A NEW FAMILY OF MAPPINGS OF INFINITELY DIVISIBLE DISTRIBUTIONS RELATED TO THE GOLDIE–STEUTEL–BONDESSON CLASS

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Abstract Let $\{X_t^{(\mu)}\}$ be a Lévy process on $\mathbb{R}^d$ whose distribution at time 1 is a d-dimensional infinitely distribution μ. It is known that the set of all infinitely divisible distributions on $\mathbb{R}^d$, each of which is represented by the law of a stochastic integral $\int_0^1 \log \frac{1}{t} dX_t^{(\mu)}$ for some infinitely divisible distribution on $\mathbb{R}^d$, coincides with the Goldie-Steutel-Bondesson class, which, in one dimension, is the smallest class that contains all mixtures of exponential distributions and is closed under convolution and weak convergence. Our purpose of this paper is to study the class of infinitely divisible distributions which are represented as the law of $\int_0^1 \left(\log \frac{1}{t}\right)^{1/\alpha} dX_t^{(\mu)}$ for general $\alpha > 0$. These stochastic integrals define a new family of mappings of infinitely divisible distributions. We first study properties of these mappings and their ranges. Then we characterize some subclasses of the range by stochastic integrals with respect to some compound Poisson processes. Finally, we investigate the limit of the ranges of the iterated mappings.

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1. Introduction

Throughout this paper, $\mathcal{L}(X)$ denotes the law of an $\mathbb{R}^d$-valued random variable X and $\widehat{\mu}(z), z \in \mathbb{R}^d$, denotes the characteristic function of a probability distribution μ on $\mathbb{R}^d$. Also $I(\mathbb{R}^d)$ denotes the class of all infinitely divisible distributions on $\mathbb{R}^d$, $I_{\mathrm{sym}}(\mathbb{R}^d) = \{\mu \in I(\mathbb{R}^d) : \mu \text{ is symmetric on } \mathbb{R}^d\}$, $I_{\mathrm{ri}}(\mathbb{R}^d) = \{\mu \in I(\mathbb{R}^d) : \mu \text{ is rotationally invariant on } \mathbb{R}^d\}$, $I_{\log}(\mathbb{R}^d) = \{\mu \in I(\mathbb{R}^d) : \int_{|x|>1} \log|x|\mu(dx) < \infty\}$ and $I_{\log^m}(\mathbb{R}^d) = \{\mu \in I(\mathbb{R}^d) : \int_{|x|>1} (\log|x|)^m \mu(dx) < \infty\}$, where $|x|$ is the Euclidean norm of $x \in \mathbb{R}^d$. Let $C_\mu(z), z \in \mathbb{R}^d$, be the cumulant function of $\mu \in I(\mathbb{R}^d)$. That is, $C_\mu(z)$ is the unique continuous function with $C_\mu(0) = 0$ such that $\widehat{\mu}(z) = \exp\{C_\mu(z)\}, z \in \mathbb{R}^d$. When μ is the distribution of a random variable X, we also write $C_X(z) := C_\mu(z)$.

We use the Lévy-Khinchine generating triplet (A, ν, γ) of $\mu \in I(\mathbb{R}^d)$ in the sense that

$$C_\mu(z) = -2^{-1}\langle z, Az\rangle + \mathrm{i}\langle \gamma, z\rangle + \int_{\mathbb{R}^d} \left(e^{\mathrm{i}\langle z,x\rangle} - 1 - \mathrm{i}\langle z, x\rangle(1+|x|^2)^{-1}\right)\nu(dx),\ z \in \mathbb{R}^d,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^d$, A is a symmetric nonnegative-definite $d \times d$ matrix, $\gamma \in \mathbb{R}^d$ and ν is a measure (called the Lévy measure) on $\mathbb{R}^d$ satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu(dx) < \infty$.

The polar decomposition of the Lévy measure ν of $\mu \in I(\mathbb{R}^d)$, with $0 < \nu(\mathbb{R}^d) \leq \infty$, is the following: There exist a measure λ on $S = \{\xi \in \mathbb{R}^d : |\xi| = 1\}$ with $0 < \lambda(S) \leq \infty$ and a family $\{\nu_\xi \colon \xi \in S\}$ of measures on $(0,\infty)$ such that $\nu_\xi(B)$ is measurable in ξ for each $B \in \mathcal{B}((0,\infty))$, $0 < \nu_\xi((0,\infty)) \leq \infty$ for each $\xi \in S$,

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi)\nu_\xi(dr),\ B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}). \tag{1.1}$$

Here λ and $\{\nu_\xi\}$ are uniquely determined by ν up to multiplication by a measurable function $c(\xi)$ and $c(\xi)^{-1}$ with $0 < c(\xi) < \infty$. The measure ν_ξ is a Lévy measure on $(0,\infty)$ for λ-a.e. $\xi \in S$. We say that ν has the polar decomposition (λ, ν_ξ) and ν_ξ is called the radial component of ν. (See, e.g., Lemma 2.1 of [3] and its proof.)

Remark 1.1. For $\mu \in I_{\mathrm{ri}}(\mathbb{R}^d)$, it is necessary and sufficient that λ and ν_ξ can be chosen such that λ is Lebesgue measure and ν_ξ is independent of ξ.

Let $\mu \in I(\mathbb{R}^d)$ and $\{X_t^{(\mu)}, t \geq 0\}$ denotes the Lévy process on $\mathbb{R}^d$ with μ as the distribution of time 1. For a nonrandom measurable function f on $(0,\infty)$, we define a mapping

$$\Phi_f(\mu) = \mathcal{L}\left(\int_0^\infty f(t)dX_t^{(\mu)}\right), \tag{1.2}$$

whenever the stochastic integral on the right-hand side is definable in the sense of stochastic integrals based on independently scattered random measures on $\mathbb{R}^d$ induced by $\{X_t^{(\mu)}\}$, as in Definitions 2.3 and 3.1 of Sato [15]. When the support of f is a finite interval $(0,a]$, $\int_0^\infty f(t)dX_t^{(\mu)} = \int_0^a f(t)dX_t^{(\mu)}$, and when the support of f is $(0,\infty)$, $\int_0^\infty f(t)dX_t^{(\mu)}$ is the limit in probability of $\int_0^a f(t)dX_t^{(\mu)}$ as $a \to \infty$. $\mathfrak{D}(\Phi_f)$ denotes the set of $\mu \in I(\mathbb{R}^d)$ for which the stochastic integral in (1.2) is definable. When we consider the composition of two mappings Φ_f and Φ_g, denoted by $\Phi_g \circ \Phi_f$, the domain of $\Phi_g \circ \Phi_f$ is $\mathfrak{D}(\Phi_g \circ \Phi_f) = \{\mu \in I(\mathbb{R}^d) : \mu \in \mathfrak{D}(\Phi_f) \text{ and } \Phi_f(\mu) \in \mathfrak{D}(\Phi_g)\}$. Once we define such a mapping, we can characterize a subclass of $I(\mathbb{R}^d)$ as the range of Φ_f, $\mathfrak{R}(\Phi_f)$, say.

In Barndorff-Nielsen et al. [3], they studied the Upsilon mapping

$$\Upsilon(\mu) = \mathcal{L}\left(\int_0^1 \log\frac{1}{t}dX_t^{(\mu)}\right), \tag{1.3}$$

and showed that its range $\mathfrak{R}(\Upsilon)$ is the Goldie–Steutel–Bondesson class, $B(\mathbb{R}^d)$, say, that is

$$\Upsilon(I(\mathbb{R}^d)) = B(\mathbb{R}^d). \tag{1.4}$$

It is also known that $\mu \in B(\mathbb{R}^d)$ can be characterized in terms of Lévy measures as follows: A distribution $\mu \in I(\mathbb{R}^d)$ belongs to $B(\mathbb{R}^d)$ if and only if the Lévy measure ν of μ is identically zero or in case $\nu \neq 0$, ν_ξ in (1.1) satisfies that $\nu_\xi(dr) = g_\xi(r)dr, r > 0$, where $g_\xi(r)$ is completely monotone in $r \in (0,\infty)$ for λ-a.e. ξ and measurable in ξ for each $r > 0$.

Our purpose of this paper is to generalize (1.3) to

$$\mathcal{E}_\alpha(\mu) := \mathcal{L}\left(\int_0^1 \left(\log\frac{1}{t}\right)^{1/\alpha} dX_t^{(\mu)}\right)$$

for any $\alpha > 0$, where $\mathcal{E}_1 = \Upsilon$, and investigate $\mathfrak{R}(\mathcal{E}_\alpha)$. We first generalize (1.4) and characterize $\mathcal{E}_\alpha(I(\mathbb{R}^d))$, in the sense of what should replace $B(\mathbb{R}^d)$ for general $\alpha > 0$. For that, we need a new class $E_\alpha(\mathbb{R}^d), \alpha > 0$. Namely, we say that $\mu \in I(\mathbb{R}^d)$ belongs to the class $E_\alpha(\mathbb{R}^d)$ if $\nu = 0$ or $\nu \neq 0$ and, in case $\nu \neq 0$, ν_ξ in (1.1) satisfies

$$\nu_\xi(dr) = r^{\alpha-1}g_\xi(r^\alpha)dr, \ r > 0,$$

for some function $g_\xi(r)$, which is completely monotone in $r \in (0,\infty)$ for λ-a.e. ξ, and is measurable in ξ for each $r > 0$. Then we will show that $\mathcal{E}_\alpha(I(\mathbb{R}^d)) = E_\alpha(\mathbb{R}^d)$ in Theorem 2.3.

In addition to that, we have two motivations of this generalization of the mapping. On $\mathbb{R}_+$, the Goldie–Steutel–Bondesson class $B(\mathbb{R}_+)$ is the smallest class that contains all mixtures of exponential distributions and is closed under convolution and weak convergence. In addition, we denote by $B^0(\mathbb{R}_+)$ the subclass of $B(\mathbb{R}_+)$, where all distributions do not have drift.

It is similarly extended to a class on $\mathbb{R}$, and in Barndorff-Nielsen et al. [3] it was proved that $B(\mathbb{R}^d)$ in (1.4) is the smallest class of distributions on $\mathbb{R}^d$ closed under convolution and weak convergence and containing the distributions of all elementary mixed exponential variables in $\mathbb{R}^d$. Here, an $\mathbb{R}^d$-valued random variable Ux is called an elementary mixed exponential random variable in $\mathbb{R}^d$ if x is a nonrandom nonzero vector in $\mathbb{R}^d$ and U is a real random variable whose distribution is a mixture of a finite number of exponential distributions. The first motivation is to characterize a subclass of $I(\mathbb{R}^d)$ based on a single Lévy process. This type of characterization is quite different from the characterization in terms of the range of some mapping $\mathfrak{R}(\Phi_f)$. This type of characterization is also done by James et al. [6] for the Thorin class. As to $B^0(\mathbb{R}_+)$, we have the following, which is a special case of Equation (4.18) in Theorem 4.2 as mentioned at the end of Section 4.

Theorem 1.2. *Let $Z = \{Z_t\}_{t \geq 0}$ be a compound Poisson process on $\mathbb{R}_+$ with Lévy measure $\nu_Z(dx) = e^{-x}dx, x > 0$. Then*

$$B^0(\mathbb{R}_+) = \left\{ \mathcal{L}\left(\int_0^\infty h(t)dZ_t \right), \quad h \in \mathrm{Dom}(Z) \right\},$$

where Dom(Z) is the set of nonrandom measurable functions h for which the stochastic integrals $\int_0^\infty h(t)dZ_t$ are definable.

We are going to generalize this underlying compound Process Y to other Y with Lévy measure $x^{\alpha-1}e^{-x^\alpha}dx, x > 0, \alpha > 0$, and furthermore to the two-sided case.

The second motivation is the following. In Maejima and Sato [9], they showed that the limits of nested subclasses constructed by iterations of several mappings are identical with the closure of the class of the stable distributions, where the closure is taken under convolution and weak convergence. We are going to show that this fact is also true for $\mathcal{M}$-mapping, which is defined by

$$\mathcal{M}(\mu) = \mathcal{L}\left(\int_0^\infty m^*(t)dX_t^{(\mu)}\right), \quad \mu \in I_{\log}(\mathbb{R}^d),$$

where $m(x) = \int_0^\infty u^{-1}e^{-u^2}du, x > 0$ and $m^*(t)$ is its inverse function in the sense that $m(x) = t$ if and only if $x = m^*(t)$. This mapping (in the symmetric case) was introduced in Aoyama et al. [2], as a subclass of selfdecomposable and type G distributions. In Maejima and Sato [9], $\lim_{m\to\infty} \mathcal{M}^m(I_{\log^m}(\mathbb{R}^d))$ is not treated, and we want to show that this limit is also equal to the closure of the class of the stable distributions. For the proof, we need our new mapping $\mathcal{E}_2$. Namely, the proof is based on the fact that

$$\mathcal{M}(\mu) = (\Phi \circ \mathcal{E}_2)(\mu) = (\mathcal{E}_2 \circ \Phi)(\mu), \quad \mu \in I_{\log}(\mathbb{R}^d), \tag{1.5}$$

where $\Phi(\mu) = \mathcal{L}\left(\int_0^\infty e^{-t}dX_t^{(\mu)}\right)$ with $\mathfrak{D}(\Phi) = I_{\log}(\mathbb{R}^d)$.

The paper is organized as follows. In Section 2, we show several properties of the mapping $\mathcal{E}_\alpha$. In Section 3, we show that $E_\alpha(\mathbb{R}^d) = \mathcal{E}_\alpha(I(\mathbb{R}^d))$, $\alpha > 0$. This relation has the meaning that $\widetilde{\mu} \in E_\alpha(\mathbb{R}^d)$ is characterized by stochastic integral representation with respect to Lévy process. Also we characterize $E_\alpha(\mathbb{R}^d), E_\alpha^+(\mathbb{R}^d)$ and $E_\alpha^{\mathrm{sym}}(\mathbb{R}^d) := E_\alpha(\mathbb{R}^d) \cap I_{\mathrm{sym}}(\mathbb{R}^d)$ based on one compound Poisson distribution on $\mathbb{R}$, where $E_\alpha^+(\mathbb{R}^d) = \{\mu \in E_\alpha(\mathbb{R}^d) : \mu(\mathbb{R}^d \setminus [0,\infty)^d) = 0\}$. In Section 4, we characterize

$$E_\alpha^{0,\mathrm{ri}}(\mathbb{R}^d) := \{\mu \in E_\alpha(\mathbb{R}^d) : \mu \text{ has no Gaussian part}\} \cap I_{\mathrm{ri}}(\mathbb{R}^d) \tag{1.6}$$

and certain subclasses of $E_\alpha(\mathbb{R}^1)$ which correspond to Lévy processes of bounded variation with zero drift, by (essential improper) stochastic integrals with respect to some compound Poisson processes. This gives us a new sight of the Goldie–Steutel–Bondesson class in $\mathbb{R}^1$. In Section 5, we consider the composition $\Phi \circ \mathcal{E}_\alpha$, and we apply this composition to show that $\lim_{m\to\infty}(\Phi \circ \mathcal{E}_\alpha)^m(I_{\log^m}(\mathbb{R}^d))$ is the closure of the class of the stable distributions as Maejima and Sato [9] showed for other mappings. Since we will see that $\Phi \circ \mathcal{E}_2 = \mathcal{M}$, we can answer the question mentioned in the second motivation above.

2. Several properties of the mapping $\mathcal{E}_\alpha$ and the range of $\mathcal{E}_\alpha$

We start with showing several properties of the mapping $\mathcal{E}_\alpha$.

Proposition 2.1. *Let* $\alpha > 0$.

(i) $\mathcal{E}_\alpha(\mu)$ *can be defined for any* $\mu \in I(\mathbb{R}^d)$ *and is infinitely divisible, and we have*

$\int_0^1 |C_\mu(z(\log t^{-1})^{1/\alpha})|\, dt < \infty$ and

$$C_{\mathcal{E}_\alpha(\mu)}(z) = \int_0^1 C_\mu(z\left(\log t^{-1})^{1/\alpha}\right)\, dt, \quad z \in \mathbb{R}^d.$$

(ii) The generating triplet $(\widetilde{A}, \widetilde{\nu}, \widetilde{\gamma})$ of $\widetilde{\mu} = \mathcal{E}_\alpha(\mu)$ can be calculated from (A, ν, γ) of μ by

$$\widetilde{A} = \Gamma(1 + 2/\alpha)\, A,$$

$$\widetilde{\nu}(B) = \int_0^\infty \nu(u^{-1}B)\alpha u^{\alpha-1}e^{-u^\alpha}\, du, \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}), \tag{2.1}$$

$$\widetilde{\gamma} = \Gamma(1 + 1/\alpha)\,\gamma + \int_0^\infty \alpha u^\alpha e^{-u^\alpha} du \int_{\mathbb{R}^d} x\left(\frac{1}{1+|ux|^2} - \frac{1}{1+|x|^2}\right)\nu(dx). \tag{2.2}$$

If additionally $\mu \in I(\mathbb{R}^d)$ is such that $\{X_t^{(\mu)}\}$ has bounded variation with drift γ_0, then also $\{X_t^{(\widetilde{\mu})}\}$ is of bounded variation with drift

$$\widetilde{\gamma}_0 = \Gamma(1 + 1/\alpha)\gamma_0. \tag{2.3}$$

(iii) The mapping $\mathcal{E}_\alpha : I(\mathbb{R}^d) \to I(\mathbb{R}^d)$ is one-to-one.
(iv) Let $\mu_n \in I(\mathbb{R}^d)$, $n = 1, 2, \ldots$ If μ_n converges weakly to some $\mu \in I(\mathbb{R}^d)$ as $n \to \infty$, then $\mathcal{E}_\alpha(\mu_n)$ converges weakly to $\mathcal{E}_\alpha(\mu)$ as $n \to \infty$. Conversely, if $\mathcal{E}_\alpha(\mu_n)$ converges weakly to some distribution $\widetilde{\mu}$ as $n \to \infty$, then $\widetilde{\mu} = \mathcal{E}_\alpha(\mu)$ for some $\mu \in I(\mathbb{R}^d)$ and μ_n converges weakly to μ as $n \to \infty$. In particular, the range $\mathcal{E}_\alpha(I(\mathbb{R}^d))$ is closed under weak convergence.
(v) For any $\mu \in I(\mathbb{R}^d)$ we also have

$$\mathcal{E}_\alpha(\mu) = \mathcal{L}\left(\int_0^1 \left(\log\frac{1}{1-t}\right)^{1/\alpha} dX_t^{(\mu)}\right) = \mathcal{L}\left(\lim_{s\downarrow 0}\int_s^1 \frac{1}{\alpha t}(\log t^{-1})^{1/\alpha - 1} X_t^{(\mu)}\, dt\right),$$

where the limit is almost sure.

Proof. (The proof follows along the lines of Proposition 2.4 of Barndorff-Nielsen et al. [3]. However, we give the proof for the completeness of the paper.)

(i) The function $f(t) = (\log t^{-1})^{1/\alpha}\mathbf{1}_{(0,1]}(t)$ is clearly square integrable, hence the result follows from Sato [13], see also Lemma 2.3 in Maejima [7].

(ii) By a general result (see Lemma 2.7 and Corollary 4.4 of Sato [12]) and a change of variable, we have

$$\widetilde{A} = \left(\int_0^1 (\log t^{-1})^{2/\alpha}\, dt\right) A = \left(\int_0^\infty u^{2/\alpha}e^{-u}\, du\right) A = \Gamma(1 + 2/\alpha)\, A,$$

$$\widetilde{\nu}(B) = \int_0^1 \nu((\log t^{-1})^{-1/\alpha}B)dt = \int_0^\infty \nu(u^{-1}B)\,\alpha u^{\alpha-1}e^{-u^\alpha}du,$$

$$\widetilde{\gamma} = \int_0^1 (\log t^{-1})^{1/\alpha} \left(\gamma + \int_{\mathbb{R}^d} x \left(\frac{1}{1+|(\log t^{-1})^{1/\alpha} x|^2} - \frac{1}{1+|x|^2} \right) \nu(dx) \right) dt$$
$$= \gamma \int_0^\infty v^{1/\alpha} e^{-v}\, dv + \int_0^\infty \alpha u^\alpha e^{-u^\alpha} du \int_{\mathbb{R}^d} x \left(\frac{1}{1+|ux|^2} - \frac{1}{1+|x|^2} \right) \nu(dx).$$

The additional part follows immediately from Theorem 3.15 in Sato [15].

(iii) By (i), we have for each $z \in \mathbb{R}^d$,

$$C_{\mathcal{E}_\alpha(\mu)}(z) = \int_0^1 C_\mu(z(\log t^{-1})^{1/\alpha})\, dt = \int_0^\infty C_\mu(zv^{1/\alpha}) e^{-v}\, dv.$$

Hence we conclude that for each $u > 0$ and $z \in \mathbb{R}^d$,

$$\frac{1}{u} C_{\mathcal{E}_\alpha(\mu)}(u^{-1/\alpha} z) = \int_0^\infty \frac{1}{u} C_\mu \left(\left(\frac{v}{u} \right)^{1/\alpha} z \right) e^{-v}\, dv = \int_0^\infty C_\mu(w^{1/\alpha} z) e^{-uw}\, dw.$$

Hence we see that for each $z \in \mathbb{R}^d$, the function $(0,\infty) \to \mathbb{R}$, $u \mapsto u^{-1} C_{\mathcal{E}_\alpha(\mu)}(u^{-1/\alpha} z)$ is the Laplace transform of $(0,\infty) \to \mathbb{R}$, $w \mapsto C_\mu(w^{1/\alpha} z)$. Hence for each fixed $z \in \mathbb{R}^d$, $C_\mu(w^{1/\alpha} z)$ is determined by $\mathcal{E}_\alpha(\mu)$ for almost every $w \in (0,\infty)$, and by continuity for every $w > 0$. In particular for $w = 1$, we see that $C_\mu(z)$ is determined by $\mathcal{E}_\alpha(\mu)$ for every $z \in \mathbb{R}^d$.

(iv) Apart from minor adjustments, the proof is the same as that of Proposition 2.4 (v) in Barndorff-Nielsen et al. [3] and hence omitted.

(v) The first equality is clear by duality (see Sato [11], Proposition 41.8). For the second, observe that $\int_s^1 (\log t^{-1})^{1/\alpha}\, dX_t^{(\mu)}$ converges almost surely to $\int_0^1 (\log t^{-1})^{1/\alpha}\, dX_t^{(\mu)}$ as $s \downarrow 0$ by the independently scattered random measure property of $X_t^{(\mu)}$. Using partial integration, we conclude

$$\int_s^1 (\log t^{-1})^{1/\alpha}\, dX_t^{(\mu)} = -X_s^{(\mu)} (\log s^{-1})^{1/\alpha} - \int_s^1 X_t^{(\mu)} d(\log t^{-1})^{1/\alpha}.$$

But $\lim_{s\downarrow 0} X_s^{(\mu)} (\log s^{-1})^{1/\alpha} = 0$ a.s. (see Sato [11], Proposition 47.11), and the claim follows. $\square$

Corollary 2.2. *Let $\alpha > 0$. Then a distribution μ is symmetric if and only if $\mathcal{E}_\alpha(\mu)$ is symmetric.*

Proof. Note that for a random variable X with the cumulant function $C_X(z)$, $\mathcal{L}(X)$ is symmetric if and only if $C_X(z) = C_{-X}(z)$. Let X and $\widetilde{X}$ have distributions μ and $\mathcal{E}_\alpha(\mu)$, respectively. Then $C_{\widetilde{X}}(z) = \int_0^1 C_X(z(-\log t)^{1/\alpha}) dt$ and $C_{-\widetilde{X}}(z) = \int_0^1 C_{-X}(z(-\log t)^{1/\alpha}) dt$. Hence, if $C_X = C_{-X}$, then $C_{\widetilde{X}} = C_{-\widetilde{X}}$. Conversely, if $C_{\widetilde{X}} = C_{-\widetilde{X}}$, then $C_X = C_{-X}$ by the one-to-one property of $\mathcal{E}_\alpha$. $\square$

Since $\mathcal{E}_1 = \Upsilon$ and $E_1(\mathbb{R}^d) = B(\mathbb{R}^d)$, the following is an extension of the fact $E_1(\mathbb{R}^d) = \mathcal{E}_1(I(\mathbb{R}^d))$ to the case of general $\alpha > 0$.

Theorem 2.3. *For $\alpha > 0$,*

$$E_\alpha(\mathbb{R}^d) = \mathcal{E}_\alpha(I(\mathbb{R}^d)).$$

Proof (i) (Proof for that $E_\alpha(\mathbb{R}^d) \supset \mathcal{E}_\alpha(I(\mathbb{R}^d))$.) Let $\widetilde{\mu} \in \mathcal{E}_\alpha(I(\mathbb{R}^d))$. Then $\widetilde{\mu} = \mathcal{L}\left(\int_0^1 (\log t^{-1})^{1/\alpha} dX_t^{(\mu)}\right)$ for some $\mu \in I(\mathbb{R}^d)$, and hence

$$\widetilde{\nu}(B) := \nu_{\widetilde{\mu}}(B) = \alpha \int_0^\infty \nu(u^{-1}B) u^{\alpha-1} e^{-u^\alpha} du,$$

where ν is the Lévy measure of μ and ν_ξ below is the radial component of ν. Thus, the spherical component $\widetilde{\lambda}$ of $\widetilde{\nu}$ is equal to the spherical component λ of ν, and the radial component $\widetilde{\nu}_\xi$ of $\widetilde{\nu}$ satisfies that, for $B \in \mathcal{B}((0,\infty))$,

$$\begin{aligned}
\widetilde{\nu}_\xi(B) &= \alpha \int_0^\infty u^{\alpha-1} e^{-u^\alpha} du \int_0^\infty \mathbf{1}_B(xu) \nu_\xi(dx) \\
&= \alpha \int_0^\infty \nu_\xi(dx) \int_0^\infty \mathbf{1}_B(y) (y/x)^{\alpha-1} e^{-(y/x)^\alpha} x^{-1} dy \\
&=: \int_0^\infty \mathbf{1}_B(y) y^{\alpha-1} \widetilde{g}_\xi(y^\alpha) dy,
\end{aligned}$$

where

$$\widetilde{g}_\xi(r) = \int_0^\infty \alpha x^{-\alpha} e^{-r/x^\alpha} \nu_\xi(dx) = \int_0^\infty e^{-ru} \widetilde{Q}_\xi(du),$$

with the measure $\widetilde{Q}_\xi$ being defined by

$$\widetilde{Q}_\xi(B) = \alpha \int_0^\infty \mathbf{1}_B(x^{-\alpha}) x^{-\alpha} \nu_\xi(dx), \quad B \in \mathcal{B}((0,\infty)).$$

We conclude that $\widetilde{g}_\xi(\cdot)$ is completely monotone. Thus,

$$\widetilde{\nu}_\xi(dy) = y^{\alpha-1} \widetilde{g}_\xi(y^\alpha) dy$$

for some completely monotone function $\widetilde{g}_\xi$. This concludes that $\widetilde{\mu} \in E_\alpha(\mathbb{R}^d)$.
(ii) (Proof for that $E_\alpha(\mathbb{R}^d) \subset \mathcal{E}_\alpha(I(\mathbb{R}^d))$.) Let $\widetilde{\mu} \in E_\alpha(\mathbb{R}^d)$ with Lévy measure $\widetilde{\nu}$ of the form

$$\widetilde{\nu}(B) = \int_S \widetilde{\lambda}(d\xi) \int_0^\infty \mathbf{1}_B(r\xi) r^{\alpha-1} \widetilde{g}_\xi(r^\alpha) dr, \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}),$$

where $g_\xi(r)$ is completely monotone in r and measurable in ξ. For each ξ, there exists a Borel measure $\widetilde{Q}_\xi$ on $[0,\infty)$ such that $\widetilde{g}_\xi(r) = \int_{[0,\infty)} e^{-rt}\, \widetilde{Q}_\xi(dt)$ and $\widetilde{Q}_\xi(B)$

is measurable in ξ for each $B \in \mathcal{B}([0,\infty))$ (see the proof of Lemma 3.3 in Sato [10]). For $\widetilde{\nu}$ to be a Lévy measure, it is necessary and sufficient that

$$\begin{aligned}
\infty &> \int_S \widetilde{\lambda}(d\xi) \int_0^1 r^{\alpha+1} \widetilde{g}_\xi(r^\alpha)\, dr + \int_S \widetilde{\lambda}(d\xi) \int_1^\infty r^{\alpha-1} \widetilde{g}_\xi(r^\alpha)\, dr \\
&= \int_S \widetilde{\lambda}(d\xi) \int_0^1 r^{\alpha+1} dr \int_{[0,\infty)} e^{-r^\alpha t} \widetilde{Q}_\xi(dt) \\
&\qquad + \int_S \widetilde{\lambda}(d\xi) \int_1^\infty r^{\alpha-1} dr \int_{[0,\infty)} e^{-r^\alpha t} \widetilde{Q}_\xi(dt) \\
&= \int_S \widetilde{\lambda}(d\xi) \alpha^{-1} \int_{[0,\infty)} t^{-1-2/\alpha} \widetilde{Q}_\xi(dt) \int_0^t u^{2/\alpha} e^{-u}\, du \\
&\qquad + \int_S \widetilde{\lambda}(d\xi) \alpha^{-1} \int_{[0,\infty)} t^{-1} e^{-t} \widetilde{Q}_\xi(dt),
\end{aligned}$$

where we have used Fubini's theorem and the substitution $u = r^\alpha t$. ¿From this it is easy to see that $\widetilde{\nu}$ is a Lévy measure if and only if $\int_S \widetilde{\lambda}(d\xi)\widetilde{Q}_\xi(\{0\}) = 0$ (which we shall assume without comment from now on) and

$$\int_S \widetilde{\lambda}(d\xi) \int_0^1 t^{-1} \widetilde{Q}_\xi(dt) < \infty, \quad \int_S \widetilde{\lambda}(d\xi) \int_1^\infty t^{-1-2/\alpha} \widetilde{Q}_\xi(dt) < \infty. \tag{2.4}$$

In part (i) we have defined $\widetilde{Q}_\xi = U(\rho_\xi)$ as the image measure of ρ_ξ under the mapping $U : (0,\infty) \to (0,\infty), r \mapsto r^{-\alpha}$, where ρ_ξ has density $r \mapsto \alpha r^{-\alpha}$ with respect to ν_ξ. Denoting by $V : r \mapsto r^{-1/\alpha}$, the inverse of U, it follows that ρ_ξ is the image measure of $\widetilde{Q}_\xi$ under the mapping V. Hence, given $\widetilde{Q}_\xi$, we define ν_ξ as having density $r \mapsto \alpha^{-1} r^\alpha$ with respect to the image measure $V(\widetilde{Q}_\xi)$ of $\widetilde{Q}_\xi$ under V, i.e.

$$\nu_\xi(B) = \alpha^{-1} \int_0^\infty \mathbf{1}_B(r^{-1/\alpha}) r^{-1} \widetilde{Q}_\xi(dr), \quad B \in \mathcal{B}((0,\infty)).$$

Define further a measure ν to have spherical component $\lambda = \widetilde{\lambda}$ and radial parts ν_ξ, i.e.

$$\nu(B) = \int_S \widetilde{\lambda}(d\xi) \int_0^\infty \mathbf{1}_B(r\xi) \nu_\xi(dr), \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}).$$

Then ν is a Lévy measure, since

$$\begin{aligned}
&\int_S \widetilde{\lambda}(d\xi) \int_0^\infty (r^2 \wedge 1) \nu_\xi(dr) \\
&\quad = \int_S \widetilde{\lambda}(d\xi) \int_0^1 r^2\, \nu_\xi(dr) + \int_S \widetilde{\lambda}(d\xi) \int_1^\infty \nu_\xi(dr) \\
&\quad = \int_S \widetilde{\lambda}(d\xi) \int_1^\infty \alpha^{-1} r^{-2/\alpha} r^{-1} \widetilde{Q}_\xi(dr) + \int_S \widetilde{\lambda}(d\xi) \int_0^1 \alpha^{-1} r^{-1} \widetilde{Q}_\xi(dr),
\end{aligned}$$

which is finite by (2.4). If μ is any infinitely divisible distribution with Lévy measure ν, then part (i) of the proof shows that $\mathcal{E}_\alpha(\mu)$ has the given Lévy measure $\widetilde{\nu}$, and from the transformation of the generating triplet in Proposition 2.1 we see that $\mu_0 \in I(\mathbb{R}^d)$ can be chosen such that $\mathcal{E}_\alpha(\mu_0) = \widetilde{\mu}$. $\square$

3. The class $E_\alpha(\mathbb{R}^d)$ and its subclasses

The first result below shows that the classes $E_\alpha(\mathbb{R}^d)$ are increasing as α increases.

Theorem 3.1. *For any* $0 < \alpha < \beta$,

$$E_\alpha(\mathbb{R}^d) \subset E_\beta(\mathbb{R}^d).$$

Proof. Let $0 < \alpha < \beta$. Then if $\mu \in E_\alpha(\mathbb{R}^d)$, ν_ξ of μ is

$$\nu_\xi(dr) = r^{\alpha-1} g_\xi(r^\alpha) dr = r^{\beta-1} \frac{g_\xi\left((r^{\alpha/\beta})^\beta\right)}{r^{\beta-\alpha}} dr = r^{\beta-1} \frac{g_\xi\left((r^{\alpha/\beta})^\beta\right)}{\left(r^{(\beta-\alpha)/\beta}\right)^\beta} dr.$$

Let

$$h_\xi(x) = \frac{g_\xi(x^{\alpha/\beta})}{x^{(\beta-\alpha)/\beta}}.$$

Note that if g is completely monotone and ψ a nonnegative function such that ψ' is completely monotone, then the composition $g \circ \psi$ is completely monotone (see, e.g., Feller [5], page 441, Corollary 2), and if g and f are completely monotone then gf is completely monotone. Thus $g_\xi(x^{\alpha/\beta})$ is completely monotone and then $h_\xi(x)$ is also completely monotone, and we have

$$\nu_\xi(dr) = r^{\beta-1} h_\xi(r^\beta).$$

Hence $\mu \in E_\beta(\mathbb{R}^d)$. $\square$

In the following, we shall call a class F of distributions in $\mathbb{R}^d$ *closed under scaling* if for every $\mathbb{R}^d$-valued random variable X such that $\mathcal{L}(X) \in F$ it also holds that $\mathcal{L}(cX) \in F$ for every $c > 0$. If F is a class of infinitely divisible distributions in $\mathbb{R}^d$ and satisfies that $\mu \in F$ implies $\mu^{s*} \in F$ for any $s > 0$, where μ^{s*} is the distribution with characteristic function $(\widehat{\mu}(z))^s$, we shall call F *closed under taking of powers.* Recall that a class F of infinitely divisible distributions on $\mathbb{R}^d$ is called *completely closed in the strong sense* (abbreviated as c.c.s.s.) if it is closed under convolution, weak convergence, scaling, taking of powers, and additionally contains $\mu * \delta_b$ for any $\mu \in F$ and $b \in \mathbb{R}^d$.

Denote

$$S_+ := \{\xi = (\xi_1, \dots, \xi_d) \in S : \xi_1, \dots, \xi_d \ge 0\}.$$

Theorem 3.2. *Let $\alpha > 0$ and $Y_1^{(\alpha)}$ and $Z_1^{(\alpha)}$ be compound Poisson distributions on $\mathbb{R}$ with Lévy measures $\nu_{Y_1^{(\alpha)}}(dx) = |x|^{\alpha-1} e^{-x^\alpha}\, dx$ and $\nu_{Z_1^{(\alpha)}}(dx) = x^{\alpha-1} e^{-x^\alpha} \mathbf{1}_{(0,\infty)}(x)\, dx$, respectively. Then we have the following.*
(i) The class $E_\alpha(\mathbb{R}^d)$ is the smallest class of infinitely divisible distributions on $\mathbb{R}^d$ which is closed under convolution, weak convergence, scaling, taking of powers and contains each of the distributions $\mathcal{L}(Z_1^{(\alpha)}\xi)$ with $\xi \in S$. Further, $E_\alpha(\mathbb{R}^d)$ is c.c.s.s.
(ii) The class $E_\alpha^+(\mathbb{R}^d)$ is the smallest class of infinitely divisible distributions on $\mathbb{R}^d$ which is closed under convolution, weak convergence, scaling, taking of powers and contains each of the distributions $\mathcal{L}(Z_1^{(\alpha)}\xi)$ with $\xi \in S_+$.
(iii) The class $E_\alpha^{\mathrm{sym}}(\mathbb{R}^d)$ is the smallest class of infinitely divisible distributions on $\mathbb{R}^d$ which is closed under convolution, weak convergence, scaling, taking of powers and contains each of the distributions $\mathcal{L}(Y_1^{(\alpha)}\xi)$ with $\xi \in S$.

Proof. By the definition it is clear that all the classes under consideration are closed under convolution, scaling and taking of powers. The class $E_\alpha(\mathbb{R}^d)$ is closed under weak convergence by Proposition 2.1 (iv) and Theorem 2.3, and hence so are $E_\alpha^+(\mathbb{R}^d)$ and $E_\alpha^{\mathrm{sym}}(\mathbb{R}^d)$. Further, it is easy to see that all the given classes contain the specified distributions, since the Lévy measure of $\mathcal{L}(Z_1^{(\alpha)}\xi)$ for $\xi \in S$ has polar decomposition $\lambda = \delta_\xi$ and $\nu_\xi(dr) = r^{\alpha-1} g_\xi(r^\alpha)\, dr$ with $g_\xi(r) = e^{-r}$, and a similar argument works for $\mathcal{L}(Y_1^{(\alpha)}\xi)$. Finally, $E_\alpha(\mathbb{R}^d)$ contains all Dirac measures, which shows that it is c.c.s.s. So it only remains to show that the given classes are the smallest classes among all classes with the specified properties.

(i) Let F be the smallest class of infinitely divisible distributions which is closed under convolution, weak convergence, scaling, taking of powers and which contains $\mathcal{L}(Z_1^{(\alpha)}\xi)$ for every $\xi \in S$. As already shown, this implies $F \subset E_\alpha(\mathbb{R}^d)$. Recall from Theorem 2.3 that $\mathcal{E}_\alpha$ defines a bijection from $I(\mathbb{R}^d)$ onto $E_\alpha(\mathbb{R}^d)$, and let $G := \mathcal{E}_\alpha^{-1}(F)$. Then G is closed under convolution, weak convergence, scaling and taking of powers. This follows from the corresponding properties of F and the definition of $\mathcal{E}_\alpha$ for the third property, and Proposition 2.1 (ii) and (iv) for the first, fourth and second property, respectively.

It is easy to see from Proposition 2.1 (ii) that for $\xi \in S$, $\mu_\xi := \mathcal{E}_\alpha^{-1}(Z_1^{(\alpha)}\xi)$ has generating triplet $(A = 0, \nu = \alpha^{-1}\delta_\xi, \gamma)$ for some $\gamma \in \mathbb{R}^d$, so that $\{X_t^{(\mu_\xi)}\}$ has bounded variation, and its drift is zero by (2.3) since $\{X_t^{\mathcal{L}(Z_1^{(\alpha)}\xi)}\}$ has zero drift. This shows that $\mu_\xi = \mathcal{L}(N_1\xi)$ where $\{N_t\}_{t\geq 0}$ is a Poisson process with parameter $1/\alpha$, and we have $\mu_\xi \in G$ by assumption. Since G is closed under convolution and scaling this implies that $\mathcal{L}(n^{-1}N_n\xi) \in G$ for each $n \in \mathbb{N}$ and hence $E(N_1)\xi \in G$ by the strong law of large numbers since G is closed under weak convergence. Since $E(N_1) > 0$ and G is closed under taking of powers this shows that $\delta_c \in G$ for all $c \in \mathbb{R}^d$. Hence G contains every infinitely divisible distribution with Gaussian part zero and Lévy measure $\alpha^{-1}\delta_\xi$ with $\xi \in S$. Since G is closed under convolution, scaling and taking of powers it also contains all infinitely divisible distributions with Gaussian part zero and Lévy measures of the form $\nu = \sum_{i=1}^n a_i\delta_{c_i}$ with $n \in \mathbb{N}$, $a_i \geq 0$ and $c_i \in \mathbb{R}^d \setminus \{0\}$. Since every finite Borel measure on $\mathbb{R}^d$ is the weak limit of a sequence of measures of the form $\sum_{i=1}^n a_i\delta_{c_i}$, it follows from Theorem 8.7 in Sato [11] and the fact that G is closed under weak convergence that G contains all compound Poisson distributions, and hence all infinitely divisible distributions by Corollary 8.8 in [11]. This shows $G = I(\mathbb{R}^d)$ and hence $F = E_\alpha(\mathbb{R}^d)$ by Theorem 2.3.

(ii) and (iii) follow in analogy to the proof of (i), where for (iii) observe that $\mathcal{E}_\alpha^{-1}(Y_1^{(\alpha)}\xi)$ has characteristic triplet $(A = 0, \nu = \alpha^{-1}\delta_\xi + \alpha^{-1}\delta_{-\xi}, \gamma = 0)$, so that, by an argument similar to the proof of (i), every symmetric compound Poisson distribution is in $\mathcal{E}_\alpha^{-1}(F)$ and hence so every symmetric infinitely divisible distribution is. Here F is the smallest class of infinitely divisible distributions on $\mathbb{R}^d$ which is closed under convolution, weak convergence, scaling, taking of powers and contains each of the distributions $\mathcal{L}(Y_1^{(\alpha)}\xi)$ with $\zeta \in S$. Corollary 2.2 and Theorem 2.3 then imply $F = E_\alpha^{\mathrm{sym}}(\mathbb{R}^d)$. □

Remark 3.3. In Introduction, it was mentioned that $B(\mathbb{R}^d)$ is the smallest class of distributions on $\mathbb{R}^d$ closed under convolution and weak convergence and containing the distributions of all elementary mixed exponential random variables in $\mathbb{R}^d$. Theorem 3.2 for $\alpha = 1$ gives a new interpretation of $B(\mathbb{R}^d)$, since it is based on a compound Poisson distribution, not cased on an exponential distribution.

Remark 3.4. Once we are given a mapping $\mathcal{E}_\alpha$, we can construct nested classes of $E_\alpha(\mathbb{R}^d)$ by the iteration of the mapping $\mathcal{E}_\alpha$, which is $\mathcal{E}_\alpha^m = \mathcal{E}_\alpha \circ \cdots \circ \mathcal{E}_\alpha$ (m-times composition). It is easy to see that $\mathfrak{D}(\mathcal{E}_\alpha^m) = I(\mathbb{R}^d)$ for any $m \in \mathbb{N}$. Then we can characterize $\mathcal{E}_\alpha^m(I(\mathbb{R}^d))$ as the smallest class of infinitely divisible distributions which is closed under convolution, weak convergence, scaling and taking of powers and contains $\mathcal{E}_\alpha^m(N_1\xi)$ for all $\xi \in S$ and N_1 being a Poisson distribution with mean $1/\alpha$. The same proof of Theorem 3.4 works, but we do not go into the details here.

4. Characterization of subclasses of $E_\alpha(\mathbb{R}^d)$ by stochastic integrals with respect to some compound Poisson processes

For any Lévy process $Y = \{Y_t\}_{t\ge 0}$ on $\mathbb{R}^d$, denote by $\mathbf{L}_{(0,\infty)}(Y)$ the class of locally Y-integrable, real valued functions on $(0,\infty)$ (cf. Sato [15], Definition 2.3), and let

$$\mathrm{Dom}(Y) = \left\{ h \in \mathbf{L}_{(0,\infty)}(Y) \ : \ \int_0^\infty h(t)dY_t \text{ is definable} \right\},$$
$$\mathrm{Dom}^{\downarrow}(Y) = \{h \in \mathrm{Dom}(Y) : h \text{ is a left-continuous and decreasing function such that } \lim_{t\to\infty} h(t) = 0\}.$$

Here, following Definition 3.1 of Sato [15], by saying that the (improper stochastic integral) $\int_0^\infty h(t)dY_t$ is definable we mean that $\int_p^q h(t)dY_t$ converges in probability as $p \downarrow 0$, $q \to \infty$, with the limit random variable being denoted by $\int_0^\infty h(t)dY_t$.

The property of h belonging to $\mathrm{Dom}(Y)$ can be characterized in terms of the generating triplet (A_Y, ν_Y, γ_Y) of Y and assumptions on h, cf. Sato [15], Theorems 2.6, 3.5 and 3.10. In particular, if $A_Y = 0$, then $h \in \mathrm{Dom}(Y)$ if and only if h is measurable,

$$\int_0^\infty ds \int_{\mathbb{R}^d} (|h(s)x|^2 \wedge 1)\, \nu_Y(dx) < \infty, \tag{4.1}$$

$$\int_p^q \left| h(s)\gamma_Y + \int_{\mathbb{R}^d} h(s)x \left(\frac{1}{1+|h(s)x|^2} - \frac{1}{1+|x|^2} \right) \nu_Y(dx) \right| ds < \infty \tag{4.2}$$

for all $0 < p < q < \infty$ and

$$\lim_{p\downarrow 0, q\to\infty} \int_p^q \left(h(s)\gamma_Y + \int_{\mathbb{R}^d} h(s)x \left(\frac{1}{1+|h(s)x|^2} - \frac{1}{1+|x|^2} \right) \nu_Y(dx) \right) ds \quad \text{exists in } \mathbb{R}^d. \tag{4.3}$$

In this case, $\int_0^\infty h(t)\, dY_t$ is infinitely divisible without Gaussian part and its Lévy measure $\nu_{Y,h}$ is given by

$$\nu_{Y,h}(B) = \int_0^\infty ds \int_{\mathbb{R}^d} \mathbf{1}_B(h(s)x)\, \nu_Y(dx), \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}). \tag{4.4}$$

If ν_Y is symmetric and $\gamma_Y = 0$, then (4.2) and (4.3) are automatically satisfied, so that $h \in \mathrm{Dom}(Y)$ if and only if (4.1) is satisfied, in which case $\gamma_{Y,h}$ in the generating triplet of $\int_0^\infty h(t)\, dY_t$ is 0.

Recall what was mentioned in Introduction. Let $Y = \{Y_t\}_{t\ge 0}$ be a compound Poisson process on $\mathbb{R}_+$ with Lévy measure $\nu_Y(dx) = e^{-x}dx, x > 0$. Then

$$B(\mathbb{R}_+) = \left\{ \mathcal{L}\left(\int_0^\infty h(t)dY_t \right), \quad h \in \mathrm{Dom}(Y) \right\}.$$

Also recall the definition of $E_\alpha^{0,\mathrm{ri}}(\mathbb{R}^d)$ from (1.6). The next theorem characterizes $E_\alpha^{0,\mathrm{ri}}(\mathbb{R}^d)$ as the class of distributions which arise as improper stochastic integrals over $(0,\infty)$ with respect to some fixed rotationally invariant compound Poisson process on $\mathbb{R}^d$.

Theorem 4.1. *Let $\alpha > 0$ and denote by $Y^{(\alpha)} = \{Y_t^{(\alpha)}\}_{t\geq 0}$ a compound Poisson process on $\mathbb{R}^d$ with Lévy measure $\nu_{Y^{(\alpha)}}(B) = \int_S d\xi \int_0^\infty 1_B(r\xi) r^{\alpha-1} e^{-r^\alpha} dr$, equivalently*

$$\nu_{Y^{(\alpha)}}(d\xi dr) = d\xi r^{\alpha-1} e^{-r^\alpha} dr,\ \xi \in S,\ r > 0 \tag{4.5}$$

(*without drift*). *Then*

$$E_\alpha^{0,\mathrm{ri}}(\mathbb{R}^d) = \left\{ \mathcal{L}\left(\int_0^\infty h(t) dY_t^{(\alpha)} \right) \ :\ h \in \mathrm{Dom}(Y^{(\alpha)}) \right\} \tag{4.6}$$

$$= \left\{ \mathcal{L}\left(\int_0^\infty h(t) dY_t^{(\alpha)} \right) \ :\ h \in \mathrm{Dom}^{\downarrow}(Y^{(\alpha)}) \right\}. \tag{4.7}$$

The function $h \in \mathrm{Dom}^{\downarrow}(Y^{(\alpha)})$ in representation (4.7) *is uniquely determined by $\mu \in E_\alpha^{0,\mathrm{ri}}(\mathbb{R}^d)$.*

Proof. Let $\mu \in E_\alpha^{0,\mathrm{ri}}(\mathbb{R}^d)$. By definition and Remark 1.1, the Lévy measure ν of μ has the polar decomposition (λ, ν_ξ) given by

$$\nu_\xi(dr) = r^{\alpha-1} g(r^\alpha) dr,\ r > 0,\ \lambda(d\xi) = d\xi, \tag{4.8}$$

and g is independent of ξ and completely monotone. (If $\mu = \delta_0$ we define $g_\xi = 0$ and shall also call (λ, ν_ξ) a polar decomposition, even if ν_ξ is not strictly positive here). Since g is completely monotone, there exists a Borel measure Q on $[0,\infty)$ such that $g(y) = \int_{[0,\infty)} e^{-yt} Q(dt)$. By (2.4), since ν_ξ satisfies $\int_0^\infty (r^2 \wedge 1) \nu_\xi(dr) < \infty$, we see that

$$Q(\{0\}) = 0, \quad \int_0^1 t^{-1}\, Q(dt) < \infty \quad \text{and} \quad \int_1^\infty t^{-1-2/\alpha}\, Q(dt) < \infty. \tag{4.9}$$

Observe that under this condition, we have for each $r > 0$,

$$\begin{aligned} \nu_\xi([r,\infty)) &= \int_r^\infty y^{\alpha-1} g(y^\alpha) dy = \int_0^\infty (\alpha t)^{-1} Q(dt) \int_r^\infty \alpha t y^{\alpha-1} e^{-y^\alpha t} dy \\ &= \int_0^\infty (\alpha t)^{-1} e^{-r^\alpha t} Q(dt). \end{aligned}$$

Next, observe that since $Y^{(\alpha)}$ is rotationally invariant without Gaussian part, we have by (4.1) that a measurable function h is in $\mathrm{Dom}(Y^{(\alpha)})$ if and only if

$$\int_0^\infty ds \int_0^\infty \left(|h(s) r|^2 \wedge 1 \right) r^{\alpha-1}\, e^{-r^\alpha}\, dr\ < \infty, \tag{4.10}$$

in which case $\int_0^\infty h(t)\,dY_t^{(\alpha)}$ is infinitely divisible with the generating triplet $(A_{Y,h}=0,\nu_{Y,h},\gamma_{Y,h}=0)$ and the Lévy measure $\nu_{Y,h}$ is rotationally invariant. Suppose $B=C\times[r,\infty)$, where $C\in\mathcal{B}(S)$ and $r>0$. Then by (4.4) and (4.5),

$$\begin{aligned}\nu_{Y,h}(B)=\nu_{Y,h}(C\times[r,\infty))&=\int_0^\infty ds\int_S 1_C(\xi)d\xi\int_{r/|h(s)|}^\infty x^{\alpha-1}e^{-x^\alpha}dx\\&=\alpha^{-1}|C|\int_0^\infty e^{-r^\alpha/|h(s)|^\alpha}ds\end{aligned}\tag{4.11}$$

for every $r>0$, where $|C|$ is the Lebesgue measure of C on S. Hence, in order to prove (4.6) and (4.7), it is enough to prove the following:

(a) For each Borel measure Q on $[0,\infty)$ satisfying (4.9) there exists a function $h\in\mathrm{Dom}^{\downarrow}(Y^{(\alpha)})$ such that

$$\int_0^\infty t^{-1}\,e^{-r^\alpha t}\,Q(dt)=\int_0^\infty e^{-r^\alpha/|h(s)|^\alpha}\,ds\quad\text{for every } r>0.\tag{4.12}$$

(b) For each $h\in\mathrm{Dom}(Y^{(\alpha)})$ there exists a Borel measure Q on $[0,\infty)$ satisfying (4.9) such that (4.12) holds.

To show (a), let Q satisfy (4.9), and denote

$$F(x):=\int_{(0,x]}t^{-1}\,Q(dt),\quad x\in[0,\infty),$$

and by

$$F^{\leftarrow}(t)=\inf\{y\geq 0:F(y)\geq t\},\quad t\in[0,\infty),$$

its left-continuous inverse, with the usual convention $\inf\emptyset=+\infty$. Now define

$$h=h_Q:(0,\infty)\to[0,\infty),\quad t\mapsto\left(F^{\leftarrow}(t)\right)^{-1/\alpha}.$$

Then h is left-continuous, decreasing, and satisfies $\lim_{t\to\infty}h(t)=0$. Denote Lebesgue measure on $(0,\infty)$ by m_1, and consider the function

$$T:(0,\infty)\to(0,\infty],\quad s\mapsto h(s)^{-\alpha}=F^{\leftarrow}(s).\tag{4.13}$$

Then $(T(m_1))_{|(0,\infty)}$, the image measure of m_1 under the mapping T, when restricted to $(0,\infty)$, satisfies

$$(T(m_1))_{|(0,\infty)}(dt)=t^{-1}\,Q_{|(0,\infty)}(dt).\tag{4.14}$$

Hence it follows that for every $r>0$,

$$\begin{aligned}\int_{(0,\infty)}e^{-r^\alpha/h(s)^\alpha}\,m_1(ds)&=\int_{(0,\infty)\cap\{s:T(s)\neq\infty\}}e^{-r^\alpha T(s)}\,m_1(ds)\\&=\int_{(0,\infty)}e^{-r^\alpha t}\,(T(m_1))(dt),\end{aligned}\tag{4.15}$$

yielding (4.12). To show (4.10), namely that $h \in \mathrm{Dom}(Y^{(\alpha)})$, observe that

$$\begin{aligned}
&\int_0^\infty ds \int_0^\infty \big(|h(s)r|^2 \wedge 1\big)\, |r|^{\alpha-1} e^{-r^\alpha} dr \\
&\qquad = \int_0^\infty r^{\alpha+1} e^{-r^\alpha} dr \int_{\{s:h(s)\le 1/r\}} h(s)^2\, ds \,+ \int_0^\infty ds \int_{1/h(s)}^\infty r^{\alpha-1} e^{-r^\alpha}\, dr \\
&\qquad = \int_0^\infty r^{\alpha+1} e^{-r^\alpha} dr \int_{\{s:T(s)\ge r^\alpha\}} T(s)^{-2/\alpha}\, ds \,+ 2\alpha^{-1} \int_0^\infty e^{-T(s)}\, ds \\
&\qquad = \int_0^\infty r^{\alpha+1} e^{-r^\alpha} dr \int_{\{t\ge r^\alpha\}} t^{-1-2/\alpha} Q(dt) \,+ \alpha^{-1} \int_0^\infty e^{-t} t^{-1}\, Q(dt)
\end{aligned}$$

by (4.14). The second of these terms is clearly finite by (4.9). To estimate the first, observe that

$$\begin{aligned}
&\int_0^\infty r^{\alpha+1} e^{-r^\alpha} dr \int_{r^\alpha}^\infty t^{-1-2/\alpha}\, Q(dt) \\
&\qquad \le \int_1^\infty r^{\alpha+1} e^{-r^\alpha} dr \int_1^\infty t^{-1-2/\alpha}\, Q(dt) \,+ \int_0^1 r^{\alpha+1} dr \int_1^\infty t^{-1-2/\alpha}\, Q(dt) \\
&\qquad\quad + \int_0^1 r^{\alpha+1} dr \int_{r^\alpha}^1 t^{-1-2/\alpha}\, Q(dt),
\end{aligned}$$

and the first two summands are finite by (4.9), while the last summand is equal to

$$\int_0^1 t^{-1-2/\alpha} Q(dt) \int_0^{t^{1/\alpha}} r^{\alpha+1}\, dr \,= (\alpha+2)^{-1} \int_0^1 t^{1+2/\alpha} t^{-1-2/\alpha}\, Q(dt)$$

and hence also finite. This shows (4.10) for h and hence (a).

To show (b), let $h \in \mathrm{Dom}(Y^{(\alpha)})$ and assume first that h is nonnegative. Let $T : (0,\infty) \to (0,\infty]$ be defined by $T(s) = h(s)^{-\alpha}$ as in (4.13), and consider the image measure $T(m_1)$. Define the measure Q on $[0,\infty)$ by $Q(\{0\}) = 0$ and equality (4.14). Since $\int_0^\infty h(t)\, dY_t^{(\alpha)}$ is automatically infinitely divisible with Lévy measure $\nu_{Y,h}$ given by (4.11), we have as in the proof of (a) for every $C \in \mathcal{B}(S)$ and $r > 0$,

$$|C| \int_{(0,\infty)} e^{-r^\alpha t} (\alpha t)^{-1}\, Q(dt) = \alpha^{-1} |C| \int_0^\infty e^{-r^\alpha / h(s)^\alpha}\, ds = \nu_{Y,h}(C \times [r,\infty)).$$

In particular, Q must be a Borel measure and (4.12) holds. Since the left hand side of this equation converges and the right hand side is known to be the tail integral of a Lévy measure, it follows from the proof of (2.4) that (4.9) must hold. Hence we have seen that $\mathcal{L}(\int_0^\infty h(t) dY_t^{(\alpha)}) \in E_\alpha^{0,\mathrm{ri}}(\mathbb{R}^d)$ for nonnegative $h \in \mathrm{Dom}(Y^{(\alpha)})$. For general $h \in \mathrm{Dom}(Y^{(\alpha)})$, write $h = h^+ - h^-$ with $h^+ := h \vee 0$ and $h^- := (-h) \vee 0$. Then $h^+, h^- \in \mathrm{Dom}(Y^{(\alpha)})$ by (4.10), and Equation (4.4) and the discussion following it show that $\int_0^\infty h(t) dY_t^{(\alpha)}$ has no Gaussian part, gamma part 0 and satisfies $\nu_{Y,h} =$

$\nu_{Y,h^+} + \nu_{Y,h^-}$. The corresponding Borel measure Q is given by $Q = Q^+ + Q^-$, where Q^+ and Q^- are constructed from h^+ and h^-, respectively, completing the proof of (b).

Finally, to show uniqueness of $h \in \mathrm{Dom}^{\downarrow}(Y^{(\alpha)})$ in the representation (4.7), let $h_1, h_2 \in \mathrm{Dom}^{\downarrow}(Y^{(\alpha)})$ such that

$$\mathcal{L}\left(\int_0^\infty h_1(t)dY_t^{(\alpha)}\right) = \mathcal{L}\left(\int_0^\infty h_2(t)dY_t^{(\alpha)}\right).$$

Define the functions $T_1, T_2 : (0,\infty) \to (0,\infty]$ by $T_1(s) := h_1(s)^{-\alpha}$ and $T_2(s) := h_2(s)^{-\alpha}$. It then follows from (4.11) that

$$\int_0^\infty e^{-r^\alpha/|h_1(s)|^\alpha} ds = \int_0^\infty e^{-r^\alpha/|h_2(s)|^\alpha} ds < \infty$$

for all $r > 0$, which using the argument of (4.15) can be written as

$$\int_{(0,\infty)} e^{-r^\alpha t}\,(T_1(m_1))(dt) = \int_{(0,\infty)} e^{-r^\alpha t}\,(T_2(m_1))(dt) < \infty, \quad r > 0. \tag{4.16}$$

Observe that T_1 and T_2 are left-continuous increasing functions with $\lim_{s\to\infty} T_1(s) = \lim_{s\to\infty} T_2(s) = \infty$. Hence $T_i(m_1)((0,b]) < \infty$ for all $b \in (0,\infty)$, $i = 1, 2$, and it follows from (4.16) and the uniqueness theorem for Laplace transforms of Borel measures on $[0,\infty)$ that

$$(T_1(m_1))_{|(0,\infty)} = (T_2(m_1))_{|(0,\infty)}.$$

In other words we have for every $b \in (0,\infty)$ that

$$m_1(\{s \in (0,\infty) : T_1(s) \le b\}) = m_1(\{s \in (0,\infty) : T_2(s) \le b\}) < \infty.$$

Since T_1 and T_2 are left-continuous and increasing, this clearly implies $T_1 = T_2$ and hence $h_1 = h_2$, completing the proof of the uniqueness assertion in representation (4.7). $\square$

Next, we assume $d = 1$ and we ask whether every distribution in $E_\alpha^0(\mathbb{R}^1) := \{\mu \in E_\alpha(\mathbb{R}^d) : \mu \text{ has no Gaussian part}\}$ can be represented as a stochastic integral with respect to the compound Poisson process $Z^{(\alpha)}$ having Lévy measure $\nu_{Z^{(\alpha)}}(dx) = x^{\alpha-1}e^{-x^\alpha}\mathbf{1}_{(0,\infty)}(x)\,dx$ (without drift) plus some constant. We shall prove that such a statement is true e.g. for those distributions in $E_\alpha^0(\mathbb{R}^1)$ which correspond to Lévy processes of bounded variation, but that not every distribution in $E_\alpha^0(\mathbb{R}^1)$ can be represented in this way. However, every distribution in $E_\alpha^0(\mathbb{R}^1)$ appears as an essential limit of locally $Z^{(\alpha)}$-integrable functions. Following Sato [15], Definition 3.2, for a Lévy process $Y = \{Y_t\}_{t\ge 0}$ and a locally Y-integrable function h over $(0,\infty)$ we say that *the essential improper stochastic integral on* $(0,\infty)$ *of* h *with respect to* Y *is definable* if for every $0 < p < q < \infty$ there are real constants $\tau_{p,q}$ such that $\int_p^q h(t)dY_t - \tau_{p,q}$ converges in probability as $p \downarrow 0$, $q \to \infty$. We write $\mathrm{Dom}_{\mathrm{es}}(Y)$ for the class of all locally Y-integrable functions h on $(0,\infty)$ for which the essential improper stochastic integral with respect to Y is definable, and for each $h \in \mathrm{Dom}_{\mathrm{es}}(Y)$ we denote the class of distributions arising as possible limits $\int_p^q h(t)dY_t - \tau_{p,q}$ as $p \downarrow 0$, $q \to \infty$ by $\Phi_{h,\mathrm{es}}(Y)$ (the limit is not unique, since different sequences $\tau_{p,q}$ may give different limit random variables). As for $\mathrm{Dom}(Y)$, the property of belonging to $\mathrm{Dom}_{\mathrm{es}}(Y)$ can be expressed in terms of the characteristic triplet (A_Y, ν_Y, γ_Y) of Y. In particular, if $A_Y = 0$, then a function h on $(0,\infty)$ is in $\mathrm{Dom}_{\mathrm{es}}(Y)$ if and only if h is measurable and (4.1) and (4.2) hold, and in that case $\Phi_{h,\mathrm{es}}(Y)$ consists of all infinitely divisible distributions μ with characteristic triplet $(A_{Y,h} = 0, \nu_{Y,h}, \gamma)$, where $\nu_{Y,h}$ is given by (4.4) and $\gamma \in \mathbb{R}$ is arbitrary (cf. [15], Theorems 3.6 and 3.11).

Recall $E_\alpha^+(\mathbb{R}^1) = \{\mu \in E_\alpha(\mathbb{R}^1) : \mu((-\infty,0)) = 0\}$ and denote

$$\begin{aligned}
E_\alpha^{+,0}(\mathbb{R}^1) &:= \{\mu \in E_\alpha^+(\mathbb{R}^1) : \{X_t^{(\mu)}\} \text{ has zero drift}\},\\
E_\alpha^{BV}(\mathbb{R}^1) &:= \{\mu \in E_\alpha(\mathbb{R}^1) : \{X_t^{(\mu)}\} \text{ is of bounded variation}\},\\
E_\alpha^{BV,0}(\mathbb{R}^1) &:= \{\mu \in E_\alpha^{BV}(\mathbb{R}^1) : \{X_t^{(\mu)}\} \text{ has zero drift}\}.
\end{aligned}$$

We then have:

Theorem 4.2. *Let* $\alpha > 0$ *and denote by* $Z^{(\alpha)} = \{Z_t^{(\alpha)}\}_{t\ge 0}$ *a compound Poisson process on* $\mathbb{R}$ *with Lévy measure* $\nu_{Z^{(\alpha)}}(dx) = x^{\alpha-1}e^{-x^\alpha}\mathbf{1}_{(0,\infty)}(x)\,dx$ *(without drift). Then it holds:*

(i) The class of distributions arising as limits of essential improper stochastic integrals with respect to $Z^{(\alpha)}$ *is* $E_\alpha^0(\mathbb{R}^1)$ *:*

$$E_\alpha^0(\mathbb{R}^1) = \bigcup_{h\in \mathrm{Dom}_{\mathrm{es}}(Z^{(\alpha)})} \Phi_{h,\mathrm{es}}(Z^{(\alpha)}). \tag{4.17}$$

(ii) Distributions in $E_\alpha^{BV,0}(\mathbb{R}^1)$ *and* $E_\alpha^{+,0}(\mathbb{R}^1)$ *can be expressed as improper stochastic integrals over* $(0,\infty)$ *with respect to* $Z^{(\alpha)}$*. More precisely*

$$E_\alpha^{+,0}(\mathbb{R}^1) = \left\{\mathcal{L}\left(\int_0^\infty h(t)dZ_t^{(\alpha)}\right) : h \in \mathrm{Dom}(Z^{(\alpha)}), h \ge 0\right\}, \tag{4.18}$$

$$E_\alpha^{BV,0}(\mathbb{R}^1) = \left\{\mathcal{L}\left(\int_0^\infty h(t)dZ_t^{(\alpha)}\right) : h \in \mathrm{Dom}(Z^{(\alpha)}) \text{ such that } \int_0^\infty ds \int_{\mathbb{R}} (|h(s)x| \wedge 1)\nu_{Z^{(\alpha)}}(dx) < \infty\right\}. \tag{4.19}$$

In particular,

$$E_\alpha^+(\mathbb{R}^1) = \left\{ \mathcal{L}\left(\int_0^\infty h(t) dZ_t^{(\alpha)} + b \right) : h \in \mathrm{Dom}(Z^{(\alpha)}),\ h \geq 0,\ b \in [0,\infty) \right\}. \tag{4.20}$$

(iii) Not every distribution in $E_\alpha^0(\mathbb{R}^1)$ *can be represented as an improper stochastic integral over* $(0,\infty)$ *with respect to* $Z^{(\alpha)}$ *plus some constant. It holds*

(4.21)

$$E_\alpha^{BV}(\mathbb{R}^1) \cup E_\alpha^{0,\mathrm{sym}}(\mathbb{R}^1) \subsetneqq \left\{ \mathcal{L}\left(\int_0^\infty h(t) dZ_t^{(\alpha)} + b \right) : b \in \mathbb{R}, h \in \mathrm{Dom}(Z^{(\alpha)}) \right\} \subsetneqq E_\alpha^0(\mathbb{R}^1).$$

Proof. (i) Let $h \in \mathrm{Dom}_{\mathrm{es}}(Z^{(\alpha)})$ and $\mu \in \Phi_{h,\mathrm{es}}(Z^{(\alpha)})$ and write $h = h^+ - h^-$ with h^+ and h^- being the positive and negative parts of h, respectively. Then μ is infinitely divisible without Gaussian part and by (4.4) its Lévy measure $\nu_{Z,h}$ satisfies

$$\nu_{Z,h,1}([r,\infty)) := \nu_{Z,h}([r,\infty)) = \alpha^{-1} \int_0^\infty e^{-r^\alpha / h^+(s)^\alpha}\, ds,$$

$$\nu_{Z,h,-1}([r,\infty)) := \nu_{Z,h}((-\infty,-r]) = \alpha^{-1} \int_0^\infty e^{-r^\alpha / h^-(s)^\alpha}\, ds$$

for every $r > 0$. Define the mappings $T_1, T_{-1} : (0,\infty) \to (0,\infty]$ by $T_1(s) = (h^+(s))^{-\alpha}$ and $T_{-1}(s) = (h^-(s))^{-\alpha}$ and the measures Q_1 and Q_{-1} on $[0,\infty)$ by

$$Q_\xi(\{0\}) = 0 \quad \text{and} \quad (T_\xi(m_1))_{|(0,\infty)}(dt) = t^{-1} Q_{\xi|(0,\infty)}(dt), \qquad \xi \in \{-1,1\}.$$

Then as in the proof of Theorem 4.1,

$$\int_{(0,\infty)} e^{-r^\alpha t} (\alpha t)^{-1} Q_\xi(dt) = \nu_{Z,h,\xi}([r,\infty)), \quad r > 0, \quad \xi \in \{-1,1\},$$

and Q_1 and Q_{-1} satisfy (4.9) and we conclude that $\nu_{Z,h,\xi}(dr) = r^{\alpha-1} g_\xi(r^\alpha) dr$ for completely monotone functions g_1 and g_{-1}, so that $\Phi_{h,\mathrm{es}}(Z^{(\alpha)}) \subset E_\alpha^0(\mathbb{R}^1)$, giving the inclusion "$\supset$" in equation (4.17).

Now let $\mu \in E_\alpha^0(\mathbb{R}^1)$ with Lévy measure ν, and define the Lévy measures ν_1 and ν_{-1} supported on $[0,\infty)$ by

$$\nu_1(B) := \nu(B), \quad \nu_{-1}(B) := \nu(-B), \quad B \in \mathcal{B}((0,\infty)). \tag{4.22}$$

Then

$$\nu_\xi([r,\infty)) = \int_0^\infty (\alpha t)^{-1} e^{-r^\alpha t} Q_\xi(dt), \quad r > 0, \quad \xi \in \{-1,1\}, \tag{4.23}$$

for some Borel measures Q_1 and Q_{-1} satisfying (4.9). As in the proof of (a) in Theorem 4.1, we find nonnegative and decreasing functions $h_1, h_{-1} : (0,\infty) \to [0,\infty)$ such that (4.10) (i.e. (4.1) with $\nu_{Z^{(\alpha)}}$ in place of ν_Y) and (4.12) hold. Since h_1, h_{-1} are bounded on compact subintervals of $(0,\infty)$ and since $Z^{(\alpha)}$ has bounded variation, it follows that h_1 and h_{-1} satisfy also (4.2), so that $h_1, h_{-1} \in \mathrm{Dom}_{\mathrm{es}}(Z^{(\alpha)})$ and the Lévy measures of $\widetilde{\mu}_1 \in \Phi_{h_1,\mathrm{es}}(Z^{(\alpha)})$ and $\widetilde{\mu}_{-1} \in \Phi_{h_{-1},\mathrm{es}}(Z^{(\alpha)})$ are given by ν_1 and ν_{-1}, respectively. Now define the function $h : (0,\infty) \to \mathbb{R}$ by

$$h(t) = \begin{cases} h_1(t-n), & t \in (2n, 2n+1], \quad n \in \{1,2,\ldots\}, \\ -h_{-1}(t-n-1), & t \in (2n+1, 2n+2], \quad n \in \{1,2,\ldots\}, \\ h_1(t - 2^{-k-1}), & t \in (2^{-k}, 2^{-k} + 2^{-k-1}], \quad k \in \{0,1,2,\ldots\}, \\ -h_{-1}(t-2^{-k}), & t \in (2^{-k} + 2^{-k-1}, 2^{-k+1}], \quad k \in \{0,1,2,\ldots\}. \end{cases} \tag{4.24}$$

Then also $h \in \mathrm{Dom}_{\mathrm{es}}(Z^{(\alpha)})$ and any $\widetilde{\mu} \in \Phi_{h,\mathrm{es}}(Z^{(\alpha)})$ has Lévy measure ν, showing the inclusion "$\subset$" in equation (4.17).

(ii) Let $h \in \mathrm{Dom}(Z^{(\alpha)})$. Then $\int_0^\infty h(t)\, dZ_t^{(\alpha)} \in E_\alpha^0(\mathbb{R}^1)$ by (i). Further, by Theorem 3.15 in Sato [15], $\int_0^\infty h(t) dZ_t^{(\alpha)}$ is the distribution at time 1 of a Lévy process of bounded variation if and only if

$$\int_0^\infty ds \int_{\mathbb{R}} (|h(s)x| \wedge 1) \nu_{Z^{(\alpha)}}(dx) < \infty, \tag{4.25}$$

in which case this Lévy process will have zero drift. Since $\mathcal{L}(\int_0^\infty h(t) dZ_t^{(\alpha)})$ has trivially support contained in $[0,\infty)$ if $h \geq 0$, this gives the inclusion "$\supset$" in (4.18) and (4.19).

Now suppose that $\mu \in E_\alpha^{BV,0}(\mathbb{R}^1)$ with Lévy measure ν, define ν_1 and ν_{-1} by (4.22) and choose Borel measures Q_1 and Q_{-1} such that (4.23) holds. Then it can be shown in complete analogy to the proof leading to (2.4) that for $\xi \in \{-1, 1\}$, ν_ξ satisfies $\int_0^\infty (1 \wedge x) \nu_\xi(dx) < \infty$ if and only if

$$Q_\xi(\{0\}) = 0, \quad \int_0^1 t^{-1} Q_\xi(dt) < \infty \quad \text{and} \quad \int_1^\infty t^{-1-1/\alpha} Q_\xi(dt) < \infty. \tag{4.26}$$

For $\xi \in \{-1,1\}$ and $x \in [0,\infty)$ define $F_\xi(x) := \int_{(0,x]} t^{-1} Q_\xi(dt)$, $h_\xi = (F_\xi^{\leftarrow})^{-1/\alpha}$ and $T_\xi = (h_\xi)^{-\alpha} = F_\xi^{\leftarrow}$. Then it follows in complete analogy to the proof of (a) of Theorem 4.1, using (4.26), that (4.12) and (4.25) hold for h_ξ and Q_ξ. By Theorem 3.15 in Sato [15] this then shows that $h_\xi \in \mathrm{Dom}(Z^{(\alpha)})$ for $\xi \in \{-1,1\}$. Now if $\mu \in E_\alpha^{+,0}(\mathbb{R}^1)$, define $h(t) := h_1(t)$, and for general $\mu \in E_\alpha^{BV,0}$, define $h(t)$ by (4.24). In each case h satisfies (4.25), $h \in \mathrm{Dom}(Z^{(\alpha)})$, and $\mu = \mathcal{L}(\int_0^\infty h(t) dZ_t^{(\alpha)})$, giving the inclusions "$\subset$" in (4.18) and (4.19).

(iii) Let $\mu \in E_\alpha^{0,\mathrm{sym}}(\mathbb{R}^1)$. By Theorem 4.1 there exists $f \in \mathrm{Dom}^\downarrow(Y^{(\alpha)})$ such that $\mu = \mathcal{L}(\int_0^\infty f(t) dY_t^{(\alpha)})$. Write $h_1 = h_{-1} := f$ and define the function $h : (0,\infty) \to \mathbb{R}$ by (4.24). We claim that $h \in \mathrm{Dom}(Z^{(\alpha)})$. To see this, observe that h clearly satisfies (4.1) with respect to $\nu_{Z^{(\alpha)}}$ since f has the corresponding property with respect to $\nu_{Y^{(\alpha)}}$. Next, since $|h(s)x|(1 + |h(s)x|^2)^{-1}$ is bounded by $1/2$ and $\nu_{Z^{(\alpha)}}(\mathbb{R})$ is finite, it follows that

$$\int_0^q \left| \int_0^\infty \frac{h(s)x}{1+|h(s)x|^2} x^{\alpha-1} e^{-x^\alpha} dx \right| ds < \infty \quad \forall\, q > 0. \tag{4.27}$$

But since $Z^{(\alpha)}$ has the generating triplet

$$\left(A_{Z^{(\alpha)}} = 0,\ \nu_{Z^{(\alpha)}},\ \gamma_{Z^{(\alpha)}} = \int_0^\infty \frac{x}{1+x^2} x^{\alpha-1} e^{-x^\alpha} dx \right),$$

(4.27) shows that (4.2) is satisfied for h with respect to $\nu_{Z^{(\alpha)}}$. Finally, by the definition of h, for

$$\gamma_{Z,h,0,q} := \int_0^q \left(\int_0^\infty \frac{h(s)x}{1+|h(s)x|^2} x^{\alpha-1} e^{-x^\alpha}\, dx \right) ds, \quad q > 0,$$

we have $\gamma_{Z,h,0,q} = 0$ for $q = 2, 4, 6, \ldots$, and since $\lim_{t\to\infty} h(t) = 0$ it follows that $\lim_{q\to\infty} \gamma_{Z,h,0,q}$ exists and is equal to 0. We conclude that (4.3) is satisfied, so that $h \in \mathrm{Dom}(Z^{(\alpha)})$. By (4.4) we clearly have $\mathcal{L}\left(\int_0^\infty h(t) dZ_t^{(\alpha)}\right) = \mathcal{L}\left(\int_0^\infty f(t) dY_t^{(\alpha)}\right) = \mu$. Together with (4.17) and (4.19) and this shows (4.21) apart from the fact that the inclusions are proper.

To show that the first inclusion in (4.21) is proper, let $\mu \in E_\alpha^{0,\mathrm{sym}}(\mathbb{R}^1) \setminus E_\alpha^{BV}(\mathbb{R}^1)$. The latter set is nonempty since by (4.9) and (4.26) it suffices to find a Borel measure Q on $[0,\infty)$ such that (4.9) holds but $\int_1^\infty t^{-1-1/\alpha} Q(dt) = \infty$. As already shown, there exists $h \in \mathrm{Dom}(Z^{(\alpha)})$ such that $\mu = \mathcal{L}(\int_0^\infty h(t) dZ_t^{(\alpha)})$. Then $h + \mathbf{1}_{[1,2]} \in \mathrm{Dom}(Z^{(\alpha)})$, and $\mathcal{L}(\int_0^\infty (h(t) + \mathbf{1}_{[1,2]}(t) dZ_t^{(\alpha)})$ is clearly neither symmetric nor of finite variation.

To see that the second inclusion in (4.21) is proper, let $\mu \in E_\alpha^0(\mathbb{R}^1)$ with Lévy measure ν being supported on $[0,\infty)$ such that $\int_0^1 x\, \nu(dx) = \infty$. Suppose there are $b \in \mathbb{R}$ and $h \in \mathrm{Dom}(Z^{(\alpha)})$ such that $\mu = \mathcal{L}(\int_0^\infty h(t) dZ_t^{(\alpha)} + b)$. Since ν is supported on $[0,\infty)$, we must have $h \geq 0$ Lebesgue almost surely, so that we can suppose that $h \geq 0$ everywhere. Then we have from (4.1) and (4.3) that

$$\int_0^\infty ds \int_0^\infty (|h(s)x|^2 \wedge 1)\, \nu_{Z^{(\alpha)}}(dx) < \infty$$

and

$$\int_0^\infty ds \int_0^\infty \frac{h(s)x}{1+h(s)x} \nu_{Z^{(\alpha)}}(dx) < \infty.$$

Together these two equations imply

$$\int_0^\infty ds \int_0^\infty (|h(s)x| \wedge 1) \nu_{Z^{(\alpha)}}(dx) < \infty,$$

so that $\mu \in E_\alpha^{BV}(\mathbb{R}^1)$ by (4.19), contradicting $\int_0^1 x\, \nu(dx) = \infty$. This completes the proof of (4.21). $\square$

Proof of Theorem 1.2. This is an immediate consequence of Equation (4.18) since $B^0(\mathbb{R}_+) = E_1^{+,0}(\mathbb{R}^1)$. $\square$

5. The composition of Φ with $\mathcal{E}_\alpha$ and its application

In this section we study the composition $\Phi \circ \mathcal{E}_\alpha$. We start with the following proposition.

Proposition 5.1. *Let $\alpha > 0$, $m \in \{1, 2, \ldots\}$ and $\mu \in I(\mathbb{R}^d)$. Then $\mu \in I_{\log^m}(\mathbb{R}^d)$ if and only if $\mathcal{E}_\alpha(\mu) \in I_{\log^m}(\mathbb{R}^d)$.*

Proof. Let ν and $\widetilde{\nu}$ denote the Lévy measures of μ and $\mathcal{E}_\alpha(\mu)$, respectively. By (2.1), we conclude that

$$\int_{\mathbb{R}^d} \varphi(x)\, \widetilde{\nu}(dx) = \int_{\mathbb{R}^d} \nu(dx) \int_0^\infty \varphi(ux) \alpha u^{\alpha-1} e^{-u^\alpha}\, du$$

for every measurable nonnegative function $\varphi : \mathbb{R}^d \to [0, \infty]$. In particular, we have

$$\begin{aligned}
\int_{|x|>1} (\log |x|)^m\, \widetilde{\nu}(dx) &= \int_{\mathbb{R}^d} \nu(dx) \int_{1/|x|}^\infty (\log(u|x|))^m\, \alpha u^{\alpha-1} e^{-u^\alpha}\, du \\
&= \int_{\mathbb{R}^d} \nu(dx) \sum_{n=0}^m \binom{m}{n} (\log |x|)^{m-n} \int_{1/|x|}^\infty (\log u)^n \alpha u^{\alpha-1} e^{-u^\alpha}\, du \\
&=: \int_{\mathbb{R}^d} h(x) \nu(dx), \quad \text{say.}
\end{aligned}$$

Then it is easy to see that $h(x) = o(|x|^2)$ as $|x| \downarrow 0$ and that $\lim_{|x|\to\infty} h(x)/(\log |x|)^m = \int_0^\infty \alpha u^{\alpha-1} e^{-u^\alpha}\, du = 1$. Hence, $\int_{|x|>1} (\log |x|)^m\, \widetilde{\nu}(dx) < \infty$ if and only if $\int_{|x|>1} (\log |x|)^m \nu(dx) < \infty$, giving the claim. $\square$

Theorem 5.2. *Let $\alpha > 0$ and*

$$n_\alpha(x) = \int_x^\infty u^{-1} e^{-u^\alpha}\, du, \quad x > 0.$$

Let $x = n_\alpha^*(t), t > 0$, be its inverse function, and define the mapping $\mathcal{N}_\alpha : I_{\log}(\mathbb{R}^d) \to I(\mathbb{R}^d)$ by

$$\mathcal{N}_\alpha(\mu) = \mathcal{L}\left(\int_0^\infty n_\alpha^*(t)\, dX_t^{(\mu)} \right), \quad \mu \in I_{\log}(\mathbb{R}^d).$$

It then holds

$$\Phi \circ \mathcal{E}_\alpha = \mathcal{E}_\alpha \circ \Phi = \mathcal{N}_\alpha, \tag{5.1}$$

including the equality of the domains. In particular, we have

$$\Phi \circ \mathcal{E}_2 = \mathcal{E}_2 \circ \Phi = \mathcal{M}. \tag{5.2}$$

Proof. We first note that $\mathfrak{D}(\mathcal{N}_\alpha)$ is independent of the value of α and equals $I_{\log}(\mathbb{R}^d)$, shown in Theorem 2.3 of [8], (essentially in Theorem 2.4 (i) of [14].)

As mentioned right after Equation (1.5), $\mathfrak{D}(\Phi) = I_{\log}(\mathbb{R}^d)$. Thus it follows from Proposition 5.1 that both $\Phi \circ \mathcal{E}_\alpha$ as well as $\mathcal{E}_\alpha \circ \Phi$ are well defined on $I_{\log}(\mathbb{R}^d)$ and that they have the same domain. Note that

$$C_{\mathcal{E}_\alpha(\mu)}(z) = \int_0^1 C_\mu\left((\log t^{-1})^{1/\alpha} z\right) dt = \int_0^\infty C_\mu(u^{1/\alpha} z) e^{-u} du$$

and

$$C_{\Phi(\mu)}(z) = \int_0^\infty C_\mu\left(e^{-t} z\right) dt.$$

Then, if we are allowed to exchange the order of the integrals by Fubini's theorem, we have

$$\begin{aligned} C_{(\mathcal{E}_\alpha \circ \Phi)(\mu)}(z) &= \int_0^\infty e^{-s} ds \int_0^\infty C_\mu(s^{1/\alpha} e^{-t} z) dt \\ &= \int_0^\infty \alpha u^{\alpha-1} C_\mu(uz) \int_0^\infty e^{\alpha t - u^\alpha e^{\alpha t}} dt\, du \\ &= \int_0^\infty C_\mu(uz) u^{-1} e^{-u^\alpha} du \\ &= -\int_0^\infty C_\mu(uz) dn_\alpha(u) \\ &= \int_0^\infty C_\mu(n_\alpha^*(t) z) dt, \end{aligned} \tag{5.3}$$

and the same calculation can be carried out for $C_{(\Phi \circ \mathcal{E}_\alpha)(\mu)}(z) = \int_0^\infty C_\mu(n_\alpha^*(t)z)dt$.

In order to assure the exchange of the order of the integrations by Fubini's theorem, it is enough to show that

$$\int_0^\infty e^{-s} ds \int_0^\infty \left|C_\mu(s^{1/\alpha} e^{-t} z)\right| dt < \infty. \tag{5.4}$$

This is Equation (4.5) in Barndorff-Nielsen et al. [3] with the replacement of s by $s^{1/\alpha}$. Hence, the proof of (4.5) in Barndorff-Nielsen et al. [3] works also here and concludes (5.4). So, we omit the detailed calculation. Thus, the calculation in (5.3) is verified, and we have that

$$C_{(\Phi \circ \mathcal{E}_\alpha)(\mu)}(z) = C_{(\mathcal{E}_\alpha \circ \Phi)(\mu)}(z) = \int_0^\infty C_\mu(n_\alpha^*(t) z)\, dt = C_{\mathcal{N}_\alpha(\mu)}(z), \quad z \in \mathbb{R}^d,$$

and that $\Phi \circ \mathcal{E}_\alpha = \mathcal{E}_\alpha \circ \Phi = \mathcal{N}_\alpha$. Since $\mathcal{N}_2 = \mathcal{M}$, this shows in particular (5.2). □

It is well known that $\Phi(I_{\log}(\mathbb{R}^d)) = L(\mathbb{R}^d)$, the class of selfdecomposable distributions on $\mathbb{R}^d$. An immediate consequence of Theorem 5.2 is the following.

Theorem 5.3. *Let $\alpha > 0$. Then*

$$\Phi(E_\alpha(\mathbb{R}^d) \cap I_{\log}(\mathbb{R}^d)) = \mathcal{E}_\alpha(L(\mathbb{R}^d)) = \mathcal{N}_\alpha(I_{\log}(\mathbb{R}^d)).$$

We conclude this section with an application of the relation (5.1) to characterize the limit of certain subclasses obtained by the iteration of the mapping $\mathcal{N}_\alpha$. We need some lemmas. In the following, $\mathcal{N}_\alpha^m$ is defined recursively as $\mathcal{N}_\alpha^{m+1} = \mathcal{N}_\alpha^m \circ \mathcal{N}_\alpha$.

Lemma 5.4. *Let $\alpha > 0$. For $m = 1, 2, \ldots$, we have*

$$\mathfrak{D}(\mathcal{N}_\alpha^m) = I_{\log^m}(\mathbb{R}^d) \quad \textit{and} \quad \mathcal{N}_\alpha^m = \Phi^m \circ \mathcal{E}_\alpha^m = \mathcal{E}_\alpha^m \circ \Phi^m.$$

Proof. By Proposition 5.1, we have $\mu \in I_{\log^m}(\mathbb{R}^d)$ if and only if $\mathcal{E}_\alpha(\mu) \in I_{\log^m}(\mathbb{R}^d)$. As shown in the proof of Lemma 3.8 in [9], we also have that $\mu \in I_{\log^{m+1}}(\mathbb{R}^d)$ if and only if $\mu \in I_{\log}(\mathbb{R}^d)$ and $\Phi(\mu) \in I_{\log^m}(\mathbb{R}^d)$, and thus $\mathfrak{D}(\Phi^m) = I_{\log^m}(\mathbb{R}^d)$. Since $\mathcal{N}_\alpha = \Phi \circ \mathcal{E}_\alpha = \mathcal{E}_\alpha \circ \Phi$, we conclude that

(5.5) $\qquad \mu \in I_{\log^{m+1}}(\mathbb{R}^d)$ if and only if $\mu \in I_{\log}(\mathbb{R}^d)$ and $\mathcal{N}_\alpha(\mu) \in I_{\log^m}(\mathbb{R}^d)$.

Now we prove $\mathfrak{D}(\mathcal{N}_\alpha^m) = I_{\log^m}(\mathbb{R}^d)$ inductively. For $m = 1$ this is known, so assume that $\mathfrak{D}(\mathcal{N}_\alpha^m) = I_{\log^m}(\mathbb{R}^d)$ for some $m \geq 1$. If $\mu \in \mathfrak{D}(\mathcal{N}_\alpha^{m+1})$, then $\mathcal{N}_\alpha^{m+1}(\mu) = \mathcal{N}_\alpha^m(\mathcal{N}_\alpha(\mu))$ is well-defined. Thus, $\mathcal{N}_\alpha(\mu) \in \mathfrak{D}(\mathcal{N}_\alpha^m) = I_{\log^m}(\mathbb{R}^d)$ by assumption, so that $\mu \in I_{\log^{m+1}}(\mathbb{R}^d)$ by (5.5). Conversely, if $\mu \in I_{\log^{m+1}}(\mathbb{R}^d)$, then $\mu \in I_{\log}(\mathbb{R}^d)$ and $\mathcal{N}_\alpha(\mu) \in I_{\log^m}(\mathbb{R}^d)$ by (5.5), so that $\mathcal{N}_\alpha^m(\mathcal{N}_\alpha(\mu))$ is well-defined by assumption. This shows $\mathfrak{D}(\mathcal{N}_\alpha^{m+1}) = I_{\log^{m+1}}(\mathbb{R}^d)$. That $\mathcal{N}_\alpha^m = \Phi^m \circ \mathcal{E}_\alpha^m = \mathcal{E}_\alpha^m \circ \Phi^m$ for every m then follows easily from (5.1), Proposition 5.1 and $\mathfrak{D}(\Phi^m) = I_{\log^m}(\mathbb{R}^d)$. $\square$

Let $S(\mathbb{R}^d)$ be the class of all stable distributions on $\mathbb{R}^d$, and for $m = 0, 1, \ldots$ denote $L_m(\mathbb{R}^d) = \Phi^{m+1}(I_{\log^{m+1}}(\mathbb{R}^d))$, $L_\infty(\mathbb{R}^d) = \cap_{m=0}^\infty L_m(\mathbb{R}^d)$, $N_{\alpha,m}(\mathbb{R}^d) = \mathcal{N}_\alpha^{m+1}(I_{\log^{m+1}}(\mathbb{R}^d))$ and $N_{\alpha,\infty}(\mathbb{R}^d) = \cap_{m=0}^\infty N_{\alpha,m}(\mathbb{R}^d)$. Lemma 5.4 implies that $N_{\alpha,m}(\mathbb{R}^d) \supset N_{\alpha,m+1}(\mathbb{R}^d)$, so that the family $N_{\alpha,m}$, $m = 0, 1, \ldots$, is nested. It is known (cf. Sato [10]) that $L_\infty(\mathbb{R}^d) = \overline{S(\mathbb{R}^d)}$, where the closure is taken under weak convergence and convolution. In order to show that also $N_{\alpha,\infty}(\mathbb{R}^d) = \overline{S(\mathbb{R}^d)}$, we need two further lemmas.

Lemma 5.5. *For $\alpha > 0$, $\mathcal{E}_\alpha$ maps $S(\mathbb{R}^d)$ bijectively onto $S(\mathbb{R}^d)$, namely*

$$\mathcal{E}_\alpha(S(\mathbb{R}^d)) = S(\mathbb{R}^d).$$

This is an immediate consequence of Proposition 2.1 (ii).

Lemma 5.6. *Let $\alpha > 0$. For $m = 0, 1, \dots,$ $N_{\alpha,m}(\mathbb{R}^d)$ is closed under convolution and weak convergence, and*

$$S(\mathbb{R}^d) \subset N_{\alpha,m}(\mathbb{R}^d) = \mathcal{E}_\alpha^{m+1}(L_m(\mathbb{R}^d)) \subset L_m(\mathbb{R}^d). \tag{5.6}$$

Proof. By Lemma 5.4,

$$N_{\alpha,m}(\mathbb{R}^d) = \mathcal{N}_\alpha^{m+1}(I_{\log^{m+1}}(\mathbb{R}^d)) = (\mathcal{E}_\alpha^{m+1} \circ \Phi^{m+1})(I_{\log^{m+1}}(\mathbb{R}^d)) = \mathcal{E}_\alpha^{m+1}(L_m(\mathbb{R}^d)),$$

hence $S(\mathbb{R}^d) \subset N_{\alpha,m}(\mathbb{R}^d)$ by Lemma 5.5 and the fact that $S(\mathbb{R}^d) \subset L_m(\mathbb{R}^d)$. Further,

$$N_{\alpha,m}(\mathbb{R}^d) = (\Phi^{m+1} \circ \mathcal{E}_\alpha^{m+1})(I_{\log^{m+1}}(\mathbb{R}^d)) \subset \Phi^{m+1}(I_{\log^{m+1}}(\mathbb{R}^d)) = L_m(\mathbb{R}^d).$$

Next observe that $\mathcal{E}_\alpha$ and hence $\mathcal{E}_\alpha^{m+1}$ clearly respect convolution. Since $L_m(\mathbb{R}^d)$ is closed under convolution and weak convergence (see the proof of Theorem D in [3]), it follows from (5.6) and Proposition 2.1 (iv) that $N_{\alpha,m}(\mathbb{R}^d)$ is closed under convolution and weak convergence, too. $\square$

We can now characterize $N_{\alpha,\infty}(\mathbb{R}^d)$ as the closure of $S(\mathbb{R}^d)$ under convolution and weak convergence:

Theorem 5.7. *Let $\alpha > 0$. It holds*

$$L_\infty(\mathbb{R}^d) = N_{\alpha,\infty}(\mathbb{R}^d) = \overline{S(\mathbb{R}^d)}.$$

In particular,

$$\lim_{m\to\infty} \mathcal{M}^m(I_{\log^m}(\mathbb{R}^d)) = \overline{S(\mathbb{R}^d)}.$$

Proof. By (5.6) we have

$$\overline{S(\mathbb{R}^d)} = L_\infty(\mathbb{R}^d) \supset N_{\alpha,\infty}(\mathbb{R}^d) \supset S(\mathbb{R}^d).$$

But since each $N_{\alpha,m}(\mathbb{R}^d)$ is closed under convolution and weak convergence, so must be the intersection $N_{\alpha,\infty}(\mathbb{R}^d) = \bigcap_{m=0}^\infty N_{\alpha,m}(\mathbb{R}^d)$, and together with $\mathcal{M} = \mathcal{N}_2$ the assertions follow. $\square$

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