h,\varepsilon}$ represents all non-diagonal terms of the sample autocovariance.

The following lemma is a generalisation of Lemma 5.5 in [23]. Note that our proof is somewhat easier as we refer to results in [21]. In fact, we only need the result for the case $\varepsilon = 1$ in this section, which follows from Lemma 5.5 in [23], but we need the result for general $\varepsilon > 0$ in section 3 and we state it already here for convenience.

Lemma 3.2.2. *Let all assumptions of Theorem 3.2.1 apart from $\mathbb{E}[L_1^4] = \eta\sigma^4 < \infty$ be fulfilled. Let $H \in \mathbb{N}_0$. Then*

$$N^{1-2d}(r_{N,0,\varepsilon},\ldots,r_{N,H,\varepsilon}) \xrightarrow{d} C_d^2\sigma^2 U_d(1)(1,\ldots,1) \text{ as } N\to\infty.$$

Proof. Without loss of generality, we can assume that $\sigma^2 = 1$ and $C_d = 1$.

Define

$$g(x_1,x_2) := \int_0^1 (t-\varepsilon x_1)_+^{d-1}(t-\varepsilon x_2)_+^{d-1}\,dt. \tag{3.10}$$

One can show that $g \in L^2(\mathbb{R}^2)$ in a similar fashion to equation (14.3.38) on page 544 in Giraitis et al. [21].

We will use Propositions 14.3.2 and 14.3.3 in Giraitis et al. [21]. To this end, we define

$$g_N(k,k') = N^{-2d}\sum_{t=1}^{N} f(t-k\varepsilon)f(t+h-\varepsilon k') \tag{3.11}$$

for $k \neq k'$ and $g_N(k,k) = 0$ with $k,k' \in \mathbb{Z}$ and

$$\tilde{g}_N(x_1,x_2) = N g_N(\lfloor Nx_1\rfloor, \lfloor Nx_2\rfloor), \quad x_1,x_2\in\mathbb{R}. \tag{3.12}$$

Then $\frac{Z_k}{\varepsilon}$ corresponds to ζ_k and $N^{1-2d}r_{N,h,\varepsilon}$ to $\varepsilon Q_2(g_N)$ in the notation of Proposition 14.3.2 in [21]. According to the next lemma, $\tilde{g}_N \xrightarrow{L^2(\mathbb{R}^2)} g$. Note that by substitution, it is easy to see that

$$\int_0^1 (v-\varepsilon x_1)_+^{d-1}(v-\varepsilon x_2)_+^{d-1}\,dv = \varepsilon^{2d-1}\int_0^{\frac{1}{\varepsilon}} (v-x_1)_+^{d-1}(v-x_2)_+^{d-1}\,dv.$$

Hence by Proposition 14.3.3 in [21], $N^{1-2d}(r_{N,0,\varepsilon},\ldots,r_{N,H,\varepsilon}) \xrightarrow{d} \varepsilon^{2d}U_d(\frac{1}{\varepsilon})(1,...,1)$ as $N\to\infty$. The Rosenblatt process $(U_d(t))_{t\in\mathbb{R}}$ is self-similar with index $2d$, see [21], p. 544, Proposition 14.3.7, hence $\varepsilon^{2d}U_d(\frac{1}{\varepsilon}) \stackrel{d}{=} U_d(1)$. □

The following lemma was needed in the proof of Lemma 3.2.2:

Lemma 3.2.3. *With the definitions in equations (3.10), (3.11), (3.12) and the assumption $C_d = 1$,*

$$\tilde{g}_N \stackrel{L^2(\mathbb{R}^2)}{\longrightarrow} g \text{ as } N \to \infty$$

holds.

Proof. Expressing the sum as an integral of a step function, we see that

$$\begin{aligned}\tilde{g}_N(x_1, x_2) &= N^{1-2d} \sum_{t=1}^{N} f(t - \varepsilon \lfloor Nx_1 \rfloor) f(t + h - \varepsilon \lfloor Nx_2 \rfloor) \\ &= N^{2-2d} \int_0^1 f(\lfloor Nt \rfloor + 1 - \varepsilon \lfloor Nx_1 \rfloor) f(\lfloor Nt \rfloor + 1 + h - \varepsilon \lfloor Nx_2 \rfloor)\, dt,\end{aligned}$$

for $x_1, x_2 \in \mathbb{R}$ with $x_1 \neq x_2$ and $\tilde{g}_N(x, x) = 0$.

1. For our further calculations, we need estimates for $f(\lfloor Nt \rfloor + 1 + h - \varepsilon \lfloor Nx_2 \rfloor)$ with $t \in [0, 1]$. We consider the three cases $t > \varepsilon x_2$, $t < \varepsilon x_2 - \frac{1+h+\varepsilon}{N}$ and $\varepsilon x_2 - \frac{1+h+\varepsilon}{N} \leq t \leq \varepsilon x_2$.

If $t > \varepsilon x_2$ and $t \in [0, 1]$, then $\lfloor Nt \rfloor + 1 + h > \varepsilon \lfloor Nx_2 \rfloor$ and hence

$$f(\lfloor Nt \rfloor + 1 + h - \varepsilon \lfloor Nx_2 \rfloor) \leq K(\lfloor Nt \rfloor + 1 + h - \varepsilon \lfloor Nx_2 \rfloor)^{d-1} \leq KN^{d-1}(t - \varepsilon x_2)^{d-1}.$$

If $t < \varepsilon x_2 - \frac{1+h+\varepsilon}{N}$ and $t \in [0, 1]$, then

$$\lfloor Nt \rfloor + 1 + h < \varepsilon \lfloor Nx_2 \rfloor$$

and hence

$$f(\lfloor Nt \rfloor + 1 + h - \varepsilon \lfloor Nx_2 \rfloor) = 0.$$

Since f is bounded by K by our assumptions, we also have

$$|f(\lfloor Nt \rfloor + 1 + h - \varepsilon \lfloor Nx_2 \rfloor)| \leq K,$$

especially in the third case $\varepsilon x_2 - \frac{1+h+\varepsilon}{N} \leq t \leq \varepsilon x_2$.

2. It suffices to show $\tilde{g}_N 1_{\{x_1 < x_2\}} \stackrel{L^2(\mathbb{R}^2)}{\longrightarrow} g 1_{\{x_1 < x_2\}}$ and $\tilde{g}_N 1_{\{x_1 > x_2\}} \stackrel{L^2(\mathbb{R}^2)}{\longrightarrow} g 1_{\{x_1 > x_2\}}$. We only show $\tilde{g}_N 1_{\{x_1 < x_2\}} \stackrel{L^2(\mathbb{R}^2)}{\longrightarrow} g 1_{\{x_1 < x_2\}}$, the other convergence follows in an analogous manner. For the first convergence, we further decompose the function $\tilde{g}_N 1_{\{x_1 < x_2\}}$ into three parts according to the cases in the last paragraph with respect to x_2. For $x_2 > x_1$ we now have

$$\begin{aligned}\tilde{g}_N(x_1, x_2) &= N^{2-2d} \int_{[0,1] \cap (\varepsilon x_2, \infty)} f(\lfloor Nt \rfloor + 1 - \varepsilon \lfloor Nx_1 \rfloor) f(\lfloor Nt \rfloor + 1 + h - \varepsilon \lfloor Nx_2 \rfloor)\, dt \\ &+ N^{2-2d} \int_{[0,1] \cap (-\infty, \varepsilon x_2 - \frac{1+h+\varepsilon}{N})} f(\lfloor Nt \rfloor + 1 - \varepsilon \lfloor Nx_1 \rfloor) \underbrace{f(\lfloor Nt \rfloor + 1 + h - \varepsilon \lfloor Nx_2 \rfloor)}_{=0}\, dt \\ &+ N^{2-2d} \int_{[0,1] \cap [\varepsilon x_2 - \frac{1+h+\varepsilon}{N}, \varepsilon x_2]} f(\lfloor Nt \rfloor + 1 - \varepsilon \lfloor Nx_1 \rfloor) f(\lfloor Nt \rfloor + 1 + h - \varepsilon \lfloor Nx_2 \rfloor)\, dt \\ &=: G_N^{(1)}(x_1, x_2) + G_N^{(2)}(x_1, x_2)\end{aligned}$$

with

$$G_N^{(1)}(x_1,x_2) := N^{2-2d} \int_{[0,1]\cap(\varepsilon x_2,\infty)} f(\lfloor Nt \rfloor + 1 - \varepsilon \lfloor Nx_1 \rfloor) f(\lfloor Nt \rfloor + 1 + h - \varepsilon \lfloor Nx_2 \rfloor)\, dt$$

and

$$G_N^{(2)}(x_1,x_2) := N^{2-2d} \int_{[0,1]\cap[\varepsilon x_2 - \frac{1+h+\varepsilon}{N}, \varepsilon x_2]} f(\lfloor Nt \rfloor + 1 - \varepsilon \lfloor Nx_1 \rfloor) f(\lfloor Nt \rfloor + 1 + h - \varepsilon \lfloor Nx_2 \rfloor)\, dt.$$

3. Let us again assume $x_1 < x_2$:
For $t \in [0,1] \cap (\varepsilon x_2, \infty)$ we also have $t > \varepsilon x_1$ since we assume $x_1 < x_2$ and hence

$$f(\lfloor Nt \rfloor + 1 - \varepsilon \lfloor Nx_1 \rfloor) \le K N^{d-1} (t - \varepsilon x_1)^{d-1}$$

and

$$f(\lfloor Nt \rfloor + 1 + h - \varepsilon \lfloor Nx_2 \rfloor) \le K N^{d-1} (t - \varepsilon x_2)^{d-1}.$$

Hence we have

$$N^{2-2d} |f(\lfloor Nt \rfloor + 1 - \varepsilon \lfloor Nx_1 \rfloor) f(\lfloor Nt \rfloor + 1 + h - \varepsilon \lfloor Nx_2 \rfloor)| \le K^2 (t - \varepsilon x_1)^{d-1} (t - \varepsilon x_2)^{d-1}$$

and hence

$$G_N^{(1)}(x_1,x_2) \le K^2 g(x_1,x_2) = K^2 \int_0^1 (t - \varepsilon x_1)^{d-1} (t - \varepsilon x_2)^{d-1}\, dt < \infty.$$

Because of $t > \varepsilon x_1$, $t > \varepsilon x_2$, we have $\lfloor Nt \rfloor + 1 - \varepsilon \lfloor Nx_1 \rfloor \to \infty$ and $\lfloor Nt \rfloor + 1 + h - \varepsilon \lfloor Nx_2 \rfloor \to \infty$ as $N \to \infty$ as well as $\frac{\lfloor Nt \rfloor + 1 - \varepsilon \lfloor Nx_1 \rfloor}{N(t - \varepsilon x_1)} \to 1$ and $\frac{\lfloor Nt \rfloor + 1 + h - \varepsilon \lfloor Nx_2 \rfloor}{N(t - \varepsilon x_2)} \to 1$. Hence

$$\begin{aligned} & \lim_{N\to\infty} N^{2-2d} f(\lfloor Nt \rfloor + 1 - \varepsilon \lfloor Nx_1 \rfloor) f(\lfloor Nt \rfloor + 1 + h - \varepsilon \lfloor Nx_2 \rfloor) \\ = & \lim_{N\to\infty} N^{2-2d} (\lfloor Nt \rfloor + 1 - \varepsilon \lfloor Nx_1 \rfloor)^{d-1} (\lfloor Nt \rfloor + 1 + h - \varepsilon \lfloor Nx_2 \rfloor)^{d-1} \\ = & (t - \varepsilon x_1)^{d-1} (t - \varepsilon x_2)^{d-1}. \end{aligned}$$

With Lebesgue's convergence theorem, we conclude

$$\begin{aligned} \lim_{N\to\infty} G_N^{(1)}(x_1,x_2) &= \int_{[0,1]\cap(\varepsilon x_2,\infty)} (t - \varepsilon x_1)_+^{d-1} (t - \varepsilon x_2)_+^{d-1}\, dt \\ &= \int_{[0,1]} (t - \varepsilon x_1)_+^{d-1} (t - \varepsilon x_2)_+^{d-1}\, dt = g(x_1,x_2). \end{aligned}$$

Since $G_N^{(1)}(x_1,x_2) 1_{\{x_1<x_2\}}$ is bounded by $K^2 g$ which is square integrable and $G_N^{(1)}(x_1,x_2) 1_{\{x_1<x_2\}}$ converges pointwise towards $g 1_{\{x_1<x_2\}}$, $G_N^{(1)}$ converges to $g 1_{\{x_1<x_2\}}$ in $L^2(\mathbb{R}^2)$ by Lebesgue's theorem.

4. For the L^2-convergence, it suffices to show that $G_N^{(1)} 1_{\{x_1<x_2\}} \stackrel{L^2(\mathbb{R}^2)}{\longrightarrow} g 1_{\{x_1<x_2\}}$ and $G_N^{(2)} 1_{\{x_1<x_2\}} \stackrel{L^2(\mathbb{R}^2)}{\longrightarrow} 0$. That $G_N^{(1)} 1_{\{x_1<x_2\}} \stackrel{L^2(\mathbb{R}^2)}{\longrightarrow} g 1_{\{x_1<x_2\}}$ was already shown in the last

paragraph. Define $G_N^{(3)} := G_N^{(2)} 1_{\{0<x_2-x_1\leq 1\}}$ and $G_N^{(4)} := G_N^{(2)} 1_{\{x_2-x_1>1\}}$. We show $G_N^{(3)} \overset{L^2(\mathbb{R}^2)}{\longrightarrow} 0$ and $G_N^{(4)} \overset{L^2(\mathbb{R}^2)}{\longrightarrow} 0$. Note that $G_N^{(2)}(x_1, x_2)$ vanishes for $x_2 < 0$ and $\varepsilon x_2 > 3 \geq 1 + \frac{1+h+\varepsilon}{N}$ (which is true for N sufficiently large).

5. Define

$$A(\varepsilon) := \{t \in [0,1] \cap [\varepsilon x_2 - \frac{1+h+\varepsilon}{N}, \varepsilon x_2] : \lfloor Nt \rfloor + 1 + h - \varepsilon \lfloor N x_2 \rfloor > 0\}$$

and

$$B(\varepsilon) := \{t \in [0,1] \cap [\varepsilon x_2 - \frac{1+h+\varepsilon}{N}, \varepsilon x_2] : \lfloor Nt \rfloor + 1 + h - \varepsilon \lfloor N x_2 \rfloor \leq 0\}.$$

Then $f(\lfloor Nt \rfloor + 1 + h - \varepsilon \lfloor N x_2 \rfloor)$ vanishes for $t \in B(\varepsilon)$. Further, denoting by λ the one-dimensional Lebesgue measure, we see that $\lambda(A(\varepsilon)) \leq \frac{1+h+\varepsilon}{N}$. For $t \in A(\varepsilon)$, we have

$$\lfloor Nt \rfloor + 1 + h - \varepsilon \lfloor N x_2 \rfloor \geq \varepsilon$$

and hence

$$\lfloor Nt \rfloor + 1 - \varepsilon \lfloor N x_1 \rfloor \geq \varepsilon - h + \varepsilon(\lfloor N x_2 \rfloor - \lfloor N x_1 \rfloor).$$

With this estimate and the fact that $f(x)$ is bounded by $K(\max(x,1))^{d-1}$, we obtain

$$\begin{aligned} |G_N^{(2)}(x_1, x_2)| &= N^{2-2d} \int_{A(\varepsilon)} f(\lfloor Nt \rfloor + 1 - \varepsilon \lfloor N x_1 \rfloor) f(\lfloor Nt \rfloor + 1 + h - \varepsilon \lfloor N x_2 \rfloor)\, dt \\ &\leq \lambda(A(\varepsilon)) N^{2-2d} K^2 \max(\varepsilon - h + \varepsilon(\lfloor N x_2 \rfloor - \lfloor N x_1 \rfloor), 1)^{d-1} \\ &\leq (1+h+\varepsilon) K^2 N^{1-2d} \max(\varepsilon - h + \varepsilon(\lfloor N x_2 \rfloor - \lfloor N x_1 \rfloor), 1)^{d-1}. \end{aligned}$$

One can see that

$$\lambda(\{x_1 \in [x_2 - 1, x_2] : \lfloor N x_2 \rfloor - \lfloor N x_1 \rfloor = i\}) \leq \frac{1}{N}$$

for $i \in \mathbb{N}_0$ and that $\varepsilon - h + \varepsilon i > 1$ for $i > \frac{1+h}{\varepsilon} - 1$. Hence

$$\begin{aligned} ||G_N^{(3)}||^2_{L^2(\mathbb{R}^2)} &= \int_0^{\frac{3}{\varepsilon}} \int_{x_2-1}^{x_2} |G_N^{(2)}(x_1, x_2)|^2\, dx_1 dx_2 \\ &\leq (1+h+\varepsilon)^2 K^4 N^{1-4d} \frac{3}{\varepsilon} (\frac{h+1}{\varepsilon} + \sum_{i=\frac{h+1}{\varepsilon}}^{\infty} ((\varepsilon - h + \varepsilon i)_+^{2d-2})) \\ &\overset{N\to\infty}{\to} 0. \end{aligned}$$

6. Now let $t \in [0,1] \cap [\varepsilon x_2 - \frac{1+h+\varepsilon}{N}, \varepsilon x_2]$ and $x_2 - x_1 > 1$. Then $t > \varepsilon x_1$ for $N > \frac{1+h+\varepsilon}{\varepsilon}$. Hence we have

$$N^{2-2d} |f(\lfloor Nt \rfloor + 1 - \varepsilon \lfloor N x_1 \rfloor) f(\lfloor Nt \rfloor + 1 + h - \varepsilon \lfloor N x_2 \rfloor)| \leq K(t - \varepsilon x_1)^{d-1} K N^{1-d}$$

and hence

$$
\begin{aligned}
|G_N^{(2)}(x_1,x_2)| &\leq K^2 N^{1-d} \int_{[0,1]\cap[\varepsilon x_2 - \frac{1+h+\varepsilon}{N}, \varepsilon x_2]} (t-\varepsilon x_1)^{d-1}\, dt \\
&\leq (1+h+\varepsilon) K^2 (\varepsilon x_2 - \frac{1+h+\varepsilon}{N} - \varepsilon x_1)^{d-1} N^{-d}.
\end{aligned}
$$

Let N be sufficiently large such that $\frac{1+h+\varepsilon}{N} < \frac{\varepsilon}{2}$. We then have

$$
\begin{aligned}
\int_{-\infty}^{x_2-1} |G_N^{(2)}(x_1,x_2)|^2\, dx_1 &\leq K^4(1+h+\varepsilon)^2 N^{-2d} \int_{-\infty}^{x_2-1} (\varepsilon x_2 - \frac{1+h+\varepsilon}{N} - \varepsilon x_1)^{2d-2}\, dx_1 \\
&\leq K^4(1+h+\varepsilon)^2 N^{-2d} \int_{-\infty}^{x_2-1} (\varepsilon x_2 - \frac{\varepsilon}{2} - \varepsilon x_1)^{2d-2}\, dx_1 \\
&= K^4(1+h+\varepsilon)^2 N^{-2d} \varepsilon^{2d-2} \underbrace{\int_{\frac{1}{2}}^{\infty} v^{2d-2}\, dv}_{<\infty}
\end{aligned}
$$

and hence

$$
\begin{aligned}
||G_N^{(4)}||^2_{L^2(\mathbb{R}^2)} &= \int_0^{\frac{3}{\varepsilon}} \int_{-\infty}^{x_2-1} |G_N^{(2)}(x_1,x_2)|^2\, dx_1 dx_2 \\
&\leq K^4(1+h+\varepsilon)^2 N^{-2d} \varepsilon^{2d-2} \frac{3}{\varepsilon} \int_{\frac{1}{2}}^{\infty} v^{2d-2}\, dv \\
&\overset{N\to\infty}{\to} 0.
\end{aligned}
$$

□

Returning to the proof Theorem 3.2.1, we define for the process $(X_t)_{t\in\mathbb{R}}$ similar random variables that correspond to the increments and the non-diagonal parts of the autocovariance function of the process $(X_t^{(m)})_{t\in\mathbb{R}}$ defined in (3.4) and (3.9). Note that the diagonal part $d_{N,h,\varepsilon}$ is only needed in section 3, which is defined in (3.23). Define

$$
\bar{Z}_{k,t,h} := \int_{\varepsilon(k-1)}^{\varepsilon k} f(t+h-s)\, dL_s, \quad k \in \mathbb{Z}, \tag{3.13}
$$

where we suppress the dependence on ε for notational convenience,

$$
\bar{r}_{N,h,\varepsilon} := \frac{1}{N} \sum_{t=1}^{N} \sum_{k\neq k'} \bar{Z}_{k,t,0} \bar{Z}_{k',t,h}, \quad h \in \mathbb{N}_0, \tag{3.14}
$$

and

$$
\bar{d}_{N,h,\varepsilon} := \frac{1}{N} \sum_{t=1}^{N} \sum_{k} \bar{Z}_{k,t,0} \bar{Z}_{k,t,h}, \quad h \in \mathbb{N}_0. \tag{3.15}
$$

$\bar{r}_{N,h,\varepsilon}$ can be seen to converge unconditionally in $L^2(\mathbb{P})$ like $r_{N,h,\varepsilon}$. By the next estimates, we see that $\bar{d}_{N,h,\varepsilon}$ converges in $L^1(\mathbb{P})$ unconditionally, where we use Cauchy's equality for square integrable random variables in the first inequality and Cauchy's equality for square summable sequences in the second inequality and the fact that $a_k := \sqrt{\int_{\varepsilon(k-1)}^{\varepsilon k} f(t-s)^2\, ds}$ and $b_k := \sqrt{\int_{\varepsilon(k-1)}^{\varepsilon k} f(t+h-s)^2\, ds}$ for $k \in \mathbb{Z}$ define two sequences in $l^2(\mathbb{Z})$ and hence

$$\sum_k |a_k b_k| \le \sqrt{\sum_k a_k^2}\sqrt{\sum_k b_k^2} = \sum_k a_k^2$$

and consequently

$$\begin{aligned}
&\sum_k \mathbb{E}[|\bar{Z}_{k,t,0}\bar{Z}_{k,t,h}|] \\
\le\ & \sum_k \sqrt{\mathbb{E}[|\bar{Z}_{k,t,0}|^2]}\sqrt{\mathbb{E}[|\bar{Z}_{k,t,h}|^2]} \\
=\ & \sigma^2 \sum_k \sqrt{\int_{\varepsilon(k-1)}^{\varepsilon k} f(t-s)^2\, ds}\sqrt{\int_{\varepsilon(k-1)}^{\varepsilon k} f(t+h-s)^2\, ds} \\
\le\ & \sigma^2 \sum_k \int_{\varepsilon(k-1)}^{\varepsilon k} f(t-s)^2\, ds \\
=\ & \sigma^2 \int_{-\infty}^{\infty} f(t-s)^2\, ds < \infty.
\end{aligned}$$

Note that $X_{t+h} = \sum_{k=-\infty}^{\infty} \bar{Z}_{k,t,h}$ converges unconditionally in $L^2(\mathbb{P})$. Then by Lemma 3.2.1 we see that

$$\hat{\gamma}_N(h) = \bar{r}_{N,h,\varepsilon} + \bar{d}_{N,h,\varepsilon} \tag{3.16}$$

is true as we showed the equality of (3.5) and (3.6). We want to prove the theorem by using Slutsky's lemma. As we have already seen

$$N^{1-2d}(r_{N,0,\varepsilon}, \dots, r_{N,H,\varepsilon}) \xrightarrow{d} C_d^2 \sigma^2 U_d(1)(1, \dots, 1),$$

it suffices to show for all $\delta > 0$

$$\lim_{N\to\infty} \mathbb{P}[|N^{1-2d}(\hat{\gamma}_N(h) - \gamma(h) - r_{N,h,\varepsilon})| > \delta] = 0.$$

We split $\hat{\gamma}_N(h) - \gamma(h) - r_{N,h,\varepsilon} = (\bar{r}_{N,h,\varepsilon} - r_{N,h,\varepsilon}) + (\bar{d}_{N,h,\varepsilon} - \gamma(h))$. Then we conclude the proof of Theorem 3.2.1 by using the next two lemmas.

Lemma 3.2.4. *Let all assumptions of Theorem 3.2.1 apart from $\mathbb{E}[L_1^4] = \eta\sigma^4 < \infty$ be fulfilled. Then for all $\delta > 0$*

$$\lim_{N\to\infty} \mathbb{P}[|N^{1-2d}(\bar{r}_{N,h,\varepsilon} - r_{N,h,\varepsilon})| > \delta] = 0.$$

Proof. We show that

$$\mathbb{E}[(N(\bar{r}_{N,h,\varepsilon} - r_{N,h,\varepsilon}))^2] = o(N^{4d})$$

and the claim then follows by Markov's inequality. To this end, we mimic the proof of Lemma 5.5 in [23] in the following. One can see by the inequality $(a-b)^2 \leq 2(a^2+b^2)$ that

$$(N(\bar{r}_{N,h,\varepsilon} - r_{N,h,\varepsilon}))^2 \leq 2[S_1(N,\varepsilon) + S_2(N,\varepsilon)],$$

where we define $S_1(N,\varepsilon)$ and $S_2(N,\varepsilon)$ by

$$S_1(N,\varepsilon) := \Big[\sum_{t=1}^{N}\sum_{k\neq k'} f(t-\varepsilon k)Z_k((f(t+h-\varepsilon k')Z_{k'} - \bar{Z}_{k',t,h})\Big]^2,$$

$$S_2(N,\varepsilon) := \Big[\sum_{t=1}^{N}\sum_{k\neq k'}(f(t-\varepsilon k)Z_k - \bar{Z}_{k,t,0})\bar{Z}_{k',t,h}\Big]^2.$$

Consequently, we evaluate $\mathbb{E}[S_1(N,\varepsilon)]$ and $\mathbb{E}[S_2(N,\varepsilon)]$. We obtain

$$\begin{aligned} &\mathbb{E}[S_1(N,\varepsilon)] \\ = &\sum_{t,s=1}^{N}\sum_{k\neq k'}\sum_{i\neq i'} \\ &f(t-\varepsilon k)f(s-\varepsilon i)\mathbb{E}[Z_k(f(t+h-\varepsilon k')Z_{k'} - \bar{Z}_{k',t,h})Z_i(f(s+h-\varepsilon i')Z_{i'} - \bar{Z}_{i',s,h})]. \end{aligned} \tag{3.17}$$

Observe that $f(t+h-\varepsilon k)Z_k$ can be written as a stochastic integral:

$$f(t+h-\varepsilon k)Z_k = \int_{\varepsilon(k-1)}^{\varepsilon k} f(t+h-\varepsilon k)\,dL_s.$$

Hence by Itô's isometry, we have

$$\begin{aligned} \mathrm{Var}[(f(t+h-\varepsilon k)Z_k - \bar{Z}_{k,t,h})] &= \sigma^2 \int_{\varepsilon(k-1)}^{\varepsilon k} (f(t+h-\varepsilon k) - f(t+h-s))^2\,ds \\ &= \sigma^2 \int_0^{\varepsilon} (f(t+h-\varepsilon k) - f(t+h-\varepsilon k+s))^2\,ds. \end{aligned}$$

Define $\tilde{g}(x) := \sqrt{\int_0^\varepsilon (1-\frac{f(x+s)}{f(x)})^2\,ds}$ for $x \in (0,\infty)$, where we assume for our calculations without loss of generality that f does not vanish on $(0,\infty)$.

We obtain

$$\sqrt{\int_0^\varepsilon (f(x) - f(x+s))^2\,ds} = x^{d-1}\tilde{g}(x)\frac{|f(x)|}{x^{d-1}} = O(x^{d-1}\tilde{g}(x)) \text{ as } x \to \infty$$

since $\lim_{x\to\infty}\frac{f(x)}{x^{d-1}} = C_d$.

We now show that $\lim_{x\to\infty} \tilde{g}(x) = 0$. Let $\mu \in (0, C_d)$. Since $\lim_{x\to\infty} \frac{f(x)}{x^{d-1}} = C_d$, there is an N_μ such that $|\frac{f(x)}{x^{d-1}} - C_d| < \mu$ for $x \geq N_\mu$. Hence we find for $x \geq \max(N_\mu, 1)$ and $s \geq 0$

$$\begin{aligned} |1 - \frac{f(x+s)}{f(x)}| &= |\frac{f(x) - f(x+s)}{f(x)}| \\ &\leq \left|\frac{(C_d+\mu)x^{d-1} - (C_d-\mu)(x+s)^{d-1}}{(C_d-\mu)x^{d-1}}\right| \\ &\leq \left|\frac{C_d}{C_d-\mu}(1 - (\frac{x+s}{x})^{d-1})\right| + 2\mu. \end{aligned}$$

Hence

$$\limsup_{x\to\infty} \tilde{g}(x)^2 \leq \limsup_{x\to\infty} \int_0^\varepsilon (\left|\frac{C_d}{C_d-\mu}(1 - (\frac{x+s}{x})^{d-1})\right| + 2\mu)^2 \, ds = \varepsilon 4\mu^2$$

by Lebesgue's dominated convergence Theorem. Letting $\mu \to 0$, we obtain $\lim_{x\to\infty} \tilde{g}(x) = 0$.

We can further replace $\tilde{g}$ by a bounded and decreasing function g on $[1, \infty)$, e.g. by $g(x) := \sup_{y \geq x} \tilde{g}(y)$.

We define $v(t) := C(g(t)t^{d-1} 1_{(1,\infty)}(t) + 1_{[-\varepsilon,1]}(t))$ with C large enough. Then

$$\sqrt{\int_{\varepsilon(k-1)}^{\varepsilon k} (f(t+h-\varepsilon k) - f(t+h-s))^2 \, ds} \leq v(t+h-\varepsilon k).$$

In the case $t + h - \varepsilon k < -\varepsilon$, the integral vanishes. f is bounded by K in the case $-\varepsilon \leq t + h - \varepsilon k < 1$.

All summands in (3.17) vanish except for the cases (a) $k = i$ and $k' = i'$ and (b) $k = i'$ and $k' = i$. By Cauchy's inequality, we get the estimate $\mathbb{E}[S_1(N, \varepsilon)] \leq \sigma^2(E_{11} + E_{12})$ with

$$E_{11} := \sum_{t,s=1}^{N} \sum_{k \neq k'} |f(t-\varepsilon k) f(s-\varepsilon k)| v(t+h-\varepsilon k') v(s+h-\varepsilon k')$$

and

$$E_{12} := \sum_{t,s=1}^{N} \sum_{k \neq k'} |f(t-\varepsilon k) f(s-\varepsilon k')| v(t+h-\varepsilon k') v(s+h-\varepsilon k).$$

We consider here E_{12}. Similar calculations show that the same asymptotics also holds for E_{11} and $\mathbb{E}[S_2(N, \varepsilon)]$. We substitute $\varepsilon i = t - \varepsilon k$ and $\varepsilon i' = t + h - \varepsilon k'$. Then

$$|E_{12}| \leq \sum_{t,s=1}^{N} \sum_{i} |f(\varepsilon i) v(s-t+h+\varepsilon i)| \sum_{i'} |v(\varepsilon i') f(s-t-h+\varepsilon i')|.$$

Like in the proof of Lemma 5.5 in [23], we denote with E_{120} the summands where $s - t = 0$, and E_{12+} and E_{12-} where $s - t > 0$ or $s - t < 0$ respectively. We obtain the upper estimates

$$|E_{120}| = N \sum_{i} |f(\varepsilon i) v(\varepsilon i + h)| \sum_{i'} |v(\varepsilon i') f(\varepsilon i' - h)| = O(N) \leq o(N^{4d}),$$

since $d > \frac{1}{4}$ and

$$|E_{12+}| \leq \sum_{n=1}^{N}(N-n)\sum_{i}|f(\varepsilon i)v(n+h+\varepsilon i)|\sum_{i'}|v(\varepsilon i')f(n-h+\varepsilon i')|.$$

By the integral convergence test, we obtain for $s - t = n \geq 1$

$$\begin{aligned}\sum_{i}|f(\varepsilon i)v(n+h+\varepsilon i)| &\leq Kv(n+h+\varepsilon) \\ &\quad +CK\int_{1}^{\infty}(\varepsilon x)^{d-1}(n+h+\varepsilon x)^{d-1}g(n+h+\varepsilon x)\,dx \\ &\leq O(n^{d-1}g(n)) + CKn^{2d-1}\varepsilon^{2d-2}g(n)\int_{\frac{1}{n}}^{\infty}y^{d-1}(\frac{1}{\varepsilon}+y)^{d-1}\,dy\end{aligned}$$

and

$$\begin{aligned}\sum_{i'}|v(\varepsilon i')f(n-h+\varepsilon i')| &\leq \sum_{i'=-1}^{(1+h)/\varepsilon}|v(\varepsilon i')f(n-h+\varepsilon i')| \\ &\quad + \sum_{i'=(1+h)/\varepsilon+1}^{\infty}CKg(\varepsilon i')(\varepsilon i')^{d-1}(n-h+\varepsilon i')^{d-1} \\ &\leq O(n^{d-1}) + CKg(1)\int_{(1+h)/\varepsilon}^{\infty}(\varepsilon x)^{d-1}(n-h+\varepsilon x)^{d-1}\,dx \\ &\leq O(n^{d-1}) + CKn^{2d-1}\varepsilon^{2d-2}g(1)\int_{\frac{1}{n}}^{\infty}y^{d-1}(\frac{1}{\varepsilon}+y)^{d-1}\,dy.\end{aligned}$$

Since $\varepsilon \leq 1$, we have

$$\int_{\frac{1}{n}}^{\infty}y^{d-1}(\frac{1}{\varepsilon}+y)^{d-1}\,dy \leq \int_{0}^{\infty}y^{d-1}(1+y)^{d-1}\,dy < \infty.$$

Hence

$$\sum_{i}|f(\varepsilon i)v(n+h+\varepsilon i)| = O(n^{2d-1}g(n))$$

and

$$\sum_{i'}|v(\varepsilon i')f(n-h+\varepsilon i')| = O(n^{2d-1}).$$

This gives

$$E_{12+} = O(N\sum_{n=1}^{N}n^{4d-2}g(n)) = o(N^{4d}),$$

since

$$\lim_{N\to\infty}\frac{1}{N^{4d-1}}\sum_{j=1}^{N}j^{4d-2}g(j) \leq g(M)\lim_{N\to\infty}\frac{1}{N^{4d-1}}\sum_{j=M}^{N}j^{4d-2} = \frac{g(M)}{4d-1}$$

for all $M \in \mathbb{N}$ and letting $M \to \infty$, we see that indeed $E_{12+} = o(N^{4d})$ is true. Note that for $n \geq 1$

$$\sum_i |f(\varepsilon i)v(-n+h+\varepsilon i)| = \sum_j |f(n-h+\varepsilon j)v(\varepsilon j)|$$

and

$$\sum_{i'} |v(\varepsilon i')f(-n-h+\varepsilon i')| = \sum_{j'} |v(n+h+\varepsilon j')f(\varepsilon j')|.$$

Hence we can see by the calculuations above that $E_{12-} = o(N^{4d})$ as well. □

Lemma 3.2.5. *Let all assumptions of Theorem 3.2.1 be fulfilled, especially* $\mathbb{E}[L_1^4] = \eta\sigma^4 < \infty$. *Then for all* $\delta > 0$

$$\lim_{N\to\infty} \mathbb{P}[|N^{1-2d}(\bar{d}_{N,h,\varepsilon} - \gamma(h))| > \delta] = 0,$$

where $\bar{d}_{N,h,\varepsilon}$ *is defined in (3.15).*

Proof. Recall $\bar{Z}_{k,t,h} := \int_{\varepsilon(k-1)}^{\varepsilon k} f(t+h-s)\, dL_s$ for $k \in \mathbb{Z}$. Define

$$z_{k,t,0,h} := \sigma^2 \int_{\varepsilon(k-1)}^{\varepsilon k} f(t-s)f(t+h-s)\, ds = \mathbb{E}[\bar{Z}_{k,t,0}\bar{Z}_{k,t,h}], \quad k \in \mathbb{Z},$$

and

$$S_N := \sum_{t=1}^N \xi_t, \quad N \in \mathbb{N},$$

with

$$\xi_t := \sum_k (\bar{Z}_{k,t,0}\bar{Z}_{k,t,h} - z_{k,t,0,h}), t \in \mathbb{N}.$$

Note that by the calculations below in the case $N = 1$ we see that ξ_t converges unconditionally in $L^2(\mathbb{P})$. Hence by definition $N(\bar{d}_{N,h,\varepsilon} - \gamma(h)) = S_N$. We want to show

$$\lim_{N\to\infty} \mathbb{P}[|N^{1-2d}(\bar{d}_{N,h,\varepsilon} - \gamma(h))| > \delta] = 0.$$

We do this by showing that $\mathrm{Var}(S_N) = \mathbb{E}[S_N^2] = o(N^{4d})$ and using then Chebyshev's inequality. In fact we show $\mathbb{E}[S_N^2] = O(N)$ which we will also need in the proof of Lemma 3.3.5. Since $\bar{Z}_{k,t,0}\bar{Z}_{k,t,h} - z_{k,t,0,h}$ and $\bar{Z}_{l,s,0}\bar{Z}_{l,s,h} - z_{l,s,0,h}$ have expectation zero and are independent for $k \neq l$, we obtain

$$\begin{aligned}
\mathbb{E}[S_N^2] = \mathbb{E}[\sum_{t,s=1}^N \xi_t\xi_s] &= \mathbb{E}[\sum_{t,s=1}^N (\sum_k \bar{Z}_{k,t,0}\bar{Z}_{k,t,h} - z_{k,t,0,h})(\sum_l \bar{Z}_{l,s,0}\bar{Z}_{l,s,h} - z_{l,s,0,h})] \\
&= \sum_{t,s=1}^N \sum_k \mathbb{E}[(\bar{Z}_{k,t,0}\bar{Z}_{k,t,h} - z_{k,t,0,h})(\bar{Z}_{k,s,0}\bar{Z}_{k,s,h} - z_{k,s,0,h})] \\
&\leq \sum_{t,s=1}^N \sum_k \sqrt{\mathbb{E}[(\bar{Z}_{k,t,0}\bar{Z}_{k,t,h} - z_{k,t,0,h})^2]}\sqrt{\mathbb{E}[(\bar{Z}_{k,s,0}\bar{Z}_{k,s,h} - z_{k,s,0,h})^2]}
\end{aligned}$$

and further

$$\begin{aligned}
& \mathbb{E}[(\bar{Z}_{k,t,0}\bar{Z}_{k,t,h} - z_{k,t,0,h})^2] \\
= \; & \mathrm{Var}(\bar{Z}_{k,t,0}\bar{Z}_{k,t,h}) \\
\leq \; & \mathbb{E}[\bar{Z}_{k,t,0}^2\bar{Z}_{k,t,h}^2] \\
\leq \; & \sqrt{\mathbb{E}[\bar{Z}_{k,t,0}^4]}\sqrt{\mathbb{E}[\bar{Z}_{k,t,h}^4]} \\
\leq \; & \sqrt{[w(t-\varepsilon k)]^2}\sqrt{[w(t+h-\varepsilon k)]^2} \leq [w(t-\varepsilon k)]^2
\end{aligned}$$

with $w(t) := C(t^{2d-2}1_{(1,\infty)}(t) + 1_{[-\varepsilon,1]}(t))$ with C large enough and the following calculations:

By Lemma 3.2 in Cohen and Lindner [11], we obtain

$$\mathbb{E}[\bar{Z}_{k,t,h}^4] = (\eta-3)\sigma^4\int_{\varepsilon(k-1)}^{\varepsilon k} f(t+h-s)^4\,ds + 3\sigma^4(\int_{\varepsilon(k-1)}^{\varepsilon k} f(t+h-s)^2\,ds)^2.$$

If $t+h-\varepsilon k \geq 1$, then

$$\int_{\varepsilon(k-1)}^{\varepsilon k} f(t+h-s)^4\,ds \leq \varepsilon K^4(t+h-\varepsilon k)^{4d-4}$$

and

$$\int_{\varepsilon(k-1)}^{\varepsilon k} f(t+h-s)^2\,ds \leq \varepsilon K^2(t+h-\varepsilon k)^{2d-2}.$$

If $t+h-\varepsilon k < -\varepsilon$, then the integrals vanish. In the in-between case $-\varepsilon \leq t+h-\varepsilon k < 1$ the integrands are bounded by K^2 and K^4 respectively.

Hence we need to consider $\sum_{s,t=1}^N\sum_k w(t-\varepsilon k)w(s-\varepsilon k)$. Substituting $\varepsilon i = t-\varepsilon k$, we get $\sum_{s,t=1}^N\sum_i w(\varepsilon i)w(\varepsilon i+s-t)$. Adopting the notation E_0,E_+ and E_- where we split the sum accordingly to the cases $n=s-t=0$, $n=s-t>0$ and $n=s-t<0$, we have $E_+ = E_-$ since $\sum_i w(\varepsilon i)w(\varepsilon i+n) = \sum_i w(\varepsilon i)w(\varepsilon i-n)$ for $n\in\mathbb{N}$ and

$$E_0 = N\sum_i w(\varepsilon i)w(\varepsilon i) = O(N).$$

For $n \geq 1$, we get

$$\begin{aligned}
& \sum_i w(\varepsilon i)w(\varepsilon i+n) \\
\leq \; & w(-\varepsilon)w(n-\varepsilon) + w(0)w(n) + C^2\int_1^\infty (\varepsilon x)^{2d-2}(\varepsilon x+n)^{2d-2}\,dx \\
= \; & O(n^{2d-2}) + n^{2d-2}C^2\int_1^\infty(\varepsilon x)^{2d-2}(\frac{\varepsilon x}{n}+1)^{2d-2}\,dx \\
= \; & O(n^{2d-2}).
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
E_+ &= O(N\sum_{n=1}^N n^{2d-2}) \\
&= O(N^{2d})
\end{aligned}$$

and $\mathbb{E}[S_N^2] = O(N)$, since $d < \frac{1}{2}$. $\square$

3.3 Theorem for regularly varying tails

In this section we show that the sample autocovariance of $X_t = \int_{-\infty}^{\infty} f(t-s)\, dL_s$ is asymptotically Rosenblatt or stable distributed (Theorem 3.3.1), if L_1 is regularly varying with index $\alpha \in (2,4)$. This parallels Theorem 3.1 in [23]. We assume that $f(t) = 0$ for $t \leq 0$, f is bounded and $f(t) \sim C_d t^{d-1}$ as $t \to \infty$ with $d \in (0, \frac{1}{2})$ and $C_d > 0$. Recall that a function $l : (0,\infty) \to (0,\infty)$ is called regularly varying with index ρ, if $\lim_{t\to\infty} \frac{l(tx)}{l(t)} = x^\rho$ for all $x > 0$. We call a random variable X regularly varying with index α, if the tail function $\bar{F}(x) := \mathbb{P}[|X| > x]$ is regularly varying with index $-\alpha$. We say that X fulfils a tail balance condition, if there is a $p \in [0,1]$ such that

$$\lim_{x\to\infty} \frac{\mathbb{P}[X > x]}{\mathbb{P}[|X| > x]} = p. \tag{3.18}$$

If X is symmetric, then p equals $\frac{1}{2}$.

For $\alpha \in (0,2]$ we denote by $S_\alpha(\tau,\beta,\mu)$ an α-stable distribution with $\tau \geq 0$ as scale parameter, $\beta \in [-1,1]$ as skewness and $\mu \in \mathbb{R}$ as location parameter, see (1.1.6), p. 9, in [34].

Let $(L_t)_{t\in\mathbb{R}}$ be a two-sided Lévy process. Assume that L_1 is regularly varying with index $\alpha \in (2,4)$ and fulfils the tail balance condition (3.18). Define

$$a_N := \inf\{y : \mathbb{P}[|L_1| > y] < \frac{1}{N}\} \text{ and } b_N := \mathbb{E}[L_1^2 1_{\{|L_1| \leq a_N\}}].$$

Note that

$$a_N^2 = \inf\{y^2 : \mathbb{P}[|L_1|^2 > y^2] < \frac{1}{N}\} = \inf\{x : \mathbb{P}[|L_1|^2 > x] < \frac{1}{N}\}.$$

Hence by Propositions 2.2.13 and 2.2.14 in [19], there is a stable distribution $S_{\frac{\alpha}{2}}(\tau,\beta,\mu)$ such that

$$\frac{1}{a_N^2} \sum_{t=1}^{N} ((L_t - L_{t-1})^2 - b_N) \xrightarrow{d} S_{\frac{\alpha}{2}}(\tau,\beta,\mu). \tag{3.19}$$

By Karamata's theorem, see Theorem 1.6.5 in Bingham et al. [3], one can show that

$$\lim_{N\to\infty} \frac{N}{a_N^2}(\sigma^2 - b_N) = \frac{\alpha}{\alpha-2}.$$

Hence

$$\frac{1}{a_N^2} \sum_{t=1}^{N} ((L_t - L_{t-1})^2 - \sigma^2) \xrightarrow{d} S_{\frac{\alpha}{2}}(\tau,\beta,\mu - \frac{\alpha}{\alpha-2}). \tag{3.20}$$

Define two Lévy processes

$$(K_s)_{s\in[0,1]} \text{ with } K_1 \stackrel{d}{=} S_{\frac{\alpha}{2}}(\tau,\beta,\mu) \tag{3.21}$$

and

$$(M_s)_{s\in[0,1]} \text{ with } M_s := K_s - s\frac{\alpha}{\alpha-2}, \quad s \in [0,1]. \tag{3.22}$$

Theorem 3.3.1. *Let $(L_t)_{t\in\mathbb{R}}$ be a two-sided Lévy process such that L_1 is symmetric about zero and has no Gaussian part and $\mathrm{Var}(L_1) = \sigma^2 \in (0,\infty)$. Assume that L_1 is regularly varying with index $\alpha \in (2,4)$. Define a continuous-time moving average process $(X_t)_{t\in\mathbb{R}}$ by*

$$X_t := \int_{-\infty}^{\infty} f(t-s)\, dL_s, \quad t \in \mathbb{R},$$

where we assume that $f(t) = 0$ for $t \le 0$, f is bounded and $f(t) \sim C_d t^{d-1}$ as $t \to \infty$ with $d \in (0, \frac{1}{2})$ and $C_d > 0$. Let $H \in \mathbb{N}_0$. Define f_m as in (3.2), i.e.

$$f_m = \sum_{k=0}^{\infty} f(\varepsilon k) 1_{[k\varepsilon,(k+1)\varepsilon)},$$

with $\varepsilon = \frac{1}{m}$. For $h \in \mathbb{N}_0$ define

$$G_h(s) := \sum_{i=-\infty}^{\infty} f(i+s) f(i+h+s), \quad s \in [0,1],$$

$$G_{m,h}(s) := \sum_{i=-\infty}^{\infty} f_m(i+s) f_m(i+h+s), \quad s \in [0,1]$$

and

$$a_N := \inf\{y : \mathbb{P}[|L_1| > y] < \frac{1}{N}\}.$$

Let $\hat\gamma(h)$ and $\gamma(h)$ denote the sample and actual covariance of $(X_t)_{t\in\mathbb{Z}}$ as in Section 2.

If $\frac{1}{\alpha} > d$ and $G_{m,h}$ converges in $L^{\frac{\alpha}{2}}([0,1])$ to G_h, then

$$\frac{N}{a_N^2}\Big(\hat\gamma_N(0)-\gamma(0),\dots,\hat\gamma_N(H)-\gamma(H)\Big) \xrightarrow{d} \Big(\int_0^1 G_0(s)\, dM_s, \dots, \int_0^1 G_H(s)\, dM_s\Big) \text{ as } N \to \infty,$$

where the stochastic integrals with respect to $(M_s)_{s\in[0,1]}$ defined in (3.22) are defined by convergence in probability, see e.g. Section 3.4 in [34]. Observe that G_h is bounded, hence it is in $L^{\frac{\alpha}{2}}([0,1])$ and the stochastic integrals are well-defined.

If $\frac{1}{\alpha} < d$, then

$$N^{1-2d}(\hat\gamma_N(0)-\gamma(0),\dots,\hat\gamma_N(H)-\gamma(H)) \xrightarrow{d} C_d^2 \sigma^2 U_d(1)(1,\dots,1) \text{ as } N \to \infty,$$

where $U_d(1)$ is the marginal distribution of the Rosenblatt process at time 1 defined in (3.2).

Remark 3.3.1. 1. The assumption that the sequence $G_{m,h}$ converges in $L^{\frac{\alpha}{2}}([0,1])$ to G_h is for example fulfilled if f is left-continuous. The sequence f_m then converges pointwise to f. We can assume that f_m is bounded by $K(1_{[0,1]}(t) + t^{d-1}1_{(1,\infty)}(t))$ for K large enough. Hence by Lebesgue's convergence theorem $G_{m,h}$ converges to G_h pointwise, since $K^2(1 + \sum_{t=1}^{\infty} t^{2(d-1)})$ is finite. Since the sequence $G_{m,h}$ is bounded by $K^2(1 + \sum_{t=1}^{\infty} t^{2(d-1)})$, it converges by using Lebesgue's convergence theorem a second time in $L^{\alpha/2}([0,1])$ as well.
2. The assumption that $G_{m,h}$ converges in $L^{\frac{\alpha}{2}}([0,1])$ to G_h can also be weakened by assuming that f coincides with a left-continuous function, say $\tilde f$, apart from a Lebesgue nullset. For then the processes $(X_t)_{t\in\mathbb{R}}$ based on f and $\tilde f$ coincide, as do the quantities G_h based on f and $\tilde f$ in $L^{\frac{\alpha}{2}}([0,1])$. Hence by proving the theorem for $\tilde f$, it also holds for f.

Remark 3.3.2. Note that Theorem 3.3.1 holds as well if we assume that $\mathbb{E}[L_1] = 0$, L_1 fulfils the tail balance condition (3.18), L has no Gaussian part and $\mathbb{E}[L_1^{\leq, a_N}] = 0$ for all $N \in \mathbb{N}$ as we point out in Remark 3.3.3 where also $L_1^{\leq, a_N}$ is defined.

The rest of this section is devoted to the proof of Theorem 3.3.1. As in section 2, we approximate f by f_m and consequently approximate $(X_t)_{t\in\mathbb{R}}$ by $(X_t^{(m)})_{t\in\mathbb{R}}$ defined by

$$X_t^{(m)} := \int_{-\infty}^{\infty} f_m(t-s)\, dL_s = \sum_{k=0}^{\infty} f(\varepsilon k)(L_{t-\varepsilon k} - L_{t-\varepsilon(k+1)}) = \sum_{k=0}^{\infty} f(\varepsilon k) Z_{mt-k}, \quad t \in \mathbb{Z}.$$

We split the autocovariance function again in diagonal and non-diagonal parts. The first step of the proof is to show that the squares of the smaller increments $(L_{t-\varepsilon k} - L_{t-\varepsilon(k+1)})$ are in the domain of attraction of the m-th convolution root of $S_{\frac{\alpha}{2}}(\tau, \beta, \mu)$. To this end, we need the following lemma:

Lemma 3.3.1. *Let $(L_t)_{t\in\mathbb{R}}$ be a two-sided Lévy process. Assume that L_1 is regularly varying with index $\alpha \in (0,\infty)$ and fulfils the tail balance condition (3.18) with $p \in [0,1]$. Then L_ε is also regularly varying with index α and fulfils (3.18) with p as well. If we define the norming sequences $(a_N)_{N\in\mathbb{N}}$ and $(c_N)_{N\in\mathbb{N}}$ by*

$$a_N := \inf\{y : \mathbb{P}[|L_1| > y] < \frac{1}{N}\} \text{ and } c_N := \inf\{y : \mathbb{P}[|L_\varepsilon| > y] < \frac{1}{N}\},$$

then

$$\lim_{N\to\infty} \frac{a_N}{c_N} = m^{1/\alpha}.$$

Proof. It is well-known, cf. e.g. Hult and Lindskog [25] Proposition 3.1, that an infinitely divisible distribution fulfilling a tail balance condition and its Lévy measure have the same behaviour as regularly varying measures. Note that this result originally goes back to Embrechts et al. [18] for subexponential measures on $(0,\infty)$. Hence, since the Lévy measure of L_ε is ε times the Lévy measure of L_1, we see that L_ε is regularly varying with index α and that it satisfies the same tail balance condition. Now define $G(x) := \mathbb{P}[|L_1| > x]$ and $H(x) := \mathbb{P}[|L_\varepsilon| > x]$. By the subexponentiality of the tails, see [20] section 1.1.1, we know that $\lim_{x\to\infty} \frac{G(x)}{H(x)} = m$. We set $U_1 := \frac{1}{G}$ and $U_2 := \frac{1}{H}$. U_1 and U_2 fulfil the conditions of Proposition 2.6 vi) of [33]. Hence we obtain $U_1^{\leftarrow}(x) \sim m^{1/\alpha} U_2^{\leftarrow}(x)$ as $x \to \infty$, where $U_1^{\leftarrow}$ denotes the left-continuous inverse of U_1, which is defined by

$$U_1^{\leftarrow}(x) := \inf\{s : U_1(s) \geq x\} = \inf\{s : G(s) \leq \frac{1}{x}\}.$$

Hence because of $\frac{a_{N-1}}{a_N} \leq \frac{U_1^{\leftarrow}(N)}{a_N} \leq 1$, we see $a_N \sim U_1^{\leftarrow}(N)$ as $N \to \infty$. We see $c_N \sim U_2^{\leftarrow}(N)$ in the same manner. This shows that $\lim_{N\to\infty} \frac{a_N}{c_N} = m^{1/\alpha}$. □

Now we have the following lemma:

Lemma 3.3.2. *Let $(L_t)_{t\in\mathbb{R}}$ be a two-sided Lévy process. Assume that L_1 is regularly varying with index $\alpha \in (0,4)$ and fulfils the tail balance condition (3.18). Let $(a_N)_{N\in\mathbb{N}}$ be defined by $a_N := \inf\{y : \mathbb{P}[|L_1| > y] < \frac{1}{N}\}$ and define $b_N := \mathbb{E}[L_1^2 1_{\{|L_1|\le a_N\}}]$ such that*

$$\frac{1}{a_N^2}\sum_{t=1}^{N}((L_t - L_{t-1})^2 - b_N) \xrightarrow{d} S_{\alpha/2}(\tau,\beta,\mu)$$

for constants τ,β and μ.

Then

$$\frac{1}{a_N^2}\sum_{t=1}^{N}((L_t - L_{t-\varepsilon})^2 - \varepsilon b_N) \xrightarrow{d} S_{\frac{\alpha}{2}}(\varepsilon^{2/\alpha}\tau,\beta,\varepsilon\mu).$$

Proof. Define $c_N := \inf\{y : \mathbb{P}[|L_\varepsilon| > y] < \frac{1}{N}\}$ and $d_N := \mathbb{E}[L_\varepsilon^2 1_{\{|L_\varepsilon|\le c_N\}}]$. Then as in (3.19), we have

$$\frac{1}{c_N^2}\sum_{t=1}^{N}((L_t - L_{t-\varepsilon})^2 - d_N) \xrightarrow{d} T$$

for a stable law T. By Karamata's Theorem, we obtain $\lim_{N\to\infty}\frac{N}{a_N^2}(\sigma^2 - b_N) = \frac{\alpha}{\alpha-2}$ and $\lim_{N\to\infty}\frac{N}{c_N^2}(\varepsilon\sigma^2 - d_N) = \frac{\alpha}{\alpha-2}$. Hence by Lemma 3.3.1, $\lim_{N\to\infty}\frac{N}{c_N^2}(\varepsilon\sigma^2 - \varepsilon b_N) = \varepsilon^{1-\frac{2}{\alpha}}\frac{\alpha}{\alpha-2}$ and hence in turn $\lim_{N\to\infty}\frac{N}{c_N^2}(\varepsilon b_N - d_N) = (1-\varepsilon^{1-\frac{2}{\alpha}})\frac{\alpha}{\alpha-2}$. Hence we can replace $(d_N)_{N\in\mathbb{N}}$ by $(\varepsilon b_N)_{N\in\mathbb{N}}$ and still obtain a stable limit. By Lemma 3.3.1 we can replace $(c_N)_{N\in\mathbb{N}}$ by $(a_N)_{N\in\mathbb{N}}$ and obtain a stable limit as well. We conclude that

$$\frac{1}{a_N^2}\sum_{t=1}^{N}((L_t - L_{t-\varepsilon})^2 - \varepsilon b_N) \xrightarrow{d} S$$

for a stable law S. We now show that $S = S_{\frac{\alpha}{2}}(\varepsilon^{2/\alpha}\tau,\beta,\varepsilon\mu)$.

We split the normed partial sums in the following manner:

$$\begin{aligned}\frac{1}{a_N^2}\sum_{t=1}^{N}((L_t - L_{t-1})^2 - b_N) &= \frac{1}{a_N^2}\sum_{t=1}^{N}\Big(\Big(\sum_{i=1}^{m}(L_{t+\varepsilon i} - L_{t+\varepsilon(i-1)})\Big)^2 - b_N\Big)\\ &= \underbrace{\sum_{i=1}^{m}\underbrace{\frac{1}{a_N^2}\sum_{t=1}^{N}\Big((L_{t+\varepsilon i} - L_{t+\varepsilon(i-1)})^2 - \varepsilon b_N\Big)}_{\to\text{stable law } S}}_{\to S^{m*}}\\ &\quad + \sum_{i,j=1 i\neq j}^{m}\underbrace{\frac{1}{a_N^2}\sum_{t=1}^{N}(L_{t+\varepsilon i} - L_{t+\varepsilon(i-1)})(L_{t+\varepsilon j} - L_{t+\varepsilon(j-1)})}_{\xrightarrow{\mathbb{P}}0},\end{aligned}$$

where the convergences are justified as follows: since we sum up random variables of independent sequences, the sum converges towards the convolution of the distributional limits. By Lemma 4.1 of [26], $|(L_{t+\varepsilon i} - L_{t+\varepsilon(i-1)})(L_{t+\varepsilon j} - L_{t+\varepsilon(j-1)})|$ is regularly varying with index α for $i \neq j$, hence

$$\frac{1}{a_N^2}\sum_{t=1}^{N} |(L_{t+\varepsilon i} - L_{t+\varepsilon(i-1)})(L_{t+\varepsilon j} - L_{t+\varepsilon(j-1)})|, \quad i \neq j,$$

converges in probability to zero by Lemma 3.3.1 and hence so does

$$\frac{1}{a_N^2}\sum_{t=1}^{N} (L_{t+\varepsilon i} - L_{t+\varepsilon(i-1)})(L_{t+\varepsilon j} - L_{t+\varepsilon(j-1)}), \quad i \neq j.$$

Now it is obvious (see e.g. [34], Property 1.2.1, p.10) that $S = S_{\frac{\alpha}{2}}(\varepsilon^{2/\alpha}\tau, \beta, \varepsilon\mu)$. □

We now show that the diagonal parts of the autocovariance function of the approximated process $(X_t^{(m)})_{t\in\mathbb{R}}$ converge to a stable law expressed as a stochastic integral with respect to a stable Lévy process.

Lemma 3.3.3. *Let $(L_t)_{t\in\mathbb{R}}$ be a two-sided Lévy process. Assume that L_1 is regularly varying with index $\alpha \in (2,4)$ and fulfils the tail balance condition (3.18). Let f and f_m be as in Theorem 3.3.1. Define*

$$a_N := \inf\{y : \mathbb{P}[|L_1| > y] < \frac{1}{N}\} \text{ and } b_N := \mathbb{E}[L_1^2 1_{\{|L_1|\leq a_N\}}].$$

Define

$$G_{m,h}(s) := \sum_{i=-\infty}^{\infty} f_m(i+s) f_m(i+h+s), \quad s \in [0,1]$$

and

$$d_{N,h,\varepsilon} := \frac{1}{N}\sum_{t=1}^{N}\sum_{i=0}^{\infty} f(\varepsilon i) f(\varepsilon i + h)((Z_{mt-i})^2 - \varepsilon b_N), \tag{3.23}$$

where $(Z_i)_{i\in\mathbb{Z}}$ was defined in (3.4). Then $d_{N,h,\varepsilon}$ converges absolutely almost surely and in $L^1(\mathbb{P})$. Further,

$$\frac{N}{a_N^2}(d_{N,0,\varepsilon}, \ldots, d_{N,H,\varepsilon}) \xrightarrow{d} (\int_0^1 G_{m,0}(s)\, dK_s, \ldots, \int_0^1 G_{m,H}(s)\, dK_s),$$

where $(K_s)_{s\in[0,1]}$ is defined in (3.21). Observe that $G_{m,h}$ is bounded, hence it is in $L^{\frac{\alpha}{2}}([0,1])$ and the stochastic integrals are well-defined.

Proof. The almost sure absolute convergence and convergence in $L^1(\mathbb{P})$ of $d_{N,h,\varepsilon}$ are clear. By Lemma 3.3.2, we know that

$$\frac{1}{a_N^2}\sum_{t=1}^{N}((L_t - L_{t-\varepsilon})^2 - \varepsilon b_N) \xrightarrow{d} S_{\frac{\alpha}{2}}(\varepsilon^{2/\alpha}\tau, \beta, \varepsilon\mu).$$

By rearranging one sees that

$$d_{N,h,\varepsilon} = \frac{1}{N}\sum_{t=1}^{N}\sum_{j\in\{0,\dots,m-1\}}\sum_{i=0}^{\infty} f(i+\varepsilon j)f(i+h+\varepsilon j)((L_{t-i-\varepsilon j} - L_{t-i-\varepsilon(j-1)})^2 - \varepsilon b_N).$$

By Theorem 4.1 of Davis and Resnick [14] and using the technique used in the proof of Lemma 5.1 in [23], we obtain with

$$d_{N,h,\varepsilon,j} := \frac{1}{N}\sum_{t=1}^{N}\sum_{i=0}^{\infty} f(i+\varepsilon j)f(i+h+\varepsilon j)((L_{t-i-\varepsilon j} - L_{t-i-\varepsilon(j-1)})^2 - \varepsilon b_N)$$

that

$$\frac{N}{a_N^2}\Big(d_{N,0,\varepsilon,j},\dots,d_{N,H,\varepsilon,j}\Big) \xrightarrow{d} \Big(G_{m,0}(\varepsilon j),\dots,G_{m,H}(\varepsilon j)\Big)S_{\frac{\alpha}{2}}(\varepsilon^{2/\alpha}\tau,\beta,\varepsilon\mu)$$

for each $j \in \{0,\dots,m-1\}$. Since the sequences are independent for different j, the convolution of the limits equals the limit of the sums and the claimed convergence follows. Observe that we use the fact that $G_{m,h}$ is an equidistant step function. □

The convergence of the non-diagonal parts of the autocovariance function to the Rosenblatt distribution was already established in section 2. Hence we can state the following lemma:

Lemma 3.3.4. *Let the assumptions of Lemma 3.3.3 be fulfilled. Define $r_{N,h,\varepsilon}$ as in (3.9) and $d_{N,h,\varepsilon}$ as in (3.23). If $\frac{1}{\alpha} > d$, then*

$$\frac{N}{a_N^2}((d_{N,0,\varepsilon} + r_{N,0,\varepsilon}),\dots,(d_{N,H,\varepsilon} + r_{N,H,\varepsilon})) \xrightarrow{d} (\int_0^1 G_{m,0}(s)\,dK_s,\dots,\int_0^1 G_{m,H}(s)\,dK_s).$$

If $\frac{1}{\alpha} < d$, then

$$N^{1-2d}((d_{N,0,\varepsilon} + r_{N,0,\varepsilon}),\dots,(d_{N,H,\varepsilon} + r_{N,H,\varepsilon})) \xrightarrow{d} C_d^2\sigma^2 U_d(1)(1,\dots,1).$$

Proof. Note that a_N^2 is regularly varying with index $\frac{2}{\alpha}$. Hence $N^{2d} = o(a_N^2)$ if $\frac{1}{\alpha} > d$ and $a_N^2 = o(N^{2d})$ if $\frac{1}{\alpha} < d$. The lemma then follows by Slutsky's lemma using Lemma 3.2.2 and Lemma 3.3.3. □

Remark 3.3.3. Now let us assume that the Lévy process is symmetric and has no Gaussian part. We decompose the Lévy process into two independent Lévy processes: let $L^{\le,a_N}$ have the Lévy measure of L restricted to $[-a_N,a_N]$ and $L^{>,a_N}$ have the Lévy measure of L restricted to $[-a_N,a_N]^c$. Note that $L^{\le,a_N}$ has the variance $\mathrm{Var}(L_1^{\le,a_N}) = \int_{-a_N}^{a_N} x^2\nu(dx)$, see Example 25.12 in [35], hence $\mathbb{E}[(L^{\le,a_N})^2] = \int_{-a_N}^{a_N} x^2\nu(dx) + \mathbb{E}[L_1^{\le,a_N}]^2 = \int_{-a_N}^{a_N} x^2\nu(dx)$, while $\mathbb{E}[(L_1 1_{\{|L_1|\le a_N\}})^2] = \int_{-a_N}^{a_N} x^2\mu(dx)$ where μ denotes the distribution of L_1 and ν denotes its Lévy measure. Of course μ and ν are in general not equal but $\mathbb{E}[(L_1)^2] = \int_{-\infty}^{\infty} x^2\mu(dx) = \int_{-\infty}^{\infty} x^2\nu(dx) + \mathbb{E}[L_1]^2 = \int_{-\infty}^{\infty} x^2\nu(dx)$ is true, see Example 25.12 in [35]. We know additionally that both μ and ν have the same tail behaviour, i.e. they are both regularly varying with the same index and the same tail balance condition, see for example Hult and Lindskog [25] Proposition 3.1. Hence by Karamata's theorem, by the tail equivalence of μ and ν and by the symmetry of μ and ν,

$$\begin{aligned}\lim_{N\to\infty}\frac{N}{a_N^2}(\sigma^2-\int_{-a_N}^{a_N}x^2\,\nu(dx)) &= \lim_{N\to\infty}\frac{N}{a_N^2}2\int_{a_N}^{\infty}x^2\,\nu(dx)\\ =\frac{\alpha}{\alpha-2} &= \lim_{N\to\infty}\frac{N}{a_N^2}(\sigma^2-\int_{-a_N}^{a_N}x^2\,\mu(dx)).\end{aligned}$$

Hence we can replace the centring sequence $b_N := \mathbb{E}[L_1^2 1_{\{|L_1|\le a_N\}}]$ by $\tilde{b}_N := \mathbb{E}[(L_1^{\le,a_N})^2]$ without changing the limit in (3.19). Note that $\lim_{N\to\infty}\frac{N}{a_N^2}(b_N-\tilde{b}_N)=0$.

Note finally that this works as well if we assume that $\mathbb{E}[L_1]=0$, L has no Gaussian part and $\mathbb{E}[L_1^{\le,a_N}]=0$ for all $N\in\mathbb{N}$.

Lemma 3.3.5. *Let the assumptions of Lemma 3.3.3 be fulfilled. Further assume that L_1 is symmetric about zero and has no Gaussian part. If $\frac{1}{\alpha}>d$, then for all $\delta>0$*

$$\lim_{\varepsilon\to 0}\limsup_{N\to\infty}\mathbb{P}[\Big|\frac{N}{a_N^2}\Big(\hat{\gamma}_N(h)-b_N\int_{-\infty}^{\infty}f(s)f(s+h)\,ds-(d_{N,h,\varepsilon}+r_{N,h,\varepsilon})\Big)\Big|>\delta]=0.$$

If $\frac{1}{\alpha}<d$, then for all $\delta>0$

$$\lim_{N\to\infty}\mathbb{P}[\Big|N^{1-2d}\Big(\hat{\gamma}_N(h)-b_N\int_{-\infty}^{\infty}f(s)f(s+h)\,ds-(d_{N,h,\varepsilon}+r_{N,h,\varepsilon})\Big)\Big|>\delta]=0.$$

Proof. We first show the first claim. Note that by (3.16)

$$\begin{aligned}&\hat{\gamma}_N(h)-b_N\int_{-\infty}^{\infty}f(s)f(s+h)\,ds-(d_{N,h,\varepsilon}+r_{N,h,\varepsilon})\\ =\;&\bar{d}_{N,h,\varepsilon}-b_N\int_{-\infty}^{\infty}f(s)f(s+h)\,ds-d_{N,h,\varepsilon}-(r_{N,h,\varepsilon}-\bar{r}_{N,h,\varepsilon}).\end{aligned}$$

We denote $Z_i^{\le,a_N}:=L_{\varepsilon i}^{\le,a_N}-L_{\varepsilon(i-1)}^{\le,a_N}$ and $Z_i^{>,a_N}:=L_{\varepsilon i}^{>,a_N}-L_{\varepsilon(i-1)}^{>,a_N}$ in analogy to (3.4), where we use the notation of Remark 3.3.3. In the same manner we define in analogy to (3.13), (3.15) and (3.23)

$$\bar{Z}_{k,t,h}^{\le,a_N}:=\int_{\varepsilon(k-1)}^{\varepsilon k}f(t+h-s)\,dL_s^{\le,a_N},\quad k\in\mathbb{Z},\tag{3.24}$$

$$\bar{Z}_{k,t,h}^{>,a_N}:=\int_{\varepsilon(k-1)}^{\varepsilon k}f(t+h-s)\,dL_s^{>,a_N},\quad k\in\mathbb{Z},\tag{3.25}$$

$$\bar{d}_{N,h,\varepsilon}^{\le,a_N}:=\frac{1}{N}\sum_{t=1}^{N}\sum_{k}\bar{Z}_{k,t,0}^{\le,a_N}\bar{Z}_{k,t,h}^{\le,a_N},\quad h\in\mathbb{N}_0\tag{3.26}$$

and

$$d_{N,h,\varepsilon}^{\le,a_N}:=\frac{1}{N}\sum_{t=1}^{N}\sum_{i=0}^{\infty}f(\varepsilon i)f(\varepsilon i+h)((Z_{mt-i}^{\le,a_N})^2-\varepsilon\tilde{b}_N).\tag{3.27}$$

$\bar{d}^{\leq,a_N}_{N,h,\varepsilon}$ and $d^{\leq,a_N}_{N,h,\varepsilon}$ can be seen to converge unconditionally as in (3.15) and (3.23), where we use the fact that $\mathbb{E}[L_1^{\leq,a_N}] = 0$ for all $N \in \mathbb{N}$ by our symmetry assumption.

We consider the upper estimate

$$\begin{aligned}
& \mathbb{P}[|\frac{N}{a_N^2}(\bar{d}_{N,h,\varepsilon} - b_N(\int_{-\infty}^{\infty} f(s)f(s+h)\,ds) - d_{N,h,\varepsilon} - (r_{N,h,\varepsilon} - \bar{r}_{N,h,\varepsilon}))| > \delta] \\
\leq\; & \mathbb{P}[|\frac{N}{a_N^2}(\bar{d}^{\leq,a_N}_{N,h,\varepsilon} - \tilde{b}_N(\int_{-\infty}^{\infty} f(s)f(s+h)\,ds))| > \frac{\delta}{7}] \\
+\; & \mathbb{P}[|\frac{N}{a_N^2}(d^{\leq,a_N}_{N,h,\varepsilon})| > \frac{\delta}{7}] \\
+\; & \mathbb{P}[|\frac{N}{a_N^2}(\bar{r}_{N,h,\varepsilon} - r_{N,h,\varepsilon})| > \frac{\delta}{7}] \\
+\; & \mathbb{P}[|\frac{N}{a_N^2}(\frac{1}{N}\sum_{t=1}^{N}\sum_{i=0}^{\infty} f(\varepsilon i)f(\varepsilon i+h)(Z^{>,a_N}_{mt-i})^2 - \frac{1}{N}\sum_{t=1}^{N}\sum_{k}(\bar{Z}^{>,a_N}_{k,t,0})(\bar{Z}^{>,a_N}_{k,t,h}))| > \frac{\delta}{7}] \\
+\; & \mathbb{P}[|\frac{N}{a_N^2}(\frac{1}{N}\sum_{t=1}^{N}\sum_{i=0}^{\infty} f(\varepsilon i)f(\varepsilon i+h)(Z^{\leq,a_N}_{mt-i})(Z^{>,a_N}_{mt-i}) - \frac{1}{N}\sum_{t=1}^{N}\sum_{k}(\bar{Z}^{\leq,a_N}_{k,t,0})(\bar{Z}^{>,a_N}_{k,t,h}))| > \frac{\delta}{7}] \\
+\; & \mathbb{P}[|\frac{N}{a_N^2}(\frac{1}{N}\sum_{t=1}^{N}\sum_{i=0}^{\infty} f(\varepsilon i)f(\varepsilon i+h)(Z^{\leq,a_N}_{mt-i})(Z^{>,a_N}_{mt-i}) - \frac{1}{N}\sum_{t=1}^{N}\sum_{k}(\bar{Z}^{>,a_N}_{k,t,0})(\bar{Z}^{\leq,a_N}_{k,t,h}))| > \frac{\delta}{7}] \\
+\; & \mathbb{P}[|\frac{N}{a_N^2}(b_N - \tilde{b}_N)(\int_{-\infty}^{\infty} f(s)f(s+h)\,ds - \varepsilon\sum_{i=0}^{\infty} f(\varepsilon i)f(\varepsilon i+h))| > \frac{\delta}{7}].
\end{aligned}$$

The last term vanishes for N large enough since $\lim_{N\to\infty}\frac{N}{a_N^2}(b_N - \tilde{b}_N) = 0$ by Remark 3.3.3. $\mathrm{Var}(N(\bar{d}^{\leq,a_N}_{N,h,\varepsilon} - \tilde{b}_N(\int_{-\infty}^{\infty} f(s)f(s+h)\,ds)))$ is in $O(NE[(L_1^{\leq,a_N})^2]) \leq O(N)$ by the calculations of Lemma 3.2.5. Hence

$$\limsup_{N\to\infty}\mathbb{P}[|\frac{N}{a_N^2}(\bar{d}^{\leq,a_N}_{N,h,\varepsilon} - \tilde{b}_N(\int_{-\infty}^{\infty} f(s)f(s+h)\,ds))| > \frac{\delta}{7}] = 0,$$

since a_N^2 is a regularly varying sequence with index $2/\alpha$ and $\alpha < 4$. This also applies to the second term by using f_m instead of f as the corresponding function. The third term is negligible by the calculations in Lemma 3.2.4, which show that $\mathbb{E}[(N(\bar{r}_{N,h,\varepsilon} - r_{N,h,\varepsilon}))^2] = o(a_N^4)$ if $d < \frac{1}{\alpha}$. Now we consider the fourth term. To this end, define

$$\xi_t := \sum_k (f(t-\varepsilon k)f(t+h-\varepsilon k)(Z_k^{>,a_N})^2 - \bar{Z}^{>,a_N}_{k,t,0}\bar{Z}^{>,a_N}_{k,t,h}).$$

ξ_t can be seen to converge unconditionally in $L^1(\mathbb{P})$. Then

$$\sum_{t=1}^{N}\sum_{i=0}^{\infty} f(\varepsilon i)f(\varepsilon i+h)(Z^{>,a_N}_{mt-i})^2 - \sum_{t=1}^{N}\sum_{k}(\bar{Z}^{>,a_N}_{k,t,0})(\bar{Z}^{>,a_N}_{k,t,h}) = \sum_{t=1}^{N}\xi_t.$$

We split the ξ_t in two parts:

$$\begin{aligned}
&\sum_k (f(t-\varepsilon k)f(t+h-\varepsilon k)(Z_k^{>a_N})^2 - \bar{Z}_{k,t,0}^{>a_N}\bar{Z}_{k,t,h}^{>a_N}) \\
= &\sum_k ((\int_{\varepsilon(k-1)}^{\varepsilon k} f(t-\varepsilon k)\, dL_s^{>a_N})(\int_{\varepsilon(k-1)}^{\varepsilon k} f(t+h-\varepsilon k)\, dL_s^{>a_N}) \\
&-(\int_{\varepsilon(k-1)}^{\varepsilon k} f(t-s)\, dL_s^{>a_N})(\int_{\varepsilon(k-1)}^{\varepsilon k} f(t+h-s)\, dL_s^{>a_N})) \\
= &\sum_k ((\int_{\varepsilon(k-1)}^{\varepsilon k} f(t-\varepsilon k)\, dL_s^{>a_N})(\int_{\varepsilon(k-1)}^{\varepsilon k} (f(t+h-\varepsilon k) - f(t+h-s))\, dL_s^{>a_N}) \\
&+(\int_{\varepsilon(k-1)}^{\varepsilon k} (f(t-\varepsilon k) - f(t-s))\, dL_s^{>a_N})(\int_{\varepsilon(k-1)}^{\varepsilon k} f(t+h-s)\, dL_s^{>a_N})) \\
=: &\ A_t + B_t.
\end{aligned}$$

We show that $\limsup_{N\to\infty} \mathbb{E}[\frac{1}{a_N^2}\sum_{t=1}^N |A_t|] = O(\varepsilon)$ as $\varepsilon \to 0$. The same argument applies to B_t. The calculations also show that the series defining A_t and B_t converge a.s. absolutely and unconditionally in $L^1(\mathbb{P})$ and hence they are well-defined.

We define a third Lévy process $|L^{>a_N}|$ which is defined by $|L^{>a_N}|_t := \sum_{0<s\le t} |\Delta L_s^{>a_N}|$ for $t \ge 0$ and $|L^{>a_N}|_t := -\sum_{t<s<0} |\Delta L_s^{>a_N}|$ for $t < 0$. Define

$$u(t) := C(t^{d-1}1_{(1,\infty)}(t) + 1_{[-\varepsilon,1]}(t))$$

with C large enough. We obtain

$$\begin{aligned}
&\mathbb{E}[|(\int_{\varepsilon(k-1)}^{\varepsilon k} f(t-\varepsilon k)\, dL_s^{>a_N})(\int_{\varepsilon(k-1)}^{\varepsilon k} (f(t+h-\varepsilon k) - f(t+h-s))\, dL_s^{>a_N})|] \\
\le &\ \mathbb{E}[(\int_{\varepsilon(k-1)}^{\varepsilon k} |f(t-\varepsilon k)|\, d|L^{>a_N}|_s)(\int_{\varepsilon(k-1)}^{\varepsilon k} |(f(t+h-\varepsilon k) - f(t+h-s))|\, d|L^{>a_N}|_s)] \\
\le &\ \varepsilon^2 |f(t-\varepsilon k)| u(t+h-\varepsilon k)\mathbb{E}[(|L_1^{>a_N}|)^2] \\
\le &\ \varepsilon^2 |f(t-\varepsilon k)| u(t-\varepsilon k)\mathbb{E}[(|L^{>a_N}|_1)^2],
\end{aligned}$$

since $u(t-\varepsilon k) \ge u(t+h-\varepsilon k)$ if $t-\varepsilon k \ge -\varepsilon$ and $f(t-\varepsilon k) = 0$ otherwise. Note that we use $|f(t+h-\varepsilon k) - f(t+h-s)| \le u(t+h-\varepsilon k)$ by the triangle inequality for C large enough. Note further

$$\begin{aligned}
\sum_k |f(t-\varepsilon k)| u(t-\varepsilon k) &= \sum_{i=1}^{\frac{1}{\varepsilon}} |f(\varepsilon i) u(\varepsilon i)| + \sum_{1+\frac{1}{\varepsilon}}^{\infty} |f(\varepsilon i) u(\varepsilon i)| \\
&\le KC\frac{1}{\varepsilon} + KC\int_{\frac{1}{\varepsilon}}^{\infty} (\varepsilon x)^{2d-2}\, dx \\
&= KC\frac{1}{\varepsilon} + KC\frac{1}{\varepsilon}\int_1^{\infty} y^{2d-2}\, dy = O(\frac{1}{\varepsilon}) \text{ as } \varepsilon \to 0.
\end{aligned}$$

By Karamata's theorem, see Theorem 1.6.5 in Bingham et al. [3],

$$\lim_{N\to\infty} \frac{N}{a_N^2} 2\int_{a_N}^{\infty} x^2\,\nu(dx)$$

exists and is finite. Also by Karamata's theorem, $\lim_{N\to\infty}\frac{N}{a_N}2\int_{a_N}^{\infty} x\,\nu(dx)$ exists and is finite, hence $\lim_{N\to\infty}\frac{N}{a_N^2}(2\int_{a_N}^{\infty} x\,\nu(dx))^2 = 0$. Thus

$$\lim_{N\to\infty} \frac{N}{a_N^2}\mathbb{E}[|L^{>a_N}|_1^2] = \lim_{N\to\infty} \frac{N}{a_N^2}[2\int_{a_N}^{\infty} x^2\,\nu(dx) + (2\int_{a_N}^{\infty} x\,\nu(dx))^2]$$

exists and is finite. Hence $\limsup_{N\to\infty}\mathbb{E}[\frac{1}{a_N^2}\sum_{t=1}^N |A_t|] = O(\varepsilon)$ as $\varepsilon\to 0$. In the same fashion one can show that $\limsup_{N\to\infty}\mathbb{E}[\frac{1}{a_N^2}\sum_{t=1}^N |B_t|] = O(\varepsilon)$ as $\varepsilon\to 0$. By Markov's inequality, this gives convergence of the fourth term to 0 when letting first $N\to\infty$ and then $\varepsilon\to 0$.

Finally, we consider the fifth and sixth term. They are dealt with in the same manner. We use the same reasoning as for the fourth term and consider $\sum_{t=1}^N \xi_t$ where we replace $(Z_{mt-i}^{>,a_N})^2$ by $(Z_{mt-i}^{>,a_N})(Z_{mt-i}^{\le,a_N})$ and replace the first or second factor of $(\bar{Z}_{k,t,0}^{>,a_N})(\bar{Z}_{k,t,h}^{>,a_N})$. We then consider A_t and see that it is negligible by the following calculations:

By using the independence of $L^{>a_N}$ and $L^{\le a_N}$, Jensen's inequality and Itô's isometry, we obtain

$$\begin{aligned}
&\mathbb{E}[|(\int_{\varepsilon(k-1)}^{\varepsilon k} f(t-\varepsilon k)\,dL_s^{\le a_N})(\int_{\varepsilon(k-1)}^{\varepsilon k}(f(t+h-\varepsilon k)-f(t+h-s))\,dL_s^{>a_N})|]\\
\le\ &\mathbb{E}[|(\int_{\varepsilon(k-1)}^{\varepsilon k} f(t-\varepsilon k)\,dL_s^{\le a_N})|]\mathbb{E}[|(\int_{\varepsilon(k-1)}^{\varepsilon k}(f(t+h-\varepsilon k)-f(t+h-s))\,dL_s^{>a_N})|]\\
\le\ &\sqrt{\mathbb{E}[((\int_{\varepsilon(k-1)}^{\varepsilon k} f(t-\varepsilon k)\,dL_s^{\le a_N}))^2]}\mathbb{E}[|(\int_{\varepsilon(k-1)}^{\varepsilon k}|(f(t+h-\varepsilon k)-f(t+h-s))|\,d|L^{>a_N}|_s)|]\\
\le\ &\sqrt{(\int_{\varepsilon(k-1)}^{\varepsilon k} f(t-\varepsilon k)^2\,ds)}\,\sigma u(t+h-\varepsilon k)\varepsilon\mathbb{E}[|L^{>a_N}|_1]\\
\le\ &\varepsilon^{\frac{3}{2}}|f(t-\varepsilon k)|\sigma u(t-\varepsilon k)\mathbb{E}[|L^{>a_N}|_1].
\end{aligned}$$

By Karamata's theorem, $\lim_{N\to\infty}\frac{N}{a_N}\mathbb{E}[|L^{>a_N}|_1]$ exists and is finite, hence

$$\lim_{N\to\infty}\frac{N}{a_N^2}\mathbb{E}[|L^{>a_N}|_1] = 0$$

and in turn $\limsup_{N\to\infty}\mathbb{E}[\frac{1}{a_N^2}\sum_{t=1}^N |A_t|] = 0$ for all $\varepsilon > 0$. In this case similar calculations show that $\limsup_{N\to\infty}\mathbb{E}[\frac{1}{a_N^2}\sum_{t=1}^N |B_t|] = 0$ as well.

The second claim follows from the calculations for the first case and the fact that $a_N^2 = o(N^{2d})$ if $d > \frac{1}{\alpha}$. □

Returning to the proof Theorem 3.3.1, We first conclude the proof for the case $\frac{1}{\alpha} > d$: We show

$$\begin{aligned} &\frac{N}{a_N^2}\Big(\hat\gamma_N(0) - b_N\int_{-\infty}^{\infty} f(s)f(s)\,ds, \dots, \hat\gamma_N(H) - b_N\int_{-\infty}^{\infty} f(s)f(s+H)\,ds\Big) \\ \xrightarrow{d}\ &\Big(\int_0^1 G_0(s)\,dK_s, \dots, \int_0^1 G_H(s)\,dK_s\Big), \text{ as } N\to\infty. \end{aligned}$$

Since the sequence $G_{m,h}$ converges in $L^{\frac{\alpha}{2}}([0,1])$ to G_h, it follows that $\int_0^1 G_{m,h}(s)\,dK_s \xrightarrow{d} \int_0^1 G_h(s)\,dK_s$, see Proposition 3.5.1 in [34]. Hence for the one-dimensional result by Theorem 3.2 in [2] together with Lemma 3.3.4, it suffices to check for all $\delta > 0$

$$\lim_{\varepsilon\to 0}\limsup_{N\to\infty} \mathbb{P}[|\frac{N}{a_N^2}(\hat\gamma_N(h) - b_N(\int_{-\infty}^{\infty} f(s)f(s+h)\,ds) - (d_{N,h,\varepsilon} + r_{N,h,\varepsilon}))| > \delta] = 0,$$

which has been proved in Lemma 3.3.5. The multidimensional results also follows by the simple fact that a vector converges in probability if its components converge in probability.

By Karamata's theorem, see Theorem 1.6.5 in Bingham et al., we see

$$\begin{aligned} \lim_{N\to\infty}\frac{N}{a_N^2}(\sigma^2 - b_N) &= \lim_{N\to\infty}\frac{N}{a_N^2}(\sigma^2 - \int_{-a_N}^{a_N} x^2\,\mu(dx)) \\ &= \lim_{N\to\infty}\frac{N}{a_N^2}2\int_{a_N}^{\infty} x^2\,\mu(dx) = \frac{\alpha}{\alpha-2}. \end{aligned} \tag{3.28}$$

By (3.28), we have

$$\begin{aligned} \lim_{N\to\infty}\frac{N}{a_N^2}\Big(b_N\int_{-\infty}^{\infty} f(s)f(s+h)\,ds - \gamma(h)\Big) &= \lim_{N\to\infty}\frac{N}{a_N^2}(b_N - \sigma^2)\int_{-\infty}^{\infty} f(s)f(s+h)\,ds \\ &= -\frac{\alpha}{\alpha-2}\int_{-\infty}^{\infty} f(s)f(s+h)\,ds. \end{aligned}$$

Hence we conclude

$$\frac{N}{a_N^2}\Big(\hat\gamma_N(0) - \gamma(0), \dots, \hat\gamma_N(H) - \gamma(H)\Big) \xrightarrow{d}$$

$$(\int_0^1 G_0(s)\,dK_s - \frac{\alpha}{\alpha-2}\int_{-\infty}^{\infty} f(s)f(s)\,ds, \dots, \int_0^1 G_H(s)\,dK_s - \frac{\alpha}{\alpha-2}\int_{-\infty}^{\infty} f(s)f(s+H)\,ds).$$

Note that $\int_{-\infty}^{\infty} f(s)f(s+h)\,ds = \int_0^1 G_h(s)\,ds$. Hence we finally conclude

$$\frac{N}{a_N^2}\Big(\hat\gamma_N(0)-\gamma(0), \dots, \hat\gamma_N(H)-\gamma(H)\Big) \xrightarrow{d} \Big(\int_0^1 G_0(s)\,dM_s, \dots, \int_0^1 G_H(s)\,dM_s\Big) \text{ as } N\to\infty.$$

We now consider the case $\frac{1}{\alpha} < d$: Since $\lim_{N\to\infty}\frac{N}{a_N^2}(\sigma^2 - b_N) = \frac{\alpha}{\alpha-2}$, we have

$$\lim_{N\to\infty} N^{1-2d}(\sigma^2 - b_N) = 0.$$

Hence we can replace $\gamma(h)$ without loss of generality by $b_N(\int_{-\infty}^{\infty} f(s)f(s+h)\,ds)$ without changing the limit. Hence for the one-dimensional result by Slutsky's Lemma together with Lemma 3.3.4, we have to check that for all $\delta > 0$

$$\lim_{N\to\infty} \mathbb{P}[|N^{1-2d}(\hat{\gamma}_N(h) - b_N(\int_{-\infty}^{\infty} f(s)f(s+h)\,ds) - (d_{N,h,\varepsilon} + r_{N,h,\varepsilon}))| > \delta] = 0,$$

which is the statement of Lemma 3.3.5.

3.4 Remarks

Remark 3.4.1. If we assume that $\mathbb{E}[L_1^4] = \eta\sigma^4 < \infty$ and that $f(t) = 0$ for $t \leq 0$, f is bounded and $f(t) \sim C_d t^{d-1}$ as $t \to \infty$ with $d \in (0, \frac{1}{4})$ and $C_d > 0$, then the conditions of Theorem 3.5 (a) in Cohen and Lindner [11] are fulfilled, i.e. the sample autocovariance is asymptotically normal distributed. More precisely,

$$\sqrt{N}(\hat{\gamma}_N(0) - \gamma(0), \ldots, \hat{\gamma}_N(H) - \gamma(H)) \xrightarrow{d} N(0, V),$$

where the covariance matrix $V = (v_{pq})_{p,q=0,\ldots,H}$ is given by

$$v_{pq} = (\eta - 3)\sigma^4 \int_0^1 G_p(u)G_q(u)\,du + \sum_{k=-\infty}^{\infty} [\gamma(k)\gamma(k-p+q) + \gamma(k+q)\gamma(k-p)]$$

and G_p is as defined in Theorem 3.3.1. This corresponds to Theorem 3.5 (a) in [23].

Proof. We have to check the assumptions of Proposition 3.1 and Theorem 3.5 (a) in [11]. It is easy to see that $f \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$, since $|f(t)| \leq K\max(1, t^{d-1})$ for a constant K. We next check that the function $[0,1] \to \mathbb{R}, u \mapsto \sum_{k=-\infty}^{\infty} f(u+k)^2$ is in $L^2([0,1])$, which is in our notation the function G_0. Observe that G_0 is bounded by $K^2(1+\sum_{k=1}^{\infty} k^{2d-2}) < \infty$. Hence it even is in $L^\infty([0,1])$. We need not check (3.3) in [11] since (3.11) in [11] is stronger than (3.3). Finally, we turn to (3.11) in [11], i.e.

$$\sum_{h=1}^{\infty} (\int_{-\infty}^{\infty} |f(s)||f(s+h)|\,ds)^2 < \infty. \tag{3.29}$$

By our assumptions on the kernel function, we obtain

$$\int_{-\infty}^{\infty} |f(s)||f(s+h)|\,ds \leq K^2 \int_0^{\infty} t^{d-1}(t+h)^{d-1}\,dt.$$

By substitution, we obtain for $h > 0$

$$\int_0^{\infty} t^{d-1}(t+h)^{d-1}\,dt = h^{2d-1} \int_0^{\infty} s^{d-1}(s+1)^{d-1}\,ds.$$

Then (3.29) is an immediate consequence of $d < \frac{1}{4}$ and the result follows from Theorem 3.5 in [11]. □

The following definition of a FICARMA process goes back to Brockwell, see [5] and [6].

Definition 3.4.1. Let $a(z) = z^p + a_1 z^{p-1} + \cdots + a_p$ and $b(z) = b_0 + b_1 z + \cdots b_q z^q$ be polynomials with real coefficients with $a_p \neq 0$, $b_q \neq 0$ and $q < p$. Let $d \in (0, \frac{1}{2})$. Let $(L_t)_{t \in \mathbb{R}}$ be a two-sided Lévy process with $\mathbb{E}[L_1] = 0$ and $\operatorname{Var}(L_1) = \sigma^2 \in (0, \infty)$. If the roots of $a(z)$ all have negative real parts, then a FICARMA(p,d,q) process $(X_t)_{t \in \mathbb{R}}$ is defined by $X_t := \int_{-\infty}^{\infty} f(t-s)\, dL_s$, where the kernel function f is defined by $f(t) = 0$ for $t \leq 0$ and

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\lambda} (i\lambda)^{-d} \frac{b(i\lambda)}{a(i\lambda)}\, d\lambda \quad \text{for } t > 0.$$

The next proposition shows that the kernel function of a FICARMA process fulfils the assumption on f in the assumptions of Theorem 3.2.1 and Theorem 3.3.1 if $b_0 \neq 0$. Hence depending on the Lévy process, one of these theorems can be applied.

Proposition 3.4.1. *The kernel function of a FICARMA process fulfils* $f(t) \sim \frac{t^{d-1}}{\Gamma(d)} \cdot \frac{b(0)}{a(0)}$ *as* $t \to \infty$. *Further,* f *is infinitely often differentiable on* $\mathbb{R}^+$ *and* $\lim_{t \searrow 0} f(t) = 0$, *hence* f *is bounded.*

Proof. Brockwell [5] shows that $f(t) \sim \frac{t^{d-1}}{\Gamma(d)} \cdot \frac{b(0)}{a(0)}$ by refering to Theorem 37.1 on page 254 in [15] and states this as equation (4.6) in [5]. He rewrites the kernel function as $\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{tz} z^{-d} \frac{b(z)}{a(z)}\, dz$. By the discussion on page 250 in [15], the path through the origin can be replaced by a path $\mathfrak{W}$ with angle ψ as in figure 37 on page 250 in [15]. Since $a(z)$ has only finitely many roots, we find $\psi > \frac{\pi}{2}$ small enough such that all singularities lie on the left-hand side of $\mathfrak{W}$. We have $\lim_{|z| \to \infty} z^{-d} \frac{b(z)}{a(z)} = 0$ since $|z^{-d}| = |z|^{-d}$ and $q < p$. Thus the conditions of Theorem 37.1 in [15] are fulfilled. To show that it is differentiable, we now assume without loss of generality that we use the path $\mathfrak{W}$, hence

$$f(t) = \frac{1}{2\pi i} \int_{\mathfrak{W}} e^{tz} z^{-d} \frac{b(z)}{a(z)}\, dz.$$

By Theorem 36.1 in [15], f is analytic on the right-hand side of $\mathfrak{W}$ with derivative

$$f'(t) = \frac{1}{2\pi i} \int_{\mathfrak{W}} e^{tz} z^{-d+1} \frac{b(z)}{a(z)}\, dz.$$

Note that the choice $\mathfrak{W}$ depends on t.

Since f is analytic on the right-hand side of $\mathfrak{W}$, it is especially infinitely often differentiable on the positive real line. The kernel function can be equivalently expressed as $f(t) = \int_0^t g(t-u) \frac{u^{d-1}}{\Gamma(d)}\, du$, see [5] equation (4.4), which is the Riemann-Liouville fractional integral of the kernel g of a CARMA process with polynomials $a(z)$ and $b(z)$. Hence by equation (2.5) on page 46 in [32], $\lim_{t \searrow 0} f(t) = 0$. □

Theorem 3.2.1 and Theorem 3.3.1 gave limit theorems for the sample autocovariance function of $(X_t)_{t \in \mathbb{Z}}$. Using the delta-method, it is easy to obtain limit theorems for the sample autocorrelation defined by $\hat{\rho}_N(h) := \frac{\hat{\gamma}_N(h)}{\hat{\gamma}_N(0)}$ for $h \in \mathbb{N}_0$. The autocorrelation function is denoted by $\rho(h) := \frac{\gamma(h)}{\gamma(0)}$ for $h \in \mathbb{N}_0$.

Corollary 3.4.1. *If*

$$N^{1-2d}(\hat{\gamma}_N(0)-\gamma(0),\dots,\hat{\gamma}_N(H)-\gamma(H)) \xrightarrow{d} C_d^2\sigma^2 U_d(1)(1,\dots,1) \text{ as } N\to\infty,$$

where $U_d(1)$ *is the marginal distribution of the Rosenblatt process at time* 1 *defined in (3.2), then*

$$N^{1-2d}(\hat{\rho}_N(h)-\rho(h)) \xrightarrow{d} C_d^2\sigma^2 U_d(1)\frac{1-\rho(h)}{\gamma(0)} \text{ as } N\to\infty.$$

If

$$\frac{N}{a_N^2}(\hat{\gamma}_N(0)-\gamma(0),\dots,\hat{\gamma}_N(H)-\gamma(H)) \xrightarrow{d} (\int_0^1 G_0(s)\,dM_s,\dots,\int_0^1 G_H(s)\,dM_s) \text{ as } N\to\infty,$$

then

$$\frac{N}{a_N^2}(\hat{\rho}_N(h)-\rho(h)) \xrightarrow{d} \frac{1}{\gamma(0)}\int_0^1 (G_h(s)-\rho(h)G_0(s))\,dM_s \text{ as } N\to\infty.$$

Proof. This follows from the delta-method, see Theorem 3.1. in [43], with the function $\varphi(x,y)=\frac{y}{x}$. □

We conclude this section by considering fractional Lévy noise as in [11]. One way of defining a fractional Lévy process $(M_t)_{t\in\mathbb{R}}$ is to set

$$M_t := \frac{1}{\Gamma(d+1)}\int_{-\infty}^{\infty}(t-s)_+^d-(-s)_+^d\,dL_s, \quad t\in\mathbb{R},$$

where $d\in(0,\frac{1}{2})$, $(L_t)_{t\in\mathbb{R}}$ is a two-sided Lévy process with $\mathbb{E}[L_1]=0$ and $\mathrm{Var}(L_1)<\infty$, see [29]. One obtains fractional Lévy noise $(Z_t)_{t\in\mathbb{Z}}$ by defining

$$Z_t := M_t - M_{t-1}, \quad t\in\mathbb{Z}.$$

Hence $Z_t=\frac{1}{\Gamma(d+1)}\int_{-\infty}^{\infty}(t-s)_+^d-(t-1-s)_+^d dL_s$. By the mean value theorem, one obtains that $\frac{1}{\Gamma(d+1)}(t_+^d-(t-1)_+^d)\sim\frac{d}{\Gamma(d+1)}t^{d-1}$ as $t\to\infty$. Cohen and Lindner derive that

$$\hat{d} := \frac{1}{2}\frac{\log(\hat{\rho}_N(1)+1)}{\log 2}$$

is a strongly consistent estimator for d, see (4.2) in [11], where $\hat{\rho}$ is the sample autocorrelation of the fractional Lévy noise. Proposition 4.1 in [11] states that $\hat{d}$ is asymptotically normal distributed, if $d\in(0,\frac{1}{4})$ and $\mathbb{E}[L_1^4]<\infty$. By the delta-method and the last corollary, we can see that for example in the case $d\in(\frac{1}{4},\frac{1}{2})$ and $\mathbb{E}[L_1^4]<\infty$, $\hat{d}$ is asymptotically Rosenblatt distributed. This complements the results in [11].

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Bibliography

[1] P. C. Besse, H. Cardot, D.B. Stephenson Autoregressive forecasting of some functional climatic variations *Scand. J. Statist.* **27**:673-687, 2000

[2] P. Billingsley *Convergence of Probability Measures*, John Wiley, 1999

[3] N. H. Bingham, C. M. Goldie, J. L. Teugels *Regular Variation*, Cambridge University Press, 1987

[4] D. Bosq *Linear Processes in Function Spaces* Springer-Verlag, New York 2000

[5] P. J. Brockwell Representations of continuous-time ARMA processes, J. Appl. Probab. **41A**, 375-382, 2004

[6] P. J. Brockwell, T. Marquardt Lévy-driven and fractionally integrated ARMA processes with continuous time parameter, Statistica Sinica **15**(2005), 477-494

[7] P. J. Brockwell, R. A. Davis *Time Series: Theory and Methods* Second Edition Springer-Verlag, New York 2006

[8] P. J. Brockwell, A. Lindner Existence and uniqueness of stationary Lévy driven CARMA processes *Stoch. Proc. Appl.* **119**, 2625-2644, 2009

[9] P. J. Brockwell, A. Lindner Strictly Stationary Solutions of Autoregressive Moving Average Equations *Biometrika* **97**, 765-772, 2010

[10] P. J. Brockwell, A. Lindner, B. Vollenbröker Strictly stationary solutions of multivariate ARMA equations with i.i.d. noise *Ann. Inst. Statist. Math.*, **64**, 1089-1119, 2012

[11] S. Cohen, A. Lindner A central limit theorem for the sample autocorrelations of a Lévy driven continuous time moving average process, Journal of Statistical Planning and Inference 143, 1295-1306, 2013

[12] J. B. Conway *Functions of One Complex Variable* Springer-Verlag, New York 1975

[13] J. B. Conway *A Course in Functional Analysis* Springer-Verlag, New York 1985

[14] R. Davis, S. Resnick Limit Theory for Moving Averages of Random Variables with Regularly Varying Tail Probabilities, The Annals of Probability, **13**,(1) 179-195, 1985

[15] G. Doetsch *Introduction to the Theory and Application of the Laplace Transform*, Springer Verlag, Berlin 1974

[16] T. Eisner *Stability of Operators and Operator Semigroups* Birkhäuser, Basel 2010

[17] K.-J. Engel, R. Nagel *A Short Course on Operator Semigroups*, Springer-Verlag, New York 2006

[18] P. Embrechts, C. Goldie, N. Veraverbeeke *Subexponentiality and infinite divisibility*, Z. Wahrschein. Verw. Geb. **49**, 335-347

[19] P. Embrechts, C. Klüppelberg, T. Mikosch *Modelling Extremal Events*, Springer Verlag, Springer Verlag, Berlin 1997

[20] V. Fasen *Extremes of Lévy Driven Moving Average Processes with Applications in Finance*, PhD thesis, TU München, 2004

[21] L. Giraitis, H. L. Koul, D. Surgailis *Large Sampe Inference for Long Memory Processes* Imperial College Press, London 2012

[22] S. Hörmann, P. Kokoszka Functional Time Series, in *Handbooks of Statistics*, Eds. Subbu Rao, Subbu Rao, Rao, **30**, Elsevier, Amsterdam 2012

[23] L. Horváth, P. Kokoszka, Sample autocovariances of long-memory time series Bernoulli **2**, 2008, 405-418

[24] L. Horváth, P. Kokoszka *Inference for Functional Data with Applications*, Springer-Verlag, New York 2012

[25] H. Hult, F. Lindskog On regular variation for infinitely divisible random vectors and additive processes, Adv. Appl. Probl. **38**, 134-148, 2006

[26] A. H. Jessen, T. Mikosch *Regularly Varying Functions* Publications de l'institut mathématique Nouvelle série, tome 80(94) 171-192 2006

[27] T. Kato *Pertubation Theory for Linear Operators* Springer-Verlag, New York 1966

[28] M. Ledoux, M. Talagrand *Probability in Banach Spaces* First Reprint, Springer-Verlag, New York 2002

[29] T. Marquardt *Fractional Lévy processes with an application to long memory moving average processes*, Bernoulli Volume 12, Number 6, 1099-1126, 2006

[30] A. Mas, B. Pumo Linear processes for functional data, in *The Oxford Handbook of Functional Data*, Ferraty and Romain Eds., Oxford 2010

[31] T. Mikosch, D. Straumann Quasi-MLE in heteroscedastic time series: a stochastic recurrence equations approach *Annals of Statistics* **34**, 2449-2495, 2006

[32] K. S. Miller, B. Ross *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley, New York 1993

[33] S. Resnick *Heavy-Tail Phenomena*, Springer 2006

[34] G. Samorodnitsky, M. Taqqu *Stable Non-Gaussian Random Processes*, Chapman & Hall, 1994

[35] K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge Studies in Advanced Mathematics, 1999

[36] T. Meinguet, J. Segers Regularly varying time series in Banach spaces *Preprint* 2010

[37] R. Sen, C. Klüppelberg Time series of functional data *Preprint* 2010

[38] F. Spangenberg Strictly stationary solutions of ARMA equations in Banach spaces *Journal of Multivariate Analysis*, Volume 121, Pages 127-138, 2013

[39] F. Spangenberg A central limit theorem for the sample autocovariance of a continuous-time moving average process with long memory, Preprint 2015

[40] M. S. Taqqu The Rosenblatt Process R.A. Davis et al. (eds.), *Selected Works of Murray Rosenblatt*, Selected Works in Probability and Statistics, Springer 2011

[41] N. N. Vakhania, V. I. Tarieladze, S. A. Chobanyan *Probability Distributions on Banach Spaces* D. Reidel Publishing Company, Dordrecht 1987

[42] M. S. Veillette, M. S. Taqqu Properties and numerical evaluation of the Rosenblatt distribution Bernoulli **19**(3), 982-1005, 2013

[43] A. W. van der Vaart *Asymptotic Statistics*, Cambridge Press, 1998