Multivariate Generalized Ornstein-Uhlenbeck Processes

Anita Behme* and Alexander Lindner*

Abstract

De Haan and Karandikar [12] introduced generalized Ornstein–Uhlenbeck processes as one-dimensional processes $(V_t)_{t\geq 0}$ which are basically characterized by the fact that for each $h > 0$ the equidistantly sampled process $(V_{nh})_{n\in\mathbb{N}_0}$ satisfies the random recurrence equation $V_{nh} = A_{(n-1)h,nh}V_{(n-1)h} + B_{(n-1)h,nh}$, $n \in \mathbb{N}$, where $(A_{(n-1)h,nh}, B_{(n-1)h,nh})_{n\in\mathbb{N}}$ is an i.i.d. sequence with positive $A_{0,h}$ for each $h > 0$. We generalize this concept to a multivariate setting and use it to define multivariate generalized Ornstein–Uhlenbeck (MGOU) processes which occur to be characterized by a starting random variable and some Lévy process (X, Y) in $\mathbb{R}^{m\times m} \times \mathbb{R}^m$. The stochastic differential equation an MGOU process satisfies is also derived. We further study invariant subspaces and irreducibility of the models generated by MGOU processes and use this to give necessary and sufficient conditions for the existence of strictly stationary solutions of MGOU processes under some extra conditions.

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1 Introduction

Let $(\xi, \eta) = (\xi_t, \eta_t)_{t\geq 0}$ be a bivariate Lévy process and V_0 a random variable, independent of (ξ, η). Then, following De Haan and Karandikar [12] and Carmona et al. [6], the one-dimensional process $(V_t)_{t\geq 0}$, given by

$$V_t = e^{-\xi_t}\left(V_0 + \int_{(0,t]} e^{\xi_{s-}}d\eta_s\right), \quad t \geq 0, \tag{1.1}$$

*Technische Universität Braunschweig, Institut für Mathematische Stochastik, Pockelsstraße 14, D-38106 Braunschweig, Germany, email: a.behme@tu-bs.de or a.lindner@tu-bs.de, tel.:+49/531/3917562 or +49/531/3917575, fax:+49/531/3917564

is called a *generalized Ornstein–Uhlenbeck* (GOU) *process*. We refer to Maller et al. [19] for further information and references regarding GOU processes. A key feature of these processes is that for any $h > 0$, the random sequence $(V_{nh})_{n\in\mathbb{N}_0}$ satisfies the random recurrence equation $V_{nh} = A_{(n-1)h,nh}V_{(n-1)h} + B_{(n-1)h,nh}$, $n \in \mathbb{N}$, where $(A_{(n-1)h,nh}, B_{(n-1)h,nh})_{n\in\mathbb{N}}$ is an i.i.d. (independent and identically distributed) sequence with $A_{0,h} > 0$ almost surely. Without assuming independence of V_0 and (ξ, η), processes of the form (1.1) are the only processes having this property for any $h > 0$ and which satisfy some natural extra conditions, as shown by De Haan and Karandikar [12]. In the present paper we extend the setting of De Haan and Karandikar [12] to random matrices with real valued entries, i.e. we aim to construct a process

$$(V_t)_{t\geq 0}, \text{ with } V_t = (V_t^{(i,j)})_{\substack{1\leq i\leq m\\ 1\leq j\leq l}} \in \mathbb{R}^{m\times l}$$

in continuous time which fulfills the random recurrence equation

$$V_t = A_{s,t}V_s + B_{s,t} \quad \text{a.s.}, \ 0 \leq s \leq t, \tag{1.2}$$

for random functionals $(A_{s,t})_{0\leq s\leq t}$, $(B_{s,t})_{0\leq s\leq t}$ such that $A_{s,t} \in \mathbb{R}^{m\times m}$ and $B_{s,t} \in \mathbb{R}^{m\times l}$, the $A_{s,t}$ are supposed to be non-singular and $(A_{(n-1)h,nh}, B_{(n-1)h,nh})$, $n \in \mathbb{N}$, are i.i.d. for all $h > 0$. We also aim to characterize all processes in continuous time which have this property and satisfy some natural extra conditions. The obtained solutions will be called *multivariate generalized Ornstein–Uhlenbeck* (MGOU) *processes* since they extend the key feature of one-dimensional generalized Ornstein–Uhlenbeck processes canonically. Observe that the question of when a solution of (1.2) exists can be treated separately for each column of $(V_t)_{t\geq 0}$. Thus, if not stated otherwise, for simplicity we set $l = 1$ throughout this paper, hence V_t and $B_{s,t}$ are elements in $\mathbb{R}^m$.

To motivate the mentioned extra conditions, following the lines of De Haan and Karandikar [12] observe that the condition of (1.2) to hold for all $0 \leq s \leq t$ yields

$$A_{u,t}V_u + B_{u,t} = V_t = A_{s,t}V_s + B_{s,t} = A_{s,t}A_{u,s}V_u + A_{s,t}B_{u,s} + B_{s,t}, \quad 0 \leq u \leq s \leq t.$$

Assuming that $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ is unique now leads to Assumption 1(a) given below while extending the i.i.d. property of $(A_{(n-1)h,nh}, B_{(n-1)h,nh})$, $n \in \mathbb{N}$, for all $h > 0$ into the continuous time setting yields the requirements 1(b) and (c). Finally, it is natural to impose that $(A_{0,t})_{t\geq 0}$ and $(B_{0,t})_{t\geq 0}$ are continuous in probability at 0 since this, together with 1(a),(b) and (c), implies the existence of càdlàg modifications of the processes

$$(A_t)_{t\geq 0} := (A_{0,t})_{t\geq 0} \quad \text{and} \quad (B_t)_{t\geq 0} := (B_{0,t})_{t\geq 0}$$

as will be shown in Lemma 2.1 below. This motivates Assumption 1(d) below. We denote the set of all invertible real $m \times m$-matrices by $\mathrm{GL}(\mathbb{R}, m)$, the identity matrix by I and by 0 the vector (or matrix) having only zero entries. We write "$\overset{d}{=}$" for equality in distribution and "P- lim" for limits in probability.

Assumption 1. *Suppose the* $\mathrm{GL}(\mathbb{R}, m) \times \mathbb{R}^m$*-valued random functional* $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ *with* $A_{t,t} = I$ *and* $B_{t,t} = 0$ *a.s. for all* $t \geq 0$ *satisfies the following four conditions.*

(a) *For all* $0 \le u \le s \le t$ *almost surely*

$$A_{u,t} = A_{s,t} A_{u,s} \quad \text{and} \quad B_{u,t} = A_{s,t} B_{u,s} + B_{s,t}. \tag{1.3}$$

(b) *For all* $0 \le a \le b \le c \le d$ *the families of random matrices* $\{(A_{s,t}, B_{s,t}), a \le s \le t \le b\}$ *and* $\{(A_{s,t}, B_{s,t}), c \le s \le t \le d\}$ *are independent.*

(c) *For all* $0 \le s \le t$ *it holds*

$$(A_{s,t}, B_{s,t}) \stackrel{d}{=} (A_{0,t-s}, B_{0,t-s}). \tag{1.4}$$

(d) *It holds*

$$\text{P-}\lim_{t \downarrow 0} A_{0,t} = I \quad \text{and} \quad \text{P-}\lim_{t \downarrow 0} B_{0,t} = 0. \tag{1.5}$$

The first main result of the paper will be a characterization of all random functionals $(A_{s,t}, B_{s,t})_{0 \le s \le t}$ which satisfy Assumption 1, in terms of appropriate driving Lévy processes. This will be achieved in Theorem 3.1 and then be used to define MGOU processes as processes which satisfy (1.2) with $(A_{s,t}, B_{s,t})_{0 \le s \le t}$ subject to Assumption 1. It will be also shown in Section 3 that MGOU processes satisfy the stochastic differential equation (SDE) $dV_t = dU_t V_{t-} + dL_t$ for appropriate Lévy processes U and L if the starting random variable V_0 is independent of $(A_{0,t}, B_{0,t})_{t \ge 0}$, extending a corresponding one-dimensional result of De Haan and Karandikar [12].

A new aspect compared to the one-dimensional GOU process is the possibility of the existence of affine subspaces H of $\mathbb{R}^m$ which are *invariant* under the model (1.2) in the sense that $A_{s,t} H + B_{s,t} \subseteq H$ holds for all $0 \le s \le t$. In Section 4 we give necessary and sufficient conditions for the existence of an invariant affine subspace of the model (1.2) and show that given the existence of a d-dimensional invariant affine subspace H, after an appropriate orthogonal transformation of the underlying space, the MGOU process with $V_0 \in H$ consists of an $(m-d)$-dimensional constant process and an $\mathbb{R}^d$-valued MGOU process. Subsequently in Section 5 strictly stationary solutions of MGOU processes are treated. Under some extra conditions we give necessary and sufficient conditions for their existence and determine their form, extending corresponding one-dimensional results of Behme et al. [3] and Lindner and Maller [18]. The proofs for the results of Sections 3–5 are given in Sections 6–8. A crucial ingredient for the derivation of the necessary and sufficient conditions for stationarity are the results on stationary solutions of random recurrence equations by Bougerol and Picard [4]. Section 8 also contains several auxiliary results about multivariate stochastic exponentials. Some preliminary results are collected in Section 2, where we also set further notation used throughout the paper.

Random recurrence equations have many applications in finance, biology or fractal images, to name just a few, see e.g. Wong and Li [27], Tong [26], or Diaconis and Freedman [7]. Hence multivariate generalized Ornstein–Uhlenbeck processes as their continuous time counterparts have considerable potential for applications. In one dimension, various applications of the GOU process are known. For example, the volatility of the COGARCH(1,1) process of Klüppelberg et al. [16] or the risk process of Paulsen [20] are one-dimensional GOU processes. As an example of an application of the MGOU process to finance, we present in Example 3.6 the state vector process of the volatility process of the COGARCH(q,p) model of Brockwell et al. [5] as a special case of an MGOU process. Further applications of MGOU processes as multivariate volatility models seem possible, but we shall not pursue this topic further in this paper but leave it to future research.

Finally, we mention that major parts of the results of this paper have been obtained in the first named author's doctoral thesis [2, Chapter 5].

2 Preliminaries

Throughout this paper for any matrix $M \in \mathbb{R}^{m\times n}$ we write $M^\perp$ for its transpose and let $M^{(i,j)}$ denote the component in the ith row and jth column of M. Limits in distribution will be denoted by "d- lim" or "$\xrightarrow{d}$", limits in probability by "P- lim" or "$\xrightarrow{P}$", and "almost surely" will be abbreviated by "a.s.". The law of a random matrix Y will be denoted by $\mathcal{L}(Y)$. We write $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and $\log^+(x) := \log\max\{x, 1\}$ for $x \in \mathbb{R}$. Jumps of a matrix valued càdlàg process $X = (X_t)_{t\geq 0}$ will be denoted by $\Delta X_t := X_t - X_{t-}$ with $X_{t-} := \lim_{s\uparrow t} X_s$ for $t > 0$ and the convention $X_{0-} := 0$.

Multiplicative Lévy processes

Recall that an (additive) Lévy process $X = (X_t)_{t\geq 0}$ with values in $\mathbb{R}^{m\times l}$ is a process with stationary and independent (additive) increments which has almost surely càdlàg paths and starts at 0. Here, an increment of X is given by $X_t - X_s$ for $s \leq t$. We refer to Applebaum [1] or Sato [23] for further information regarding Lévy processes. In the following it will be also necessary to consider multiplicative Lévy processes with values in the general linear group $\mathrm{GL}(\mathbb{R}, m)$ of order m, where the group operation is matrix multiplication. For that, remark that the group structure allows us to define *(multiplicative) left increments* $X_t X_s^{-1}$ and *(multiplicative) right increments* $X_s^{-1} X_t$ for $0 \leq s \leq t < \infty$ of a $\mathrm{GL}(\mathbb{R}, m)$-valued process. We say that the process $(X_t)_{t\geq 0}$ in $\mathrm{GL}(\mathbb{R}, m)$ has *independent left increments* if for any $n \in \mathbb{N}$, $0 < t_1 < \ldots < t_n$, the random variables $X_0, X_{t_1} X_0^{-1}, \ldots, X_{t_n} X_{t_{n-1}}^{-1}$ are independent. The process has *stationary left increments* if $X_t X_s^{-1} \stackrel{d}{=} X_{t-s} X_0^{-1}$ holds for all $s < t$. Stationarity and independence of right increments is understood analogously. Now following the notations in the book of Liao [17] a càdlàg process $(X_t)_{t\geq 0}$ in $\mathrm{GL}(\mathbb{R}, m)$, $m \geq 1$, with $X_0 = I$ a.s. is called a *(multiplicative) left Lévy process*, if it has independent and stationary right increments. Similarly, a càdlàg process $(X_t)_{t\geq 0}$ in $\mathrm{GL}(\mathbb{R}, m)$, $m \geq 1$, with $X_0 = I$ a.s. is called a *(multiplicative) right Lévy process*, if it has independent and stationary left increments. Given a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, a left Lévy process $(X_t)_{t\geq 0}$ in $\mathrm{GL}(\mathbb{R}, m)$ is called a *left* $\mathbb{F}$*-Lévy process*, if it is adapted to $\mathbb{F}$ and for any $s < t$ the right increment $X_s^{-1} X_t$ is independent of $\mathcal{F}_s$. *Right* $\mathbb{F}$*-Lévy processes* and *(additive)* $\mathbb{F}$*-Lévy processes* are defined similarly.

The following lemma gives the connection between the random functionals $A_{s,t}$ satisfying Assumption 1 and multiplicative Lévy processes.

Lemma 2.1.

(a) *For any* $(A_{s,t})_{0\le s\le t}$ *fulfilling Assumption 1 the process* $(A_t)_{t\ge 0} = (A_{0,t})_{t\ge 0}$ *has a càdlàg modification which is a right Lévy process in* $\mathrm{GL}(\mathbb{R}, m)$*. Conversely, if* $(A_t)_{t\ge 0}$ *is a right Lévy process in* $\mathrm{GL}(\mathbb{R}, m)$*, then* $(A_{s,t})_{0\le s\le t}$ *defined by* $A_{s,t} = A_t A_s^{-1}$ *fulfills Assumption 1.*

(b) *For any* $(A_{s,t}, B_{s,t})_{0\le s\le t}$ *fulfilling Assumption 1 the process* $(A_t, B_t)_{t\ge 0} = (A_{0,t}, B_{0,t})_{t\ge 0}$ *has a càdlàg modification.*

Proof. (a) Since by Assumption 1(a) we have $A_t A_s^{-1} = A_{s,t}$ it follows directly from Assumption 1(b) and (c), that $(A_t)_{t\ge 0}$ is a stochastic process in $\mathrm{GL}(\mathbb{R}, m)$ with stationary and independent left increments. It is everywhere continuous in probability from the right since by 1(a), (c) and (d)

$$\mathrm{P}\text{-}\lim_{h\downarrow 0} A_{t+h} = \mathrm{P}\text{-}\lim_{h\downarrow 0} A_{t,t+h} A_t = A_t, \quad t \ge 0.$$

Similarly due to

$$\mathrm{P}\text{-}\lim_{h\downarrow 0} A_{t-h} = \mathrm{P}\text{-}\lim_{h\downarrow 0} A_t A_{t-h,t}^{-1} = A_t \cdot \mathrm{P}\text{-}\lim_{h\downarrow 0} A_h^{-1} = A_t, \quad t \ge 0,$$

it is also continuous in probability from the left such that by [25, Theorem V.3] a càdlàg modification exists which is a right Lévy process in $\mathrm{GL}(\mathbb{R}, m)$ as specified above.
The converse is true by the definition of right Lévy processes.
(b) Since $B_{t+h} = A_{t+h} A_t^{-1} B_t + B_{t,t+h}$ the process $(B_t)_{t\ge 0}$ is by Assumption 1(c) and (d) everywhere continuous in probability from the right and similarly from the left. Hence it admits a càdlàg modification which can be shown by a simple extension of the proof in the one-dimensional case given in [12, Lemma 2.1]. □

Since every set $(A_{s,t}, B_{s,t})_{0\le s\le t}$ of random functionals satisfying Assumption 1 admits a càdlàg modification $(A_t, B_t)_{t\ge 0}$ by the preceding lemma, we may and do restrict attention to such functionals with càdlàg paths.

Matrix valued stochastic integrals

Given a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\ge 0}$ satisfying the usual hypotheses (cf. [21, p. 3]), a matrix-valued stochastic process $M = (M_t)_{t\ge 0}$ is called an $\mathbb{F}$-semimartingale or simply a semimartingale if every component $(M_t^{(i,j)})_{t\ge 0}$ is a semimartingale with respect to the filtration $\mathbb{F}$. For a semimartingale M in $\mathbb{R}^{m\times n}$ and a locally bounded predictable process H in $\mathbb{R}^{l\times m}$ the $\mathbb{R}^{l\times n}$-valued (left) stochastic integral $I = \int H dM$ is given by $I^{(i,j)} = \sum_{k=1}^m \int H^{(i,k)} dM^{(k,j)}$ and in the same way for $M \in \mathbb{R}^{l\times m}$, $H \in \mathbb{R}^{m\times n}$, the $\mathbb{R}^{l\times n}$-valued stochastic (right) integral $J = \int dM H$ is given by $J^{(i,j)} = \sum_{k=1}^m \int H^{(k,j)} dM^{(i,k)}$. Stochastic integrals of the form $\int H dM\, H'$ for locally bounded predictable processes H and H' are defined similarly in the obvious way.

Given two semimartingales M and N in $\mathbb{R}^{l\times m}$ and $\mathbb{R}^{m\times n}$ the quadratic variation $[M, N]$ in $\mathbb{R}^{l\times n}$ is defined by its components via $[M, N]^{(i,j)} = \sum_{k=1}^m [M^{(i,k)}, N^{(k,j)}]$. Similarly its continuous part $[M,N]^c$ is given by $([M,N]^c)^{(i,j)} = \sum_{k=1}^m [M^{(i,k)}, N^{(k,j)}]^c$. With these notations, for two semimartingales M and N in $\mathbb{R}^{m\times m}$ and two locally bounded predictable processes G and H in $\mathbb{R}^{m\times m}$ we have the following a.s. equalities as stated e.g. in Karandikar [15]

$$\left[\int_{(0,\cdot]} G_s dM_s, \int_{(0,\cdot]} dN_s H_s\right]_t = \int_{(0,t]} G_s d[M,N]_s H_s, \quad t \geq 0, \tag{2.1}$$

$$\left[M, \int_{(0,\cdot]} G_s dN_s\right]_t = \left[\int_{(0,\cdot]} dM_s G_s, N\right]_t, \quad t \geq 0, \tag{2.2}$$

and the integration by parts formula takes the form

$$(MN)_t = \int_{(0,t]} M_{s-} dN_s + \int_{(0,t]} dM_s\, N_{s-} + [M,N]_t, \quad t \geq 0. \tag{2.3}$$

The multivariate stochastic exponential

Stochastic exponentials of $\mathbb{R}^{m\times m}$-valued Lévy processes will play a crucial rule in our considerations. We first recall the definition of left and right stochastic exponentials from [21, p. 325-326].

Definition 2.2. *Let $(X_t)_{t\geq 0}$ be a semimartingale in $\mathbb{R}^{m\times m}$. Then its* left stochastic exponential $\overleftarrow{\mathcal{E}}(X)_t$ *is defined as the unique $\mathbb{R}^{m\times m}$-valued, adapted, càdlàg solution of the integral equation*

$$Z_t = I + \int_{(0,t]} Z_{s-} dX_s, \quad t \geq 0, \tag{2.4}$$

while the unique adapted, càdlàg solution of

$$Z_t = I + \int_{(0,t]} dX_s\, Z_{s-}, \quad t \geq 0, \tag{2.5}$$

will be called right stochastic exponential *and denoted by $\overrightarrow{\mathcal{E}}(X)_t$. Both $\overleftarrow{\mathcal{E}}(X)$ and $\overrightarrow{\mathcal{E}}(X)$ are semimartingales.*

Unfortunately, unlike for one-dimensional stochastic exponentials as e.g. in [21, Theorem II.37], no closed form expression is available for general multivariate stochastic exponentials, which makes their treatment more difficult. The SDE of the stochastic exponential for processes with values in arbitrary Lie groups has been studied by Estrade [10].

Remark that replacing Z and X by their transposes in (2.4) leads to the SDE (2.5) and vice versa. Hence we have

$$\overleftarrow{\mathcal{E}}(X)^{\perp} = \overrightarrow{\mathcal{E}}(X^{\perp}). \tag{2.6}$$

As has been observed by Karandikar [15] a necessary and sufficient condition for non-singularity of the left stochastic exponential of an $\mathbb{R}^{m\times m}$-valued process X at time t, is to claim that $(I+\Delta X_s)$ is invertible for all $0<s\leq t$. Due to the above stated relationship between left and right exponential this result holds true also for right exponentials and hence any stochastic exponential is invertible for all $t\geq 0$ if and only if

$$\det(I+\Delta X_t)\neq 0 \quad \text{for all} \quad t\geq 0. \tag{2.7}$$

For $\mathrm{GL}(\mathbb{R},m)$-valued semimartingales, the stochastic logarithm is defined as follows.

Definition 2.3. *Let $(Z_t)_{t\geq 0}$ be a $\mathrm{GL}(\mathbb{R},m)$-valued semimartingale with $Z_0=I$. Then the* left stochastic logarithm $\overleftarrow{\mathrm{Log}}\, Z$ *and* right stochastic logarithm $\overrightarrow{\mathrm{Log}}\, Z$ *of Z are defined by*

$$\overleftarrow{\mathrm{Log}}\,(Z_t)=\int_{(0,t]} Z_{s-}^{-1}dZ_s, \quad \text{and} \quad \overrightarrow{\mathrm{Log}}\,(Z_t)=\int_{(0,t]} dZ_s\, Z_{s-}^{-1}, \quad t\geq 0, \tag{2.8}$$

respectively.

It is clear from the defining SDE $dZ_t=Z_{t-}dX_t$ for left stochastic exponentials that if X is a semimartingale satisfying (2.7) with $X_0=0$, then $\overleftarrow{\mathrm{Log}}\,\overleftarrow{\mathcal{E}}(X)=X$ and X is the unique semimartingale Y satisfying $Y_0=0$ and $\overleftarrow{\mathcal{E}}(Y)=\overleftarrow{\mathcal{E}}(X)$. The same is true for right stochastic exponentials and right stochastic logarithms.

The following one-to-one relation between multiplicative Lévy processes and stochastic exponentials of additive Lévy processes is a key observation for the investigations in this paper.

Proposition 2.4. *Let $\mathbb{F}=(\mathcal{F}_t)_{t\geq 0}$ be a filtration satisfying the usual hypotheses. Then for every $\mathbb{F}$-Lévy process $(X_t)_{t\geq 0}$ in $\mathbb{R}^{m\times m}$ fulfilling* (2.7)*, the stochastic exponential $Z_t=\overleftarrow{\mathcal{E}}(X)_t$ (resp. $Z_t=\overrightarrow{\mathcal{E}}(X)_t$) is a left (resp. right) $\mathbb{F}$-Lévy process in* $\mathrm{GL}(\mathbb{R},m)$*. Conversely, if $Z=(Z_t)_{t\geq 0}$ is a left (resp. right) $\mathbb{F}$-Lévy process in* $\mathrm{GL}(\mathbb{R},m)$*, then Z is an $\mathbb{F}$-semimartingale and $\overleftarrow{\mathrm{Log}}\, Z$ (resp. $\overrightarrow{\mathrm{Log}}\, Z$) is an additive Lévy process in $\mathbb{R}^{m\times m}$ satisfying* (2.7)*.*

Sketch of Proof. The first part follows by simple calculations using the Markov property of X, and we refer to [2, Prop. 5.5] for a complete proof. The converse has been observed by Holevo [13] as a conclusion of results by Skorokhod [24]. Actually, there it is only observed that Z is a semimartingale with respect to its augmented natural filtration, $\mathbb{H}$ say, and that $\overleftarrow{\mathrm{Log}}\, Z$ and $\overrightarrow{\mathrm{Log}}\, Z$, resp., are $\mathbb{H}$-Lévy processes, but it is easy to see that then $\overleftarrow{\mathrm{Log}}\, Z$ and $\overrightarrow{\mathrm{Log}}\, Z$ are even $\mathbb{F}$-Lévy processes, and since $\overleftarrow{\mathcal{E}}(\overleftarrow{\mathrm{Log}}\, Z)=Z$ and $\overrightarrow{\mathcal{E}}(\overrightarrow{\mathrm{Log}}\, Z)=Z$, resp., it follows that Z is an $\mathbb{F}$-semimartingale. Again we refer to [2, Prop. 5.5] for detailed calculations. □

Since the inverse and the transpose of a left Lévy process in $\mathrm{GL}(\mathbb{R},m)$ are right Lévy processes and vice versa, for any additive Lévy process $(X_t)_{t\geq 0}$ fulfilling (2.7) the process $(\overleftarrow{\mathcal{E}}(X)_t^{-1})_{t\geq 0}$ is a right Lévy process and hence by the above proposition it is the right stochastic exponential of another Lévy process $(U_t)_{t\geq 0}$. In fact (see [15, Theorem 1]) if $(X_t)_{t\geq 0}$ is a semimartingale such that (2.7) is fulfilled, then it holds

$$\overleftarrow{\mathcal{E}}(X)_t^{-1} = [\overleftarrow{\mathcal{E}}(U^{\perp})_t]^{\perp} = \overrightarrow{\mathcal{E}}(U)_t, \quad t \geq 0$$

with

$$U_t := -X_t + [X,X]_t^c + \sum_{0<s\leq t} \left((I+\Delta X_s)^{-1} - I + \Delta X_s\right), \quad t \geq 0. \tag{2.9}$$

Remark that it follows from (2.9) by standard calculations that the processes U and X fulfill the relation

$$U_t = -X_t - [X,U]_t, \quad t \geq 0, \tag{2.10}$$

and that if X is a Lévy process, then so is U and vice versa.

3 Multivariate Generalized Ornstein-Uhlenbeck Processes

In this section we will characterize all families of random functionals $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ satisfying Assumption 1 and then will use this to define multivariate generalized Ornstein-Uhlenbeck processes. Further, we show that every multivariate generalized Ornstein-Uhlenbeck process $(V_t)_{t\geq 0}$ is a solution of the SDE

$$dV_t = dU_t\, V_{t-} + dL_t$$

for a suitable $\mathbb{R}^{m\times m} \times \mathbb{R}^m$-valued Lévy process (U, L). Conversely, provided that V_0 is $\mathcal{F}_0$-measurable for some filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual hypotheses such that the Lévy process (U, L) is a semimartingale with respect to $\mathbb{F}$, the solution to this SDE is a multivariate generalized Ornstein–Uhlenbeck process. The proofs for the results of this section are given in Section 6.

The following theorem characterizes all choices of random functionals $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ fulfilling Assumption 1. Recall that $A_t = A_{0,t}$, $B_t = B_{0,t}$ and that by Lemma 2.1 we can restrict to càdlàg versions of $(A_t, B_t)_{t\geq 0}$.

Theorem 3.1. *Suppose that $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ satisfies Assumption 1 and that $(A_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$ are chosen to be càdlàg. Then there is a unique Lévy process (X, Y) in $\mathbb{R}^{m\times m} \times \mathbb{R}^m$ such that X satisfies* (2.7) *and such that*

$$\begin{pmatrix} A_{s,t} \\ B_{s,t} \end{pmatrix} = \begin{pmatrix} \overleftarrow{\mathcal{E}}(X)_t^{-1} \overleftarrow{\mathcal{E}}(X)_s \\ \overleftarrow{\mathcal{E}}(X)_t^{-1} \int_{(s,t]} \overleftarrow{\mathcal{E}}(X)_{u-} dY_u \end{pmatrix} \quad a.s., \quad 0 \leq s \leq t. \tag{3.1}$$

The Lévy process (X, Y) is given by

$$\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \begin{pmatrix} \overleftarrow{\operatorname{Log}}\, A_t^{-1} \\ \int_{(0,t]} A_{u-}\, d(A_u^{-1} B_u) \end{pmatrix}, \quad t \geq 0, \tag{3.2}$$

where the integral is defined as a stochastic integral with respect to the natural augmented filtration of $(A_t, B_t)_{t\geq 0}$*, for which* $(A_t)_{t\geq 0}$ *and* $(B_t)_{t\geq 0}$ *are semimartingales.*
Conversely, if (X, Y) *is a Lévy process in* $\mathbb{R}^{m\times m} \times \mathbb{R}^m$ *such that* X *satisfies* (2.7)*, then* $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ *defined by the right hand side of* (3.1) *satisfies Assumption 1.*

Since a multivariate generalized Ornstein-Uhlenbeck process $(V_t)_{t\geq 0}$ was supposed to satisfy (1.2) with $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ satisfying Assumption 1, Theorem 3.1 motivates the following definition.

Definition 3.2. *Let* $(X, Y) = (X_t, Y_t)_{t\geq 0}$ *be a Lévy process in* $\mathbb{R}^{m\times m} \times \mathbb{R}^m$ *such that* X *satisfies* (2.7) *and let* V_0 *be a random variable in* $\mathbb{R}^m$*. Then the* $\mathbb{R}^m$*-valued process* $(V_t)_{t\geq 0}$*, given by*

$$V_t := \overleftarrow{\mathcal{E}}(X)_t^{-1} \left(V_0 + \int_{(0,t]} \overleftarrow{\mathcal{E}}(X)_{s-} dY_s \right), \quad t \geq 0, \tag{3.3}$$

will be called multivariate generalized Ornstein-Uhlenbeck (MGOU) process *driven by* $(X_t, Y_t)_{t\geq 0}$*. The MGOU process will be called* causal *or* non-anticipative*, if* V_0 *is independent of* (X, Y)*, and* strictly non-causal *if* V_t *is independent of* $(X_s, Y_s)_{0\leq s<t}$ *for all* $t \geq 0$*.*

It is easy to see that an MGOU process indeed satisfies (1.2). Remark that even for $m = 1$ Definition 3.2 is generalizing the standard definition of a generalized Ornstein-Uhlenbeck process since we do not assume a priori that V_0 is independent of $(X_t, Y_t)_{t\geq 0}$ and also the condition of $\mathcal{E}(X)_t^{-1}$ to be strictly positive is dropped. Nevertheless it seems natural to us to include these cases in the class of generalized Ornstein-Uhlenbeck processes. Observe that any MGOU process with starting random variable V_0 independent of (X, Y) is a time-homogeneous Markov process.

Example 3.3.

(a) If $X_t = \Lambda t$ for some $\Lambda \in \mathbb{R}^{m\times m}$ is a pure drift process then $\overleftarrow{\mathcal{E}}(X)_t = \overrightarrow{\mathcal{E}}(X)_t = e^{\Lambda t}$ and the MGOU process

$$V_t = e^{-\Lambda t} \left(V_0 + \int_0^t e^{\Lambda s}\, dY_s \right), \quad t \geq 0,$$

driven by (X, Y) is the usual multivariate Ornstein–Uhlenbeck type process driven by Y as introduced in [22].

(b) If $(X_t, Y_t) = (\mathrm{diag}(X_t^{(1,1)}, \ldots, X_t^{(m,m)}), (Y_t^{(1)}, \ldots, Y_t^{(m)})^{\perp})$, i.e. if X is a Lévy process concentrated on the diagonal matrices, and X satisfies condition (2.7), then $\overleftarrow{\mathcal{E}}(X)_t = \overrightarrow{\mathcal{E}}(X)_t = \mathrm{diag}(\mathcal{E}(X^{(1,1)})_t, \ldots, \mathcal{E}(X^{(m,m)})_t)$, where $\mathcal{E}(\cdot)$ denotes the usual one-dimensional stochastic exponential, and the ith component $V^{(i)}$ of the MGOU process $(V_t)_{t\geq 0}$ driven by (X, Y) satisfies

$$V_t^{(i)} = \mathcal{E}(X^{(i,i)})_t^{-1} \left(V_0^{(i)} + \int_{(0,t]} \mathcal{E}(X^{(i,i)})_{s-} dY_s^{(i)} \right), \quad t \geq 0,\ i = 1, \ldots, m.$$

It follows that $V^{(i)}$ is a one-dimensional MGOU process driven by $(X^{(i,i)}, Y^{(i)})$. If additionally $X^{(i,i)}$ does not have jumps of size less than or equal to -1 and if $V_0^{(i)}$ is independent of $(X^{(i,i)}, Y^{(i)})$, then $V^{(i)}$ is a GOU process. Observe that in general components of MGOU processes are no MGOU processes if X is not concentrated on the diagonal matrices.

An MGOU process can also be characterized by the stochastic differential equation it satisfies.

Theorem 3.4.

(a) *Let (X, Y) be a Lévy process in $\mathbb{R}^{m\times m} \times \mathbb{R}^m$ such that* (2.7) *holds, and let $(V_t)_{t\geq 0}$ be the MGOU process driven by (X, Y) with starting random variable V_0. Let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be some filtration satisfying the usual hypotheses such that (X, Y) is a semimartingale with respect to $\mathbb{F}$ and V_0 is $\mathcal{F}_0$-measurable. Then $(V_t)_{t\geq 0}$ solves the SDE*

$$dV_t = dU_t V_{t-} + dL_t, \tag{3.4}$$

where (U, L) is the Lévy process in $\mathbb{R}^{m\times m} \times \mathbb{R}^m$ with U as defined in (2.9) *and L given by*

$$L_t = Y_t + \sum_{0<s\leq t} \left((I + \Delta X_s)^{-1} - I\right) \Delta Y_s - [X, Y]^c_t, \quad t \geq 0. \tag{3.5}$$

The process U satisfies

$$\det(I + \Delta U_t) \neq 0 \quad \text{for all} \quad t \geq 0. \tag{3.6}$$

(b) *Conversely, if (U, L) is a Lévy process in $\mathbb{R}^{m\times m} \times \mathbb{R}^m$ such that U satisfies* (3.6), *$\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is a filtration satisfying the usual hypotheses such that (U, L) is an $\mathbb{F}$-semimartingale and V_0 is an $\mathbb{R}^m$-valued $\mathcal{F}_0$-measurable starting random variable, then the solution to* (3.4) *is an MGOU process driven by (X, Y), where (X, Y) is the Lévy process defined by*

$$\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \begin{pmatrix} \overleftarrow{\mathrm{Log}}\,(\overrightarrow{\mathcal{E}}(U)_t^{-1}) \\ L_t + [\overleftarrow{\mathrm{Log}}\,(\overrightarrow{\mathcal{E}}(U)^{-1}), L]_t \end{pmatrix}, \quad t \geq 0, \tag{3.7}$$

and X satisfies (2.7).

Observe that under the natural assumption that V_0 is independent of (X, Y) (i.e. for a causal MGOU process), the smallest filtration $\mathbb{F}$ which satisfies the usual hypotheses and is such that V_0 is $\mathcal{F}_0$ measurable and (X, Y) is adapted to $\mathbb{F}$ is a filtration such that X, Y, U and L are semimartingales with respect to it (cf. Corollary 1 of Theorem VI.11 in [21]), as required in the statement of (a). A similar remark holds for (b) if V_0 is independent of (U, L).

In the following proposition we state some cross-relations between (X, Y) and (U, L) defined by (2.9) and (3.5).

Proposition 3.5. *Let (X, Y) be a Lévy process in $\mathbb{R}^{m\times m} \times \mathbb{R}^m$ such that X satisfies (2.7) and let (U, L) be defined by (2.9) and (3.5). Then*

$$L_t = Y_t + [U, Y]_t, \quad t \geq 0, \tag{3.8}$$

and

$$Y_t = L_t + [X, L]_t, \quad t \geq 0. \tag{3.9}$$

Finally, we show in the next example that the state vector of the COGARCH(q, m) volatility process is an m-dimensional MGOU process.

Example 3.6. Let $m, q \in \mathbb{N}$, $q \leq m$, $c_1, \ldots, c_m, d_0, \ldots, d_{m-1} \in \mathbb{R}$ with $c_m \neq 0$ and $d_{q-1} \neq 0, d_q = \ldots = d_{m-1} = 0$. Denote

$$C = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -c_m & -c_{m-1} & -c_{m-2} & \cdots & -c_1 \end{pmatrix}, \ \mathbf{e} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \ \mathbf{d} = \begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ d_{m-2} \\ d_{m-1} \end{pmatrix}$$

with $C \in \mathbb{R}^{m\times m}$, $\mathbf{e}, \mathbf{d} \in \mathbb{R}^m$, and let M be a one-dimensional Lévy process with non-trival Lévy measure. Let $\beta > 0$. Then, as defined in [5], the COGARCH(q, m) process, driven by M and with parameters C, β and $\mathbf{d}$ has (right-continuous) *volatility process* $(S_t)_{t\geq 0}$ given by

$$S_t = \beta + \mathbf{d}^\perp V_t, \quad t \geq 0, \tag{3.10}$$

where the state vector process $V = (V_t)_{t\geq 0}$ is the unique càdlàg solution of the stochastic differential equation

$$dV_t = CV_{t-}\,dt + \mathbf{e}S_{t-}\,d[M, M]_t^{(d)} = CV_{t-}\,dt + \mathbf{e}(\beta + \mathbf{d}^\perp V_{t-})\,d[M, M]_t^{(d)}, \quad t \geq 0, \tag{3.11}$$

with initial value V_0, independent of $(M_t)_{t\geq 0}$. Here, $[M, M]_t^{(d)} = \sum_{0<s\leq t}(\Delta M_s)^2$ denotes the discrete part of the quadratic variation of M. If the process $(S_t)_{t\geq 0}$ is non-negative almost surely, conditions for which are given in Section 5 of [5], then $G = (G_t)_{t\geq 0}$, defined by

$$G_0 = 0, \quad dG_t = \sqrt{S_{t-}}\,dM_t,$$

is called a *COGARCH(q, m) process* with parameters C, $\mathbf{d}$, β and driving Lévy process M.

It follows from [5, Theorem 3.3] and its proof that the state vector process $(V_t)_{t\geq 0}$ satisfies (1.2) with random functionals $(A_{s,t}, B_{s,t})$ which satisfy Assumption 1, so that $(V_t)_{t\geq 0}$ is an MGOU process. Using the SDE (3.11) and Theorem 3.4, we get another proof of this, observing that

$$\begin{aligned} dV_t &= CV_{t-}\,dt + \beta\mathbf{e}\mathbf{d}^\perp V_{t-}\,d[M, M]_t^{(d)} + \beta\mathbf{e}\,d[M, M]_t^{(d)} \\ &= (C\,dt + \beta\mathbf{e}\mathbf{d}^\perp\,d[M, M]_t^{(d)})V_{t-} + \beta\mathbf{e}\,d[M, M]_t^{(d)} \\ &= dU_t\,V_{t-} + dL_t, \end{aligned}$$

where

$$U_t = Ct + \beta[M,M]_t^{(d)}\mathbf{ed}^\perp \quad \text{and} \quad L_t = \beta[M,M]_t^{(d)}\mathbf{e}. \tag{3.12}$$

Since the jumps of $[M,M]^{(d)}$ are non-negative, it follows that U satisfies condition (3.6) and hence that V is a causal MGOU process by Theorem 3.4.

4 MGOU Processes Carried by Affine Subspaces

In this section we will classify MGOU processes which are carried by affine subspaces of $\mathbb{R}^m$. To do that, we introduce the notion of irreducibility which we mainly adopt from Bougerol and Picard [4] who studied generalized autoregressive models in discrete time. The proofs for the results of this section are given in Section 7.

Definition 4.1. *Suppose $(X_t, Y_t)_{t\geq 0}$ is a Lévy process in $\mathbb{R}^{m\times m}\times\mathbb{R}^m$ such that X satisfies* (2.7) *and define $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ by* (3.1)*. Then an affine subspace H of $\mathbb{R}^m$ is called* invariant *under the autoregressive model* (1.2) *if $A_{s,t}H + B_{s,t} \subseteq H$, almost surely, holds for all $0 \leq s \leq t$. If $\mathbb{R}^m$ is the only invariant affine subspace, the model* (1.2) *is called* irreducible.

Obviously, by Assumption 1(c), it is enough to require the above condition for $s = 0$ and all $t \geq 0$. Remark that the given definition of invariant subspaces is more restrictive than the one in [4], since e.g. setting $Y_t = B_t = 0$ and letting A_t be a rotation operator with angle $2\pi t$ implies that in the discrete time model $V_n = A_{n-1,n}V_{n-1} + B_{n-1,n}$, $n \in \mathbb{N}$, every point is a zero-dimensional invariant affine subspace, while only the rotation axis is invariant for all $t \geq 0$.
Accordingly, irreducibility of the continuous time model does not directly imply that for all $h > 0$ the discrete time model $V_{nh} = A_{(n-1)h,nh}V_{(n-1)h} + B_{(n-1)h,nh}$, $n \in \mathbb{N}$, is irreducible in the sense of [4]. But we can show the following proposition which states that at least there is some $h > 0$ for which the corresponding discrete time model is irreducible. This will be an important ingredient when proving Theorems 5.3, 5.4 and 5.7 below on the existence of strictly stationary solutions.

Proposition 4.2. *Suppose $(X_t, Y_t)_{t\geq 0}$ is a Lévy process in $\mathbb{R}^{m\times m}\times\mathbb{R}^m$ such that X satisfies* (2.7) *and define $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ by* (3.1)*. Suppose that the autoregressive model* (1.2) *is irreducible. Then there exists $h > 0$ for which the discrete-time autoregressive model*

$$V_{nh} = A_{(n-1)h,nh}V_{(n-1)h} + B_{(n-1)h,nh},\ n \in \mathbb{N}, \tag{4.1}$$

is irreducible in the sense that there exists no affine subspace H of $\mathbb{R}^m$, $H \neq \mathbb{R}^m$, such that for all $n \in \mathbb{N}$, $A_{(n-1)h,nh}H + B_{(n-1)h,nh} \subseteq H$ almost surely.

The next theorem treats MGOU processes where the corresponding autoregressive model admits a d-dimensional invariant affine subspace H. It turns out that in this case we can split up the process carried by H in a constant part and an $\mathbb{R}^{m-d}$-valued MGOU process. For convenience we first assume that H is parallel to the axes.

Theorem 4.3. *Suppose $(V_t)_{t\geq 0}$ is an MGOU process with starting random variable V_0, driven by the Lévy process $(X_t, Y_t)_{t\geq 0}$ in $\mathbb{R}^{m\times m}\times\mathbb{R}^m$, where X fulfills (2.7), and let $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ as defined in (3.1).*

(a) *Assume that $H = \{(k_1,\ldots,k_d,h_{d+1},\ldots,h_m)^\perp, h_{d+1},\ldots,h_m\in\mathbb{R}\}$ with $1\leq d\leq m$ and constants $k_1,\ldots,k_d\in\mathbb{R}$ is an invariant, affine subspace of $\mathbb{R}^m$ with respect to the model (1.2). Then, given that $V_0\in H$ a.s., it holds $V_t = \begin{pmatrix} K \\ \mathcal{V}_t\end{pmatrix}\in H$ a.s. for each $t\geq 0$ with $K=(k_1,\ldots,k_d)^\perp$ and $\mathcal{V}_t\in\mathbb{R}^{m-d}$, and the Lévy processes X and Y satisfy for all $t\geq 0$*

$$X_t = \begin{pmatrix} \mathcal{X}_t^1 & 0 \\ \mathcal{X}_t^2 & \mathcal{X}_t^3\end{pmatrix} \quad \text{a.s. where} \quad \mathcal{X}_t^1\in\mathbb{R}^{d\times d} \quad \text{and} \tag{4.2}$$

$$Y_t = \begin{pmatrix} \mathcal{Y}_t^1 \\ \mathcal{Y}_t^2\end{pmatrix} = \begin{pmatrix} \mathcal{X}_t^1 K \\ \mathcal{Y}_t^2\end{pmatrix} \quad \text{a.s. where} \quad \mathcal{Y}_t^1\in\mathbb{R}^d. \tag{4.3}$$

The process $(\mathcal{V}_t)_{t\geq 0}$ is an MGOU process driven by the Lévy process

$$\left(\mathcal{X}_t^3, \mathcal{Y}_t^2 - \mathcal{X}_t^2 K\right)_{t\geq 0} \tag{4.4}$$

in $\mathbb{R}^{(m-d)\times(m-d)}\times\mathbb{R}^{m-d}$. Further, if (U,L) is defined as in (2.9) and (3.5), and if V_0 is $\mathcal{F}_0$-measurable for a filtration $\mathbb{F}=(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual assumptions such that U and L are semimartingales with respect to $\mathbb{F}$ (hence $(V_t)_{t\geq 0}$ solves the SDE (3.4) by Theorem 3.4), then we have a.s. for each $t\geq 0$

$$U_t = \begin{pmatrix} \mathcal{U}_t^1 & 0 \\ \mathcal{U}_t^2 & \mathcal{U}_t^3\end{pmatrix} \quad \text{and} \quad L_t = \begin{pmatrix} \mathcal{L}_t^1 \\ \mathcal{L}_t^2\end{pmatrix} \quad \text{with} \quad \mathcal{U}_t^1\in\mathbb{R}^{d\times d},\ \mathcal{L}_t^1\in\mathbb{R}^d, \tag{4.5}$$

where $\mathcal{L}^1 = -\mathcal{U}^1 K$ a.s. and $(\mathcal{V}_t)_{t\geq 0}$ solves the SDE

$$d\mathcal{V}_t = d\mathcal{U}_t^3\mathcal{V}_{t-} + d(\mathcal{L}_t^2 + \mathcal{U}_t^2 K), \quad t\geq 0. \tag{4.6}$$

(b) *Conversely, if (4.2) and (4.3) hold for $K=(k_1,\ldots,k_d)^\perp\in\mathbb{R}$ constant, then the affine subspace $H=\{(k_1,\ldots,k_d,h_{d+1},\ldots,h_m)^\perp, h_{d+1},\ldots,h_m\in\mathbb{R}\}$ of $\mathbb{R}^m$ is invariant with respect to the model (1.2) and for any starting random variable $V_0\in H$ the MGOU process defined by (3.3) can be written as $V_t=\begin{pmatrix} K \\ \mathcal{V}_t\end{pmatrix}$ a.s., where $(\mathcal{V}_t)_{t\geq 0}$ is an MGOU process driven by the Lévy process (4.4).*

Remark 4.4. *Observe that if in the setting of Theorem 4.3 the invariant affine subspace H is not parallel to the axes, then there exists an orthogonal transformation matrix O, such that OH fulfills the assumptions of Theorem 4.3 for the transformed MGOU process $V'=OV$. The process $(V_t')_{t\geq 0}$ fulfills the random recurrence equation $V_t' = A_{s,t}'V_s' + B_{s,t}'$ for $0\leq s\leq t$ where $A_{s,t}' = OA_{s,t}O^{-1}$ and $B_{s,t}' = OB_{s,t}$ and hence by Theorem 3.1 it is an MGOU process driven by $(OX_tO^{-1}, OY_t)_{t\geq 0}$. Thus the study of arbitrary invariant affine subspaces reduces to the case treated in Theorem 4.3.*

This observation and Theorem 4.3 imply the following characterization of irreducibility of the model (1.2).

Corollary 4.5. *Suppose $(X_t, Y_t)_{t\geq 0}$ in $\mathbb{R}^{m\times m}\times\mathbb{R}^m$ is a Lévy process such that X fulfills* (2.7)*. Then the autoregressive model* (1.2) *with $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ as defined in* (3.1) *is irreducible if and only if there exists no pair (O, K) of an orthogonal transformation $O\in\mathbb{R}^{m\times m}$ and a constant $K = (k_1,\dots,k_d)^\perp\in\mathbb{R}^d$, $1\leq d\leq m$, such that a.s.*

$$OX_tO^{-1} = \begin{pmatrix}\mathcal{X}_t^1 & 0\\ \mathcal{X}_t^2 & \mathcal{X}_t^3\end{pmatrix} \quad \text{and} \quad OY_t = \begin{pmatrix}\mathcal{X}_t^1 K\\ \mathcal{Y}_t^2\end{pmatrix} \quad \text{where} \quad \mathcal{X}_t^1\in\mathbb{R}^{d\times d}, \quad t\geq 0. \tag{4.7}$$

With $(U_t, L_t)_{t\geq 0}$ as defined in (2.9) *and* (3.5)*, Equation* (4.7) *is further equivalent to*

$$OU_tO^{-1} = \begin{pmatrix}\mathcal{U}_t^1 & 0\\ \mathcal{U}_t^2 & \mathcal{U}_t^3\end{pmatrix} \quad \text{and} \quad OL_t = \begin{pmatrix}-\mathcal{U}_t^1 K\\ \mathcal{L}_t^2\end{pmatrix} \quad \text{a.s. with} \quad \mathcal{U}_t^1\in\mathbb{R}^{d\times d}. \tag{4.8}$$

5 Stationary Solutions of MGOU Processes

In this section we investigate conditions for the existence of strictly stationary solutions of multivariate generalized Ornstein-Uhlenbeck processes. The proofs of the results are given in Section 8.

Given some extra information on the limit behaviour of $\overleftarrow{\mathcal{E}}(X)$ our first theorem provides necessary and sufficient conditions for the existence of stationary solutions of MGOU processes. Before we state it we give the following lemma on stochastic exponentials which is interesting in its own right.

Lemma 5.1. *Let $(X_t)_{t\geq 0}$ be a Lévy process in $\mathbb{R}^{m\times m}$. Then for any $t\geq 0$ fixed we have that*

$$\overleftarrow{\mathcal{E}}(X)_t \overset{d}{=} \overrightarrow{\mathcal{E}}(X)_t.$$

In particular this implies

$$\text{P-}\lim_{t\to\infty}\overleftarrow{\mathcal{E}}(X)_t = 0 \quad \Leftrightarrow \quad \text{P-}\lim_{t\to\infty}\overrightarrow{\mathcal{E}}(X)_t = 0. \tag{5.1}$$

Since $\overleftarrow{\mathcal{E}}(U)_t = \overrightarrow{\mathcal{E}}(X)_t^{-1}$, the condition $\text{P-}\lim_{t\to\infty}\overleftarrow{\mathcal{E}}(U)_t = 0$ appearing in Theorem 5.2(a) below is equivalent to $\text{P-}\lim_{t\to\infty}\overleftarrow{\mathcal{E}}(X)_t^{-1} = 0$. Hence, Theorem 5.2 gives necessary and sufficient conditions for stationarity if either $\text{P-}\lim_{t\to\infty}\overleftarrow{\mathcal{E}}(X)_t^{-1} = 0$ or $\text{P-}\lim_{t\to\infty}\overleftarrow{\mathcal{E}}(X)_t = 0$ and thus extends [3, Theorem 2.1].

Theorem 5.2. *Suppose $(V_t)_{t\geq 0}$ is an MGOU process driven by the Lévy process $(X_t, Y_t)_{t\geq 0}$ in $\mathbb{R}^{m\times m}\times\mathbb{R}^m$ such that X satisfies* (2.7)*. Let $(U_t, L_t)_{t\geq 0}$ be the Lévy process defined in* (2.9) *and* (3.5)*.*

(a) *Suppose* $\lim_{t\to\infty} \overleftarrow{\mathcal{E}}(U)_t = 0$ *in probability. Then a finite random variable* V_0 *can be chosen such that* $(V_t)_{t\geq 0}$ *is strictly stationary if and only if the integral* $\int_{(0,t]} \overleftarrow{\mathcal{E}}(U)_{s-} dL_s$ *converges in distribution for* $t \to \infty$ *to a finite random variable. In this case, the distribution of the strictly stationary process* $(V_t)_{t\geq 0}$ *is uniquely determined and is obtained by choosing* V_0 *independent of* $(X_t, Y_t)_{t\geq 0}$ *with* $V_0 \stackrel{d}{=} \text{d-}\lim_{t\to\infty} \int_{(0,t]} \overleftarrow{\mathcal{E}}(U)_{s-} dL_s$.

(b) *Suppose* $\lim_{t\to\infty} \overleftarrow{\mathcal{E}}(X)_t = 0$ *in probability. Then a finite random variable* V_0 *can be chosen such that* $(V_t)_{t\geq 0}$ *is strictly stationary if and only if the integral* $\int_{(0,t]} \overleftarrow{\mathcal{E}}(X)_{s-} dY_s$ *converges in probability to a finite random variable as* $t \to \infty$. *In this case the strictly stationary solution is unique and given by*

$$V_t = -\overleftarrow{\mathcal{E}}(X)_t^{-1} \int_{(t,\infty)} \overleftarrow{\mathcal{E}}(X)_{s-} dY_s \quad \textit{a.s. for all} \quad t \geq 0.$$

Observe that the solution obtained in Theorem 5.2(a) is causal and that the one in (b) is strictly non-causal. By adding the assumption of irreducibility of the underlying model, as characterized in Corollary 4.5, the above theorem can be sharpened as follows.

Theorem 5.3. *Suppose* $(X_t, Y_t)_{t\geq 0}$ *is a Lévy process in* $\mathbb{R}^{m\times m} \times \mathbb{R}^m$ *such that* X *satisfies* (2.7) *and such that the corresponding autoregressive model* (1.2) *with* $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ *as defined in* (3.1) *is irreducible. Let* $(V_t)_{t\geq 0}$ *be the MGOU process driven by* $(X_t, Y_t)_{t\geq 0}$ *and let* $(U_t, L_t)_{t\geq 0}$ *be the Lévy process defined in* (2.9) *and* (3.5).

(a) *A finite random variable* V_0, *independent of* $(X_t, Y_t)_{t\geq 0}$, *can be chosen such that* $(V_t)_{t\geq 0}$ *is strictly stationary if and only if* $\lim_{t\to\infty} \overleftarrow{\mathcal{E}}(U)_t = 0$ *in probability and the integral* $\int_{(0,t]} \overleftarrow{\mathcal{E}}(U)_{s-} dL_s$ *converges in distribution for* $t \to \infty$ *to a finite random variable.*

(b) *A finite random variable* V_0 *can be chosen such that* $(V_t)_{t\geq 0}$ *is strictly stationary and strictly non-causal if and only if* $\lim_{t\to\infty} \overleftarrow{\mathcal{E}}(X)_t = 0$ *in probability and the integral* $\int_{(0,t]} \overleftarrow{\mathcal{E}}(X)_{s-} dY_s$ *converges in probability as* $t \to \infty$.

Given that the processes U and L have a finite log-moment Theorem 5.3 can be sharpenend to obtain a necessary and sufficient condition for the existence of strictly stationary solutions of MGOU processes in terms of the driving Lévy process as stated in Theorem 5.4. To explain its conditions (iv) and (v) and relate it to the corresponding discrete time results, let $\|\cdot\|$ be a fixed, submultiplicative matrix norm. Recall that the *top Lyapunov exponent* of an $\mathbb{R}^{m\times m}$-valued i.i.d. sequence $(C_n)_{n\in\mathbb{N}}$ with $E[\log^+ \|C_1\|]$ is given by

$$\gamma := \inf_{n\in\mathbb{N}} \frac{1}{n} E[\log \|C_1 \cdots C_n\|]. \tag{5.2}$$

It is independent of the specific submultiplicative matrix norm used and it holds

$$\gamma = \lim_{n\to\infty} \frac{1}{n} \log \|C_1 \cdots C_n\| = \lim_{n\to\infty} \frac{1}{n} \log \|C_n \cdots C_1\| \quad \text{a.s.,} \tag{5.3}$$

cf. Furstenberg and Kesten [11] and Bougerol and Picard [4]. In [4, Theorem 2.5] it is also shown that if the discrete time model $W_n = C_n W_{n-1} + D_n$, $n \in \mathbb{Z}$, is irreducible, where $(C_n, D_n)_{n\in\mathbb{Z}}$ is an i.i.d. $\mathbb{R}^{m\times m} \times \mathbb{R}^m$-valued sequence with $E[\log^+ \|C_1\|] < \infty$ and $E[\log^+ \|D_1\|] < \infty$, then the discrete model admits a strictly stationary causal solution if and only if the top Lyapunov exponent of the sequence $(C_n)_{n\in\mathbb{N}}$ is strictly negative.

Now if X and U are as in Theorem 5.4 (v), then $E[\log^+ \|U_1\|] < \infty$ implies $E[\log^+ \|\overleftarrow{\mathcal{E}}(U)_t\|] < \infty$ for every $t > 0$ as will be shown in Proposition 8.4 below. Since for each $h > 0$ the sequence $(A_{(n-1)h,nh} = \overrightarrow{\mathcal{E}}(U)_{nh}\overrightarrow{\mathcal{E}}(U)^{-1}_{(n-1)h})_{n\in\mathbb{N}}$ is i.i.d. and $\overrightarrow{\mathcal{E}}(U)_{nh} = A_{(n-1)h,nh}\cdots A_{0,h}$, it follows that there is $h > 0$ such that the top Lyapunov exponent of $(A_{(n-1)h,nh})_{n\in\mathbb{N}}$ is strictly negative if and only if there is $t_0 > 0$ such that $E[\log^+ \|\overrightarrow{\mathcal{E}}(U)_{t_0}\|] < \infty$, which is equivalent to condition (iv) below by Lemma 5.1.

Theorem 5.4. *Suppose $(X_t, Y_t)_{t\geq 0}$ is a Lévy process in $\mathbb{R}^{m\times m} \times \mathbb{R}^m$ such that X satisfies* (2.7) *and that the corresponding autoregressive model* (1.2) *with $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ as defined in* (3.1) *is irreducible. Let $(V_t)_{t\geq 0}$ be the MGOU process driven by $(X_t, Y_t)_{t\geq 0}$ and let $(U_t, L_t)_{t\geq 0}$ be the Lévy process defined in* (2.9) *and* (3.5)*. Suppose that $E[\log^+ \|U_1\|] < \infty$ and $E[\log^+ \|L_1\|] < \infty$. Then the following are equivalent:*

- (i) *A finite random variable V_0, independent of $(X_t, Y_t)_{t\geq 0}$, can be chosen such that $(V_t)_{t\geq 0}$ is strictly stationary.*
- (ii) *It holds $\lim_{t\to\infty} \overleftarrow{\mathcal{E}}(U)_t = 0$ in probability and the integral $\int_{(0,t]} \overleftarrow{\mathcal{E}}(U)_{s-} dL_s$ converges in distribution for $t \to \infty$ to a finite random variable.*
- (iii) *It holds $\lim_{t\to\infty} \overleftarrow{\mathcal{E}}(U)_t = 0$ a.s. and the integral $\int_{(0,t]} \overleftarrow{\mathcal{E}}(U)_{s-} dL_s$ converges a.s. for $t \to \infty$ to a finite random variable.*
- (iv) *There exists $t_0 > 0$ such that $E[\log \|\overleftarrow{\mathcal{E}}(U)_{t_0}\|] < 0$.*

If additionally U is a compound Poisson process with jump heights $(S_k)_{k\in\mathbb{N}}$, then the above conditions (i) to (iv) are further equivalent to

- (v) *The top Lyapunov exponent of the sequence $(I + S_k)_{k\in\mathbb{N}}$ is strictly negative.*

Remark 5.5.

- (a) *A similar result as Theorem 5.4 also holds true for strictly non-causal strictly stationary solutions of MGOU processes in the irreducible case.*
- (b) *The proof of Theorem 5.4 given in Section 8 shows that the implications "*(iv) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i)*" and "*(v) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i)*" also hold without assuming irreducibility of the underlying model.*

Example 5.6. Consider the state vector process $(V_t)_{t\geq 0}$ of the COGARCH(q, m)-volatility process $(S_t)_{t\geq 0}$ as defined in Example 3.6 with $dV_t = dU_t\, V_{t-} + dL_t$ and $(U_t, L_t)_{t\geq 0}$ given by (3.12). Suppose that $m = 2$. Then it follows from Corollary 4.5 by a straightforward but tedious calculation, using that $c_2 \neq 0$, that the corresponding autoregressive model (1.2) is irreducible. In particular, by Theorem 5.3(a), a strictly stationary (causal) COGARCH$(q,2)$-volatility state vector process exists if and only if $\lim_{t\to\infty} \overleftarrow{\mathcal{E}}(U)_t = 0$ in probability and $\int_{(0,t]} \overleftarrow{\mathcal{E}}(U)_{s-}\, dL_s = \beta \sum_{0<s\le t} \overleftarrow{\mathcal{E}}(U)_{s-} \mathbf{e}(\Delta M_s)^2$ converges in distribution to a finite random variable as $t \to \infty$. If in addition $\int_{|x|>1} \log |x|\, \nu_M(dx) < \infty$, where ν_M denotes the Lévy measure of M, then $E \log^+ \|U_1\| < \infty$ and $E \log^+ \|L_1\| < \infty$, and by Theorem 5.4 the above conditions are equivalent to $E \log^+ \|\overleftarrow{\mathcal{E}}(U)_{t_0}\| < 0$ for some $t_0 > 0$. That the latter condition is sufficient for a (causal) strictly stationary state vector to exist was already observed in Remark 3.4(a) of [5], but having the irreducibility of the model we now also know that it is necessary under the finite log-moment assumption on ν_M. Observe however that the volatility process $(S_t)_{t\ge0}$ defined in (3.10) may be strictly stationary even without $(V_t)_{t\ge0}$ being strictly stationary, since it is only a specific linear combination of $(V_t)_{t\ge0}$ plus a constant. We shall not pursue the issue of strict stationarity of $(S_t)_{t\ge0}$ further. Also, we have not investigated if the autoregressive model (1.2) for the COGARCH(q,m) volatility process with $m \ge 3$ is always irreducible.

In the case that the underlying model is not irreducible, $\text{P-}\lim_{t\to\infty} \overleftarrow{\mathcal{E}}(U)_t = 0$ is not necessary for the existence of a causal strictly stationary solution as shown in the following theorem.

Theorem 5.7. *Suppose $(X_t, Y_t)_{t\ge0}$ is a Lévy process in $\mathbb{R}^{m\times m} \times \mathbb{R}^m$ such that X satisfies (2.7) and let $(V_t)_{t\ge0}$ be the MGOU process driven by $(X_t, Y_t)_{t\ge0}$ satisfying the autoregressive model (1.2) with $(A_{s,t}, B_{s,t})_{0\le s\le t}$ as defined in (3.1). Define $(U_t, L_t)_{t\ge0}$ via (2.9) and (3.5).*

Then a finite random variable V_0 can be chosen such that $(V_t)_{t\ge0}$ is strictly stationary and causal if and only if there exists a pair (O, K) of an orthogonal transformation $O \in \mathbb{R}^{m\times m}$ and a constant $K = (k_1, \dots, k_d)^\perp$, $0 \le d \le m$ such that (4.7) and hence (4.8) hold and such that $\text{P-}\lim_{t\to\infty} \overleftarrow{\mathcal{E}}(\mathfrak{U}^3)_t = 0$ and $\int_{(0,t]} \overleftarrow{\mathcal{E}}(\mathfrak{U}^3)_{s-} d(\mathfrak{L}_s^2 + \mathfrak{U}_s^2 K)$ converges in distribution to a finite random variable as $t \to \infty$.

If these conditions are satisfied a strictly stationary solution can be obtained by choosing V_0 independent of $(X_t, Y_t)_{t\ge0}$ with the same distribution as the distributional limit as $t \to \infty$ of

$$O^{-1} \begin{pmatrix} K \\ \int_{(0,t]} \overleftarrow{\mathcal{E}}(\mathfrak{U}^3)_{s-} d(\mathfrak{L}_s^2 + \mathfrak{U}_s^2 K) \end{pmatrix}.$$

If $d = 0$ in the above conditions then $\mathfrak{L}_s^2 + \mathfrak{U}_s^2 K$ has to be interpreted as $\mathfrak{L}_s^2$, and if $d = m$ then $\mathfrak{U}^3$ is zero-dimensional and the convergence conditions regarding $\overleftarrow{\mathcal{E}}(\mathfrak{U}^3)_t$ and $\int_{(0,t]} \overleftarrow{\mathcal{E}}(\mathfrak{U}^3)_{s-} d(\mathfrak{L}_s^2 + \mathfrak{U}_s^2 K)$ do not appear.

Remark 5.8. *Using arguments as in the proof of Theorem 5.3(b) a similar result as Theorem 5.7 for strictly noncausal strictly stationary solutions of MGOU processes can be obtained, too.*

Remark 5.9. *The results in Sections 3 and 5 remain valid if we treat an MGOU process* $(V_t)_{t\ge 0}$ *with* $V_t \in \mathbb{R}^{m\times l}$ *and drop the condition of* $l = 1$*. As the value of* l *has no influence on the proofs we can simply replace the vector valued processes* $(Y_t)_{t\ge0}$ *and* $(L_t)_{t\ge0}$ *by* $\mathbb{R}^{m\times l}$*-valued processes. Theorem 4.3 may be applied column-by-column or, alternatively, it is possible to interpret the MGOU process* $(V_t)_{t\ge0}$ *in* $\mathbb{R}^{m\times l}$ *driven by* $(X_t, Y_t)_{t\ge0}$*,* $X_t \in \mathbb{R}^{m\times m}$*,* $Y_t = (Y_t^1, \dots, Y_t^l) \in \mathbb{R}^{m\times l}$ *as an MGOU process in* $\mathbb{R}^{ml}$ *driven by the Lévy process*

$$\left(\begin{pmatrix} X_t & & 0 \\ & \ddots & \\ 0 & & X_t \end{pmatrix}, \begin{pmatrix} Y_t^1 \\ \vdots \\ Y_t^l \end{pmatrix} \right)_{t\ge0} \quad \textit{in} \quad \mathbb{R}^{ml\times ml} \times \mathbb{R}^{ml}.$$

6 Proofs for Section 3

Before proving Theorem 3.1 we give the following proposition which establishes in particular the semimartingale property of $(A_t)_{t\ge0}$ and $(B_t)_{t\ge0}$.

Proposition 6.1. *Suppose* $(A_{s,t}, B_{s,t})_{0\le s\le t}$ *is a process satisfying Assumption 1 and such that* $(A_t, B_t)_{t\ge0}$ *is càdlàg. Let* $\mathbb{H}$ *be the natural augmented filtration of* $(A_t, B_t)_{t\ge0}$*. Then* $(A_t)_{t\ge0}$ *and* $(B_t)_{t\ge0}$ *are* $\mathbb{H}$*-semimartingales. Further, the* $\mathbb{R}^{m\times m}\times\mathbb{R}^m\times\mathbb{R}^{m\times m}\times\mathbb{R}^m$*- valued process* $(U_t, L_t, X_t, Y_t)_{t\ge0}$ *defined by*

$$\begin{pmatrix} U_t \\ L_t \\ X_t \\ Y_t \end{pmatrix} = \begin{pmatrix} \overrightarrow{\operatorname{Log}}\, A_t = \int_{(0,t]} dA_s\, A_{s-}^{-1} \\ B_t - \int_{(0,t]} dA_s\, A_{s-}^{-1} B_{s-} \\ \overleftarrow{\operatorname{Log}}\, A_t^{-1} = \int_{(0,t]} A_{s-}\, dA_s^{-1} \\ \int_{(0,t]} A_{s-}\, d(A_s^{-1} B_s) \end{pmatrix}, \quad t \ge 0, \tag{6.4}$$

is an $\mathbb{H}$*-Lévy process.*

Proof. Observe that $(A_t)_{t\ge0}$ is a right $\mathbb{H}$-Lévy process by Assumption 1 and Lemma 2.1 and hence an $\mathbb{H}$-semimartingale by Proposition 2.4. It follows that $(U_t, L_t, X_t)_{t\ge0}$ as given in (6.4) is well defined. By computations similar to those in the proof of [12, Theorem 2.2] one can show that for $0 \le s \le t$

$$\begin{pmatrix} U_t - U_s \\ L_t - L_s \\ X_t - X_s \end{pmatrix} = \begin{pmatrix} \int_{(s,t]} d(A_{s,\cdot})_u\, A_{s,u-}^{-1} \\ B_{s,t} - \int_{(s,t]} d(A_{s,\cdot})_u\, A_{s,u-}^{-1} B_{s,u-} \\ \int_{(s,t]} A_{s,u}\, d(A_{s,\cdot}^{-1})_u \end{pmatrix}, \quad 0 \le s \le t. \tag{6.5}$$

By Assumption 1(b,c) we observe that $(A_{s,s+u}, B_{s,s+u})_{u\ge0} \overset{d}{=} (A_{0,u}, B_{0,u})_{u\ge0}$ and thus we obtain from (6.5) that (U, L, X) has stationary increments. By Assumption 1(b), $(U_t - U_s, L_t - L_s, X_t - X_s)$ is independent from $\mathcal{H}_s$ for $0 \le s \le t$, where $\mathbb{H} = (\mathcal{H}_t)_{t\ge0}$. We also know that $(U_0, L_0, X_0) = 0$ a.s., that the paths of (U, L, X) are càdlàg since that held true for $(A_t, B_t)_{t\ge0}$, and that clearly (U, L, X) is adapted to $\mathbb{H}$. Hence $(U_t, L_t, X_t)_{t\ge0}$ is an $\mathbb{H}$-Lévy process. In particular, L is an $\mathbb{H}$-semimartingale, so that by (6.4),

$$(B_t)_{t\ge0} = (L_t + \int_{(0,t]} dA_s\, A_{s-}^{-1} B_{s-})_{t\ge0}$$

is an $\mathbb{H}$-semimartingale, too. Consequently Y as given in (6.4) is well defined. For $0 \le s \le t$ we then have from Assumption 1(a) that

$$\begin{aligned} Y_t - Y_s &= \int_{(s,t]} A_{u-}\, d(A_u^{-1} B_u) \\ &= \int_{(s,t]} A_{s,u-} A_s\, d(A_s^{-1} A_{s,\cdot}^{-1} A_{s,\cdot} B_s + A_s^{-1} A_{s,\cdot}^{-1} B_{s,\cdot})_u \\ &= \int_{(s,t]} A_{s,u-}\, d(A_{s,\cdot}^{-1} B_{s,\cdot})_u. \end{aligned}$$

It now follows in complete analogy to the reasoning given above that (U, L, X, Y) is an $\mathbb{H}$-Lévy process. □

Proof of Theorem 3.1. Let $(A_{s,t}, B_{s,t})_{0\le s\le t}$ satisfy Assumption 1. By Proposition 6.1, $(A_t)_{t\ge 0}$ and $(B_t)_{t\ge 0}$ are semimartingales with respect to their natural augmented filtration, and (X, Y) defined by (3.2) is a Lévy process. Clearly, X satisfies (2.7), and for $0 \le s \le t$ it holds

$$A_{s,t} = A_t A_s^{-1} = (A_t^{-1})^{-1} A_s^{-1} = \overleftarrow{\mathcal{E}}(X)_t^{-1} \overleftarrow{\mathcal{E}}(X)_s.$$

Furthermore, $A_{t-}^{-1} dY_t = d(A_t^{-1} B_t)$ from (3.2), so that

$$B_t = A_t \int_{(0,t]} A_{u-}^{-1}\, dY_u = \overleftarrow{\mathcal{E}}(X)_t^{-1} \int_{(0,t]} \overleftarrow{\mathcal{E}}(X)_{u-}\, dY_u,$$

giving

$$\begin{aligned} B_{s,t} &= B_t - A_{s,t} B_s \\ &= \overleftarrow{\mathcal{E}}(X)_t^{-1} \int_{(0,t]} \overleftarrow{\mathcal{E}}(X)_{u-}\, dY_u - \overleftarrow{\mathcal{E}}(X)_t^{-1} \overleftarrow{\mathcal{E}}(X)_s \overleftarrow{\mathcal{E}}(X)_s^{-1} \int_{(0,s]} \overleftarrow{\mathcal{E}}(X)_{u-}\, dY_u \\ &= \overleftarrow{\mathcal{E}}(X)_t^{-1} \int_{(s,t]} \overleftarrow{\mathcal{E}}(X)_{u-}\, dY_u. \end{aligned}$$

This is (3.1). The uniqueness of (X, Y) is clear from (3.1).

For the converse, let (X, Y) a Lévy process in $\mathbb{R}^{m\times m} \times \mathbb{R}^m$ such that X satisfies (2.7). Let $\mathbb{F}$ be the augmented natural filtration of $(X_t, Y_t)_{t\ge 0}$, then $(A_{s,t}, B_{s,t})_{0\le s\le t}$ as given in (3.1) is well defined with respect to $\mathbb{F}$ and we know from Proposition 2.4 that $\overleftarrow{\mathcal{E}}(X)_t^{-1}$ is a right $\mathbb{F}$-Lévy process in $\mathrm{GL}(\mathbb{R}, m)$ whose left increments are given by $A_{s,t}$. Thus we have that for all $0 \le s \le u \le t$ almost surely $A_{s,t} = A_{u,t} A_{s,u}$ holds. Also it follows directly from the definitions of $A_{s,t}$ and $B_{s,t}$ that $B_{s,t} = A_{u,t} B_{s,u} + B_{u,t}$ a.s. such that Assumption 1(a) is fulfilled.

For the common process $(A_{s,t}, B_{s,t})_{0\le s\le t}$ observe that for $0 \le s \le t$ we have

$$\begin{pmatrix} A_{s,t} \\ B_{s,t} \end{pmatrix} = \begin{pmatrix} \overleftarrow{\mathcal{E}}(X)_t^{-1} \overleftarrow{\mathcal{E}}(X)_s \\ \overleftarrow{\mathcal{E}}(X)_t^{-1} \overleftarrow{\mathcal{E}}(X)_s \int_{(s,t]} \overleftarrow{\mathcal{E}}(X)_s^{-1} \overleftarrow{\mathcal{E}}(X)_{u-} dY_u \end{pmatrix}.$$

Since $(A_t)_{t\geq 0}$ is a right $\mathbb{F}$-Lévy process the common increments $(\overleftarrow{\mathcal{E}}(X)_t^{-1}\overleftarrow{\mathcal{E}}(X)_s, Y_t - Y_s)_{t\geq s}$ are independent of $(X_u, Y_u)_{0\leq u\leq s}$. Hence it follows that $\{(A_{s,t}, B_{s,t}), a \leq s \leq t \leq b\}$ and $\{(A_{s,t}, B_{s,t}), c \leq s \leq t \leq d\}$ with $b \leq c$ are independent. Similarly we conclude that

$$\begin{pmatrix} A_{s,t} \\ B_{s,t} \end{pmatrix} \overset{d}{=} \begin{pmatrix} \overleftarrow{\mathcal{E}}(X)_{t-s}^{-1}\overleftarrow{\mathcal{E}}(X)_0 \\ \overleftarrow{\mathcal{E}}(X)_{t-s}^{-1}\overleftarrow{\mathcal{E}}(X)_0 \int_{(0,t-s]} \overleftarrow{\mathcal{E}}(X)_0^{-1}\overleftarrow{\mathcal{E}}(X)_{u-}dY_u \end{pmatrix} = \begin{pmatrix} A_{0,t-s} \\ B_{0,t-s} \end{pmatrix}$$

which yields Assumption 1(c).

The continuity in probability at 0 of $A_t = A_{0,t}$ is clear, while for $B_t = B_{0,t}$ it follows from that of A_t and Y_t and the continuity of the integral. □

Proof of Theorem 3.4. (a) It is easy to see that (U, L) as constructed in (2.9) and (3.5) is a Lévy process and that U satisfies (3.6). Define $A_t = A_{0,t}$ and $B_t = B_{0,t}$ for $t \geq 0$ by the right hand side of (3.1). Then $V_t = A_t V_0 + B_t$. By the definition of U, we further have that $dA_t = dU_t\, A_{t-}$. Hence, denoting $L'_t := B_t - \int_{(0,t]} dA_u\, A_{u-}^{-1} B_{u-}$ as in (6.4), we obtain

$$\begin{aligned} dV_t &= dA_t V_0 + dB_t = dU_t A_{t-} V_0 + dB_t \\ &= dU_t(A_{t-}V_0 + B_{t-}) + dB_t - dU_t B_{t-} = dU_t V_{t-} + dB_t - dA_t\, A_{t-}^{-1} B_{t-} \\ &= dU_t V_{t-} + dL'_t. \end{aligned}$$

It remains to show that $L' = L$. Using the integration by parts formula (2.3) and (2.2), we obtain

$$\begin{aligned} L'_t &= B_t - \int_{(0,t]} dA_s\, A_{s-}^{-1} B_{s-} \\ &= \overleftarrow{\mathcal{E}}(X)_t^{-1} \int_{(0,t]} \overleftarrow{\mathcal{E}}(X)_{s-} dY_s - \int_{(0,t]} d(\overleftarrow{\mathcal{E}}(X)_s^{-1}) \int_{(0,s)} \overleftarrow{\mathcal{E}}(X)_{u-} dY_u \\ &= \overleftarrow{\mathcal{E}}(X)_t^{-1} \int_{(0,t]} \overleftarrow{\mathcal{E}}(X)_{s-} dY_s - \overleftarrow{\mathcal{E}}(X)_t^{-1} \int_{(0,t]} \overleftarrow{\mathcal{E}}(X)_{s-} dY_s \\ &\quad + \int_{(0,t]} \overleftarrow{\mathcal{E}}(X)_{s-}^{-1} d\left(\int_{(0,s]} \overleftarrow{\mathcal{E}}(X)_{u-} dY_u\right) + \left[\overleftarrow{\mathcal{E}}(X)^{-1}, \int_{(0,\cdot]} \overleftarrow{\mathcal{E}}(X)_{u-} dY_u\right]_t \\ &= \int_{(0,t]} \overleftarrow{\mathcal{E}}(X)_{s-}^{-1}\overleftarrow{\mathcal{E}}(X)_{s-} dY_u + \left[\int_{(0,\cdot]} d(\overleftarrow{\mathcal{E}}(X)_s^{-1})\overleftarrow{\mathcal{E}}(X)_{s-}, Y\right]_t \\ &= Y_t + \left[\int_{(0,\cdot]} dA_s\, A_{s-}^{-1}, Y\right]_t \\ &= Y_t + [U, Y]_t. \end{aligned}$$

That $L' = L$ then follows from the definition of U in (2.9) since

$$\begin{aligned} [U, Y]_t &= -[X, Y]_t + [[X, X]^c, Y]_t + \left[\sum_{0<s\leq\cdot} \left((I + \Delta X_s)^{-1} - I + \Delta X_s\right), Y\right]_t \\ &= -[X, Y]_t^c + \sum_{0<s\leq t} \left((I + \Delta X_s)^{-1} - I\right) \Delta Y_s. \end{aligned}$$

(b) Is is clear that (X,Y) as defined in (3.7) is a Lévy process with X satisfying (2.7). Observe that the given definition of $(X_t)_{t\geq 0}$ is equivalent to (2.9) and that from the definition of $(Y_t)_{t\geq 0}$ we deduce

$$\begin{aligned} Y_t &= L_t + [X,L]_t \\ &= L_t + \sum_{0<s\leq t} (\Delta X_s \Delta L_s) + [X,L]_t^c \\ &= L_t - \sum_{0<s\leq t} \left((I+\Delta X_s)^{-1} - I\right)(I+\Delta X_s)\Delta L_s + [X,L]_t^c, \quad t\geq 0. \end{aligned}$$

Hence $\Delta Y_t = (I+\Delta X_t)\Delta L_t$ and $[X,Y]_t^c = [X,L]_t^c$ and we conclude that

$$Y_t = L_t - \sum_{0<s\leq t} \left((I+\Delta X_s)^{-1} - I\right)\Delta Y_s + [X,Y]_t^c, \quad t\geq 0,$$

which is equivalent to (3.5). Thus the MGOU process $\left(\overleftarrow{\mathcal{E}}(X)_t^{-1}\left(V_0 + \int_{(0,t]} \overleftarrow{\mathcal{E}}(X)_{s-} dY_s\right)\right)_{t\geq 0}$ solves the SDE (3.4) by part (a), giving the claim by the uniqueness of the solution to (3.4). □

Proof of Proposition 3.5. Equation (3.8) has been established when showing that $L' = L$ in the proof of Theorem 3.4(a). Equation (3.9) follows from the fact that by (3.5), $\Delta L_t = (I+\Delta X_t)^{-1}\Delta Y_t$ and $[X,L]_t^c = [X,Y]_t^c$, so that by the same calculation as in the proof of Theorem 3.4(b),

$$L_t + [X,L]_t = L_t - \sum_{0<s\leq t} \left((I+\Delta X_s)^{-1} - I\right)\Delta Y_s + [X,Y]_t^c, \quad t\geq 0.$$

By (3.5) this implies (3.9). □

7 Proofs for Section 4

In this section we give the proofs of the results of Section 4.

Proof of Proposition 4.2. For $\nu \in \mathbb{N}_0$ let $C_\nu := A_{0,2^{-\nu}}$, $D_\nu := B_{0,2^{-\nu}}$ and consider the random affine transformation $f_\nu : \mathbb{R}^m \to \mathbb{R}^m, x \mapsto C_\nu x + D_\nu$, $\nu \in \mathbb{N}_0$. For $0 \leq d < m$, a d-dimensional affine subspace H of $\mathbb{R}^m$ will be called an *affine d-flat*, and it is *f_ν-invariant* if $f_\nu(H) = C_\nu H + D_\nu \subseteq H$ a.s., which by Assumption 1(c) is equivalent to saying that H is invariant for the discrete time model $V_{nh} = A_{(n-1)h,nh} V_{(n-1)h} + B_{(n-1)h,nh}$, $n \in \mathbb{N}_0$, as defined in (4.1) with $h = 2^{-\nu}$. Since by (1.3) any subspace which is invariant for the model (4.1) for some $h > 0$ is also invariant for the model (4.1) for every $h' := kh$ with $k \in \mathbb{N}$, it is clear that any f_ν invariant affine d-flat is also $f_{\nu-1}$-invariant. Hence, denoting the set of all f_ν-invariant affine d-flats by $\mathcal{H}_\nu^d$, $0 \leq d < m$, it follows that $\mathcal{H}_{\nu+1}^d \subseteq \mathcal{H}_\nu^d$

for all $\nu \in \mathbb{N}_0$, $0 \leq d < m$, such that $\mathcal{H}^d_\infty := \lim_{n\to\infty} \mathcal{H}^d_\nu = \bigcap_{\nu=0}^\infty \mathcal{H}^d_\nu$ can be defined for $0 \leq d < m$. We further denote

$$\mathcal{H}_\nu := \bigcup_{d=0}^{m-1} \mathcal{H}^d_\nu \quad \text{for} \quad \nu \in \mathbb{N}_0 \quad \text{and} \quad \mathcal{H}_\infty := \bigcup_{d=0}^{m-1} \mathcal{H}^d_\infty = \bigcap_{\nu=0}^{\infty} \mathcal{H}_\nu.$$

The proof of the proposition will be given in two steps: first it will be shown that irreducibility of the continuous time model (1.2) implies $\mathcal{H}_\infty = \emptyset$, and in a second step that $\mathcal{H}_\infty = \emptyset$ implies the existence of some $\nu_0 \in \mathbb{N}_0$ such that $\mathcal{H}_{\nu_0} = \emptyset$, i.e. that the discrete time model (4.1) is irreducible for $h = 2^{-\nu_0}$.

The first step will be shown by contradiction, i.e. we assume that $\mathcal{H}_\infty \neq \emptyset$, i.e. that there exists an affine subspace $H \neq \mathbb{R}^m$ of $\mathbb{R}^m$ which is invariant under f_ν for all $\nu \in \mathbb{N}$. Thus, as argued above, H is also invariant under the model (4.1) for all $h = k2^{-\nu}$, $k \in \mathbb{N}$, $\nu \in \mathbb{N}$. It then remains to show that $A_{0,t}H + B_{0,t} \subseteq H$ holds for all $t > 0$, i.e. that H is invariant under (1.2). But this follows easily from the fact that $(A_{0,t})_{t\geq 0}$ and $(B_{0,t})_{t\geq 0}$ have almost surely càdlàg paths, so that for every number $t > 0$ we can find a sequence $(t_n)_{n\in\mathbb{N}}$ of the form $t_n = k_n 2^{-\nu_n}$ converging from the right to t such that almost surely,

$$A_{0,t}H + B_{0,t} = \lim_{n\to\infty} (A_{0,t_n}H + B_{0,t_n}) \subseteq H.$$

Hence H is an invariant affine subspace of the continuous time model (1.2) giving the desired contradiction.

It remains to show that $\mathcal{H}_\infty = \bigcup_{d=0}^{m-1} \mathcal{H}^d_\infty = \emptyset$ implies the existence of some ν_0 such that $\mathcal{H}_{\nu_0} = \emptyset$. Since any affine 0-flat H of the form $H = \{x\}$ is f_ν-invariant if and only if $(f_\nu - I)(x) = 0$ a.s., and since $f_n - I$ is an affine linear mapping, its kernel is an affine linear subspace of $\mathbb{R}^m$, S_ν say, and we have $\mathcal{H}^0_\nu = \{\{x\}, x \in S_\nu\}$. Since $S_{\nu+1} \subseteq S_\nu$, it follows that there is $\nu_1 \in \mathbb{N}$ such that $S_{\nu_1+n} = S_{\nu_1}$ for all $n \in \mathbb{N}_0$. Hence, $\mathcal{H}_\infty = \emptyset$ implies that there is $\nu_1 \in \mathbb{N}$ such that $\mathcal{H}^0_\nu = \emptyset$ for all $\nu \geq \nu_1$, and in the following we can concentrate on invariant affine d-flats with $0 < d < m$.

Fix a family $(\mathcal{O}, \mathcal{K}) = \{(O_H, K_H), H \text{ affine } d\text{-flat with } 0 < d < m\}$ of pairs of an orthogonal transformation $O_H \in \mathbb{R}^{m\times m}$ and a constant $K_H \in \mathbb{R}^m$ such that $H' := O_H H - K_H = \{0\}^{m-d} \times \mathbb{R}^d$. Then given $\nu \in \mathbb{N}$ and some affine d-flat H we obtain by easy computations that H is invariant under $f_\nu : x \mapsto C_\nu x + D_\nu$ if and only if the subspace H' is invariant under the mapping $g_\nu : \mathbb{R}^m \to \mathbb{R}^m, x \mapsto \widetilde{C}_\nu x + \widetilde{D}_\nu$ with $\widetilde{C}_\nu = O_H C_\nu O_H^{-1}$ and $\widetilde{D}_\nu = O_H D_\nu + (O_H C_\nu O_H^{-1} - I)K_H$. Using the special structure of H' this yields that H' is invariant under g_ν if and only if it holds almost surely

$$\widetilde{C}^{(i,j)}_\nu = 0 \text{ and } \widetilde{D}^{(i)}_\nu = 0 \text{ for all } (i,j) \in J_H := \{(i,j), 1 \leq i \leq d, d < j \leq m\}.$$

This is by definition of $\widetilde{C}_\nu$ and $\widetilde{D}_\nu$ equivalent to state that, almost surely,

$$\sum_{k,l=1}^m O_H^{(i,k)} C_\nu^{(k,l)} (O_H^{-1})^{(l,j)} = \sum_{k,l=1}^m O_H^{(i,k)} O_H^{(j,l)} C_\nu^{(k,l)} = 0 \quad \text{and} \tag{7.1}$$

$$\sum_{k=1}^m O_H^{(i,k)} D_\nu^{(k)} + \sum_{q=1}^m \left(\sum_{k,l=1}^m O_H^{(i,k)} O_H^{(q,l)} C_\nu^{(k,l)} \right) K_H^{(q)} - K_H^{(i)} = 0 \text{ for all } (i,j) \in J_H. \tag{7.2}$$

By introducing the matrices $M_{H,i,j}$ and $N_{H,i}$ in $\mathbb{R}^{m\times m}$ via

$$M_{H,i,j}^{(k,l)} := O_H^{(i,k)} O_H^{(j,l)} \quad \text{and} \quad N_{H,i}^{(k,l)} := \sum_{q=1}^{m} O_H^{(i,k)} O_H^{(q,l)} K_H^{(q)}, \quad k,l = 1,\dots,m,$$

denoting the ith row of the matrix O_H by $O_H^{(i,\cdot)}$ and letting $\mathrm{vec}\,(\cdot) : \mathbb{R}^{m\times m} \to \mathbb{R}^{m^2}$ be the vectorization operator which stacks the colums of a given matrix below one another, (7.1) and (7.2) turn out to be equivalent to

$$\left\langle \begin{pmatrix} \mathrm{vec}\,(M_{H,i,j}) \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \mathrm{vec}\,(C_\nu) \\ D_\nu \\ -1 \end{pmatrix} \right\rangle = 0 = \left\langle \begin{pmatrix} \mathrm{vec}\,(N_{H,i}) \\ (O_H^{(i,\cdot)})^\perp \\ K_H^{(i)} \end{pmatrix}, \begin{pmatrix} \mathrm{vec}\,(C_\nu) \\ D_\nu \\ -1 \end{pmatrix} \right\rangle \forall (i,j) \in J_H,$$

almost surely, where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^{m^2+m+1}$. Now set

$$R_\nu := \mathrm{span} \bigcup_{H \in \mathcal{H}_\nu \setminus \mathcal{H}_\nu^0} \left\{ \begin{pmatrix} \mathrm{vec}\,(M_{H,i,j}) \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \mathrm{vec}\,(N_{H,i}) \\ (O_H^{(i,\cdot)})^\perp \\ K_H^{(i)} \end{pmatrix}; (i,j) \in J_H \right\} \subseteq \mathbb{R}^{m^2+m+1},$$

where span denotes the linear span. Then by the above we have established that R_ν is orthogonal to $(\mathrm{vec}\,(C_\nu)^\perp, D_\nu^\perp, -1)^\perp$, a.s., and that, given an affine d-flat H, $0 < d < m$, it is invariant under f_ν if and only if all corresponding vectors $(\mathrm{vec}\,(M_{H,i,j})^\perp, 0, 0)^\perp$ and $(\mathrm{vec}\,(N_{H,i})^\perp, O_H^{(i,\cdot)}, K_H^{(i)})^\perp$ for $(i,j) \in J_H$ are in R_ν.
Finally, as R_ν is a vector space and we have that $R_{\nu+1} \subseteq R_\nu$ we observe that its limit for $\nu \to \infty$ can only be empty (which is equivalent to $\bigcup_{d=1}^{m} \mathcal{H}_\infty^d = \emptyset$) if there exists some $\nu_2 \in \mathbb{N}$ such that for all $\nu \geq \nu_2$ we have $R_\nu = \emptyset$. Hence it holds $\mathcal{H}_\nu = \emptyset$ for all $\nu \geq \nu_0 := \max\{\nu_1, \nu_2\}$ so that the discrete time model (4.1) is irreducible for all $\nu \geq \nu_0$ as had to be shown. □

Proof of Theorem 4.3. (a) We start by verifying (4.2) and (4.3). Since H is an invariant affine subspace we deduce from (1.2) that for any $t \geq 0$ and all $h_{d+1}, \dots, h_m \in \mathbb{R}$ the equation

$$A_t(k_1, \dots, k_d, h_{d+1}, \dots, h_m)^\perp + B_t = (k_1, \dots, k_d, g_{d+1}, \dots, g_m)^\perp \quad \text{a.s.}$$

has to admit a solution $g_{d+1}, \dots, g_m \in \mathbb{R}$. This is equivalent to

$$\begin{aligned} \sum_{j=1}^{d} k_j A_t^{(i,j)} + \sum_{j=d+1}^{m} h_j A_t^{(i,j)} + b_i &= k_i, \quad \forall i = 1, \dots, d \\ \sum_{j=1}^{d} k_j A_t^{(i,j)} + \sum_{j=d+1}^{m} h_j A_t^{(i,j)} + b_i &= g_i, \quad \forall i = d+1, \dots, m. \end{aligned}$$

Thus we can conclude that $A_t^{(i,j)} = 0$ holds a.s. for $i \leq d, j > d$. Observe by simple algebraic calculations that if two matrices M and N in $\mathbb{R}^{m\times m}$ have a $d \times (m-d)$ block of zero entries in the upper right corner, then so do M^{-1} and MN. More detailed we have for

$$M = \begin{pmatrix} \mathcal{M}_1 & 0 \\ \mathcal{M}_2 & \mathcal{M}_3 \end{pmatrix} \in \mathrm{GL}(\mathbb{R}, m) \quad \text{and} \quad N = \begin{pmatrix} \mathcal{N}_1 & 0 \\ \mathcal{N}_2 & \mathcal{N}_3 \end{pmatrix} \in \mathbb{R}^{m\times m},\ \mathcal{M}_1, \mathcal{N}_1 \in \mathbb{R}^{d\times d},$$

that $\mathcal{M}_1$ and $\mathcal{M}_3$ are non-singular and it holds

$$M^{-1} = \begin{pmatrix} \mathcal{M}_1^{-1} & 0 \\ -\mathcal{M}_3^{-1}\mathcal{M}_2\mathcal{M}_1^{-1} & \mathcal{M}_3^{-1} \end{pmatrix} \quad \text{and} \quad MN = \begin{pmatrix} \mathcal{M}_1\mathcal{N}_1 & 0 \\ \mathcal{M}_2\mathcal{N}_1 + \mathcal{M}_3\mathcal{N}_2 & \mathcal{M}_3\mathcal{N}_3 \end{pmatrix}.$$

Now recall that $A_t = \overleftarrow{\mathcal{E}}(X)_t^{-1}$ and thus we know that $\overleftarrow{\mathcal{E}}(X)_t$ and $\overleftarrow{\mathcal{E}}(X)_t^{-1}$ a.s. admit a $d \times (m-d)$ zero block for all $t \geq 0$. Hence it follows from (2.8) that also X_t a.s. has such a zero block which is (4.2). Thus we deduce from (2.4) that

$$\overleftarrow{\mathcal{E}}(X)_t =: \begin{pmatrix} \mathcal{E}_t^1 & 0 \\ \mathcal{E}_t^2 & \mathcal{E}_t^3 \end{pmatrix} = \begin{pmatrix} I + \int_0^t \mathcal{E}_{s-}^1 d\mathcal{X}_s^1 & 0 \\ \int_0^t \mathcal{E}_{s-}^2 d\mathcal{X}_s^1 + \int_0^t \mathcal{E}_{s-}^3 d\mathcal{X}_s^2 & I + \int_0^t \mathcal{E}_{s-}^3 d\mathcal{X}_s^3 \end{pmatrix}, \quad t \geq 0, \tag{7.3}$$

and observe in particular that $\mathcal{E}_t^1 = \overleftarrow{\mathcal{E}}(\mathcal{X}^1)_t$ and $\mathcal{E}_t^3 = \overleftarrow{\mathcal{E}}(\mathcal{X}^3)_t$ hold for $t \geq 0$. Inserting the previous results in (3.3) yields for all $t \geq 0$ a.s.

$$\begin{aligned} V_t = \begin{pmatrix} K \\ \mathcal{V}_t \end{pmatrix} &= \begin{pmatrix} (\mathcal{E}_t^1)^{-1} & 0 \\ -(\mathcal{E}_t^3)^{-1}\mathcal{E}_t^2(\mathcal{E}_t^1)^{-1} & (\mathcal{E}_t^3)^{-1} \end{pmatrix} \left[\begin{pmatrix} K \\ \mathcal{V}_0 \end{pmatrix} + \int_{(0,t]} \begin{pmatrix} \mathcal{E}_{s-}^1 & 0 \\ \mathcal{E}_{s-}^2 & \mathcal{E}_{s-}^3 \end{pmatrix} d \begin{pmatrix} \mathcal{Y}_s^1 \\ \mathcal{Y}_s^2 \end{pmatrix} \right] \\ &= \begin{pmatrix} (\mathcal{E}_t^1)^{-1}K \\ -(\mathcal{E}_t^3)^{-1}\mathcal{E}_t^2(\mathcal{E}_t^1)^{-1}K + (\mathcal{E}_t^3)^{-1}\mathcal{V}_0 \end{pmatrix} \\ &\quad + \begin{pmatrix} (\mathcal{E}_t^1)^{-1} \int_0^t \mathcal{E}_{s-}^1 d\mathcal{Y}_s^1 \\ -(\mathcal{E}_t^3)^{-1}\mathcal{E}_t^2(\mathcal{E}_t^1)^{-1} \int_0^t \mathcal{E}_{s-}^1 d\mathcal{Y}_s^1 + (\mathcal{E}_t^3)^{-1}[\int_0^t \mathcal{E}_{s-}^3 d\mathcal{Y}_s^2 + \int_0^t \mathcal{E}_{s-}^2 d\mathcal{Y}_s^1] \end{pmatrix}. \end{aligned} \tag{7.4}$$

The equation for the first d entries of (7.4) is equivalent to

$$K + \int_{(0,t]} \overleftarrow{\mathcal{E}}(\mathcal{X}^1)_{s-} d\mathcal{Y}_s^1 = \overleftarrow{\mathcal{E}}(\mathcal{X}^1)_t K = K + \int_{(0,t]} \overleftarrow{\mathcal{E}}(\mathcal{X}^1)_{s-} d\mathcal{X}_s^1 K \quad \text{a.s., } t \geq 0,$$

from where we deduce (4.3). From the equation for the second $m-d$ entries of (7.4) we derive under use of (4.3), (2.4) and (7.3) that

$$\begin{aligned} &\mathcal{V}_t - \overleftarrow{\mathcal{E}}(\mathcal{X}^3)_t^{-1}\mathcal{V}_0 \\ &= \overleftarrow{\mathcal{E}}(\mathcal{X}^3)_t^{-1} \left(\int_{(0,t]} \mathcal{E}_{s-}^3 d\mathcal{Y}_s^2 + \int_{(0,t]} \mathcal{E}_{s-}^2 d\mathcal{X}_s^1 K - \mathcal{E}_t^2 (\mathcal{E}_t^1)^{-1} \left(I + \int_{(0,t]} \mathcal{E}_{s-}^1 d\mathcal{X}_s^1 \right) K \right) \\ &= \overleftarrow{\mathcal{E}}(\mathcal{X}^3)_t^{-1} \left(\int_{(0,t]} \mathcal{E}_{s-}^3 d\mathcal{Y}_s^2 + \int_{(0,t]} \mathcal{E}_{s-}^2 d\mathcal{X}_s^1 K - \mathcal{E}_t^2 \overleftarrow{\mathcal{E}}(\mathcal{X}^1)_t^{-1} \overleftarrow{\mathcal{E}}(\mathcal{X}^1)_t K \right) \\ &= \overleftarrow{\mathcal{E}}(\mathcal{X}^3)_t^{-1} \left(\int_{(0,t]} \mathcal{E}_{s-}^3 d\mathcal{Y}_s^2 + \int_{(0,t]} \mathcal{E}_{s-}^2 d\mathcal{X}_s^1 K - \left(\int_{0,t]} \mathcal{E}_{s-}^2 d\mathcal{X}_s^1 + \int_{(0,t]} \mathcal{E}_{s-}^3 d\mathcal{X}_s^2 \right) K \right) \\ &= \overleftarrow{\mathcal{E}}(\mathcal{X}^3)_t^{-1} \int_{(0,t]} \overleftarrow{\mathcal{E}}(\mathcal{X}^3)_{s-} d(\mathcal{Y}_s^2 - \mathcal{X}_s^2 K) \quad \text{a.s., } t \geq 0, \end{aligned}$$

such that (4.4) is shown.

Finally let $(U_t, L_t)_{t\geq 0}$ be the Lévy process defined in (2.9) and (3.5). By Theorem 3.4, $(V_t)_{t\geq 0}$ solves the SDE (3.4) with respect to $\mathbb{F}$. Observe that by the same argumentation as for X or alternatively by (2.9) we deduce that for all $t \geq 0$ it holds $U_t^{(i,j)} = 0$ a.s. for $i \leq d, j > d$. By inserting U and L as given in (4.5) in the SDE (3.4) we obtain $\mathcal{L}^1 = -\mathcal{U}^1 K$ in the first and (4.6) in the second component. This completes the proof.

(b) Inserting (4.2) and (4.3) in (3.3) directly gives the assumption by calculations similar as under (a). □

Proof of Corollary 4.5. By Remark 4.4, it only remains to show the equivalence of (4.7) and (4.8). For that, observe that $\overrightarrow{\mathcal{E}}(U)_t = \overleftarrow{\mathcal{E}}(X)_t^{-1}$ and thus $\overrightarrow{\mathcal{E}}(OUO^{-1})_t = \overleftarrow{\mathcal{E}}(OXO^{-1})_t^{-1}$. Hence, as shown in the proof of Theorem 4.3, OXO^{-1} has a $d\times(m-d)$ block of zero entries in the upper right corner, if and only if the same is true for $\overleftarrow{\mathcal{E}}(OXO^{-1})$, equivalently for $\overleftarrow{\mathcal{E}}(OXO^{-1})^{-1}$, and hence equivalently for $OUO^{-1} = \overrightarrow{\mathrm{Log}}\,(\overleftarrow{\mathcal{E}}(OXO^{-1})^{-1})$. It follows that OX_tO^{-1} is of the form as specified in (4.7) if and only if OU_tO^{-1} is of the form specified in (4.8), and as seen in the proof of Theorem 4.3, it further holds that $\overrightarrow{\mathcal{E}}(\mathcal{U}^1)_t = \overleftarrow{\mathcal{E}}(\mathcal{X}^1)_t^{-1}$. To see the equivalence of the relations regarding OY_t and OL_t, suppose first that Y_t satisfies (4.7). Then by (3.8),

$$OL_t = OY_t + [OUO^{-1}, OY]_t = \begin{pmatrix} \mathcal{X}_t^1 K \\ \mathcal{Y}_t^2 \end{pmatrix} + \left[\begin{pmatrix} \mathcal{U}^1 & 0 \\ \mathcal{U}^2 & \mathcal{U}^3 \end{pmatrix}, \begin{pmatrix} \mathcal{X}^1 K \\ \mathcal{Y}^2 \end{pmatrix} \right]_t,$$

and the upper d components on the right hand side of this equation are given by

$$\mathcal{X}_t^1 K + [\mathcal{U}^1, \mathcal{X}^1 K]_t = -\mathcal{U}_t^1 K,$$

where the last equation follows from (2.10) since $\overrightarrow{\mathcal{E}}(\mathcal{U}^1)_t = \overleftarrow{\mathcal{E}}(\mathcal{X}^1)_t^{-1}$. It follows that L_t satisfies (4.8). Conversely, if (4.8) holds, then it follows from (3.9) that

$$OY_t = OL_t + [OXO^{-1}, OL]_t = \begin{pmatrix} -\mathcal{U}_t^1 K \\ \mathcal{L}_t^2 \end{pmatrix} + \left[\begin{pmatrix} \mathcal{X}^1 & 0 \\ \mathcal{X}^2 & \mathcal{X}^3 \end{pmatrix}, \begin{pmatrix} -\mathcal{U}^1 K \\ \mathcal{L}^2 \end{pmatrix} \right]_t,$$

and as above it follows from this equation and (2.10) that Y_t satisfies (4.7). □

8 Proofs for Section 5

In this section we give the proofs for Section 5 along with a few results on multivariate stochastic exponentials which will be needed but are also interesting in their own right. We start by introducing an approximation of the stochastic exponential which will be a useful tool. Namely, the following result is due to Emery [8].

Lemma 8.1. *Let $\sigma = (t_0 = 0, t_1, \ldots, t_j, \ldots)$ with $t_j \to \infty$ and $|\sigma| := \sup_{j\in\mathbb{N}} |t_j - t_{j-1}| < \infty$ be a subdivision of the positive real line. Let X be a Lévy process in $\mathbb{R}^{m\times m}$. Then the processes $\overleftarrow{\mathcal{E}}(X)^\sigma$ given by $\overleftarrow{\mathcal{E}}(X)^\sigma_0 := I$ and*

$$\overleftarrow{\mathcal{E}}(X)^\sigma_t := (I + X_{t_1})(I + X_{t_2} - X_{t_1}) \cdots (I + X_{t_j} - X_{t_{j-1}})(I + X_t - X_{t_j}) \tag{8.5}$$

for $t_j < t \le t_{j+1}$ converge to $\overleftarrow{\mathcal{E}}(X)$ uniformly on compacts in probability when $|\sigma|$ tends to 0. Similarly, by (2.6) it follows that the approximating processes $\overrightarrow{\mathcal{E}}(X)^\sigma$ with $\overrightarrow{\mathcal{E}}(X)^\sigma_0 := I$ and

$$\overrightarrow{\mathcal{E}}(X)^\sigma_t = (I + X_t - X_{t_j})(I + X_{t_j} - X_{t_{j-1}}) \cdots (I + X_{t_2} - X_{t_1})(I + X_{t_1}) \tag{8.6}$$

for $t_j < t \le t_{j+1}$ converge to $\overrightarrow{\mathcal{E}}(X)$ uniformly on compacts in probability when $|\sigma|$ tends to 0.

Now we can easily prove Lemma 5.1.

Proof of Lemma 5.1. Fix $t > 0$ and for $n \in \mathbb{N}$ let $\sigma = (0, t/n, 2t/n, \ldots)$ be a subdivision of the positive real line. Then the approximations of the left and right stochastic exponential as defined in (8.5) and (8.6) are given by

$$\begin{aligned}
\overleftarrow{\mathcal{E}}(X)^\sigma_t &= (I + X_{t/n})(I + X_{2t/n} - X_{t/n}) \cdots \\
&\quad \cdots (I + X_{(n-1)t/n} - X_{(n-2)t/n})(I + X_t - X_{(n-1)t/n}) \quad \text{and} \\
\overrightarrow{\mathcal{E}}(X)^\sigma_t &= (I + X_t - X_{(n-1)t/n})(I + X_{(n-1)t/n} - X_{(n-2)t/n}) \cdots \\
&\quad \cdots (I + X_{2t/n} - X_{t/n})(I + X_{t/n}).
\end{aligned}$$

Since X is a Lévy process it has stationary and independent increments, so that $\overleftarrow{\mathcal{E}}(X)^\sigma_t \overset{d}{=} \overrightarrow{\mathcal{E}}(X)^\sigma_t$. Letting n tend to infinity yields the assumption by Lemma 8.1. □

Apart from transposition and inversion, another connection between left and right Lévy processes in $\mathrm{GL}(\mathbb{R}, m)$ is given via time reversal, which is treated in (8.7) of the following lemma. Observe that $\tilde{X}$ defined below has the same law as $(-X_s)_{0\le s\le t}$.

Lemma 8.2. *Let $t > 0$ be fixed and suppose $(X_s)_{s\ge 0}$ is a Lévy process in $\mathbb{R}^{m\times m}$. Define the time reversed process $\tilde{X} = (\tilde{X}_s)_{0\le s\le t}$ by $\tilde{X}_s := X_{(t-s)-} - X_{t-}$. Then*

$$\overrightarrow{\mathcal{E}}(X)_t \overrightarrow{\mathcal{E}}(X)^{-1}_{(t-s)-} = \overleftarrow{\mathcal{E}}(-\tilde{X})_s \quad \text{a.s. for all} \quad 0 \le s < t, \tag{8.7}$$

and

$$\overrightarrow{\mathcal{E}}(X)_{(t+s)-} \overrightarrow{\mathcal{E}}(X)^{-1}_t = \overrightarrow{\mathcal{E}}(X_{t+\cdot} - X_t)_{s-} \quad \text{a.s. for all} \quad s > 0. \tag{8.8}$$

Proof. Due to similarity we only prove (8.7). For notational simplicity assume $t = 1$. Let $\sigma = (s_0 = 0, s_1 = 1/n, s_2 = 2/n, \ldots)$, $n \in \mathbb{N}$, be a partition of the positive real line. Then for any $i = 0, \dots, n-1$ we have a.s. by (8.5)

$$\begin{aligned}
\overleftarrow{\mathcal{E}}(-\tilde{X})^{\sigma}_{1-s_i} &= (I - \tilde{X}_{s_1}) \cdots (I - \tilde{X}_{s_{n-i}} + \tilde{X}_{s_{n-i-1}}) \\
&= (I - \tilde{X}_{s_1-}) \cdots (I - \tilde{X}_{s_{n-i}-} + \tilde{X}_{s_{n-i-1}-}) \\
&= (I + X_{s_n} - X_{s_{n-1}}) \cdots (I + X_{s_{i+1}} - X_{s_i}) \\
&= (I + X_{s_n} - X_{s_{n-1}}) \cdots (I + X_{s_{i+1}} - X_{s_i}) \\
&\quad (I + X_{s_i} - X_{s_{i-1}}) \cdots (I + X_{s_1})(I + X_{s_1})^{-1} \cdots (I + X_{s_i} - X_{s_{i-1}})^{-1} \\
&= \overrightarrow{\mathcal{E}}(X)^{\sigma}_1 (\overrightarrow{\mathcal{E}}(X)^{\sigma}_{s_i})^{-1}
\end{aligned}$$

where we have used the fact that at fixed time s_i the process X and thus $\tilde{X}$ a.s. does not jump. Hence we have established that $\overleftarrow{\mathcal{E}}(-\tilde{X})^{\sigma}_{1-s} = \overrightarrow{\mathcal{E}}(X)^{\sigma}_1(\overrightarrow{\mathcal{E}}(X)^{\sigma}_s)^{-1}$ a.s. holds for all $s \in \{0, 1/n, 2/n, \dots, (n-1)/n\}$. Letting n tend to infinity gives us $\overleftarrow{\mathcal{E}}(-\tilde{X})_{1-s} = \overrightarrow{\mathcal{E}}(X)_1(\overrightarrow{\mathcal{E}}(X)_{s-})^{-1}$ a.s. for all $s \in \mathbb{Q} \cap [0, 1)$. Finally the fact that left and right exponential as multiplicative Lévy processes have càdlàg paths yields the assumption. $\square$

With the aid of Lemma 8.2 we can prove the following proposition which in turn will be needed to prove Theorem 5.2. It generalizes Proposition 2.3 in [18] and its extension Lemma 3.1 in [3] to a multivariate setting. Remark the switch of direction of the exponential in the distributional equality which results from a time change.

Proposition 8.3. *Suppose $(X_t, Y_t)_{t \geq 0}$ is a Lévy process in $\mathbb{R}^{m \times m} \times \mathbb{R}^m$ such that X satisfies* (2.7) *and define the process $(U_t, L_t)_{t \geq 0}$ by* (2.9) *and* (3.5)*. Then it holds for each $t > 0$*

$$\overrightarrow{\mathcal{E}}(U)_t \int_{(0,t]} \overrightarrow{\mathcal{E}}(U)^{-1}_{s-} dY_s \overset{d}{=} \int_{(0,t]} \overleftarrow{\mathcal{E}}(U)_{s-} dY_s + \left[\overleftarrow{\mathcal{E}}(U), Y\right]_t = \int_{(0,t]} \overleftarrow{\mathcal{E}}(U)_{s-} dL_s \quad a.s. \quad (8.9)$$

and analogously

$$\overrightarrow{\mathcal{E}}(X)_t \int_{(0,t]} \overrightarrow{\mathcal{E}}(X)^{-1}_{s-} dL_s \overset{d}{=} \int_{(0,t]} \overleftarrow{\mathcal{E}}(X)_{s-} dL_s + \left[\overleftarrow{\mathcal{E}}(X), L\right]_t = \int_{(0,t]} \overleftarrow{\mathcal{E}}(X)_{s-} dY_s \quad a.s.. \quad (8.10)$$

Proof. The almost sure equalities in (8.9) and (8.10) follow directly from (3.8) and (3.9), respectively, under use of (2.1) and (2.4), while the distributional equalities will be shown following the proof of [21, Theorem VI.22]. Due to similarity we restrict on showing (8.9). Fix $t > 0$ and define for $0 \leq s \leq t$

$$\hat{U}_s := U_t - U_{(t-s)-} \quad \text{and} \quad \hat{Y}_s := Y_t - Y_{(t-s)-}.$$

For $n \in \mathbb{N}$ let $\sigma = (0, t/n, 2t/n, \dots)$ be a partition of the positive real line, set

$$H_s := \overleftarrow{\mathcal{E}}(\hat{U})_s \quad \text{and} \quad G_s := \hat{Y}_s, \quad 0 \leq s \leq t,$$

and define the additional random variables

$$\begin{aligned} A^\sigma &:= \sum_{i=0}^{n-1} H_{t(i+1)/n}(G_{t(i+1)/n} - G_{ti/n}) \\ &= \sum_{i=0}^{n-1} H_{ti/n}(G_{t(i+1)/n} - G_{ti/n}) + \sum_{i=0}^{n-1}(H_{t(i+1)/n} - H_{ti/n})(G_{t(i+1)/n} - G_{ti/n}) \\ B^\sigma &:= -\sum_{i=0}^{n-1} H_{t(i+1)/n-}(G_{t(i+1)/n-} - G_{ti/n-}). \end{aligned}$$

Since integral and quadratic variation are defined component-by-component, letting $|\sigma|$ tend to zero, we obtain by [21, Theorems II.21 and II.23]

$$\begin{aligned} A^\sigma \overset{P}{\longrightarrow} & \int_{(0,t]} H_{s-} dG_s + [H,G]_t = \int_{(0,t]} \overleftarrow{\mathcal{E}}(\hat{U})_{s-} d\hat{Y}_s + [\overleftarrow{\mathcal{E}}(\hat{U}), \hat{Y}]_t \\ & \overset{d}{=} \int_{(0,t]} \overleftarrow{\mathcal{E}}(U)_{s-} dY_s + [\overleftarrow{\mathcal{E}}(U), Y]_t, \end{aligned}$$

where the last equality follows from the fact that $(\hat{U}_s, \hat{Y}_s)_{0\le s\le t} \overset{d}{=} (U_s, Y_s)_{0\le s\le t}$ which yields $(\overleftarrow{\mathcal{E}}(\hat{U})_s, \hat{Y}_s)_{0\le s\le t} \overset{d}{=} (\overleftarrow{\mathcal{E}}(U)_s, Y_s)_{0\le s\le t}$. On the other hand remark that by definition $G_{t(i+1)/n-} - G_{ti/n-} = Y_{t(n-i)/n} - Y_{t(n-i-1)/n}$ for $i \in \{0, \dots, n-1\}$ and since by (8.7) we have for $0 < s \le t$ that $H_{s-} = \overleftarrow{\mathcal{E}}(-\tilde{U})_{s-} = \overrightarrow{\mathcal{E}}(U)_t \overrightarrow{\mathcal{E}}(U)^{-1}_{t-s}$, it holds

$$\begin{aligned} B^\sigma &= -\sum_{i=0}^{n-1} \overrightarrow{\mathcal{E}}(U)_t \overrightarrow{\mathcal{E}}(U)^{-1}_{t(n-i-1)/n}(Y_{t(n-i)/n} - Y_{t(n-i-1)/n}) \\ &= -\overrightarrow{\mathcal{E}}(U)_t \sum_{i=1}^{n} \overrightarrow{\mathcal{E}}(U)^{-1}_{t(i-1)/n}(Y_{ti/n} - Y_{t(i-1)/n}) \\ &\overset{P}{\to} -\overrightarrow{\mathcal{E}}(U)_t \int_{(0,t]} \overrightarrow{\mathcal{E}}(U)^{-1}_{s-} dY_s, \quad |\sigma| \to 0. \end{aligned}$$

A combination of A^σ and B^σ gives

$$\begin{aligned} & A^\sigma + B^\sigma \\ &= \sum_{i=0}^{n-1} H_{t(i+1)/n}(\Delta G_{t(i+1)/n} - \Delta G_{ti/n}) + \sum_{i=0}^{n-1} \Delta H_{t(i+1)/n}(G_{t(i+1)/n-} - G_{ti/n-}) \\ &= 0 \quad \text{a.s.} \end{aligned}$$

since at fixed times G and H a.s. do not jump. Hence the limits of A^σ and B^σ add to zero which gives the assumption. □

With the above proposition at hand we can now prove the conditions for strict stationarity of MGOU processes stated in Theorem 5.2.

Proof of Theorem 5.2. (a) Assume that $\lim_{t\to\infty} \overleftarrow{\mathcal{E}}(U)_t = 0$ in probability and suppose that $(V_t)_{t\geq 0}$ is strictly stationary. Then by (5.1) we have that $\lim_{t\to\infty} \overrightarrow{\mathcal{E}}(U)_t = 0$ in probability and obtain

$$V_0 \stackrel{d}{=} \text{d-}\lim_{t\to\infty} V_t = \text{d-}\lim_{t\to\infty} \left(\overrightarrow{\mathcal{E}}(U)_t V_0 + \overrightarrow{\mathcal{E}}(U)_t \int_{(0,t]} \overrightarrow{\mathcal{E}}(U)_{s-}^{-1} dY_s \right).$$

Thus by (8.9) we conclude that $\int_{(0,t]} \overleftarrow{\mathcal{E}}(U)_{s-} dL_s \stackrel{d}{=} \overrightarrow{\mathcal{E}}(U)_t \int_{(0,t]} \overrightarrow{\mathcal{E}}(U)_{s-}^{-1} dY_s$ tends to V_0 in distribution as stated.
Conversely, assume that $\lim_{t\to\infty} \overrightarrow{\mathcal{E}}(U)_t = 0$ in probability and $\int_{(0,\infty)} \overleftarrow{\mathcal{E}}(U)_{s-} dL_s$ converges in distribution and set V_0 independent of $(U_t, L_t)_{t\geq 0}$ such that $V_0 \stackrel{d}{=} \text{d}-\lim_{t\to\infty} \int_{(0,t]} \overleftarrow{\mathcal{E}}(U)_{s-} dL_s$. Then by (8.9), letting t tend to infinity, V_t converges in distribution to V_0. Since $(V_t)_{t\geq 0}$ satisfies (1.2) with $(A_{t,t+h}, B_{t,t+h})$ independent of V_t this yields for all $h > 0$

$$V_0 \stackrel{d}{=} \text{d-}\lim_{t\to\infty} V_{t+h} = \text{d-}\lim_{t\to\infty} A_{t,t+h} V_t + B_{t,t+h} \stackrel{d}{=} A_{0,h} V_0 + B_{0,h} = V_h$$

such that $(V_t)_{t\geq 0}$ is strictly stationary, since it is a time homogeneous Markov process.

For (b) suppose that $\lim_{t\to\infty} \overleftarrow{\mathcal{E}}(X)_t = 0$ in probability and that $(V_t)_{t\geq 0}$ is strictly stationary. Then we have that $V_0 + \int_{(0,t]} \overleftarrow{\mathcal{E}}(X)_{s-} dY_s = \overleftarrow{\mathcal{E}}(X)_t V_t \to 0$ in probability as t tends to infinity. Hence $V_0 = \text{P-}\lim_{t\to\infty}(-\int_{(0,t]} \overleftarrow{\mathcal{E}}(X)_{s-} dY_s)$ showing one direction of (b).
Conversely, setting $V_0 = -\int_{(0,\infty)} \overleftarrow{\mathcal{E}}(X)_{s-} dY_s$ yields directly that

$$V_t = -\overleftarrow{\mathcal{E}}(X)_t^{-1} \int_{(t,\infty)} \overleftarrow{\mathcal{E}}(X)_{s-} dY_s = -\int_{(t,\infty)} \overrightarrow{\mathcal{E}}(U)_t \overrightarrow{\mathcal{E}}(U)_{s-}^{-1} dY_s$$

and hence by applying the inverse of (8.8) we observe that for any $t \geq 0$ it holds

$$V_t = -\int_{(0,\infty)} \overrightarrow{\mathcal{E}}(U_{t+\cdot} - U_t)_{s-}^{-1} d(Y_{t+s} - Y_t) \stackrel{d}{=} -\int_{(0,\infty)} \overrightarrow{\mathcal{E}}(U)_{s-}^{-1} dY_s = V_0.$$

Thus for any $t \geq 0$, $n \in \mathbb{N}$ and $0 \leq h_1 \leq \ldots \leq h_n$ we obtain from (1.2) with $(A_{t,t+h}^{-1}, A_{t,t+h}^{-1} B_{t,t+h})$ independent of V_{t+h} that

$$\begin{aligned}
&(V_t, V_{t+h_1}, \ldots, V_{t+h_n}) \\
&= (A_{t,t+h_n}^{-1}(V_{t+h_n} - B_{t,t+h_n}), A_{t+h_1,t+h_n}^{-1}(V_{t+h_n} - B_{t+h_1,t+h_n}), \ldots, V_{t+h_n}) \\
&\stackrel{d}{=} (A_{0,h_n}^{-1}(V_{h_n} - B_{0,h_n}), A_{h_1,h_n}^{-1}(V_{h_n} - B_{h_1,h_n}), \ldots, V_{h_n}) \\
&= (V_0, V_{h_1}, \ldots, V_{h_n})
\end{aligned}$$

such that $(V_t)_{t\geq 0}$ is strictly stationary. □

Proof of Theorem 5.3. (a) In view of Theorem 5.2 it remains to show that the existence of a strictly stationary and causal solution $(V_t)_{t\geq 0}$ implies $\text{P-}\lim_{t\to\infty} \overleftarrow{\mathcal{E}}(U)_t = 0$. For this, observe that by Proposition 4.2 there is some $h > 0$ such that the corresponding discrete time model $V_{nh} = \bar{A}_{n,h}V_{(n-1)h} + \bar{B}_{n,h}$, $n \in \mathbb{N}$, where $\bar{A}_{n,h} := A_{(n-1)h,nh} = \overrightarrow{\mathcal{E}}(U)_{nh}\overrightarrow{\mathcal{E}}(U)^{-1}_{(n-1)h}$ and $\bar{B}_{n,h} := B_{(n-1)h,nh}$, is irreducible. Since $(\bar{A}_{n,h}, \bar{B}_{n,h}, V_{(n-1)h})_{n\in\mathbb{N}}$ is strictly stationary, we can extend it to a new stationary process $(\bar{A}_{n,h}, \bar{B}_{n,h}, V_{(n-1)h})_{n\in\mathbb{Z}}$ and observe that $(V_{nh})_{n\in\mathbb{Z}}$ is a strictly stationary, causal solution of the irreducible autoregressive model $V_{nh} = \bar{A}_{n,h}V_{(n-1)h} + \bar{B}_{n,h}$, $n \in \mathbb{Z}$. Thus by Bougerol and Picard [4, Theorem 2.4] we have that a.s. the product $\bar{A}_{0,h}\bar{A}_{-1,h}\cdots\bar{A}_{-k,h}$ converges to 0 as $k \to \infty$. By the stationarity of $(\bar{A}_{n,h})_{n\in\mathbb{Z}}$ this yields that the product $\bar{A}_{k,h}\bar{A}_{k-1,h}\cdots\bar{A}_{1,h}$ tends to 0 in probability as $k \to \infty$ which is equivalent to $\text{P-}\lim_{n\to\infty}\overrightarrow{\mathcal{E}}(U)_{nh} = 0$ and by (5.1) also to $\text{P-}\lim_{n\to\infty}\overleftarrow{\mathcal{E}}(U)_{nh} = 0$. Denote by $\|\cdot\|$ some submultiplicative matrix norm and by $\lfloor x \rfloor$ for $x \in \mathbb{R}$ the largest integer which is smaller than or equal to x. Then $\text{P-}\lim_{n\to\infty}\overleftarrow{\mathcal{E}}(U)_{nh} = 0$ together with

$$\|\overleftarrow{\mathcal{E}}(U)_t\| \leq \|\overleftarrow{\mathcal{E}}(U)_{\lfloor t/h\rfloor h}\| \sup_{\lfloor t/h\rfloor h \leq s < (\lfloor t/h\rfloor+1)h} \|\overleftarrow{\mathcal{E}}(U)^{-1}_{\lfloor t/h\rfloor h}\overleftarrow{\mathcal{E}}(U)_s\|$$

and

$$\sup_{\lfloor t/h\rfloor h \leq s < (\lfloor t/h\rfloor+1)h} \|\overleftarrow{\mathcal{E}}(U)^{-1}_{\lfloor t/h\rfloor h}\overleftarrow{\mathcal{E}}(U)_s\| \overset{d}{=} \sup_{s\in[0,h]} \|\overleftarrow{\mathcal{E}}(U)_s\|$$

for $t > 0$ imply that $\text{P-}\lim_{t\to\infty}\overleftarrow{\mathcal{E}}(U)_t = 0$ by Slutsky's lemma as had to be shown.
(b) As above and with the same notations, in view of Theorem 5.2 and Proposition 4.2 we need to prove $\text{P-}\lim_{t\to\infty}\overleftarrow{\mathcal{E}}(X)_t = 0$ given the irreducibility of the underlying discrete model $V_{nh} = \bar{A}_{n,h}V_{(n-1)h} + \bar{B}_{n,h}$, $n \in \mathbb{N}$, for some $h > 0$ fixed and provided that $(V_t)_{t\geq 0}$ is strictly stationary and strictly non-causal. It can be easily seen that $(\bar{A}_{n,h}, \bar{B}_{n,h}, V_{(n-1)h})_{n\in\mathbb{N}}$ is strictly stationary and thus can again be extended to a strictly stationary process $(\bar{A}_{n,h}, \bar{B}_{n,h}, V_{(n-1)h})_{n\in\mathbb{Z}}$ where by the provided strict non-causality V_{nh} is independent of $(\bar{A}_{k,h}, \bar{B}_{k,h})_{k\leq n}$. Defining the process $(C_{n,h}, D_{n,h}, W_{nh})_{n\in\mathbb{Z}}$ by $C_{n,h} := \bar{A}^{-1}_{-n,h}$, $D_{n,h} := -\bar{A}^{-1}_{-n,h}\bar{B}_{-n,h}$ and $W_{nh} := V_{-nh}$ we see that it is strictly stationary and obtain that W_{nh} fulfills the autoregressive model

$$W_{(n+1)h} = C_{n,h}W_{nh} + D_{n,h}, \quad n \in \mathbb{Z}, \tag{8.11}$$

where W_{nh} is independent of $(C_{k,h}, D_{k,h})_{k\geq n}$ and hence it is causal. The model (8.11) is irreducible since any invariant affine subspace of the model (8.11) is also an invariant affine subspace of the initial model $V_{nh} = \bar{A}_{n,h}V_{(n-1)h} + \bar{B}_{n,h}$, $n \in \mathbb{N}$; namely, suppose there exists an invariant affine subspace H of (8.11) then we have a.s. $\bar{A}^{-1}_{-n,h}H - \bar{A}^{-1}_{-n,h}\bar{B}_{-n,h} = H$ since the mapping $x \mapsto \bar{A}^{-1}_{-n,h}x - \bar{A}^{-1}_{-n,h}\bar{B}_{-n,h}$ is bijective. Thus it follows $\bar{A}_{-n,h}H + \bar{B}_{-n,h} = H$ such that H is invariant under the initial model. Hence we can again apply [4, Theorem 2.4] and an argumentation as under (a) yields the result. □

For the proof of Theorem 5.4 we need some further preparations.

Proposition 8.4. *Let $(X_t)_{t\geq 0}$ be a Lévy process in $\mathbb{R}^{m\times m}$ such that $E[\log^+\|X_1\|] < \infty$. Then*

$$E[\sup_{0\leq s\leq t}\log^+\|\overleftarrow{\mathcal{E}}(X)_s\|] < \infty \quad \textit{and} \quad E[\sup_{0\leq s\leq t}\log^+\|\overrightarrow{\mathcal{E}}(X)_s\|] < \infty] \quad \textit{for all} \quad t \geq 0. \tag{8.12}$$

Proof. Due to similarity we will only treat left exponentials in this proof and for simplicity we fix $t = 1$.

Define the Lévy processes $(X_t^\flat)_{t\geq 0}$ and $(X_t^\sharp)_{t\geq 0}$ such that $X_t = X_t^\flat + X_t^\sharp$ with $\|\Delta X_t^\flat\| \leq 1/2$, $t \geq 0$ and $(X_t^\sharp = \sum_{k=1}^{N_t} Y_k)_{t\geq 0}$ being a compound Poisson process with parameter $\lambda > 0$, jump times T_i, $i \in \mathbb{N}$ and jump heights Y_i, $i \in \mathbb{N}$ such that $\|Y_i\| > 1/2$ for all $i \in \mathbb{N}$. Then $X^\flat$ satisfies (2.7). Define $U^\flat$ corresponding to $X^\flat$ by (2.9), i.e. such that $\overrightarrow{\mathcal{E}}(U^\flat) = \overleftarrow{\mathcal{E}}(X^\flat)^{-1}$. Then both $X^\flat$ and $U^\flat$ have bounded jumps.

It is an easy consequence of the definition of the stochastic exponential (2.4) that

$$\overleftarrow{\mathcal{E}}(X)_t = \left(\prod_{k=1}^{N_t} \overleftarrow{\mathcal{E}}(X^\flat)_{(T_{k-1},T_k]}(I + Y_k)\right) \overleftarrow{\mathcal{E}}(X^\flat)_{(T_{N_t},t]}, \quad t \geq 0,$$

where

$$\overleftarrow{\mathcal{E}}(X^\flat)_{(s,t]} := \overleftarrow{\mathcal{E}}(X^\flat)_s^{-1}\overleftarrow{\mathcal{E}}(X^\flat)_t = \overrightarrow{\mathcal{E}}(U^\flat)_s\overleftarrow{\mathcal{E}}(X^\flat)_t, \quad 0 \leq s \leq t.$$

Taking norms then implies that for each $t \in [0,1]$ (observe that $\|I\| \geq 1$),

$$\|\overleftarrow{\mathcal{E}}(X)_t\| \leq \left(\sup_{0\leq s\leq 1} \|\overrightarrow{\mathcal{E}}(U^\flat)_s\|\right)^{N_1+1} \left(\sup_{0\leq s\leq 1} \|\overleftarrow{\mathcal{E}}(X^\flat)_s\|\right)^{N_1+1} \prod_{k=1}^{N_1} (\|I\| + \|Y_k\|)\,.$$

Using the independence of N_1, $(Y_k)_{k\in\mathbb{N}}$ and $(X^\flat, U^\flat)$ then shows that

$$\begin{aligned} E[\log^+ \sup_{0\leq t\leq 1} \|\overleftarrow{\mathcal{E}}(X)_t\|] \;\leq\; & E(N_1+1)\left(E[\log^+ \sup_{0\leq s\leq 1} \|\overrightarrow{\mathcal{E}}(U^\flat)_s\|] + E[\log^+ \sup_{0\leq s\leq 1} \|\overleftarrow{\mathcal{E}}(X^\flat)_s\|]\right) \\ & + E\sum_{k=1}^{N_1} \log(\|I\| + \|Y_k\|). \end{aligned} \tag{8.13}$$

Since $X^\flat$ and $U^\flat$ have bounded jumps and hence finite second moment, it follows from Jacod et al. [14, Proposition 5.2(a)] that $\sup_{0\leq s\leq 1} \|\overrightarrow{\mathcal{E}}(U^\flat)_s\|$ and $\sup_{0\leq s\leq 1} \|\overleftarrow{\mathcal{E}}(X^\flat)_s\|$ have finite second moment and in particular finite log-moment. Since $EN_1 < \infty$ and $E\log^+ \|Y_k\| < \infty$ by assumption, an application of Wald's identity shows that the right-hand side of (8.13) is finite, which is the claim. □

Lemma 8.5. *Suppose $(X_t)_{t\geq 0}$ is a Lévy process in $\mathbb{R}^{m\times m}$ satisfying $E[\log^+ \|X_1\|] < \infty$ and (2.7) and assume there exists $t > 0$ such that*

$$E[\log^+ \|\overrightarrow{\mathcal{E}}(X)_t\|] < 0. \tag{8.14}$$

Then there exists a constant $\lambda > 0$ and an a.s. finite random time τ such that

$$\|\overrightarrow{\mathcal{E}}(X)_s\| \leq e^{-\lambda s}, \quad \textit{for all} \quad s \geq \tau. \tag{8.15}$$

In particular there exists an a.s. finite random variable C such that

$$\|\overrightarrow{\mathcal{E}}(X)_s\| \leq Ce^{-\lambda s}, \quad \textit{for all} \quad s \geq 0. \tag{8.16}$$

The above remains true if all right exponentials are replaced by left exponentials.

Proof. For simplicity we assume again that $t = 1$ and due to similarity we only prove the result for one type of stochastic exponentials, this time for right exponentials. Denote $E_n := \overrightarrow{\mathcal{E}}(X)_{(n-1,n]} = \overrightarrow{\mathcal{E}}(X)_n \overrightarrow{\mathcal{E}}(X)_{n-1}^{-1}$ for $n \in \mathbb{N}$, then $(E_n)_{n\in\mathbb{N}}$ is an i.i.d. sequence of random matrices. By (8.14) the top Lyapunov exponent γ of the sequence $(E_n)_{n\in\mathbb{N}}$ (cf. (5.2)) is strictly negative. By (5.3) this implies that

$$\lim_{n\to\infty} n^{-1} \log \|\overrightarrow{\mathcal{E}}(X)_n\| = \lim_{n\to\infty} n^{-1} \log \|E_n \cdots E_1\| = \gamma < 0 \quad \text{a.s.}$$

and thus for $\lambda' > 0$ such that $\lambda' < |\gamma|$ there exists a random time τ' such that

$$\|\overrightarrow{\mathcal{E}}(X)_n\| \leq e^{-\lambda' n}, \quad \text{for all} \quad n \geq \tau',\, n \in \mathbb{N}. \tag{8.17}$$

Define $F_n := \sup_{s\in(n,n+1]} \|\overrightarrow{\mathcal{E}}(X)_{(n,s]}\|$, $n \in \mathbb{N}_0$, where $\overrightarrow{\mathcal{E}}(X)_{(n,s]} = \overrightarrow{\mathcal{E}}(X)_s \overrightarrow{\mathcal{E}}(X)_n^{-1}$, then the sequence $(F_n)_{n\in\mathbb{N}_0}$ is i.i.d. and by Proposition 8.4 it holds $E[\log^+ F_1] < \infty$. Hence we conclude for $0 < \lambda'' < \lambda'$

$$\sum_{n=1}^{\infty} P(F_n > e^{\lambda'' n}) = \sum_{n=1}^{\infty} P(\log^+ F_1 > \lambda'' n) < \infty$$

and thus by the Borel-Cantelli Lemma we obtain that $P(\limsup_{n\to\infty}\{F_n > e^{\lambda'' n}\}) = 0$. Consequently there exists a random time τ'' such that

$$F_n \leq e^{\lambda'' n} \quad \text{for all} \quad n \geq \tau''.$$

Together with (8.17) this gives

$$\|\overrightarrow{\mathcal{E}}(X)_s\| = \|\overrightarrow{\mathcal{E}}(X)_{(\lfloor s\rfloor,s]} \overrightarrow{\mathcal{E}}(X)_{\lfloor s\rfloor}\| \leq e^{\lambda''\lfloor s\rfloor} e^{-\lambda'\lfloor s\rfloor} = e^{-(\lambda'-\lambda'')\lfloor s\rfloor} \quad \text{for all} \quad \lfloor s\rfloor \geq \max\{\tau', \tau''\}$$

and hence (8.15) and (8.16). □

Proof of Theorem 5.4. The inclusion (iii)⇒(ii) is clear and (ii)⇒(i) is Theorem 5.2(a).

To prove (iv)⇒(iii), observe that since $\overleftarrow{\mathcal{E}}(X)_t \to 0$ a.s. as $t \to \infty$ by Lemma 8.5, all what remains to show is that for all $i, j = 1, \ldots, m$ the integral $\int_{(0,t]} \overleftarrow{\mathcal{E}}(X)_{s-}^{(i,j)} dL_s^{(j)}$ converges a.s. as $t \to \infty$. Therefore observe that again by Lemma 8.5 there exists a random time τ and a constant $\lambda > 0$ such that $|\overleftarrow{\mathcal{E}}(X)_{s-}^{(i,j)}| \leq e^{-\lambda s}$ for all $s \geq \tau$. Writing $L_s = L_s^\flat + L_s^\sharp$, where each component of $L^\flat$ is a square integrable martingale with zero mean and $(L^\sharp)^{(j)}$ consists of all jumps of $L^{(j)}$ which have absolute value greater than one, exactly the same reasoning as in the proof of sufficiency of Theorem 2 in [9, pp. 84-85] shows that $\int_{(0,\infty)} \overleftarrow{\mathcal{E}}(X)_{s-}^{(i,j)} d(L^\flat)_s^{(j)}$ converges a.s.
Further, if $M_s^{(j)}$ denotes the total variation of $(L^\sharp)^{(j)}$ over $[0, s]$, then $M^{(j)}$ is a subordinator with finite log-moment and hence $\int_{(0,\infty)} e^{-\lambda s} dM_s^{(j)}$ converges a.s. by [9, Theorem 2]. By (8.16) this implies almost sure convergence of $\int_{(0,\infty)} \overleftarrow{\mathcal{E}}(X)_{s-}^{(i,j)} d(L^\sharp)_s^{(j)}$ and thus finishes the proof of (iv)⇒(iii).

To prove the inclusion (i)⇒(iv), let $(V_t)_{t\geq 0}$ be a strictly stationary causal solution. By [4, Remark 2.8] for every $h > 0$ there exist sequences $(A_{(n-1)h,nh}, B_{(n-1)h,nh})_{-n\in\mathbb{N}_0} \in \mathrm{GL}(\mathbb{R}, m) \times \mathbb{R}^m$ and $(V_{nh})_{-n\in\mathbb{N}} \in \mathbb{R}^m$ such that $(A_{(n-1)h,nh}, B_{(n-1)h,nh})_{n\in\mathbb{Z}}$ is i.i.d and $(V_{nh})_{n\in\mathbb{Z}}$ is a strictly stationary causal solution of

$$V_{nh} = A_{(n-1)h,nh}V_{(n-1)h} + B_{(n-1)h,nh}, \quad \text{for all} \quad n \in \mathbb{Z}. \tag{8.18}$$

By Proposition 4.2 there exists $h > 0$ for which the model (8.18) is irreducible. Hence by [4, Theorem 2.4] the product $A_{0,h}A_{-h,0}\cdots A_{-nh,-(n-1)h}$ converges a.s. to 0 as $n \to \infty$. Thus

$$A^{\perp}_{-nh,-(n-1)h}\cdots A^{\perp}_{-h,0}A^{\perp}_{0,h} \to 0 \quad \text{a.s. as} \quad n \to \infty$$

and by [4, Lemma 3.4] this implies that the top Lyapunov exponent of $(A^{\perp}_{-nh,-(n-1)h})_{n\in\mathbb{N}_0}$ is strictly negative. Since by the equivalence of norms there are constants $c_1, c_2 > 0$ such that $c_1\|D\| \leq \|D^{\perp}\| \leq c_2\|D\|$ for all $D \in \mathbb{R}^{m\times m}$, the top Lyapunov exponent of the sequence $(A_{-nh,-(n-1)h})_{n\in\mathbb{N}_0}$ coincides with that of $(A^{\perp}_{-nh,-(n-1)h})_{n\in\mathbb{N}_0}$ by (5.3). Hence there exists $n_0 \in \mathbb{N}_0$ such that

$$\begin{aligned} 0 &> E\log\|A_{0,h}A_{-h,0}\cdots A_{-(n_0-1)h,-(n_0-2)h}\| \\ &= E\log\|A_{(n_0-1)h,n_0h}\cdots A_{h,2h}A_{0,h}\| \\ &= E\log\|\overrightarrow{\mathcal{E}}(U)_{n_0h}\|. \end{aligned}$$

Together with Lemma 5.1 we obtain (iv).

Now suppose that U is a compound Poisson process with jump heights $(S_k)_{k\in\mathbb{N}}$ such that (v) holds. Then due to the finite log-moment of U we obtain as in the proof of Lemma 8.5 by [11, Theorem 1] that there exist $\lambda > 0$ and a random $K \in \mathbb{N}$ such that

$$\|(I + S_1)(I + S_2)\cdots(I + S_k)\| \leq e^{-\lambda k} \quad \text{for all} \quad k \geq K.$$

Hence there exists a random time τ such that

$$\|\overleftarrow{\mathcal{E}}(U)_t\| = \|(I + S_1)(I + S_2)\cdots(I + S_{N_t})\| \leq e^{-\lambda N_t} \quad \text{for all} \quad t \geq \tau.$$

This implies $\overleftarrow{\mathcal{E}}(U)_t \to 0$ a.s. as $t \to \infty$ and the a.s. convergence of $\int_{(0,\infty)}\overleftarrow{\mathcal{E}}(U)_{s-}dL_s$ as in the proof of (iv)⇒(iii). Hence we get (iii).

Conversely assume that (iii) holds, then $\overleftarrow{\mathcal{E}}(U)_{N_t} \to 0$ a.s. and hence $(I+S_1)(I+S_2)\cdots(I+S_k) \to 0$ a.s. as $k \to \infty$. By [4, Lemma 3.4] this implies that the top Lyapunov exponent of the sequence $((I + S_k)^{\perp})_{k\in\mathbb{N}}$ is strictly negative, which by (5.3) and the equivalence of norms coincides with the top Lyapunov exponent of $(I + S_k)_{k\in\mathbb{N}}$. Hence we get (v). □

Proof of Theorem 5.7. Sufficiency of the given condition as well as the stated form of the distribution follow directly from Theorem 4.3(b) together with Remark 4.4 and Theorem 5.2(a).

To prove necessity assume that $(V_t)_{t\geq 0}$ is strictly stationary and let G be the smallest affine subspace of $\mathbb{R}^m$ with $P(V_0 \in H) = 1$. Since $(V_t)_{t\geq 0}$ is strictly stationary it is clear

that G is invariant under the model (1.2). Let $H \subseteq G$ be an arbitrary invariant affine subspace of minimal dimension. Then the model (1.2) is irreducible on H in the sense that there exists no subspace $F \subsetneq H$ which is invariant under (1.2). Hence by Proposition 4.2 there exists $h_0 > 0$ such that the discrete-time model (4.1) admits no invariant affine subspace $F \subsetneq H$ for any h of the form $h = 2^{-k}h_0$ with $k \in \mathbb{N}_0$.
By [4, Proposition 2.6] the space H carries a causal, strictly stationary solution $(W_n^{(h)})_{n\in\mathbb{N}}$ of the model (4.1) for any such h. Moreover the marginal distribution of this solution is uniquely determined by [4, Theorem 2.4] as the model (4.1) is irreducible on H. Since any strictly stationary solution of the model (4.1) for $h = 2^{-k}h_0$, $k \in \mathbb{N}_0$, is strictly stationary in the model (4.1) for h_0, too, this implies $\mathcal{L}(W_0^{(2^{-k}h_0)}) = \mathcal{L}(W_0^{(h_0)})$ for all $k \in \mathbb{N}_0$. Hence a starting random variable, independent of (X, Y) and with distribution $\mathcal{L}(W_0^{(h_0)})$, yields a strictly stationary solution of the model (4.1) for any h of the form $h = 2^{-k}h_0$ with $k \in \mathbb{N}_0$. Using the fact that MGOU processes have càdlàg paths, the MGOU process $(W_t)_{t\geq 0}$ with $\mathcal{L}(W_0) = \mathcal{L}(W_0^{(h_0)})$ and W_0 independent of (X, Y) is a strictly stationary solution of the continuous-time model (1.2). Finally as the model is irreducible on H, Theorems 4.3(a) and 5.3(a) together with Remark 4.4 provide that the stated condition holds. □

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