Generalized Ornstein-Uhlenbeck Processes and Extensions

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I almost wish I hadn't
gone down that rabbit-hole
- and yet - and yet -
it's rather curious, you know,
this sort of life!

Alice in Wonderland

Abstract

The generalized Ornstein-Uhlenbeck process

$$V_t = e^{-\xi_t}\left(V_0 + \int_0^t e^{\xi_{s-}}\,d\eta_s\right), \quad t \geq 0,$$

driven by a bivariate Lévy process $(\xi_t, \eta_t)_{t\geq 0}$ with starting random variable V_0 independent of (ξ, η) fulfills the stochastic differential equation $dV_t = V_{t-}dU_t + dL_t$ for another bivariate Lévy process $(U_t, L_t)_{t\geq 0}$, which is determined completely by (ξ, η). In particular it holds $\xi_t = -\log(\mathcal{E}(U)_t)$, $t \geq 0$, where $\mathcal{E}(U)$ denotes the stochastic exponential of U.

In Chapter 2 of this work, for a given bivariate Lévy process (U, L), necessary and sufficient conditions for the existence of a strictly stationary solution of the stochastic differential equation $dV_t = V_{t-}\,dU_t + dL_t$ are obtained. Neither strict positivity of the stochastic exponential of U nor independence of V_0 and (U, L) are assumed and noncausal solutions may appear. The form of the stationary solution is determined and shown to be unique in distribution, provided it exists. For non-causal solutions, a sufficient condition for U and L to remain semimartingales with respect to the corresponding expanded filtration is given.

In Chapter 3 distributional properties of the stationary solutions of the stochastic differential equation $dV_t = V_{t-}dU_t + dL_t$ are analysed. In particular the expectation and autocorrelation function are obtained in terms of the process (U, L) and in several cases of interest the tail behaviour is described. In the case where U has jumps of size -1, necessary and sufficient conditions for the law of the solutions to be (absolutely) continuous are given.

It is known that in many cases distributions of exponential integrals of Lévy processes, as they occur as stationary solutions of generalized Ornstein-Uhlenbeck processes, are infinitely divisible and in some cases they are also selfdecomposable. In Chapter 4, we give some sufficient conditions under which distributions of exponential integrals are not only selfdecomposable but furthermore are generalized gamma convolutions. We also study exponential integrals of more general independent increment processes. Several examples are given for illustration.

Finally, in Chapter 5 a multivariate generalized Ornstein-Uhlenbeck process driven by a Lévy process $(X_t, Y_t)_{t\geq 0}$, with $(X_t, Y_t) \in \mathbb{R}^{d\times d} \times \mathbb{R}^d$, $d \geq 1$, is defined as

$$V_t = \mathcal{E}(X)_t^{-1}\left(V_0 + \int_0^t \mathcal{E}(X)_{s-}dY_s\right), \quad t \geq 0.$$

A key result for the investigations in Chapter 5 is the fact that every multidimensional stochastic exponential of a Lévy process is a multiplicative Lévy process in the general linear group and vice versa. Using this, it is shown that the process $(V_t)_{t\geq 0}$ solves the stochastic differential equation $dV_t = dU_t V_{t-} + dL_t$ for another Lévy process $(U_t, L_t)_{t\geq 0}$ in $\mathbb{R}^{d\times d} \times \mathbb{R}^d$, which is given in terms of (X, Y). Finally it is characterized when the process is carried by an affine subspace of $\mathbb{R}^d$ and, under some extra conditions on the limit behaviour of $\mathcal{E}(X)$, necessary and sufficient conditions for the existence of strictly stationary solutions are deduced.

Zusammenfassung

Der von einem bivariaten Lévyprozess $(\xi_t, \eta_t)_{t\geq 0}$ getriebene verallgemeinerte Ornstein-Uhlenbeck Prozess

$$V_t = e^{-\xi_t} \left(V_0 + \int_0^t e^{\xi_{s-}} d\eta_s \right), \quad t \geq 0,$$

mit Anfangsvariable V_0 unabhängig von (ξ, η) erfüllt die stochastische Differentialgleichung $dV_t = V_{t-} dU_t + dL_t$ für einen weiteren bivariaten Lévyprozess $(U_t, L_t)_{t\geq 0}$, welcher vollständig durch (ξ, η) bestimmt ist. Insbesondere gilt $\xi_t = -\log(\mathcal{E}(U)_t)$, $t \geq 0$, wobei $\mathcal{E}(U)$ das stochastische Exponential von U bezeichnet.
In Kapitel 2 dieser Arbeit werden für einen gegebenen bivariaten Lévyprozess (U, L) hinreichende und notwendige Bedingungen für die Existenz einer strikt stationären Lösung der stochastischen Differentialgleichung $dV_t = V_{t-} dU_t + dL_t$ entwickelt. Hierbei wird weder Positivität des Exponentials von U noch Unabhängigkeit von V_0 und (U, L) vorausgesetzt, so dass nicht-kausale Lösungen auftreten können. Die Form der stationären Lösungen wird bestimmt und es wird gezeigt, dass diese, sofern sie existieren, eindeutig in Verteilung sind. Für nicht-kausale Lösungen wird eine hinreichende Bedingung angegeben unter welcher U und L bezüglich der zugehörigen erweiterten Filtration Semimartingale bleiben.
In Kapitel 3 werden Verteilungseigenschaften der stationären Lösungen der stochastischen Differentialgleichung $dV_t = V_{t-} dU_t + dL_t$ analysiert. Insbesondere werden Erwartungswert und Autokovarianzfunktion in Abhängigkeit von U und L ermittelt sowie das Tailverhalten in verschiedenen Situationen beschrieben. Für den Fall, dass U Sprünge der Größe -1 besitzt, werden hinreichende und notwendige Bedingungen für Absolutstetigkeit und Stetigkeit der Lösungen angegeben.
Es ist bekannt, dass die Verteilung exponentieller Integrale von Lévyprozesses, wie sie als stationäre Lösungen des verallgemeinerten Ornstein-Uhlenbeck Prozesses auftreten, in vielen Fällen unendlich teilbar und in einigen Fällen sogar selbstzerlegbar ist. In Kapitel 4 werden hinreichende Bedingungen angegeben unter denen die Verteilung exponentieller Integrale nicht nur selbstzerlegbar ist, sondern eine verallgemeinerte Gamma Faltung ist. Hierbei werden auch exponentielle Integrale allgemeinerer Prozesse mit unabhängigen Zuwächsen betrachtet. Zur Anschauung werden einige Beispiele angegeben.
Schließlich wird in Kapitel 5 der von einem Lévyprozess $(X_t, Y_t)_{t\geq 0}$ mit $(X_t, Y_t) \in \mathbb{R}^{d\times d} \times \mathbb{R}^d$, $d \geq 1$, getriebene multivariate verallgemeinerte Ornstein-Uhlenbeck

Prozess als

$$V_t = \mathcal{E}(X)_t^{-1}\left(V_0 + \int_0^t \mathcal{E}(X)_{s-}dY_s\right), \quad t \geq 0$$

definiert. Von besonderer Bedeutung für die Untersuchungen in Kapitel 5 ist die Tatsache, dass jedes multidimensionale stochastische Exponential eines Lévyprozesses ein multiplikativer Lévyprozess in der allgemeinen linearen Gruppe ist und umgekehrt. Hiermit wird gezeigt, dass der Prozess $(V_t)_{t\geq 0}$ die stochastische Differentialgleichung $dV_t = dU_tV_{t-} + dL_t$ für einen weiteren Lévyprozess $(U_t, L_t)_{t\geq 0}$ in $\mathbb{R}^{d\times d} \times \mathbb{R}^d$ löst, wobei $(U_t, L_t)_{t\geq 0}$ in Abhängigkeit von (X, Y) angegeben wird. Schließlich wird charakterisiert wann der Prozess von einem affinen Unterraum des $\mathbb{R}^d$ getragen wird und es werden, unter Zusatzbedingungen an das Grenzverhalten von $\mathcal{E}(X)$, hinreichende und notwendige Bedingungen für die Existenz strikt stationärer Lösungen ermittelt.

Contents

Chapter 1: Introduction

The Ornstein-Uhlenbeck process originally has been developed to describe the motion of a free particle in a fluid. In 1905 Albert Einstein [16] modelled this movement by a Brownian motion. Twenty five years later the two physicists Leonard Ornstein and George Uhlenbeck [53] added the concept of friction to Einstein's model. This led to the following differential equation for the velocity $v_t, t \geq 0$ of a free particle in a fluid

$$m dv_t = -\lambda v_t dt + dB_t \tag{1.1}$$

where $(B_t)_{t\geq 0}$ is a Brownian motion (times a constant), m is the mass of the given particle and $\lambda > 0$ is a friction coefficient. For $\lambda = 0$ equation (1.1) reduces to the simple Brownian motion model of Einstein.
Equation (1.1) is also known as *Langevin equation* and its solution given a starting value v_0 is given by

$$v_t = e^{-\lambda t/m} v_0 + \frac{1}{m} \int_{(0,t]} e^{\lambda(s-t)/m} dB_s, \; t \geq 0.$$

This solution is called an *Ornstein-Uhlenbeck process*.

In this work we will investigate generalizations of the Ornstein-Uhlenbeck process, give necessary and sufficient conditions for the existence of stationary (i.e. time shift invariant) solutions of these generalizations and then examine properties of the (stationary) generalized Ornstein-Uhlenbeck processes. But before we can start with defining the generalized Ornstein-Uhlenbeck process, we need some prerequisites which will be given in the following section.

1.1 Preliminaries and Notations

Throughout this work we assume to be given a complete probability space $(\Omega, \mathcal{F}, P)$. We will equip this probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ to obtain a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, P)$. We say that $(\Omega, \mathcal{F}, \mathbb{F}, P)$ satisfies the *usual hypotheses* if $\mathbb{F}$ is *complete*, i.e. all P-null sets of $\mathcal{F}$ are contained in $\mathcal{F}_0$, and right continuous, i.e. $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$ for all $t \geq 0$. If a random variable X is $\mathcal{F}_t$-measurable we write

$X \in \mathcal{F}_t$. The standard Borel σ-field generated by the class of open subsets of $\mathbb{R}^d$ will be denoted by $\mathcal{B}_d$.

A random variable $T : \Omega \to [0, \infty]$ is a *stopping time* if the event $\{T \leq t\}$ is in $\mathcal{F}_t$ for all $t \in [0, \infty]$. A *stochastic process* X on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is a family of $\mathbb{R}^d$-valued random variables $(X_t)_{t \geq 0}$ in $(\Omega, \mathcal{F}, P)$. Given a stochastic process $(X_t)_{t \geq 0}$ with $X_t \in \mathcal{F}_t$ for all $t \geq 0$, we say that X is *adapted.* The filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ generated by X, i.e. $\mathcal{F}_t := \sigma(X_s, 0 \leq s \leq t)$ is called the *natural filtration.* It is the smallest filtration such that X is adapted. Extending the natural filtration such that it satisfies the usual hypothesis yields the so-called *augmented natural filtration.*

For fixed $\omega \in \Omega$ the function $[0, \infty) \to \mathbb{R}^d : t \mapsto X_t(\omega)$ is called a *path* of the process X. We say that X has *càdlàg paths* if there exists a set $\Omega_0 \in \mathcal{F}$, such that $P(\Omega_0) = 1$ and for all $\omega \in \Omega_0$ the path $t \mapsto X_t(\omega)$ is right-continuous (continue à droite) in $t \geq 0$ and has finite left limits (limites à gauche) in $t > 0$. The left limit of X at time t will be denoted by X_{t-} and $\Delta X_t := X_t - X_{t-}$ is the jump of X at time t. We write $\mathbb{D}$ for the space of all adapted processes with càdlàg paths. Analogously we say that a process has *càglàd paths* if its paths are left-continuous and have finite right limits. The space of all adapted processes with càglàd paths will be denoted by $\mathbb{L}$.

We say that a stochastic process X is *strictly stationary* (or simply *stationary*) if its finite-dimensional distributions are time shift invariant, i.e. for all $n \in \mathbb{N}$, $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n$ and $h > 0$, $(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$ and $(X_{t_1+h}, X_{t_2+h}, \ldots, X_{t_n+h})$ are equally distributed.

An adapted stochastic process $(M_t)_{t \geq 0}$ is called a *martingale* with respect to the filtration $\mathbb{F}$, if it is integrable, i.e. $E|X_t| < \infty$ for all $t \geq 0$, and it holds $E[X_t | \mathcal{F}_s] = X_s$ a.s. for $0 \leq s \leq t$.

In what follows, "$\xrightarrow{P}$" and "$\xrightarrow{d}$" will denote convergence in probability and distribution, respectively, while "$\overset{d}{=}$" denotes equality in distribution of two random variables. The abbreviation "i.i.d." stands for "independent and identically distributed" while "a.s." means "almost surely". For any $a, b \in \mathbb{R}$ we write $a \wedge b$ for the minimum $\min\{a, b\}$ and set $a \vee b = \max\{a, b\}$. The indicator function on an arbitrary space is defined as follows:

$$\mathbb{1}_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases}$$

For a random variable X its distribution will be denoted by $\mathcal{L}(X)$, its expectation by $E[X]$ and its variance by $\mathrm{Var}\,(X) = E[X^2] - (E[X])^2$. The covariance of two random variables X and Y will be written $\mathrm{Cov}\,(X, Y) = E[XY] - E[X]E[Y]$.

Lévy Processes

Definition 1.1. *A* Lévy process *with values in* $\mathbb{R}^d$, $d \in \mathbb{N}$ *is a stochastic process* $(X_t)_{t\geq 0}$ *such that the following properties hold.*

(i) $(X_t)_{t\geq 0}$ *has* independent increments, *i.e. for all* $n \in \mathbb{N}$, $0 \leq t_0 \leq t_1 \leq \ldots \leq t_n$, *the random variables* $X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ *are independent.*

(ii) *It has* stationary increments, *i.e. for all* $s, t \geq 0$ *it holds* $X_{t+s} - X_s \overset{d}{=} X_t - X_0$.

(iii) *It starts almost surely at* 0, *i.e.* $X_0 = 0$ *a.s.*

(iv) $(X_t)_{t\geq 0}$ *has a.s. càdlàg paths.*

It can be shown that every Lévy process $(X_t)_{t\geq 0}$ is continuous in probability, i.e. it holds $\lim_{s\to t} P(|X_t - X_s| > \epsilon) = 0$ for all $t \geq 0$ and $\epsilon > 0$.

Lévy processes are strongly connected with infinitely divisible distributions.

Definition 1.2. *A probability distribution* μ *of a random variable* Z *in* $\mathbb{R}^d$ *is called* infinitely divisible *if for any* $n \in \mathbb{N}$ *there exists another probability distribution* μ_n *(there exists a sequence of i.i.d. random variables* $Z_{1,n}, \ldots Z_{n,n}$ *having law* μ_n*) such that*

$$\mu = \mu_n^{*n} \quad (Z \overset{d}{=} Z_{1,n} + \ldots + Z_{n,n}.)$$

It follows directly from the definition of Lévy processes that the distribution of a Lévy process $(X_t)_{t\geq 0}$ for fixed time $t > 0$ is infinitely divisible. Conversely, given any infinitely divisible law μ, we can define a Lévy process $(X_t)_{t\geq 0}$ such that $\mathcal{L}(X_1) = \mu$.

Infinitely divisible distributions (and hence Lévy processes) can be characterized completely by their characteristic exponent. More precisely we have the following theorem.

Theorem 1.3 (Lévy-Khintchine formula)**.** *Let* $(X_t)_{t\geq 0}$ *be an* $\mathbb{R}^d$*-valued Lévy process. Then there exists a unique triplet* (A_X, Π_X, γ_X), *called the* characteristic triplet, *consisting of a symmetric, non-negative* $d \times d$*-matrix* A_X, *a measure* Π_X *on* $\mathbb{R}^d \backslash \{0\}$ *fulfilling*

$$\int_{\mathbb{R}^d} \min\{|x|^2, 1\} \Pi_X(dx) < \infty \tag{1.2}$$

and a constant $\gamma_X \in \mathbb{R}^d$, *such that*

$$E[e^{izX_t}] = e^{t\psi_X(z)}, \tag{1.3}$$

where

$$\psi_X(z) = -\frac{1}{2}\langle z, A_X z\rangle + i\langle \gamma_X, z\rangle + \int_{\mathbb{R}^d} (1 - e^{i\langle z,x\rangle} + i\langle z, x\rangle \mathbb{1}_{|x|\leq 1}) \Pi_X(dx) \tag{1.4}$$

is the characteristic exponent *of* X.

Conversely, given a triplet (A_X, Π_X, γ_X), *consisting of a symmetric, non-negative*

$d \times d$-matrix A_X, a measure Π_X on $\mathbb{R}^d \backslash \{0\}$ fulfilling (1.2) and a constant $\gamma_X \in \mathbb{R}^d$, there exists a Lévy process $(X_t)_{t \geq 0}$ on $\mathbb{R}^d$, unique up to identity in law, such that (1.3) and (1.4) hold.

The class of Lévy processes on $\mathbb{R}^d$ includes every Brownian motion, linear deterministic processes of the form $[0, \infty) \to \mathbb{R}^d : t \mapsto \gamma t$ for $\gamma \in \mathbb{R}^d$ and all compound Poisson processes with parameter $\lambda > 0$ and jump size distribution ρ, i.e. all processes of the form

$$X_t = \sum_{i=1}^{N_t} Y_i,$$

where $(N_t)_{t \geq 0}$ is a Poisson process with parameter $\lambda > 0$ and $(Y_i)_{i \in \mathbb{N}}$ is an i.i.d. sequence of random variables with distribution ρ such that $\rho(\{0\}) = 0$, independent of $(N_t)_{t \geq 0}$. In particular we have the following theorem.

Theorem 1.4 (Lévy-Itô decomposition)**.** *Let $X = (X_t)_{t \geq 0}$ be an $\mathbb{R}^d$-valued Lévy process with characteristic triplet (A_X, Π_X, γ_X). Then we can write X as the independent sum of three Lévy processes $X^{(1)}, X^{(2)}$ and $X^{(3)}$ where*

(i) *$X^{(1)}$ is a Brownian motion with drift having characteristic triplet $(A_X, 0, \gamma_{X^{(1)}})$.*
(ii) *$X^{(2)}$ is a compound Poisson process with $|\Delta X_t^{(2)}| > 1$, $\forall t \geq 0$, having characteristic triplet $(0, \Pi_X|_{\{x \in \mathbb{R}^d : |x| > 1\}}, 0)$.*
(iii) *$X^{(3)}$ is a purely discontinuous (i.e. orthogonal in the L^2 sense to the stable subspace generated by continuous L^2 martingales), square integrable martingale with $|\Delta X_t^{(3)}| \leq 1$, $\forall t \geq 0$, having characteristic triplet $(0, \Pi_X|_{\{x \in \mathbb{R}^d : |x| \leq 1\}}, \gamma_{X^{(3)}})$.*

The measure Π_X in the characteristic triplet of a Lévy process $(X_t)_{t \geq 0}$ is called the *Lévy measure* of X. In particular it follows from the Lévy-Itô decomposition that for all Borel sets Λ in $\mathbb{R}^d$ such that $0 \notin \Lambda$ it holds

$$\Pi_X(\Lambda) = E \sum_{0 < s \leq 1} \mathbb{1}_\Lambda(\Delta X_s).$$

As another consequence of the Lévy-Itô decomposition the matrix A_X in the characteristic triplet of X can be identified as the covariance matrix of the Brownian motion component $X^{(1)}$ of X at time $t = 1$ and hence will be called *Gaussian covariance matrix* of X. In the case of a one-dimensional Lévy process this motivates the use of the notation σ_X^2 instead of A_X.

For further information on Lévy processes we refer to the books of Applebaum [2], Bertoin [7], Kyprianou [44] or Sato [58].

Some Words on Stochastic Integration Theory

Since in this work we will investigate stochastic differential equations, we give a very brief introduction to stochastic integration in the following lines. For more details and a deeper understanding consult the books by Applebaum [2], Bichteler [10], Kallenberg [35] or Protter [55].

Although a simple extension of the Lebesgue-Stieltjes integral to integrators with paths of infinite variation like e.g. the Brownian motion is not possible, one can define a stochastic integral with respect to an even larger class of processes than Lévy processes. To do so, we start by defining a class of "nice" integrands.

Definition 1.5. *A stochastic process H is called* simply predictable *if it has a representation*

$$H_t = H_0 \mathbb{1}_0(t) + \sum_{i=1}^{n} H_i \mathbb{1}_{(T_i, T_{i+1}]}(t) \tag{1.5}$$

where $0 = T_1 \leq T_2 \leq \ldots \leq T_{n+1} < \infty$ is a sequence of stopping times and $H_i \in \mathcal{F}_{T_i}$ with $|H_i| < \infty$ a.s., $0 \leq i \leq n$. The space of all simply predictable processes is denoted by $\mathcal{S}$.

We can now define the stochastic integral of the simply predictable process H given in (1.5) with respect to a càdlàg process X as

$$I_X(H) = H_0 X_0 + \sum_{i=1}^{n} H_i (X^{T_{i+1}} - X^{T_i}), \tag{1.6}$$

where $X^T = (X_{t \wedge T})_{t \geq 0}$ is the process X stopped at time T. The mapping $I_X(H) : \mathcal{S} \to \mathbb{D}$ is linear and does not depend on the choice of the representation of H.

Since we want to widen up the class of possible integrands, remark that the space $\mathcal{S}$ is dense in $\mathbb{L}$ under the so-called *ucp*-topology. Here we say that a sequence of processes $\{H^n\}_{n \geq 1}$ converges to a process H *uniformly on compacts in probability* (*ucp* in short) if, for each $t > 0$, $\sup_{0 \leq s \leq t} |H^n_s - H_s|$ converges to 0 in probability.

The class of possible integrators shall be chosen as wide as possible, too, leading to the class of semimartingales which can be characterized as follows.

Definition 1.6. *A stochastic process X is called a* semimartingale with respect to a filtration $\mathbb{F}$, *where $\mathbb{F}$ satisfies the usual hypotheses, if it can be written as a sum $X_t = X_0 + M_t + A_t$ where*

(i) *M is a local martingale, i.e. it is adapted, càdlàg and there exists a sequence of increasing stopping times $(T_n)_{n \in \mathbb{N}}$ with $T_n \to \infty$ a.s., $n \to \infty$, such that the stopped process $(M_{t \wedge T_n} \mathbb{1}_{T_n > 0})_{t \geq 0}$ is a uniformly integrable martingale for each n.*

(ii) *A is a càdlàg, adapted process with paths of finite variation on compacts, starting in* 0.

In particular it follows from the Lévy-Itô decomposition 1.4 that all Lévy processes are semimartingales with respect to the augmented natural filtration.

For any semimartingale X the mapping $I_X : \mathcal{S}_{ucp} \to \mathbb{D}_{ucp}$ is continuous and this fact allows us to define the stochastic integral with respect to a semimartingale as follows.

Definition 1.7. *Let X be a semimartingale. The continuous linear mapping $I_X : \mathbb{L}_{ucp} \to \mathbb{D}_{ucp}$ obtained as the extension of $I_X : \mathcal{S} \to \mathbb{D}$ is called the* stochastic integral. *Given a process $H \in \mathbb{L}$ and a semimartingale X we write*

$$I_X(H) = \int H dX.$$

Evaluating the process $I_X(H)$ at some time $t \geq 0$ we get $\int_{[0,t]} H_s dX_s$. In case that we want to exclude 0 in the integral we write $\int_{(0,t]} H_s dX_s$.

Remark that for a process $H \in \mathbb{L}$ and a semimartingale X with paths of finite variation on compacts, the stochastic integral $\int H dX$ is indistinguishable from the path-by-path computed Lebesgue-Stieltjes integral.

We end up this short introduction to stochastic integration theory by noting some important properties of stochastic integrals which will be used frequently throughout this work.

For a semimartingale X and a process $H \in \mathbb{L}$, $I_X(H)$ is itself a semimartigale. In particular, given another process $G \in \mathbb{L}$, we have

$$\int_{(0,t]} G_s d\left(\int_{(0,s]} H_u dX_u\right) = \int_{(0,t]} G_s H_s dX_s, \quad t \geq 0.$$

Given two semimartingales X, Y their *quadratic covariation* is defined by

$$[X,Y]_t = X_t Y_t - \int_{(0,t]} X_{s-} dY_s - \int_{(0,t]} Y_{s-} dX_s, \quad t \geq 0. \tag{1.7}$$

Its path-by-path continuous part will be denoted by $[X,Y]^c$. In particular, if L is a one-dimensional Lévy process, we have that

$$[L,L]_t = [L,L]_t^c + \sum_{0<s\leq t} (\Delta L_s)^2 = t \cdot \sigma_L^2 + \sum_{0<s\leq t} (\Delta L_s)^2, \quad t \geq 0.$$

Reordering (1.7) leads to the integration by parts formula of stochastic integrals

$$(XY)_t = \int_{(0,t]} X_{s-} dY_s + \int_{(0,t]} Y_{s-} dX_s + [X,Y]_t, \quad t \geq 0, \tag{1.8}$$

for two semimartingales X and Y and with two additional processes $H, G \in \mathbb{L}$ we have that

$$\left[\int_{(0,\cdot]} H_s dX_s, \int_{(0,\cdot]} G_s dY_s\right]_t = \int_{(0,t]} H_s G_s d[X,Y]_s, \quad t \geq 0.$$

The Stochastic Exponential

A semimartingale which will be of special importance throughout the following investigations is the *stochastic exponential*, also known as the *Doléans-Dade exponential*.

Definition 1.8. *For a real-valued semimartingale X satisfying $X_0 = 0$ the* stochastic exponential of X, *written $\mathcal{E}(X)$, is the unique semimartingale Z, such that $Z_t = 1 + \int_{(0,t]} Z_{s-} dX_s$ holds for all $t \geq 0$.*

It can be shown that for any semimartingale X with $X_0 = 0$, it holds

$$\begin{aligned} \mathcal{E}(X)_t &= e^{X_t - \frac{1}{2}[X,X]_t} \prod_{0<s\leq t} (1 + \Delta X_s) e^{-\Delta X_s + \frac{1}{2}(\Delta X_s)^2} \\ &= e^{X_t - \frac{1}{2}[X,X]^c_t} \prod_{0<s\leq t} (1 + \Delta X_s) e^{-\Delta X_s}, \quad t \geq 0, \end{aligned} \tag{1.9}$$

and for two semimartingales X, Y we have $\mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}(X + Y + [X,Y])$. The proofs of the above as well as further information on stochastic exponentials can be found e.g. in the book of Protter [55, pp. 84–86].

1.2 The Generalized Ornstein-Uhlenbeck Process: Definition and some Properties

We are now prepared to give sense to the following definition.

Definition 1.9. *The* generalized Ornstein-Uhlenbeck (GOU) process $(V_t)_{t\geq 0}$ *driven by the bivariate Lévy process $(\xi_t, \eta_t)_{t\geq 0}$ is given by*

$$V_t = e^{-\xi_t}\left(V_0 + \int_0^t e^{\xi_{s-}} d\eta_s\right), \quad t \geq 0, \tag{1.10}$$

where V_0 is a finite random variable, independent of (ξ, η).
In the case that $(\xi_t, \eta_t) = (\lambda t, \eta_t)$ with a Lévy process $(\eta_t)_{t\geq 0}$ and a constant $\lambda \neq 0$ the process $(V_t)_{t\geq 0}$ is called Lévy-driven Ornstein-Uhlenbeck process *or* Ornstein-Uhlenbeck type process. *Obviously, if additionally $(\eta_t)_{t\geq 0}$ is a Brownian motion, we get the classical Ornstein-Uhlenbeck process.*

The terminology "generalized Ornstein-Uhlenbeck process" has first been introduced by De Haan and Karandikar [27] in 1989. In that paper the authors investigate a continuous time embedding of the stochastic difference equation $V_n = A_n V_{n-1} + B_n$, $n \in \mathbb{N}$, for an i.i.d. sequence $(A_n, B_n)_{n\in\mathbb{N}}$ with $P(A_1 > 0) = 1$ and show that every solution V of the continuous time model is a GOU process as defined in (1.10). In particular, they show that the GOU process driven by (ξ, η) is the unique solution of the stochastic differential equation (SDE)

$$dV_t = V_{t-}dU_t + dL_t, \quad t \geq 0, \tag{1.11}$$

for the bivariate Lévy process $(U_t, L_t)_{t\geq 0}$ given by

$$\begin{pmatrix} U_t \\ L_t \end{pmatrix} = \begin{pmatrix} -\xi_t + \sum_{0<s\leq t}\left(e^{-\Delta\xi_s} - 1 + \Delta\xi_s\right) + t\,\sigma_\xi^2/2 \\ \eta_t + \sum_{0<s\leq t}(e^{-\Delta\xi_s} - 1)\Delta\eta_s - t\,\sigma_{\xi,\eta} \end{pmatrix}, \quad t \geq 0, \tag{1.12}$$

where σ_ξ^2 and $\sigma_{\xi,\eta}$ denote the $(1,1)$ and $(1,2)$ elements of the Gaussian covariance matrix $A_{(\xi,\eta)}$ in the Lévy-Khintchine representation of the characteristic function of (ξ, η).

The definition of U in (1.12) is equivalent to saying that $\mathcal{E}(U)_t = e^{-\xi_t}$, where $\mathcal{E}(U)$ denotes the Doléans-Dade stochastic exponential of U as given in Definition 1.8. In general the stochastic exponential may take zero or negative values, but in satisfying $\mathcal{E}(U)_t = e^{-\xi_t}$, we see that this version of $\mathcal{E}(U)$ must be strictly positive, which is equivalent to the Lévy measure of U having no mass on $(-\infty, -1]$.

In 2005 Lindner and Maller [46] developed necessary and sufficient conditions for the existence of strictly stationary solutions of the generalized Ornstein-Uhlenbeck process. In particular they gave the following theorem [46, Theorem 2.1].

Theorem 1.10. *Let $(V_t)_{t\geq 0}$ be the generalized Ornstein-Uhlenbeck process driven by the bivariate Lévy process $(\xi_t, \eta_t)_{t\geq 0}$ as defined in* (1.10) *and define the Lévy process $(L_t)_{t\geq 0}$ as in* (1.12). *Suppose the process $(V_t)_{t\geq 0}$ is strictly stationary, then one of the following two conditions (i) or (ii) is satisfied:*

(i) $\int_{(0,t]} e^{-\xi_{s-}} dL_s$ *converges a.s. to a finite random variable as $t \to \infty$.*

(ii) *There exists a constant $k \in \mathbb{R}$ such that the process $(V_t)_{t\geq 0}$ is indistinguishable from the constant process $t \mapsto k$, i.e. it holds $V_t = k$, $\forall t \geq 0$ a.s.*

Conversely, if (i) or (ii) holds, then there is a finite random variable V_∞ (unique in distribution), such that $(V_t)_{t\geq 0}$ starting with $V_0 \overset{d}{=} V_\infty$ is strictly stationary.
Furthermore, if (i) holds, then the stationary random variable V_∞ can be chosen as

$$V_\infty = \int_{(0,\infty)} e^{-\xi_{s-}} dL_s. \tag{1.13}$$

Necessary and sufficient conditions for the convergence of integrals of the form (1.13) for a bivariate Lévy process $(\xi_t, L_t)_{t\geq 0}$ were given by Erickson and Maller [19]. More precisely, in [19, Theorem 2], they state the following.

Theorem 1.11. *The exponential integral* $\int_{(0,t]} e^{-\xi_{s-}} dL_s$ *converges a.s. to a finite random variable as* $t \to \infty$ *if and only if for some* $\epsilon > 0$ *such that* $A_\xi(x) > 0$ *for all* $x > \epsilon$ *it holds*

$$\lim_{t\to\infty} \xi_t = \infty \ a.s. \ and \ I_{\xi,L} = \int_{(e^\epsilon,\infty)} \left(\frac{\log y}{A_\xi(\log y)} \right) |d\bar{\Pi}_L(y)| < \infty. \tag{1.14}$$

Here $\bar{\Pi}_L(x) = \Pi_L((-\infty,-x)) + \Pi_L((x,\infty))$ *is the tail-function of the Lévy measure of* L *while* $A_\xi(x) = \gamma_\xi + \Pi_\xi((1,\infty)) + \int_{(1,x]} \Pi_\xi((y,\infty))dy$.
In case of divergence, if $\lim_{t\to\infty} \xi_t = \infty$ *a.s. but* $I_{\xi,L} = \infty$, *it holds*

$$\left| \int_{(0,t]} e^{-\xi_{s-}} dL_s \right| \xrightarrow{P} \infty \ \ as \ \ t \to \infty \tag{1.15}$$

while if ξ *does not tend to* ∞ *as* $t \to \infty$, *then either* (1.15) *holds or there exists a constant* $k \in \mathbb{R}$ *such that*

$$P\left(\int_{(0,t]} e^{-\xi_{s-}} dL_s = k\left(1 - e^{-\xi_t}\right) \ \forall t \geq 0 \right) = 1.$$

1.3 Main Results of this Thesis

In **Chapter 2** we shall generalize the above results to the solutions of the SDE (1.11) for an arbitrary bivariate Lévy process (U, L), where to avoid trivialities neither U nor L is the constant zero process . In particular that means, that we do not assume $\Pi_U((-\infty,-1])$ to be zero, which leads to the general solution of the SDE (1.11) given in Theorem 2.6. Another generalization compared to [46] is that we allow dependence between the starting random variable V_0 and the process (U, L). This sharpens the results of [46] even in the case of a GOU process. Motivated by notations in time series, solutions with V_0 being dependent of (U, L) will be called *non-causal* while a solution with V_0 being independent of (U, L) is called a *causal* or *non-anticipative* solution. The causal solutions of (1.11) are, as the GOU process, time homogeneous Markov processes.

As generalization of Theorem 1.10 in Theorems 2.1 and 2.2 necessary and sufficient conditions for the existence of strictly stationary solutions of the SDE (1.11) are shown. In fact, given a bivariate Lévy process (U, L), a finite random variable V_0 can be chosen such that the solution $(V_t)_{t\geq 0}$ of (1.11) is strictly stationary if and only if one of the following conditions holds:

(i) There is $k \neq 0$ such that $U_t = -L_t/k$, $t \geq 0$;

(ii) The integral $\int_{(0,t]} \mathcal{E}(U)_{s-}\, dL_s$ converges almost surely to a finite random variable as $t \to \infty$;

(iii) $\Pi_U(\{-1\}) = 0$ and the integral $\int_{(0,t]} [\mathcal{E}(U)_{s-}]^{-1}\, d\eta_s$ converges almost surely to a finite random variable as $t \to \infty$, where $(\eta_t)_{t\geq 0}$ is a Lévy process given by

$$\eta_t := L_t - \sum_{\substack{0<s\leq t \\ \Delta U_s \neq -1}} \frac{\Delta U_s \Delta L_s}{1+\Delta U_s} - t\sigma_{U,L},\ t \geq 0.$$

If one of the conditions (i) to (iii) is satisfied, then the distributions of V_0 and of the corresponding strictly stationary process V are uniquely determined. Namely, given that $(V_t)_{t\geq 0}$ is a strictly stationary solution of the SDE (1.11), then one of the cases below holds:

(i) There is $k \neq 0$ such that $V_t = k$, $t \geq 0$;

(ii) $\Pi_U(\{-1\}) = 0$, the integral $\int_{(0,t]} \mathcal{E}(U)_{s-} dL_s$ converges a.s. and

$$\mathcal{L}(V_0) = \mathcal{L}\left(\int_{(0,\infty)} \mathcal{E}(U)_{s-}\, dL_s\right);$$

(iii) $\Pi_U(\{-1\}) = 0$, the integral $\int_{(0,t]} [\mathcal{E}(U)_{s-}]^{-1} d\eta_s$ converges a.s. and

$$\mathcal{L}(V_0) = \mathcal{L}\left(\int_{(0,\infty)} [\mathcal{E}(U)_{s-}]^{-1} d\eta_s\right);$$

(iv) $\Pi_U(\{-1\}) > 0$ and $\mathcal{L}(V_0) = \mathcal{L}(Z_\tau)$, for the process

$$Z_t = \mathcal{E}(\widetilde{U})_t \left(Y + \int_{(0,t]} [\mathcal{E}(\widetilde{U})_{s-}]^{-1} d\widetilde{\eta}_s\right), \quad t \geq 0,$$

with

$$\widetilde{U}_t = U_t - \sum_{\substack{0<s\leq t \\ \Delta U_s = -1}} \Delta U_s, \quad \widetilde{\eta}_t = \eta_t - \sum_{\substack{0<s\leq t \\ \Delta U_s = -1}} \Delta \eta_s$$

and a random variable $Y \overset{d}{=} \Delta L_{T_1}$ independent of (U, L). Here T_1 is the time of first jump of U of size -1. The time variable τ is exponentially distributed with expectation $\Pi_U(\{-1\})$ and independent of (U, L) and Y.

As a generalization of Theorem 1.11 in Section 2.2 necessary and sufficient conditions for the almost sure convergence of $\int_0^\infty \mathcal{E}(U)_{s-}\, dL_s$ and $\int_0^\infty [\mathcal{E}(U)_{s-}]^{-1}\, d\eta_s$ in terms of the characteristic triplets of the underlying Lévy processes are given. By the above, these are important for characterizing when a strictly stationary solution of the SDE (1.11) exists.

In the case that $\Pi_U(\{-1\}) = 0$ and the integral $\int_{(0,t]} [\mathcal{E}(U)_{s-}]^{-1} d\eta_s$ converges a.s., the stationary solution $(V_t)_{t\geq 0}$ of (1.11) is a.s. given by

$$V_t = -\mathcal{E}(U)_t \int_{(t,\infty)} [\mathcal{E}(U)_{s-}]^{-1} d\eta_s, \quad t \geq 0,$$

and hence this solution is *strictly non-causal* in the sense that V_t is independent of $(U_s, L_s)_{0\leq s<t}$. To interpret these non-causal processes as solutions of the stochastic differential equation (1.11) we have to enlarge the underlying filtration, such that the non-causal solution is adapted, and prove that the solution $(V_t)_{t\geq 0}$ still satisfies the SDE (1.11) with respect to the new filtration. In Section 2.4 we show that absolute continuity of the law of $\int_0^\infty [\mathcal{E}(U)_{s-}]^{-1} \, d\eta_s$ is a sufficient condition for this to hold and examples when this condition is satisfied are mentioned.

Another aspect of GOU processes studied by Lindner and Maller [46] are second order properties of the stationary solutions. In particular, in [46] they develop moment conditions, derive the autocovariance function and give, based on results of Goldie [22] and Kesten [38], sufficient conditions for the stationary solutions to be heavy-tailed. To be more specific, we restate some of their results in the following theorem [46, Proposition 4.1 and Theorem 4.2].

Theorem 1.12. *Let $(V_t)_{t\geq 0}$ be the causal generalized Ornstein-Uhlenbeck process driven by the bivariate Lévy process (ξ, η) as defined in (1.10). Fix $\kappa > 0$ and assume that there are $p, q > 1$ with $1/p + 1/q = 1$ such that*

$$E[e^{-\max\{1,\kappa\}p\xi_1}] < \infty \quad \textit{and} \quad E|\eta_1|^{\max\{1,\kappa\}q} < \infty. \tag{1.16}$$

Assume further that $E[e^{-\kappa\xi_1}] < 1$. Then there exists a stationary solution of the GOU process and this solution satisfies $E|V_0|^\kappa < \infty$.
In particular, if $\kappa = 2$ and $(V_t)_{t\geq 0}$ is stationary, then it holds

$$\mathrm{Cov}\,(V_s, V_t) = e^{(t-s)\log E[\exp(-\xi_1)]} \mathrm{Var}\,(V_s), \quad 0 \leq s \leq t. \tag{1.17}$$

In **Chapter 3** we investigate distributional properties of the stationary solutions of the SDE (1.11) for the general Lévy process (U, L). In particular in Section 3.1 we give the moment conditions and quote first and second moments as well as the autocorrelation function of the stationary solutions in terms of (U, L). By avoiding the use of Hölder's inequality in the proofs, even for the GOU process we obtain sharper results than given above. Namely, in Theorem 3.3 it is shown, that condition (1.16) in Theorem 1.12 can be replaced by

$$E|U_1|^{\max\{1,\kappa\}} < \infty \quad \text{and} \quad E|L_1|^{\max\{1,\kappa\}} < \infty.$$

Given that $(V_t)_{t\geq 0}$ is a strictly stationary solution of (1.11) with finite second moments, in Theorem 3.4 its autocovariance function is shown to be

$$\mathrm{Cov}\,(V_s, V_t) = -e^{E[U_1](t-s)} \frac{E\left[(U_1 E[L_1] - E[U_1]L_1)^2\right]}{(E[U_1])^2(2E[U_1] + \mathrm{Var}\,(U_1))}, \quad 0 \leq s \leq t,$$

which reduces to (1.17) in the case of a GOU process.

Applying the results of Kesten [38] and Goldie [22] to the general solutions of (1.11) leads to sufficient conditions for the stationary solutions to be heavy-tailed, analogously as in [46], although again the avoidance of Hölder's inequality sharpens the results as shown in Section 3.2. Additionally, an application of results of Goldie and Grübel [23] on the strictly stationary solutions of (1.11) leads to conditions for the appearance of power-law tails and exponentially decreasing tails. So, e.g. in the case of a non-constant, causal solution, we obtain (Propositions 3.8 and 3.9) that V_0 has at least a power-law tail, i.e.

$$\liminf_{x\to\infty} \frac{\log(P(|V_0| \geq x))}{\log x} > -\infty$$

whenever U is of infinite variation.
On the other hand, if U is of finite variation with $\Pi_U(\mathbb{R}\backslash[-2,0]) > 0$ and strictly negative drift, then the tails of $\mathcal{L}(V_0)$ decrease at least exponentially fast, i.e.

$$\limsup_{x\to\infty} x^{-1} \log(P(|V_0| \geq x)) < 0$$

if there exists $\kappa > 0$ such that $Ee^{\kappa|L_1|} < \infty$.

The law of the stationary solution of the GOU process, i.e. of the integral V_∞ in (1.13) is a pure-type measure which means that it is either absolutely continuous with respect to the Lebesgue measure, continuous singular or a Dirac measure. This follows directly from a theorem by Alsmeyer, Iksanov and Rösler [1]. Bertoin, Lindner and Maller [8] give a complete characterization of when $\mathcal{L}(V_\infty)$ has an atom in terms of the Lévy-Khintchine triplet of (ξ, L). In Section 2.4 we investigate necessary and sufficient conditions for the law of the strictly stationary solutions of the SDE (1.11) in the case $\Pi_U(\{-1\}) > 0$ to be (absolutely) continuous. It turns out that the distributions of the stationary solutions do not fulfill a pure-type theorem in this case. For example, if U and L are independent and L is a compound Poisson process with a continuous jump distribution, then $\mathcal{L}(V_0)$ is shown to have an atom and a continuous part.

To obtain more distributional informations on the stationary solution of the GOU process, we investigate the question of when the law of (1.13) is a so called Generalized Gamma Convolution (GGC) in **Chapter 4**. The class of GGCs is defined to be the smallest class containing all Gamma distributions which is closed under convolution and weak convergence. It is a subclass of the class of selfdecomposable distributions which itself is a subclass of the class of infinitely divisible distributions. All non-degenerate Generalized Gamma Convolutions are absolutely continuous.

In the case that either the process ξ or the process L is a compound Poisson process while the other is a general Lévy process and they are independent, we derive

sufficient conditions for $\mathcal{L}(V_\infty)$ to be in the class of GGCs and give several examples fulfilling those conditions in Section 4.2.1. For example we show that $\mathcal{L}(V_\infty)$ is a GGC if ξ is a compound Poisson process with normal distributed jump heights having positive mean and L is deterministic or a stable subordinator.
The corresponding proofs are mainly based on the definition of the class of GGCs and its relations to the class of distributions with hyperbolically monotone densities. Therefore the obtained results can easily be extended to the case that either ξ or L is a so called compound sum process and the other is a general Lévy process and they are independent, as done in Section 4.3. Here we say that a process $(X_t)_{t\geq 0}$ is a compound sum process if we have $X_t = \sum_{i=1}^{M_t} S_i$ for an i.i.d. family $\{S_i\}_{i\in\mathbb{N}}$ and a renewal (or counting) process $(M_t)_{t\geq 0}$ independent of $\{S_i\}_{i\in\mathbb{N}}$. For example it is shown that in this setting $\mathcal{L}(V_\infty)$ is a GGC if ξ is a compound sum process with normal distributed jump heights having positive mean and any GGC waiting times with finite log-moment and L is deterministic or a stable subordinator. Obviously this includes the case mentioned above of ξ being a compound Poisson process.

Lindner and Sato [47] investigated the distribution of $\int_{(0,\infty)} e^{-\xi_{s-}} dL_s$ when $\xi_t = (\log c)R_t$ for a constant $c > 1$ and possibly dependent Poisson processes R and L. They gave an explicite expression for the law and showed that it can be absolutely continuous or continuous singular, depending on c, the ratio of the rates of the Poisson processes R and L and their dependence structure.
In Section 4.2.2 we extend the setting of [47] to the case that (ξ, L) is a bivariate compound Poisson process such that the marginal process ξ is a Poisson process. Then $\mathcal{L}(V_\infty)$ is shown to be a GGC, given that the distribution of $\Delta L_s|(\Delta\xi_s = 1)$ and the compound geometric distribution of $\Delta L_s|(\Delta\xi_s = 0)$ are GGC. These assumptions are e.g. fulfilled if both distributions are exponential.
Interestingly it also turns out that $\mathcal{L}(V_\infty)$ is infinitely divisible if the distribution of $\Delta L_s|(\Delta\xi_s = 1)$ is infinitely divisible, no matter how $\Delta L_s|(\Delta\xi_s = 0)$ behaves.

As mentioned above, the GOU process has been derived by De Haan and Karandikar [27] as a continuous time analogon to the time series which fulfill the stochastic difference equation $V_{nh} = A_{n,h}V_{(n-1)h} + B_{n,h}$, $h > 0$, $n \in \mathbb{N}$, for a real-valued i.i.d. sequence $(A_{n,h}, B_{n,h})_{n\in\mathbb{N}}$ with $P(A_{1,h} > 0) = 1$. In **Chapter 5** we investigate the construction of a multidimensional generalized Ornstein-Uhlenbeck process following De Haan and Karandikar, i.e. we consider solutions of $V_{nh} = A_{n,h}V_{(n-1)h} + B_{n,h}$, $h > 0$, $n \in \mathbb{N}$, for an i.i.d. sequence $(A_{n,h}, B_{n,h})_{n\in\mathbb{N}}$ with $(A_{n,h}, B_{n,h}) \in \mathbb{R}^{d\times d}\times\mathbb{R}^d$, $n \in \mathbb{N}$, $d \geq 1$, and $A_{1,h}$ a.s. invertible. We show that the solutions $(V_t)_{t\geq 0}$, $V_t \in \mathbb{R}^d$, $t \geq 0$ of the corresponding continuous time model can be written as

$$V_t = \mathcal{E}(X)_t^{-1}\left(V_0 + \int_{(0,t]} \mathcal{E}(X)_{s-}dY_s\right), \quad t \geq 0, \tag{1.18}$$

where $(X_t, Y_t)_{t\geq 0}$ is a Lévy process in $\mathbb{R}^{d\times d} \times \mathbb{R}^d$. The process $(V_t)_{t\geq 0}$ defined in (1.18) will be called *multivariate generalized Ornstein-Uhlenbeck (MGOU) process* and it is the unique solution of the SDE $V_t = dU_t V_{t-} + dL_t$ for the $\mathbb{R}^{d\times d} \times \mathbb{R}^d$-valued Lévy process $(U_t, L_t)_{t\geq 0}$ given by

$$\begin{pmatrix} U_t \\ L_t \end{pmatrix} = \begin{pmatrix} -X_t + [X,X]_t^c + \sum_{0<s\leq t}((I+\Delta X_s)^{-1} - I + \Delta X_s) \\ Y_t + \sum_{0<s\leq t}((I+\Delta X_s)^{-1} - I)\,\Delta Y_s - [X,Y]_t^c \end{pmatrix}, \quad t \geq 0.$$

A particularly interesting aspect in the described multidimensional setting is that the stochastic exponential of an $\mathbb{R}^{d\times d}$-valued Lévy process can be interpreted as a multiplicative *left Lévy process in the general linear group* $\mathrm{GL}(\mathbb{R}, d)$ *of order* d, i.e. a stochastic process $(X_t)_{t\geq 0}$, having càdlàg paths and stationary and independent increments of the form $X_s^{-1}X_t$, which have to be multiplied from the right side, starting a.s. in I, the identity matrix. In fact (see Proposition 5.5), every multiplicative left Lévy process in $\mathrm{GL}(\mathbb{R}, d)$ is shown to fulfill the SDE of the stochastic exponential $Z_t = I + \int_{(0,t]} Z_{s-} dX_s$ for some $\mathbb{R}^{d\times d}$-valued Lévy process X. This observation is originally due to Skorokhod [59].

Compared to the real valued GOU process another new aspect when studying MGOU processes, is the possibility of the existence of affine subspaces H of $\mathbb{R}^d$ which are *invariant* under the autoregressive model

$$V_t = A_{s,t}V_s + B_{s,t}, \quad 0 \leq s \leq t, \tag{1.19}$$

in the sense that $V_0 \in H$ implies $V_t \in H$ for all $t \geq 0$. In Theorem 5.14 and Corollary 5.15 we show that an n-dimensional invariant affine subspace of the model (1.19) exists if and only if we can choose an orthogonal transformation $O \in \mathbb{R}^{d\times d}$ such that

$$OX_tO^{-1} = \begin{pmatrix} \mathcal{X}_t^1 & 0 \\ \mathcal{X}_t^2 & \mathcal{X}_t^3 \end{pmatrix} \quad \text{and} \quad OY_t = \begin{pmatrix} \mathcal{X}_t^1 K \\ \mathcal{Y}_t^2 \end{pmatrix} \quad \text{with} \quad \mathcal{X}_t^1 \in \mathbb{R}^{(d-n)\times(d-n)},\ t \geq 0, \tag{1.20}$$

holds for a constant vector $K \in \mathbb{R}^{d-n}$. Given the existence of an invariant affine subspace H of dimension n, for any $V_0 \in H$ the MGOU process $(V_t)_{t\geq 0}$ can be written as $V_t = O^{-1}\begin{pmatrix} K \\ \mathcal{V}_t \end{pmatrix}$ where $(\mathcal{V}_t)_{t\geq 0}$ is an $\mathbb{R}^n$-valued MGOU process.

Finally in Section 5.3.3 we treat strictly stationary solutions of MGOU processes and give conditions for their existence. It turns out that, provided $P-\lim_{t\to\infty} \mathcal{E}(U)_t = 0$, a finite random variable V_0 can be chosen such that the resulting MGOU process $(V_t)_{t\geq 0}$ is strictly stationary if and only if the integral $\int_{(0,t]} \mathcal{E}(U)_{s-} dL_s$ converges in distribution for $t \to \infty$. In this case it holds $V_0 \stackrel{d}{=} \int_{(0,\infty)} \mathcal{E}(U)_{s-} dL_s$ and the resulting process is causal. Given that $d \leq 3$ and the underlying model is irreducible in the sense that no invariant subspace $H \subsetneq \mathbb{R}^d$ exists, existence of a causal strictly stationary solution already implies that $P-\lim_{t\to\infty} \mathcal{E}(U)_t = 0$. On the other hand, if the model is not irreducible $P - \lim_{t\to\infty} \mathcal{E}(U)_t = 0$ is not necessary for the existence of a strictly stationary causal solution as shown in Corollary 5.15. Namely we obtain that if we can choose an orthogonal transformation $O \in \mathbb{R}^{d\times d}$ such that (1.20) holds, then we have

$$OU_tO^{-1} = \begin{pmatrix} \mathcal{U}_t^1 & 0 \\ \mathcal{U}_t^2 & \mathcal{U}_t^3 \end{pmatrix} \quad \text{and} \quad OL_t = \begin{pmatrix} -\mathcal{U}_t^1 K \\ \mathcal{L}_t^2 \end{pmatrix} \quad \text{with} \quad \mathcal{U}_t^1 \in \mathbb{R}^{(d-n)\times(d-n)},\ t \geq 0,$$

and a finite random variable V_0 can be chosen such that $(V_t)_{t\geq 0}$ is strictly stationary and causal if $P - \lim_{t\to\infty} \mathcal{E}(\mathcal{U}^3)_t = 0$ and $\int_{(0,t]} \mathcal{E}(\mathcal{U}^3)_{s-} d(\mathcal{L}_s^2 + \mathcal{U}_s^2 K)$ converges in distribution to a finite random variable as $t \to \infty$. In particular a strictly stationary solution can be obtained by choosing V_0 independent of $(X_t, Y_t)_{t\geq 0}$ as the distributional limit of

$$O^{-1} \begin{pmatrix} K \\ \int_{(0,t]} \mathcal{E}(\mathcal{U}^3)_{s-} d(\mathcal{L}_s^2 + \mathcal{U}_s^2 K) \end{pmatrix}$$

as t tends to infinity. A corresponding result for strictly non-causal solutions is given in Corollary 5.15, too.

1.4 Applications of Generalized Ornstein-Uhlenbeck Processes

Although in this work we shall not deal with specific applications we will sketch here two examples for the usage of generalized Ornstein-Uhlenbeck processes.

Paulsen's risk process

The classical Cramér-Lundberg model describes the capital of an insurance company at time t by

$$V_t = V_0 + pt - \sum_{i=1}^{N_{L,t}} S_{L,i} + B_{L,t}, \quad t \geq 0,$$

where V_0 describes the initial capital, p is a constant premium rate, $(N_{L,t})_{t\geq 0}$ is a Poisson process with $N_{L,s}$ representing the number of claims up to time s and $\{S_{L,i}\}_{i\in\mathbb{N}}$ is an i.i.d. sequence independent of $(N_{L,t})_{t\geq 0}$ where $S_{L,i} > 0$ denotes the size of the ith claim such that the claim process $t \mapsto \sum_{i=1}^{N_{L,t}} S_{L,i}$ is a compound Poisson process. The disturbance term $(B_{L,t})_{t\geq 0}$ is supposed to be a Brownian motion with variance σ_L^2 and is supposed to be independent of the claim process. Hence it holds

$$V_t = V_0 + L_t, \quad t \geq 0, \tag{1.21}$$

for the Lévy process $(L_t)_{t\geq 0}$ defined by

$$L_t = pt - \sum_{i=1}^{N_{L,t}} S_{L,i} + B_{L,t}, \quad t \geq 0.$$

Paulsen [54] proposed to combine this model with an investment model. In fact, assume that the insurance company invests its capital in a market whose evolution is given by the stochastic differential equation

$$dY_t = Y_{t-}dU_t, \quad t \geq 0, \tag{1.22}$$

with investment returns described by the Lévy process

$$U_t = rt + \sum_{i=1}^{N_{U,t}} S_{U,i} + B_{U,t}, \quad t \geq 0.$$

Here r is a constant interest rate, $t \mapsto \sum_{i=1}^{N_{U,t}} S_{U,i}$ is a compound Poisson process and $(B_{U,t})_{t\geq 0}$ is a Brownian motion with variance σ_U^2, independent of $t \mapsto \sum_{i=1}^{N_{U,t}} S_{U,i}$. The SDE (1.22) is the SDE of a stochastic exponential as in Definition 1.8 such that it follows from (1.9) that we have to assume that $S_{U,1} > -1$ a.s. to prevent Y_t from becoming negative due to a jump of U, which is a natural assumption.
Now the proposed combination of (1.21) and (1.22) leads to the SDE (1.11) of the GOU process with independent processes U and L. Since $\Pi_U((-\infty, -1]) = 0$ holds, its solution, and thus the capital process, is a generalized Ornstein-Uhlenbeck process.

The COGARCH model

To model the volatility of a financial time series, Engle [18] in 1982 proposed the ARCH (autoregressive conditionally heteroscedastic) process of which in 1986 Bollerslev [11] developed the GARCH (generalized ARCH) process. In detail, let $(\epsilon_n)_{n\in\mathbb{N}_0}$ be an i.i.d. sequence of random variables and let $\beta > 0$, $\lambda > 0$ and $\delta \geq 0$ be fixed parameters. Then a solution $(Y_n)_{n\in\mathbb{N}_0}$ of

$$\begin{aligned} Y_n &= \sigma_n \epsilon_n, \quad n \in \mathbb{N}_0, \\ \sigma_n^2 &= \beta + \lambda Y_{n-1}^2 + \delta \sigma_{n-1}^2, \quad n \in \mathbb{N}, \end{aligned}$$

such that σ_n is independent of $(\epsilon_{n+h})_{h\in\mathbb{N}_0}$ and non-negative for all $n \in \mathbb{N}_0$, is called a *GARCH(1,1) process with volatility process* $(\sigma_n)_{n\in\mathbb{N}_0}$. For $\delta = 0$ the GARCH(1,1) reduces to the ARCH(1) process.
Since in financial mathematics continuous time settings are widely used, the GARCH model has been generalized in 2004 by Klüppelberg, Lindner and Maller [39] to a continuous time model, the COGARCH (continuous-time GARCH), as following. Let $(X_t)_{t\geq 0}$ be a Lévy process with non-zero Lévy measure and let $\beta > 0$, $\lambda > 0$ and $\delta \geq 0$ be fixed parameters. Then a solution $(G_t)_{t\geq 0}$, $G_0 = 0$, of

$$\begin{aligned} dG_t &= \sigma_{t-} dX_t, \quad t \geq 0, \\ \sigma_t^2 &= e^{-\xi_t} \left(\beta \int_{(0,t]} e^{\xi_{s-}} ds + \sigma_0^2 \right), \quad t \geq 0, \end{aligned}$$

with $(\xi_t)_{t\geq 0}$ defined by

$$\xi_t = -t\log\delta - \sum_{0<s\leq t}\log(1+\frac{\lambda}{\delta}(\Delta X_s)^2), \quad t\geq 0,$$

is called *COGARCH process with volatility process* $(\sigma_t)_{t\geq 0}$ where $\sigma_t = \sqrt{\sigma_t^2}$. Remark that $(\xi_t)_{t\geq 0}$ is itself a Lévy process and hence the squared volatility process in the above definition is a generalized Ornstein-Uhlenbeck process driven by $(\xi_t, \beta t)_{t\geq 0}$.

Chapter 2: Stationary Solutions of the SDE $dV_t = V_{t-}dU_t + dL_t$ with Lévy Noise[1]

Let $(\xi, \eta) = (\xi_t, \eta_t)_{t\geq 0}$ be a bivariate Lévy process. The generalized Ornstein–Uhlenbeck process (GOU) associated with (ξ, η) is given by

$$V_t = e^{-\xi_t}\left(V_0 + \int_0^t e^{\xi_{s-}} d\eta_s\right), \quad t \geq 0,$$

where V_0 is a finite random variable, *independent* of (ξ, η). As described in Section 1.2, in [46] necessary and sufficient conditions for a GOU to be strictly stationary were obtained, and properties of the strictly stationary solution studied. It was also pointed out in Section 1.2, that the GOU in (1.10) is the unique solution of the stochastic differential equation (1.11), i.e. of $dV_t = V_{t-}dU_t + dL_t$, $t \geq 0$, where (U, L) is another bivariate Lévy process, constructed from (ξ, η) by (1.12).
The purpose of the present chapter is to extend the results of [46] to the more general setting of solutions to the stochastic differential equation (1.11), where (U, L) is an arbitrary bivariate Lévy process. In particular, we do not assume that the Lévy measure Π_U of U is concentrated on $(-1, \infty)$, but also allow jumps of size less than or equal to -1. As a second generalization, we shall allow possible dependence between the starting random variable V_0 and (U, L). Even in the case when $\Pi_U((-\infty, -1]) = 0$, this represents a sharpening of the results of [46]. As in time series analysis, we will call a solution with V_0 being independent of (U, L) a *causal* or *non-anticipative* solution. We shall see that non-causal solutions can appear in some important cases. Dealing with the non-causality is non-trivial as it introduces a possible problem regarding the filtration with respect to which the stochastic differential equation (1.11) is defined, such that U still remains a semimartingale. Hence, in the following, possible non-causal solutions (relevant in the case $\Pi_U(\{-1\}) = 0$) will be interpreted in the following sense. First, (1.11) is solved assuming that U is a semimartingale for a suitable filtration to which V is adapted. This is achieved, with the general solution given by (2.8) below. In Equation (2.8), however, the semimartingale problem is

[1]Based on [5]: A. Behme, A. Lindner and R. Maller (2011):
Stationary solutions of the stochastic differential equation $dV_t = V_{t-}dU_t + dL_t$ with Lévy noise, *Stochastic Processes and their Applications* **121**, 91–108

avoided since V_0 enters in an additive fashion there and does not have to be measurable with respect to the filtration for which the stochastic integrals are defined. The problem of finding all stationary solutions is thus reduced to finding all possible choices of V_0, without assuming independence, such that the process given by (2.8) is strictly stationary.
This we do in Theorems 2.1 and 2.2 of the next section. After that, Section 2.2 sets notation, verifies that the solution to (1.11) is as given in Equations (2.2) and (2.8) of Theorems 2.1 and 2.2, and introduces various auxiliary processes used throughout this and the following chapter. Also in Section 2.2 necessary and sufficient conditions for the almost sure convergence of the integrals $\int_0^\infty \mathcal{E}(U)_{s-}\, dL_s$ and $\int_0^\infty [\mathcal{E}(U)_{s-}]^{-1}\, d\eta_s$ in terms of the characteristic triplets of the underlying Lévy processes are given. These are essential results for characterizing the existence of a stationary solution to (1.11).
Section 2.3 gives the proofs of Theorems 2.1 and 2.2, and of two useful corollaries also stated in Section 2.1. The semimartingale problem described above is taken up again in Section 2.4. In the situation of Theorem 2.1 (b), non-causal solutions of (2.8) appear, and Section 2.4 is concerned with the question of filtration enlargements such that the non-causal solution is adapted and U remains a semimartingale with respect to it. It is shown that absolute continuity of $\int_0^\infty [\mathcal{E}(U)_{s-}]^{-1}\, d\eta_s$ is a sufficient condition for this to hold and examples when this condition is satisfied are mentioned.

As noted in the Introduction, the GOU and stationary solutions of the SDE (1.11) are important in the analysis of the COGARCH (COntinuous time GARCH model) due to Klüppelberg et al. [39]. An option pricing model based on COGARCH, and incorporating the possibility of default, has recently been proposed by Szimayer; see Klüppelberg et al. [41]. For the solution of (1.11), in a financial process setting, a jump of U of size -1 can be interpreted as the occurrence of default, and jumps of size less than -1 have interpretations when U describes the value of a certain contract, when a positive value is turned into obligations one has to pay.

2.1 Main Results

Let (U, L) be a bivariate Lévy process with characteristic triplet $\left(\begin{pmatrix} \sigma_U^2 & \sigma_{U,L} \\ \sigma_{U,L} & \sigma_L^2 \end{pmatrix},\right.$ $\left.\Pi_{U,L}, \gamma_{U,L}\right)$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$, and correspondingly denote the characteristic triplets of the coordinate processes U and L by $(\sigma_U^2, \Pi_U, \gamma_U)$ and $(\sigma_L^2, \Pi_L, \gamma_L)$, respectively. To avoid trivialities assume throughout that neither U nor L is the zero Lévy process. Let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be the smallest filtration satisfying the usual hypotheses such that both U and L are adapted. Then U and L are semimartingales with respect to $\mathbb{F}$.
The main theorems of this chapter give necessary and sufficient conditions for the existence of a strictly stationary solution of (1.11) in all cases, in particular including

$\Pi_U((-\infty,-1))\geq 0$ and $\Pi_U(\{-1\})\geq 0$. Even in the case $\Pi_U((-\infty,-1])=0$ (the only one treated in [46]) they sharpen the results of [46], since independence of V_0 and (U,L) is not assumed *a priori* in our present results, whereas it was a crucial ingredient in [46] for the proof in the oscillating case.
We first deal with the case $\Pi_U(\{-1\})=0$. Define an auxiliary process η by

$$\eta_t := L_t - \sum_{\substack{0<s\leq t\\ \Delta U_s\neq -1}} \frac{\Delta U_s \Delta L_s}{1+\Delta U_s} - t\sigma_{U,L},\ t\geq 0. \tag{2.1}$$

As will be seen in Proposition 2.6 below, the general solution to (1.11) is given by (2.8), which in the case $\Pi_U(\{-1\})=0$ simplifies to (2.2).

Theorem 2.1. *Let (U,L) be a bivariate Lévy process such that $\Pi_U(\{-1\})=0$. Let $(V_t)_{t\geq 0}$ be given by*

$$V_t = \mathcal{E}(U)_t\left(V_0 + \int_{(0,t]} [\mathcal{E}(U)_{s-}]^{-1}\, d\eta_s\right),\quad t\geq 0, \tag{2.2}$$

where the stochastic integral in (2.2) is with respect to $\mathbb{F}$.

(a) *Suppose that $\lim_{t\to\infty}\mathcal{E}(U)_t = 0$ a.s. Then a finite random variable V_0 can be chosen such that $(V_t)_{t\geq 0}$ is strictly stationary if and only if $\int_{(0,\infty)}\mathcal{E}(U)_{s-}dL_s$ converges almost surely. If this condition is satisfied, then the strictly stationary solution is unique in distribution when viewed as a random element in $\mathbb{D}[0,\infty)$, and it is obtained by choosing V_0 to be independent of (U,L) and to have the same distribution as $\int_{(0,\infty)}\mathcal{E}(U)_{s-}dL_s$.*

(b) *Suppose that $\lim_{t\to\infty}[\mathcal{E}(U)_t]^{-1}=0$ a.s. Then a finite random variable V_0 can be chosen such that $(V_t)_{t\geq 0}$ is strictly stationary if and only if $\int_{(0,\infty)}[\mathcal{E}(U)_{s-}]^{-1}d\eta_s$ converges a.s. In this case the stationary solution is unique and given by $V_t = -\mathcal{E}(U)_t\int_{(t,\infty)}[\mathcal{E}(U)_{s-}]^{-1}d\eta_s$ a.s., $t\geq 0$.*

(c) *Suppose that $\mathcal{E}(U)_t$ oscillates in the sense that*

$$0=\liminf_{t\to\infty}|\mathcal{E}(U)_t| < \limsup_{t\to\infty}|\mathcal{E}(U)_t| = +\infty \text{ a.s.}$$

Then V_t admits a strictly stationary solution if and only if there exists $k\in\mathbb{R}\backslash\{0\}$ such that $U=-L/k$. In this case the strictly stationary solution is indistinguishable from the constant process $t\mapsto k$.

The possibilities for the asymptotic behaviour of $\mathcal{E}(U)_t$ in (a), (b) and (c) of Theorem 2.1 are mutually exclusive and exhaustive; see Theorem 2.9 in Section 2.2. Conditions for the almost sure convergence of the integrals $\int_{(0,\infty)}\mathcal{E}(U)_{s-}\,dL_s$ and $\int_{(0,\infty)}[\mathcal{E}(U)_{s-}]^{-1}\,d\eta_s$ are given in Theorem 2.10 and Corollary 2.11, respectively. Observe that the solutions obtained in Theorem 2.1(a), (c) are equal in distribution to a causal solution, while the solution in part (b) is purely non-causal.

The case when $\Pi_U(\{-1\}) > 0$ is treated in the next theorem. Again, the solutions turn out to be equal in distribution to a causal solution. We will need some other auxiliary processes:

$$\widetilde{U}_t = U_t - \sum_{\substack{0<s\leq t\\ \Delta U_s=-1}} \Delta U_s \quad \text{and} \quad \widetilde{\eta}_t = \eta_t - \sum_{\substack{0<s\leq t\\ \Delta U_s=-1}} \Delta \eta_s, \quad t \geq 0, \tag{2.3}$$

and

$$K(t) := \text{number of jumps of size } -1 \text{ of } U \text{ in [0,t]}, \tag{2.4}$$
$$T(t) := \sup\{s \leq t : \Delta U_s = -1\}, \tag{2.5}$$

all for $t \geq 0$. It is easy to see that (U, L, η, K) is a Lévy process. Also, for $0 \leq s < t$ define

$$\mathcal{E}(U)_{(s,t]} := e^{(U_t - U_s) - \sigma_U^2 (t-s)/2} \prod_{s<u\leq t} (1 + \Delta U_u) e^{-\Delta U_u}, \tag{2.6}$$
$$\mathcal{E}(U)_{(s,t)} := e^{(U_{t-} - U_s) - \sigma_U^2 (t-s)/2} \prod_{s<u<t} (1 + \Delta U_u) e^{-\Delta U_u}, \tag{2.7}$$

while for $s \geq t$ let $\mathcal{E}(U)_{(s,t]} := 1$. Recall again that (2.8) gives the general solution of (1.11) as will be seen in Proposition 2.6.

Theorem 2.2. *Let (U, L) be a bivariate Lévy process such that $\Pi_U(\{-1\}) > 0$. Let η and K be as defined in (2.1) and (2.4), respectively, and let $(V_t)_{t\geq 0}$ be given by*

$$\begin{aligned} V_t \;=\; & \mathcal{E}(U)_t \left(V_0 + \int_{(0,t]} [\mathcal{E}(U)_{s-}]^{-1}\, d\eta_s \right) \mathbb{1}_{\{K(t)=0\}} \\ & + \mathcal{E}(U)_{(T(t),t]} \left(\Delta L_{T(t)} + \int_{(T(t),t]} \left[\mathcal{E}(U)_{(T(t),s)}\right]^{-1} d\eta_s \right) \mathbb{1}_{\{K(t)\geq 1\}}, \quad t \geq 0, \end{aligned} \tag{2.8}$$

where the stochastic integrals in (2.8) are with respect to $\mathbb{F}$. Then the following hold:

(a) *A finite random variable V_0 can be chosen such that $(V_t)_{t\geq 0}$ is strictly stationary. More precisely, with $\widetilde{U}$ and $\widetilde{\eta}$ as defined in (2.3), define*

$$Z_t = \mathcal{E}(\widetilde{U})_t \left(Y + \int_{(0,t]} [\mathcal{E}(\widetilde{U})_{s-}]^{-1} d\widetilde{\eta}_s \right), \quad t \geq 0, \tag{2.9}$$

where Y is a random variable, independent of (U, L), with distribution

$$P_Y(dy) = \frac{\Pi_{U,L}(\{-1\}, dy)}{\Pi_U(\{-1\})},$$

i.e., Y has the same distribution as ΔL_{T_1}, where T_1 denotes the time of the first jump of U of size -1. Let τ be an exponentially distributed random variable with parameter $\lambda := \Pi_U(\{-1\})$, independent of (U, L) and Y. Then if V_0 is chosen to be independent of (U, L) and to have the same distribution as Z_τ, the process $(V_t)_{t\geq 0}$ is strictly stationary.

(b) *Any two strictly stationary solutions $(V_t)_{t\geq 0}$ are equal in distribution when viewed as random elements of $\mathbb{D}[0,\infty)$, having the same distribution as the process specified in (a).*

The necessary and sufficient conditions for strictly stationary solutions of (1.11) in the specific cases can be summarised as follows:

Corollary 2.3. *Let (U,L) be a bivariate Lévy process, and let $(\eta_t)_{t\geq 0}$ and $V = (V_t)_{t\geq 0}$ be defined by (2.1) and (2.8). Then a finite random variable V_0 can be chosen such that V is strictly stationary if and only if one of the conditions (i), (ii) or (iii) below holds:*

(i) *There exists $k \neq 0$ such that $U_t = -L_t/k$ for $t \geq 0$;*
(ii) *The integral $\int_0^t \mathcal{E}(U)_{s-}\, dL_s$ converges almost surely to a finite random variable as $t \to \infty$;*
(iii) $\Pi_U(\{-1\}) = 0$ *and the integral $\int_0^t [\mathcal{E}(U)_{s-}]^{-1}\, d\eta_s$ converges almost surely to a finite random variable as $t \to \infty$.*

If one of the conditions (i) to (iii) is satisfied, then the distributions of V_0 and of the corresponding strictly stationary process V are unique.

A natural question is that of how close the stationary solution of Theorem 2.2 is to the stationary solution of Theorem 2.1 (a) if $\Pi_U(\{-1\})$ is small. The following shows that the stationary marginal distribution of Theorem 2.1 can be obtained as a limit of stationary marginal distributions with $\Pi_U(\{-1\}) > 0$ under certain conditions, and more generally that the corresponding stationary processes converge weakly in the J_1-Skorokhod topology, which is the topology generated by the metric d in $\mathbb{D}[0,\infty)$ defined via

$$d(f,g) = \inf_{\lambda\in\Lambda} \max\big\{ \sup_{0<s\leq t} |\lambda(s) - s|, \sup_{0<s\leq t} |f(s) - (g\circ\lambda)(s)|\big\}, \quad f,g \in \mathbb{D}[0,\infty), t > 0$$

where Λ is the space of continuous bijections on $[0,\infty)$ starting in 0. For more details on the J_1-Skorokhod topology see e.g. Jacod and Shiryaev [31].

Corollary 2.4. *Let (U,L) be a bivariate Lévy process with $\Pi_U(\{-1\}) = 0$ and such that $\int_{(0,\infty)} \mathcal{E}(U)_{s-}\, dL_s$ converges almost surely. Let $V = (V_t)_{t\geq 0}$ be the strictly stationary solution of (2.2) specified in Theorem 2.1 (a). Let $(\overline{U}^{(n)}, \overline{L}^{(n)})$ be a sequence of bivariate compound Poisson processes, independent of (U,L), with Lévy measure $\lambda_n\sigma$, where σ is a probability distribution on $\{-1\}\times\mathbb{R}$ and $\lambda_n > 0$ for each $n \in \mathbb{N}$ with $\lambda_n \to 0$ as $n \to \infty$. Let $(U^{(n)}, L^{(n)}) := (U + \overline{U}^{(n)}, L + \overline{L}^{(n)})$, and let $V^{(n)} = (V_t^{(n)})_{t\geq 0}$ be the strictly stationary solution of the process associated with $(U^{(n)}, L^{(n)})$ as specified in Theorem 2.2 (a). Then $V^{(n)}$ converges weakly to V as $n \to \infty$ when viewed as random elements in $\mathbb{D}[0,\infty)$ endowed with the J_1-Skorokhod topology.*

2.2 Preliminary Results

Solving the SDE

We begin with the following lemma, which is a generalization of Proposition 2.3 in [46] and can be proved analogously. A multivariate extension will also be shown in Chapter 5, Proposition 5.23. The integrals and quadratic covariation below are understood with respect to $\mathbb{F}$.

Lemma 2.5. *Let $(U_t, L_t)_{t\geq 0}$ be a bivariate Lévy process with $\Pi_U(\{-1\}) = 0$ and $(\eta_t)_{t\geq 0}$ defined by (2.1). Then for every $t \geq 0$, we have*

$$\int_{(0,t]} \mathcal{E}(U)_{s-} dL_s = \int_{(0,t]} \mathcal{E}(U)_{s-} d\eta_s + [\mathcal{E}(U), \eta]_t \tag{2.10}$$

and

$$\begin{pmatrix} \mathcal{E}(U)_t \\ \mathcal{E}(U)_t \int_{(0,t]} [\mathcal{E}(U)_{s-}]^{-1} \, d\eta_s \end{pmatrix} \stackrel{D}{=} \begin{pmatrix} \mathcal{E}(U)_t \\ \int_{(0,t]} \mathcal{E}(U)_{s-} dL_s \end{pmatrix}. \tag{2.11}$$

We can now verify that (2.2) and (2.8) solve the stochastic differential equation (1.11). For the case that both U and L remain semimartingales for $\mathbb{H}$ in the following Proposition, the result can be found in Exercise V.27 of Protter [55], who refers to an unpublished note by Yoeurp and Yor. For the case that additionally $\Pi_U((-\infty, -1]) = 0$ see also De Haan and Karandikar [27] or Equation (15) of Maller et al. [49]. Given that U and L are semimartingales and $\Pi_U(\{-1\}) = 0$ the result is also given in Jaschke [33, Theorem 1]. Since the result is of fundamental importance for this work, we shall give a short sketch of its proof for the case when both U and L remain semimartingales and then extend it to the case when only U remains a semimartingale.

Proposition 2.6. *Let V_0 be a finite random variable and let $\mathbb{H} = (\mathcal{H}_t)_{t\geq 0}$ be the smallest filtration satisfying the usual hypotheses which contains $\mathbb{F}$ and is such that V_0 is $\mathcal{H}_0$ measurable. Let η, K, T be as defined in (2.1), (2.4) and (2.5), respectively. Assume that U remains a semimartingale with respect to $\mathbb{H}$. Then the unique adapted càdlàg solution to (1.11), or, equivalently, to the integral equation*

$$V_t = V_0 + L_t + \int_{(0,t]} V_{s-} dU_s, \quad t \geq 0, \tag{2.12}$$

is given by (2.8). If $\Pi_U(\{-1\}) = 0$, then the unique solution is given by (2.2).

Proof. By Theorem V.7 in Protter [55], (2.12) has a unique $\mathbb{H}$-adapted càdlàg solution, so it only remains to show that the process given by (2.8) satisfies (2.12). For that, suppose first that V_0 is $\mathcal{F}_0$-measurable, so $\mathbb{H} = \mathbb{F}$, in which case the result is known from Exercise V.27 in [55], but again it is useful to give a short sketch:

since the solution of (2.12) clearly satisfies $V_t = \Delta L_t$ if $\Delta U_t = -1$, the equation renews itself with starting value ΔL_t whenever a jump in K occurs at time t, so by (2.8) it suffices to consider the case $\Pi_U(\{-1\}) = 0$, thus $K(t) = 0$. Then writing $A_t = \mathcal{E}(U)_t$ and $B_t = V_0 + \int_{(0,t]} \mathcal{E}(U)_{s-}^{-1} \, d\eta_s$, the process V given by (2.2) satisfies $V_t = A_t B_t$ and A, B, V are semimartingales with respect to $\mathbb{F}$. Partial integration then gives

$$\begin{aligned} V_t - V_0 &= \int_{(0,t]} A_{s-} dB_s + \int_{(0,t]} B_{s-} dA_s + [A,B]_t \\ &= \int_{(0,t]} d\eta_s + \int_{(0,t]} B_{s-} d\left(\mathcal{E}(U)_s\right) + \int_{(0,t]} [\mathcal{E}(U)_{s-}]^{-1} d\left([\mathcal{E}(U), \eta]_s\right) \\ &= \int_{(0,t]} dL_s + \int_{(0,t]} V_{s-} dU_s, \end{aligned}$$

where we have used that $d\mathcal{E}(U)_t = \mathcal{E}(U)_{t-} \, dU_t$ and $d[\mathcal{E}(U), \eta]_t = \mathcal{E}(U)_{t-} d(L_t - \eta_t)$ (the latter follows from (2.10)). Thus (2.12) holds.
Now suppose that V_0 is not necessarily $\mathcal{F}_0$-measurable and that U remains a semimartingale with respect to $\mathbb{H}$. Let V_t be the unique $\mathbb{H}$-adapted càdlàg solution of (2.12) and define a process V' by

$$V'_t := V_t - V_0 \mathcal{E}(U)_t \mathbb{1}_{\{K(t)=0\}} = V_t - V_0 \mathcal{E}(U)_t, \quad t \geq 0. \tag{2.13}$$

Substituting for V_t in (2.12) gives

$$\begin{aligned} V'_t &= V_0 + L_t + \int_{(0,t]} V'_{s-} \, dU_s + \int_{(0,t]} V_0 \mathcal{E}(U)_{s-} \, dU_s - V_0 \mathcal{E}(U)_t \\ &= L_t + \int_{(0,t]} V'_{s-} \, dU_s + V_0 \left(1 + \int_{(0,t]} \mathcal{E}(U)_{s-} \, dU_s - \mathcal{E}(U)_t \right) \\ &= L_t + \int_{(0,t]} V'_{s-} \, dU_s. \end{aligned}$$

Since $V'_0 = 0$ is $\mathcal{F}_0$-measurable it follows from the part already proved that V'_t is of the form (2.8) with $V'_0 = 0$, and (2.13) then shows that V_t satisfies (2.12). □

As already pointed out at the beginning of this chapter, when seeking stationary solutions of the SDE (1.11), in Theorems 2.1 and 2.2 we more conveniently look for stationary solutions of Equation (2.8), since no semimartingale problems with respect to $\mathbb{H}$ arise in (2.8), the integrals being defined in terms of $\mathbb{F}$ there. The arising semimartingale problem for the SDE (1.11) for non-causal solutions as in Theorem 2.1 (b) is taken up again in Section 2.4. In the case that V_0 is chosen independent of (U, L), as in Theorems 2.1 (a), (c) and Theorem 2.2, there are no problems with the filtration, since then, further, U, L and η all remain semimartingales for $\mathbb{H}$ by Corollary 1 to Theorem VI.11 in [55]. In that case, $(V_t)_{t \geq 0}$ is also a time homogeneous Markov process and we give its transition functions in the following lemma. Recall $\widetilde{U}$ and $\widetilde{\eta}$ defined in (2.3).

Lemma 2.7. *Let $(V_t)_{t\geq 0}$ be as defined in (2.8) and suppose that V_0 is independent of $(U_t, L_t)_{t\geq 0}$. Then $(V_t)_{t\geq 0}$ is a time homogeneous Markov process. More precisely, defining*

$$A_{s,t} := \mathcal{E}(\widetilde{U})_{(s,t]} \mathbb{1}_{\{K(t)=K(s)\}} \text{ and } B_{s,t} := \mathcal{E}(\widetilde{U})_{(s,t]} \int_{(s,t]} \left[\mathcal{E}(\widetilde{U})_{(s,u)}\right]^{-1} d\widetilde{\eta}_u \qquad (2.14)$$

for $0 \leq s < t$, with $\widetilde{U}$ and $\widetilde{\eta}$ given by (2.3), we have

$$V_t = [A_{s,t}V_s + B_{s,t}]\,\mathbb{1}_{\{K(t)-K(s)=0\}} + \left[A_{T(t),t}\Delta L_{T(t)} + B_{T(t),t}\right] \mathbb{1}_{\{K(t)-K(s)>0\}}, \quad (2.15)$$

with $(A_{s,t}, B_{s,t}, K(t)-K(s))_{t\geq s}$ being independent of $\mathcal{H}_s$ and

$$(A_{s,t}, B_{s,t}, K(t)-K(s)) \stackrel{D}{=} (A_{s+h,t+h}, B_{s+h,t+h}, K(t+h)-K(s+h)) \qquad (2.16)$$

for every $h \geq 0$ and $t \geq s$. Here, $\mathcal{H}_s$ is as defined in Proposition 2.6.

Proof. These are direct consequences of (2.8) and the strong Markov property of Lévy processes, respectively. □

Other Auxiliary Processes and their Properties

In the case that $\Pi_U(\{-1\}) = 0$ it is helpful to introduce the processes $N = (N_t)_{t\geq 0}$, $\widehat{U} = (\widehat{U}_t)_{t\geq 0}$ and $W = (W_t)_{t\geq 0}$ defined by

$$\begin{aligned} N_t &:= \text{number of jumps of size} < -1 \text{ of } U \text{ in } [0,t], && (2.17)\\ \widehat{U}_t &:= -U_t + \sigma_U^2 t/2 + \sum_{0<s\leq t} [\Delta U_s - \log|1+\Delta U_s|]\,, && (2.18)\\ W_t &:= -U_t + \sigma_U^2 t + \sum_{0<s\leq t} \frac{(\Delta U_s)^2}{1+\Delta U_s}. && (2.19) \end{aligned}$$

Then $(U, L, \eta, N, \widehat{U}, W)$ is a Lévy process. We are interested in the characteristic triplets of $\widehat{U}$ and W and their expectations when they exist, which appear in Theorem 2.9 and Corollary 2.11, respectively.

Lemma 2.8. *Let U have characteristic triplet $(\sigma_U^2, \Pi_U, \gamma_U)$ such that $\Pi_U(\{-1\}) = 0$ holds. Let N, $\widehat{U}$ and W be as defined in (2.17)–(2.19). Then we have:*

(a) *The process $\widehat{U}$ is a Lévy process satisfying*

$$\mathcal{E}(U)_t = (-1)^{N_t} e^{-\widehat{U}_t}, \quad t \geq 0, \qquad (2.20)$$

and the characteristic triplet $(\sigma_{\widehat{U}}^2, \Pi_{\widehat{U}}, \gamma_{\widehat{U}})$ of $\widehat{U}$ has $\sigma_{\widehat{U}}^2 = \sigma_U^2$, $(\Pi_{\widehat{U}})_{|\mathbb{R}\setminus\{0\}} = X(\Pi_U)_{|\mathbb{R}\setminus\{0\}}$ and

$$\begin{aligned} \gamma_{\widehat{U}} &= -\gamma_U + \sigma_U^2/2 \\ &\quad + \int_{\mathbb{R}} \left(x\mathbb{1}_{\{|x|\leq 1\}} - (\log|1+x|)\mathbb{1}_{\{x\in[-e-1,-1-e^{-1}]\cup[e^{-1}-1,e-1]\}}\right) \Pi_U(dx), \end{aligned}$$

where $X(\Pi_U)$ *is the image measure of* Π_U *under the transformation*

$$X : \mathbb{R} \setminus \{-1\} \to \mathbb{R}, \quad x \mapsto X(x) = -\log|1+x|. \tag{2.21}$$

We have $E|\widehat{U}_1| < \infty$ *if and only if*

$$\int_{|x|\geq e} \log|x|\,\Pi_U(dx) < \infty \quad \textit{and} \quad \int_{(-3/2,-1/2)} \big|\log|1+x|\big|\,\Pi_U(dx) < \infty, \tag{2.22}$$

in which case

$$E\widehat{U}_1 = -\gamma_U + \sigma_U^2/2 + \int_{\mathbb{R}} (x\mathbb{1}_{\{|x|\leq 1\}} - \log|1+x|)\Pi_U(dx). \tag{2.23}$$

(b) *The process* W *is a Lévy process satisfying*

$$[\mathcal{E}(U)_t]^{-1} = \mathcal{E}(W)_t, \quad t \geq 0, \tag{2.24}$$

and its characteristic triplet $(\sigma_W^2, \Pi_W, \gamma_W)$ *is given by* $\sigma_W^2 = \sigma_U^2$, $\Pi_W = Y(\Pi_U)$ *for the transformation*

$$Y : \mathbb{R} \setminus \{-1\} \to \mathbb{R} \setminus \{-1\}, \quad x \mapsto Y(x) = \frac{-x}{1+x},$$

and

$$\gamma_W = -\gamma_U + \sigma_U^2 + \int_{\mathbb{R}} \left(x\mathbb{1}_{\{|x|\leq 1\}} - \frac{x}{1+x}\mathbb{1}_{\{x\geq -1/2\}} \right) \Pi_U(dx).$$

We have $E|W_1| < \infty$ *if and only if*

$$\int_{(-3/2,-1/2)} |1+x|^{-1}\,\Pi_U(dx) < \infty, \tag{2.25}$$

in which case

$$EW_1 = -\gamma_U + \sigma_U^2 + \int_{[-1,1]} \frac{x^2}{1+x}\Pi_U(dx) - \int_{|x|>1} \frac{x}{1+x}\Pi_U(dx). \tag{2.26}$$

Proof. (a) Equation (2.20) is immediate from (1.9), (2.17) and (2.18). From (2.18) we obtain

$$\Delta\widehat{U}_t = -\log|1 + \Delta U_t|, \quad t \geq 0,$$

which implies $(\Pi_{\widehat{U}})_{|\mathbb{R}\setminus\{0\}} = X(\Pi_U)_{|\mathbb{R}\setminus\{0\}}$. The Brownian motion components of $\widehat{U}$ and U satisfy $B_{\widehat{U}_t} = -B_{U_t}$, so that $\sigma_{\widehat{U}}^2 = \sigma_U^2$. For the calculation of $\gamma_{\widehat{U}}$, take $\varepsilon > 0$ and let $C_\varepsilon := [-1 - e^\varepsilon, -1 - e^{-\varepsilon}]$ and $D_\varepsilon := [-1 + e^{-\varepsilon}, e^\varepsilon - 1]$. Omitting the summation index $0 < s \leq 1$ in the following calculation, it then follows from (2.18) and the Lévy-Itô decomposition ([58], Theorem 19.2) that

$$\begin{aligned}
\gamma_{\widehat{U}} + B_{\widehat{U}_1} &= \widehat{U}_1 - \lim_{\varepsilon\downarrow 0}\left(\sum_{|\Delta\widehat{U}_s|>\varepsilon}\Delta\widehat{U}_s - \int_{|x|\in(\varepsilon,1]} x\,\Pi_{\widehat{U}}(dx)\right)\\
&= \sigma_U^2/2 - U_1\\
&\quad + \lim_{\varepsilon\downarrow 0}\left(\sum_{\Delta U_s\in C_\varepsilon\cup D_\varepsilon}(\Delta U_s - \log|1+\Delta U_s|) + \sum_{\Delta U_s\notin C_\varepsilon\cup D_\varepsilon}\Delta U_s + \int_{|x|\in(\varepsilon,1]} x\,\Pi_{\widehat{U}}(dx)\right)\\
&= \sigma_U^2/2 - U_1 + \lim_{\varepsilon\downarrow 0}\left(\sum_{\Delta U_s\notin D_\varepsilon}\Delta U_s + \int_{|x|\in(\varepsilon,1]} x\,\Pi_{\widehat{U}}(dx)\right)\\
&= \sigma_U^2/2 - \gamma_U - B_{U_1} + \lim_{\varepsilon\downarrow 0}\left(\int_{x\in[-1,1]\setminus D_\varepsilon} x\,\Pi_U(dx) + \int_{|x|\in(\varepsilon,1]} x\,\Pi_{\widehat{U}}(dx)\right).
\end{aligned}$$

Together with $B_{\widehat{U}_1} = -B_{U_1}$ and $\Pi_{\widehat{U}} = X(\Pi_U)$ this implies the representation for $\gamma_{\widehat{U}}$. Next, observe that $E|\widehat{U}_1| < \infty$ if and only if $\int_{|x|>1}|x|\,\Pi_{\widehat{U}}(dx) < \infty$ ([58], Example 25.12), which is equivalent to (2.22) since $\Pi_{\widehat{U}} = X(\Pi_U)$ on $\mathbb{R}\setminus\{0\}$. Equation (2.23) then follows from the representation of $\gamma_{\widehat{U}}$ and the fact that $E\widehat{U}_1 = \gamma_{\widehat{U}} + \int_{|x|>1} x\,\Pi_{\widehat{U}}(dx)$.
(b) Equation (2.24) is a special case of a formula by Karandikar [36, Theorem 1] and has also been proven by Jaschke [33].
The remaining assertions follow similarly to the ones proved in (a). □

Similarly, it can be shown that the Lévy measure of η as defined in (2.1) is the restriction to $\mathbb{R}\setminus\{0\}$ of the image measure of $\Pi_{U,L}$ under the mapping $(\mathbb{R}\setminus\{-1\})\times\mathbb{R}\to\mathbb{R}$, $(x,y)\mapsto\frac{y}{1+x}$, and moment conditions for η can be expressed in terms of the characteristic triplet of (U,L). We omit further details here.

Convergence of $\mathcal{E}(U)_t$ and Integrals Involving It

In the case $\Pi_U(\{-1\}) = 0$ the characterization of the existence of stationary solutions in Section 2.3 will be achieved in terms of the almost sure convergence of $\int_0^\infty \mathcal{E}(U)_{s-}\,dL_s$ and $\int_0^\infty[\mathcal{E}(U)_{s-}]^{-1}\,d\eta_s$. So, finally in this section, we obtain necessary and sufficient conditions for convergence of these integrals, which are also interesting in their own right.
We need also necessary and sufficient conditions for a Lévy process to drift to $\pm\infty$ in terms of its characteristic triplet. The following is a reformulation of a result of Doney and Maller (see Theorem 4.4. in [14]) for the process $\widehat{U}$ in terms of the characteristics of U. In the case when $E|\widehat{U}_1| = \infty$, it describes in particular how the large time behaviour of $\widehat{U}$ is determined by the behaviour of Π_U around -1 and for large values.

Theorem 2.9. *Let U be a non-zero Lévy process with $\Pi_U(\{-1\}) = 0$, let $\widehat{U}$ be defined by (2.18), and recall (2.20).*

(a) *The following are equivalent:*
  (i) $\mathcal{E}(U)_t$ *converges almost surely to* 0 *as* $t \to \infty$.
  (ii) $\widehat{U}_t$ *converges almost surely to* ∞ *as* $t \to \infty$.
  (iii) $0 < E\widehat{U}_1 \le E|\widehat{U}_1| < \infty$, *or* $\int_{(-3/2,-1/2)} \big|\log|1+x|\big|\, \Pi_U(dx) = \infty$ *and*

$$\int_{\mathbb{R}\setminus[-e,e]} \frac{\log|x|\, \Pi_U(dx)}{1 + \int_{1/|x|}^{1/e} \Pi_U\big((-1-z,-1+z)\big) z^{-1} dz} < \infty.$$

(b) *The following are equivalent:*
  (i) $[\mathcal{E}(U)_t]^{-1}$ *converges almost surely to* 0 *as* $t \to \infty$.
  (ii) $\widehat{U}_t$ *converges almost surely to* $-\infty$ *as* $t \to \infty$.
  (iii) $0 < -E\widehat{U}_1 \le E|\widehat{U}_1| < \infty$, *or* $\int_{|x|\ge e} \log|x|\, \Pi_U(dx) = \infty$ *and*

$$\int_{(-1-e^{-1},-1+e^{-1})} \frac{-\log|1+x|\, \Pi_U(dx)}{1 + \int_e^{1/|1+x|} \Pi_U\big(\mathbb{R}\setminus[1-z,z-1]\big) z^{-1} dz} < \infty.$$

(c) *If none of the conditions in (a) or (b) is satisfied, then $\widehat{U}$ oscillates, equivalently,*

$$0 = \liminf_{t\to\infty} |\mathcal{E}(U)_t| < \limsup_{t\to\infty} |\mathcal{E}(U)_t| = +\infty.$$

Proof of Theorem 2.9. Let us prove (a). The equivalence of (i) and (ii) is clear from (2.20). Further, by Theorem 4.4 in [14], $\widehat{U}_t$ converges almost surely to ∞ if and only if $0 < E\widehat{U}_1 \le E|\widehat{U}_1| < \infty$, or

$$\lim_{x\to\infty} A^+_{\widehat{U}}(x) = \infty \quad \text{and} \quad \int_{-\infty}^{-1} \frac{|x|\, \Pi_{\widehat{U}}(dx)}{A^+_{\widehat{U}}(|x|)}\, dx < \infty,$$

where

$$A^+_{\widehat{U}}(x) := 1 + \int_1^x \Pi_{\widehat{U}}\big((y,\infty)\big)\, dy, \quad x \ge 1.$$

Using $\Pi_{\widehat{U}} = X(\Pi_U)$ (cf. (2.21)), it is then easy to see that this is equivalent to the condition (iii). The proof of (b) is similar, and assertion (c) is well known (e.g. [58], Theorem 48.1). □

The following is a version of Theorem 1.11 for the stochastic exponential.

Theorem 2.10. *Let (U, L) be a bivariate Lévy process such that $\Pi_U(\{-1\}) = 0$. Then the following are equivalent:*

(i) *The integral* $\int_0^t \mathcal{E}(U)_{s-}dL_s$ *converges almost surely to a finite random variable as* $t \to \infty$.

(ii) *The integral* $\int_0^t \mathcal{E}(U)_{s-}dL_s$ *converges in distribution to a finite random variable as* $t \to \infty$.

(iii) $\mathcal{E}(U)_t$ *converges almost surely to 0 as* $t \to \infty$ *and*

$$I_{U,L} := \int_{\mathbb{R}\setminus[-e,e]} \frac{\log|y|\,\Pi_L(dy)}{1+\int_{1/|y|}^{1/e}\Pi_U\big((-1-z,-1+z)\big)z^{-1}dz} < \infty. \tag{2.27}$$

In the case of divergence, we have: if $\lim_{t\to\infty}\mathcal{E}(U)_t = 0$ *a.s. but* $I_{U,L} = +\infty$, *then*

$$\left|\int_{(0,t]} \mathcal{E}(U)_{s-}dL_s\right| \xrightarrow{P} \infty, \quad t\to\infty, \tag{2.28}$$

and if $\mathcal{E}(U)_t$ *does not tend to* 0 *a.s. as* $t\to\infty$, *then* (2.28) *holds or there exists* $k \in \mathbb{R}\setminus\{0\}$ *such that*

$$P\left(\int_{(0,t]} \mathcal{E}(U)_{s-}dL_s = k(1-\mathcal{E}(U)_t)\,\forall t\geq 0\right) = 1. \tag{2.29}$$

Proof of Theorem 2.10. Using $\mathcal{E}(U)_t = (-1)^{N_t}e^{-\widehat{U}_t}$, it follows in complete analogy to the proof of Erickson and Maller [19] that $\int_0^t(-1)^{N_{s-}}e^{-\widehat{U}_{s-}}\,dL_s$ converges almost surely to a finite random variable if and only if $\widehat{U}_t$ converges almost surely to $+\infty$ as $t\to\infty$ and

$$\int_{\mathbb{R}\setminus[-e,e]} \left(\frac{\log|y|}{1+\int_1^{\log|y|}\Pi_{\widehat{U}}\big((x,\infty)\big)\,dx}\right)\Pi_L(dy) < \infty,$$

which by Lemma 2.8 can be seen to be equivalent to (iii). The remaining assertions follow similarly as in [19]. □

Corollary 2.11. *Let* (U,L) *be a bivariate Lévy process such that* $\Pi_U(\{-1\}) = 0$. *Let* (W,η) *be defined by* (2.19) *and* (2.1). *Then the following are equivalent:*

(i) *The integral* $\int_0^t[\mathcal{E}(U)_{s-}]^{-1}d\eta_s$ *converges almost surely to a finite random variable as* $t\to\infty$.

(ii) *The integral* $\int_0^t[\mathcal{E}(U)_{s-}]^{-1}d\eta_s$ *converges in distribution to a finite random variable as* $t\to\infty$.

(iii) $[\mathcal{E}(U)_t]^{-1}$ *converges almost surely to 0 as* $t\to\infty$ *and* $I_{W,\eta}<\infty$, *where* $I_{W,\eta}$ *is defined similarly to* (2.27), *with* Π_L *being replaced by* Π_η *and* Π_U *by* Π_W.

Proof. This is an immediate consequence of Theorem 2.10 since $[\mathcal{E}(U)_t]^{-1} = \mathcal{E}(W)_t$ for every $t\geq 0$ by (2.24). □

2.3 Proofs of Main Results

Proof of Theorem 2.1. (a) Suppose that $\widehat{U}_t \to \infty$ a.s. as $t \to \infty$. Then $\mathcal{E}(U)_t V_0$ converges a.s. to 0 by (2.20). Thus if a stationary solution $(V_t)_{t\geq 0}$ exists, then we know that $\mathcal{E}(U)_t \int_{(0,t]} [\mathcal{E}(U)_{s-}]^{-1}\, d\eta_s$ tends to V_0 in distribution as $t \to \infty$. By (2.11) this means that $\int_{(0,t]} \mathcal{E}(U)_{s-} dL_s \stackrel{D}{\to} V_0$ as $t \to \infty$ and hence $\int_0^\infty \mathcal{E}(U)_{s-}\, dL_s$ converges almost surely by Theorem 2.10. Let $n \in \mathbb{N}$ and $h_1, \ldots, h_n \geq 0$. Since $\lim_{t\to\infty} \mathcal{E}(U)_t = 0$ a.s., and since

$$(V_{h_1}, \ldots, V_{h_n}) \stackrel{d}{=} (V_{t+h_1}, \ldots, V_{t+h_n}), \quad t \geq 0,$$

an application of Slutsky's Lemma shows that $(V_{h_1}, \ldots, V_{h_n})$ has the same distribution as the distributional limit as $t \to \infty$ of

$$\left(\mathcal{E}(U)_{t+h_1} \int_0^{t+h_1} [\mathcal{E}(U)_{s-}]^{-1}\, d\eta_s, \ldots, \mathcal{E}(U)_{t+h_n} \int_0^{t+h_n} [\mathcal{E}(U)_{s-}]^{-1}\, d\eta_s \right).$$

This does not depend on V_0. Hence any two stationary solutions have the same finite dimensional distributions and hence the same distributions when viewed as random elements in $\mathbb{D}[0,\infty)$.
Conversely, suppose that $\int_0^\infty \mathcal{E}(U)_{s-}\, dL_s$ converges almost surely to a finite random variable and take V_0 independent of (U, L) and with the same distribution as $\int_0^\infty \mathcal{E}(U)_{s-}\, dL_s$. Then, by (2.11), V_t converges in distribution to V_0 as $t \to \infty$, since $\lim_{t\to\infty} \mathcal{E}(U)_t = 0$. Together with Lemma 2.7 this shows that

$$V_t = A_{t-h,t} V_{t-h} + B_{t-h,t} \stackrel{d}{\to} A_{0,h} V_0 + B_{0,h} = V_h, \quad t \to \infty,$$

for every $h \geq 0$. Since also $V_t \stackrel{d}{\to} V_0$ as $t \to \infty$ it follows that $V_h \stackrel{D}{=} V_0$. Since $(V_t)_{t\geq 0}$ is a Markov process by Lemma 2.7, this implies strict stationarity of $(V_t)_{t\geq 0}$.
(b) Suppose that $\widehat{U}_t \to -\infty$ and hence $[\mathcal{E}(U)_t]^{-1} \to 0$ a.s. as $t \to \infty$. Then if $(V_t)_{t\geq 0}$ is a strictly stationary solution, it follows that

$$V_0 + \int_{(0,t]} [\mathcal{E}(U)_{s-}]^{-1}\, d\eta_s = [\mathcal{E}(U)_t]^{-1} V_t \stackrel{P}{\to} 0, \quad t \to \infty.$$

Hence $-\int_0^\infty [\mathcal{E}(U)_{s-}]^{-1}\, d\eta_s$ converges almost surely to V_0 by Corollary 2.11, and this immediately yields $V_t = -\mathcal{E}(U)_t \int_{(t,\infty)} [\mathcal{E}(U)_{s-}]^{-1} d\eta_s$ a.s.
Conversely, if $\int_{(0,\infty)} [\mathcal{E}(U)_{s-}]^{-1}\, d\eta_s$ converges a.s., let $V_0 := -\int_{(0,\infty)} [\mathcal{E}(U)_{s-}]^{-1}\, d\eta_s$. Then

$$V_t = -\mathcal{E}(U)_t \int_{(t,\infty)} [\mathcal{E}(U)_{s-}]^{-1} d\eta_s = \int_{(t,\infty)} (-1)^{(N_{s-} - N_t)} e^{\widehat{U}_{s-} - \widehat{U}_t}\, d\eta_s, \quad t \geq 0,$$

which is strictly stationary since $(N, \widehat{U}, \eta)$, as a Lévy process, has stationary increments.

(c) Suppose that $\widehat{U}_t$ oscillates and let $(V_t)_{t\geq 0}$ be a strictly stationary solution of (2.2). By Theorem 2.10 this implies that (2.28) or (2.29) must hold. Suppose first that (2.28) holds. Together with (2.11) this gives $|\mathcal{E}(U)_t \int_{(0,t]} [\mathcal{E}(U)_{s-}]^{-1} d\eta_s| \xrightarrow{P} \infty$ as $t \to \infty$. Since V_t is strictly stationary this and (2.2) imply that $|V_0 \mathcal{E}(U)_t|$ and thus $|\mathcal{E}(U)_t|$ tend to ∞ in probability, too. Hence $V_0 + \int_{(0,t]} [\mathcal{E}(U)_{s-}]^{-1} d\eta_s = [\mathcal{E}(U)_t]^{-1} V_t$ converges to 0 in probability, hence in distribution, so $\int_{(0,t]} [\mathcal{E}(U)_{s-}]^{-1} d\eta_s \xrightarrow{D} -V_0$ as $t \to \infty$, contradicting Corollary 2.11 because $[\mathcal{E}(U)_t]^{-1}$ does not converge. Hence (2.28) cannot occur.
Now suppose that (2.29) holds, i.e. there is a constant $k \in \mathbb{R}\backslash\{0\}$ such that for all $t > 0$ we have $\int_{(0,t]} \mathcal{E}(U)_{s-} dL_s = k(1 - \mathcal{E}(U)_t)$ a.s. Since we know that $\mathcal{E}(U)_t = 1 + \int_{(0,t]} \mathcal{E}(U)_{s-} dU_s$ a.s., this is equivalent to $\int_{(0,t]} \mathcal{E}(U)_{s-} dL_s = -k \int_{(0,t]} \mathcal{E}(U)_{s-} dU_s$ a.s. for all $t > 0$ and hence to $U = -L/k$. From (2.1) and (2.19), this implies

$$\eta_t = -kU_t + \sum_{0<s\leq t} \frac{k\Delta U_s^2}{1+\Delta U_s} + kt\sigma_U^2 = kW_t,$$

so that $\int_{(0,t]} [\mathcal{E}(U)_{s-}]^{-1} d\eta_s = k \int_{(0,t]} \mathcal{E}(W)_{s-} dW_s = (-k)(1 - [\mathcal{E}(U)_t]^{-1})$ a.s. We conclude that

$$V_t = \mathcal{E}(U)_t \left(\int_{(0,t]} [\mathcal{E}(U)_{s-}]^{-1} d\eta_s + V_0 \right) = \mathcal{E}(U)_t (V_0 - k) + k, \quad t \geq 0, \tag{2.30}$$

so $V_t - k = \mathcal{E}(U)_t (V_0 - k)$ a.s. Since V_t was assumed to be strictly stationary this yields $|V_0 - k| \stackrel{D}{=} |\mathcal{E}(U)_t||V_0 - k| = e^{-\widehat{U}_t}|V_0 - k|$, because $\mathcal{E}(U)_t = (-1)^{N_t} e^{-\widehat{U}_t}$. Since $|\widehat{U}_t| \xrightarrow{P} \infty$, we get $V_0 - k = 0$ a.s. and hence $V_t = k$ a.s. for all $t \geq 0$. So V is indistinguishable from the constant process, since it has càdlàg paths.
Conversely, if there is a $k \in \mathbb{R} \setminus \{0\}$ such that $U = -L/k$, and $V_0 := k$, then it follows from (2.30) that $V_t = k$ for all $t \geq 0$, which is a strictly stationary solution. □

Proof of Theorem 2.2. (a) Choose V_0 to be independent of (U, L) with $V_0 \stackrel{D}{=} Z_\tau$. Then $(V_t)_{t\geq 0}$ is a Markov process by Lemma 2.7, hence it suffices to show that $V_t \stackrel{d}{=} V_0$ for every $t > 0$. Fix $t > 0$ and for $k \in \mathbb{N}_0$ let $p_k := P(K(t) = k)$ and let T_k be the time of the kth jump of size -1 of U. Then by (2.8) we get, for $x \in \mathbb{R}$,

$$\begin{aligned} & P(V_t \leq x) \\ = \;& p_0 P\left(\mathcal{E}(U)_t \left(V_0 + \int_{(0,t]} [\mathcal{E}(U)_{s-}]^{-1} d\eta_s \right) \leq x \,\middle|\, K(t) = 0 \right) \\ & + \sum_{k\geq 1} p_k P\left(\mathcal{E}(U)_{(T_k,t]} \left(\Delta L_{T_k} + \int_{(T_k,t]} [\mathcal{E}(U)_{(T_k,s)}]^{-1} d\eta_s \right) \leq x \,\middle|\, K(t) = k \right) \\ =: \;& A(x) + B(x), \quad \text{say.} \end{aligned}$$

By (2.3), $U = \widetilde{U}$ and $\eta = \widetilde{\eta}$ on $\{K(t) = 0\}$. Thus

$$A(x) = p_0 P\left(\mathcal{E}(\widetilde{U})_t \left(V_0 + \int_{(0,t]} [\mathcal{E}(\widetilde{U})_{s-}]^{-1} d\widetilde{\eta}_s\right) \leq x\right).$$

Since τ and $(\widetilde{U}, \widetilde{\eta})$ are independent, an application of the strong Markov property to the Lévy process $(\widetilde{K}, \widetilde{U}, \widetilde{\eta})$, where $\widetilde{K}$ is a Poisson process with parameter λ, independent of $(\widetilde{U}, \widetilde{\eta})$ and first jump time τ, shows that $(\widetilde{U}_{t+\tau}, \widetilde{\eta}_{t+\tau})_{t\geq 0}$ is a Lévy process with the same distribution as $(\widetilde{U}_t, \widetilde{\eta}_t)_{t\geq 0}$, independent of Z_τ and V_0. Together with $V_0 \stackrel{d}{=} Z_\tau$ this shows

$$\mathcal{E}(\widetilde{U})_t \left(V_0 + \int_{(0,t]} [\mathcal{E}(\widetilde{U})_{s-}]^{-1} d\widetilde{\eta}_s\right) \stackrel{D}{=} \mathcal{E}(\widetilde{U})_{(\tau,t+\tau]} \left(Z_\tau + \int_{(\tau,t+\tau]} [\mathcal{E}(\widetilde{U})_{(\tau,s)}]^{-1} d\widetilde{\eta}_s\right).$$

Hence we obtain for $A(x)$ recalling that $p_0 = e^{-\lambda t}$

$$\begin{aligned}
A(x) &= p_0 P\Bigg(\mathcal{E}(\widetilde{U})_{(\tau,t+\tau]} \Bigg(\mathcal{E}(\widetilde{U})_\tau \left(Y + \int_{(0,\tau]} [\mathcal{E}(\widetilde{U})_{s-}]^{-1} d\widetilde{\eta}_s\right) \\
&\qquad + \int_{(\tau,t+\tau]} [\mathcal{E}(\widetilde{U})_{(\tau,s)}]^{-1} d\widetilde{\eta}_s\Bigg) \leq x\Bigg) \\
&= p_0 P\left(\mathcal{E}(\widetilde{U})_{t+\tau} \left(Y + \int_{(0,t+\tau]} [\mathcal{E}(\widetilde{U})_{s-}]^{-1} d\widetilde{\eta}_s\right) \leq x\right) \\
&= p_0 P\left(Z_{t+\tau} \leq x\right) \\
&= e^{-\lambda t} \int_{(0,\infty)} P(Z_{t+y} \leq x)\, dP_\tau(y) \\
&= \lambda \int_{(t,\infty)} P(Z_y \leq x) e^{-\lambda y} dy.
\end{aligned}$$

For $B(x)$, recall that the times of jumps of size -1 on an interval $[0,t]$ of the Lévy process U given the value of $K(t) = k$ have the same distribution as the order statistics of k uniformly distributed random variables on $[0,t]$. In particular, $P(T_k \leq y | K(t) = k) = (y/t)^k$ for all $0 \leq y \leq t$. Defining a random variable $\upsilon(k)$ with this distribution, independent of (U, L), we conclude recalling that $p_k = e^{-\lambda t}(\lambda t)^k/k!$

$$\begin{aligned}
B(x) &= \sum_{k\geq 1} p_k P\left(\mathcal{E}(\widetilde{U})_{(T_k,t]} \left(Y + \int_{(T_k,t]} [\mathcal{E}(\widetilde{U})_{(T_k,s)}]^{-1} d\widetilde{\eta}_s\right) \leq x \,\middle|\, K(t) = k\right) \\
&= \sum_{k\geq 1} p_k P\left(\mathcal{E}(\widetilde{U})_{t-\upsilon(k)} \left(Y + \int_{(0,t-\upsilon(k)]} [\mathcal{E}(\widetilde{U})_{s-}]^{-1} d\widetilde{\eta}_s\right) \leq x\right) \\
&= \sum_{k\geq 1} p_k P(Z_{t-\upsilon(k)} \leq x) \\
&= \sum_{k\geq 1} \frac{e^{-\lambda t}(\lambda t)^k}{k!} \int_{(0,t]} P(Z_{t-y} \leq x) d(y/t)^k
\end{aligned}$$

$$= \lambda e^{-\lambda t} \int_{(0,t]} P(Z_{t-y} \le x) e^{\lambda y} dy$$
$$= \lambda \int_{(0,t]} P(Z_y \le x) e^{-\lambda y} dy.$$

Summing $A(x)$ and $B(x)$ we obtain

$$P(V_t \le x) = \lambda \int_{(0,\infty)} P(Z_y \le x) e^{-\lambda y} dy = P(Z_\tau \le x) = P(V_0 \le x),$$

so $V_t \stackrel{d}{=} V_0$, giving strict stationarity of $(V_t)_{t\ge 0}$.
(b) Let $(V_t)_{t\ge 0}$ be a strictly stationary solution of (2.8). Then for any $n \in \mathbb{N}$ and $h_1, \dots, h_n \ge 0$ we have

$$(V_{t+h_1}, \dots, V_{t+h_n}) \stackrel{D}{\to} (V_{h_1}, \dots, V_{h_n}), \quad t \to \infty,$$

and since $K(t) \to +\infty$ a.s. as $t \to \infty$, it can be seen from (2.8) that the last expression does not depend on V_0. Hence any two strictly stationary solutions have the same finite dimensional distributions and hence are equal as random elements in $\mathbb{D}[0,\infty)$. □

Proof of Corollary 2.3. To show sufficiency of each of the conditions (i)–(iii), it is enough to suppose $\Pi_U(\{-1\}) = 0$, since otherwise a strictly stationary solution exists by Theorem 2.2. Then by Theorem 2.10 and Corollary 2.11, convergence of $\int_0^\infty \mathcal{E}(U)_{s-}\, dL_s$ and that of $\int_0^\infty [\mathcal{E}(U)_{s-}]^{-1}\, d\eta_s$ imply $\lim_{t\to\infty} \mathcal{E}(U)_t = 0$ a.s. and $\lim_{t\to\infty} [\mathcal{E}(U)_t]^{-1} = 0$, respectively, so that Theorem 2.1 (a), (b) shows sufficiency of conditions (ii) and (iii). By Theorem 2.1 (c), condition (i) is sufficient if $\hat{U}$ oscillates, but its proof shows that (i) is sufficient whenever $\Pi_U(\{-1\}) = 0$, since $U = -L/k$ clearly implies Equation (2.30) by the same argument. The uniqueness assertion is clear from Theorems 2.1 and 2.2.
To see that the existence of a strictly stationary solution implies at least one of the conditions (i)–(iii), observe that this is clear from Theorem 2.1 if $\Pi_U(\{-1\}) = 0$. In the case that $\Pi_U(\{-1\}) > 0$, denote by T_1 the time of the first jump of U of size -1. Then T_1 is finite almost surely and it holds $\mathcal{E}(U)_t = 0$ for $t \ge T_1$. Hence the integral $\int_0^\infty \mathcal{E}(U)_{s-}\, dL_s$ converges almost surely, which is condition (ii). □

Proof of Corollary 2.4. In the following we denote the quantities corresponding to $(U^{(n)}, L^{(n)})$ as needed in Theorem 2.2 (a) by $\widetilde{\eta}^{(n)}, T_1^{(n)}, \tau^{(n)}$, etc. Observe that $\widetilde{U}^{(n)} = U$ and $\widetilde{\eta}^{(n)} = \eta$. Further observe that convergence of $\int_{(0,\infty)} \mathcal{E}(U)_{s-}\, dL_s$ implies that $\mathcal{E}(U)_t \to 0$ a.s. as $t \to \infty$ by Theorem 2.10. But since the distribution of $\Delta L^{(n)}_{T_1^{(n)}}$ is σ for each n, it follows that $\mathcal{E}(U)_{\tau^{(n)}} Y^{(n)} \stackrel{P}{\to} 0$ as $n \to \infty$ since $\lambda_n \to 0$. Next, observe that

$$\mathcal{E}(U)_{\tau^{(n)}} \int_{(0,\tau^{(n)}]} [\mathcal{E}(U)_{s-}]^{-1}\, d\eta_s \stackrel{d}{=} \int_{(0,\tau^{(n)}]} \mathcal{E}(U)_{s-}\, dL_s,$$

which follows from (2.11) by conditioning on $\tau^{(n)} = t$ and using that $\tau^{(n)}$ is independent of (U, L). This, together with $\mathcal{E}(U)_{\tau^{(n)}} Y^{(n)} \xrightarrow{P} 0$ implies that

$$V_0^{(n)} \stackrel{d}{=} Z^{(n)}_{\tau^{(n)}} \xrightarrow{d} \int_{(0,\infty)} \mathcal{E}(U)_{s-}\, dL_s \stackrel{d}{=} V_0, \quad n \to \infty,$$

so that the marginal stationary distributions converge weakly. By Skorokhod's theorem we can then assume that $V_0^{(n)}$ and V_0 are additionally chosen such that $V_0^{(n)} \to V_0$ a.s. as $n \to \infty$, since this does not alter the distributions of the processes $V^{(n)}$ and V, respectively, and we are only concerned with weak convergence. But since $\lambda_n \to 0$ we have $K^{(n)}(t) \xrightarrow{P} 0$ as $n \to \infty$ for fixed $t \geq 0$, and hence it follows from (2.2) and (2.8), for any $t > 0$ and $\varepsilon > 0$,

$$\lim_{n\to\infty} P\left(\sup_{0 \leq s \leq t} |V_s - V_s^{(n)}| > \varepsilon \right) = 0,$$

giving weak convergence of $V^{(n)}$ to V in the Skorokhod topology (cf. Jacod and Shiryaev [31], Lemma VI.3.31, p. 352). □

2.4 Filtration Expansions

Having determined all strictly stationary solutions of (2.8), it is natural to ask whether the strictly stationary process $(V_t)_{t\geq 0}$ still satisfies (2.12) for the smallest filtration $\mathbb{H} = (\mathcal{H}_t)_{t\geq 0}$ containing $\mathbb{F}$, satisfying the usual hypotheses and which is such that V_0 is $\mathcal{H}_0$-measurable. In other words, we pose the question: does U at least remain a semimartingale with respect to $\mathbb{H}$?

In the causal cases described in Theorem 2.1 (a),(c) and Theorem 2.2, this is indeed the case, as a consequence of Jacod's criterion (see the main result of [30]). For the non-causal cases, this is not at all evident. Clearly, if U is of bounded variation, then U remains an $\mathbb{H}$-semimartingale, but the general case is not clear. The following theorem presents a sufficient condition for all $\mathbb{F}$-semimartingales to remain $\mathbb{H}$-semimartingales. The proof is along the lines of Theorem 3.6 of Jacod [30], who considered the case $U_t = \lambda t$ with $\lambda > 0$ below, in which case the distribution of V_0 is either degenerate, or absolutely continuous.

Theorem 2.12. *Let (U, L) be a bivariate Lévy process such that $\Pi_U(\{-1\}) = 0$ and suppose that $\lim_{t\to\infty}[\mathcal{E}(U)_t]^{-1} = 0$ a.s. and that $V_0 := -\int_{(0,\infty)} [\mathcal{E}(U)_{s-}]^{-1}\, d\eta_s$ converges a.s., where η is defined by (2.1). Denote by $V_t = -\int_{(t,\infty)} [\mathcal{E}(U)_{(t,s)}]^{-1}\, d\eta_s$, $t \geq 0$, as in Theorem 2.1 (b), the unique solution of (2.2), and suppose that the distribution of V_0 is absolutely continuous or a Dirac measure. Then every $\mathbb{F}$-semimartingale is also an $\mathbb{H}$-semimartingale. In particular, U and L are $\mathbb{H}$-semimartingales and $(V_t)_{t\geq 0}$ solves (1.11) when considered as an SDE with respect to the filtration $\mathbb{H}$ and is an $\mathbb{H}$-semimartingale.*

Proof of Theorem 2.12. First observe that

$$V_0 = -\int_{(0,\infty)} [\mathcal{E}(U)_{s-}]^{-1}\, d\eta_s = -\int_{(0,t]} [\mathcal{E}(U)_{s-}]^{-1}\, d\eta_s + [\mathcal{E}(U)_t]^{-1} V_t, \qquad (2.31)$$

so that $(V_t)_{t\geq 0}$ is clearly adapted to $\mathbb{H}$, and if V_0 is a constant random variable, then $\mathbb{F} = \mathbb{H}$ and there is nothing to prove. So suppose that the law μ of V_0 is absolutely continuous. Since $\int_{(0,t]} [\mathcal{E}(U)_{s-}]^{-1}\, d\eta_s$ and $[\mathcal{E}(U)_t]^{-1}$ are measurable with respect to $\mathcal{F}_t$ but $V_t = -\int_{(t,\infty)} [\mathcal{E}(U)_{(s,t)}]^{-1}\, d\eta_s$ is independent of $\mathcal{F}_t$, and has distribution μ by stationarity of V, (2.31) shows that the regular conditional distribution of V_0 given $\mathcal{F}_t$ is given by

$$P(V_0 \in B|\mathcal{F}_t)(\omega) = \mu\big(\mathcal{E}(U)_t(\omega) B + \mathcal{E}(U)_t(\omega) \int_{(0,t]} [\mathcal{E}(U)_{s-}]^{-1}\, d\eta_s(\omega)\big)$$

for every Borel set B in $\mathbb{R}$ and $\omega \in \Omega$. Hence if the Lebesgue measure of B is zero, the Lebesgue measure of $\mathcal{E}(U)_t(\omega)B + \mathcal{E}(U)_t(\omega) \int_{(0,t]} [\mathcal{E}(U)_{s-}]^{-1}\, d\eta_s(\omega)$ is zero as well, and since μ is absolutely continuous it follows that $P(V_0 \in B|\mathcal{F}_t)(\omega) = 0$. But this means that the regular conditional distribution of V_0 given $\mathcal{F}_t$ is almost surely absolutely continuous, and hence by Jacod's criterion [30], every $\mathbb{F}$-semimartingale is an $\mathbb{H}$-semimartingale. That then also V is an $\mathbb{H}$-semimartingale follows from Theorem V.7 in [55]. □

The problem of characterizing when the law μ of $V_0 := -\int_0^\infty [\mathcal{E}(U)_{s-}]^{-1}\, d\eta_s$ appearing in Theorem 2.12 is absolutely continuous is an open question. As pointed out by Watanabe [63], it follows from Theorem 1.3 in Alsmeyer et al. [1] that μ is either absolutely continuous, continuous singular, or a Dirac measure, i.e. a pure types theorem holds for μ. Watanabe's proof is based on the fact that, by (2.31), $V_0 \overset{d}{=} V_t \overset{d}{=} \mu$ satisfies a distributional fixed point equation $V_t \overset{d}{=} V_0 = M_t V_t + Q_t$, with V_t being independent of (M_t, Q_t) and $P(M_t = 0) = 0$, for which Theorem 1.3 in [1] applies. The same pure types theorem holds by the same argument for the causal solutions of Theorem 2.1 (a).
While it follows from the arguments of Theorem 2.2 in Bertoin et al. [8] that V_0 as defined in Theorem 2.1 (b) is constant if and only if $U = kL$ for some constant $k \neq 0$ (equivalently that $W = -k\eta$ as seen in the proof of Theorem 2.1 (c)), the question of when this law is absolutely continuous or continuous singular is much more involved. Lindner and Sato [47] investigate the distribution $-\int_{(0,\infty)} [\mathcal{E}(U)_{s-}]^{-1}\, d\eta_s$ when $U_t = (c^{-1}-1)R_t$ for a constant $c > 1$ and independent Poisson processes R and η, showing that the distribution can be absolutely continuous or continuous singular, depending in an intrinsic way on c and the ratio of the rates of the Poisson processes R and η.
We conclude this chapter by mentioning that if $\Pi_U((-\infty,-1]) = 0$ and $\Pi_U \neq 0$, U and L are independent with L being of bounded variation with non-zero drift term, and $V_0 = -\int_0^\infty [\mathcal{E}(U)_{s-}]^{-1}\, d\eta_s$ converges almost surely, then it follows from Theorem 3.9 in Bertoin et al. [8] that V_0 is absolutely continuous. Further examples for absolutely continuous V_0 with independent U and L can be found in Gjessing and Paulsen [21], covering cases when U is Brownian motion with drift, or in Chapter 4.

Chapter 3: Distributional Properties of Solutions of $\mathbf{dV_t = V_{t-}dU_t + dL_t}$ with Lévy Noise[1]

For a general bivariate Lévy process $(U_t, L_t)_{t\geq 0}$ the unique solution of the SDE (1.11) is given by (see Proposition 2.6)

$$\begin{aligned} V_t &= \mathcal{E}(U)_t \left(V_0 + \int_{(0,t]} [\mathcal{E}(U)_{s-}]^{-1} \, d\eta_s \right) \mathbb{1}_{\{K(t)=0\}} \\ &\quad + \mathcal{E}(U)_{(T(t),t]} \left(\Delta L_{T(t)} + \int_{(T(t),t]} \left[\mathcal{E}(U)_{(T(t),s)}\right]^{-1} d\eta_s \right) \mathbb{1}_{\{K(t)\geq 1\}}, \quad t \geq 0, \end{aligned} \tag{3.1}$$

where $(\eta_t)_{t\geq 0}$, $K(t)$, $T(t)$ and the generalizations of the Doléans-Dade exponential $\mathcal{E}(U)_{(s,t]}$ and $\mathcal{E}(U)_{(s,t)}$ are defined by (2.1), (2.4), (2.5) and (2.6) and (2.6), respectively. If the starting random variable V_0 is independent of $(U_t, L_t)_{t\geq 0}$ the process V_t is called causal, otherwise it is called non-causal.

Necessary and sufficient conditions for the existence of stationary solutions of (3.1) have been given in Chapter 2. In this chapter we will investigate distributional properties of these stationary solutions. In particular in Section 3.1 we give the moment conditions and quote first and second moments as well as the autocorrelation function of the stationary solutions in terms of (U, L). In Section 3.2 we investigate the tail-behaviour of the stationary solutions by applying results of Kesten [38], Goldie [22] and Goldie and Grübel [23]. It shows up that, depending on properties of U and L, the resulting solutions can have a different tail behaviour like being heavy-tailed or having exponentially decreasing tails.
As mentioned in Section 2.4, one can conclude from Theorem 1.3 in Alsmeyer et al. [1] that the law of the stationary processes in the case $\Pi_U(\{-1\}) = 0$ is a pure-type measure, i.e. it is either absolutely continuous, continuous singular or a Dirac measure. In the case of generalized Ornstein-Uhlenbeck processes conditions for continuity of the stationary solutions have already been established in Bertoin et al. [8]. In Section 3.3 we shall study the case $\Pi_U(\{-1\}) > 0$. It turns out that the

[1]Based on [4]: A. Behme (2011):
Distributional properties of solutions of $dV_t = V_{t-}dU_t + dL_t$ with Lévy noise,
to appear in *Advances of Applied Probability.*

distributions of the stationary solutions do not fulfill a pure-type theorem in this case. We then give necessary and sufficient conditions for them to be (absolutely) continuous. Some examples are given for illustration. Remark that the results given in Section 2.4 hold for any solution X of a random fixed point equation $X \stackrel{d}{=} AX'+B$ with $X \stackrel{d}{=} X'$ and (A,B) being independent of X' such that $P(A=0)>0$.
Finally in Section 3.4 the proofs of the results in the prior Sections 3.1 to 3.3 are given.

3.1 Moment conditions and the autocorrelation function

Recall that by [58, Theorem 25.17] for a Lévy process $(X_t)_{t\geq 0}$ and a constant $\kappa \geq 0$, one has $Ee^{-\kappa X_1} < \infty$ if and only if $Ee^{-\kappa X_t} < \infty$ for all $t \geq 0$. In particular, if we define

$$\psi_X(\kappa) := \log E\left[e^{-\kappa X_1}\right] \tag{3.2}$$

it holds $Ee^{-\kappa X_t} = e^{t\psi_X(\kappa)}$ for all $t \geq 0$.
In order to deal with negative moments of the stochastic exponential in the case $\Pi_U(\{-1\}) = 0$ we recall the auxiliary Lévy process

$$W_t := -U_t + \sigma_U^2 t + \sum_{0<s\leq t} \frac{(\Delta U_s)^2}{1+\Delta U_s} \tag{3.3}$$

which fulfills $[\mathcal{E}(U)_t]^{-1} = \mathcal{E}(W)_t$, $t \geq 0$, by Lemma 2.8.

The following result on moments of the Doléans-Dade exponential will be needed later on. Although we expect it might be known, we were unable to find a ready reference and hence give a proof in Section 3.4.1.

Proposition 3.1. *Let $(U_t)_{t\geq 0}$ be a Lévy process and $\kappa \geq 1$.*

(a) *$|\mathcal{E}(U)_t|^\kappa$ is integrable if and only if $E|U_1|^\kappa < \infty$. In particular for $\kappa = 1$ and $\kappa = 2$ resp. it holds*

$$E[\mathcal{E}(U)_t] = e^{E[U_1]t} \tag{3.4}$$

$$\text{and} \quad \operatorname{Var}(\mathcal{E}(U)_t) = e^{2tE[U_1]}\left(e^{t\operatorname{Var}(U_1)} - 1\right). \tag{3.5}$$

(b) *Additionally suppose $\Pi_U(\{-1\}) = 0$. Then $|\mathcal{E}(U)_t|^{-\kappa} = |\mathcal{E}(W)_t|^\kappa$ is integrable if and only if*

$$\int_{(-1-e^{-1},-1+e^{-1})} |1+x|^{-\kappa}\, \Pi_U(dx) < \infty. \tag{3.6}$$

In particular for $\kappa = 1$ and $\kappa = 2$ resp. equations (3.4) and (3.5) hold true with U replaced by W, where $E[W_1]$ and $\mathrm{Var}\,(W_1)$ are given by

$$\begin{aligned} E[W_1] &= -\gamma_U + \sigma_U^2 \\ &\quad + \int_{[-1,1]} \frac{x^2}{1+x}\Pi_U(dx) - \int_{|x|>1} \frac{x}{1+x}\Pi_U(dx) \end{aligned} \tag{3.7}$$

$$\begin{aligned} \textit{and} \quad \mathrm{Var}\,(W_1) &= \sigma_U^2 + \int_{\mathbb{R}} \frac{x^2}{(1+x)^2}\Pi_U(dx) \\ &= \mathrm{Var}\,(U_1) - \int_{\mathbb{R}} \frac{x^3(2+x)}{(1+x)^2}\Pi_U(dx). \end{aligned} \tag{3.8}$$

In the following we will examine second-order properties of the stationary process $(V_t)_{t\geq 0}$. We start with a short lemma characterizing the constant solutions.

Lemma 3.2. *The process $(V_t)_{t\geq 0}$ as in (3.1) is a.s. constant equal to $k \in \mathbb{R}$ if and only if $kU_t = -L_t$ a.s. and $V_0 = k$ a.s.*

The next theorem gives us moment conditions, expectation and variance of the non-constant stationary solutions of (3.1). For $\kappa \geq 1$ the moment conditions could have been deduced from [62, Theorem 5.1]. We extend to $\kappa > 0$ and give a proof in Section 3.4.1 which is based on the proof of Proposition 4.1 in [46]. Compared to the special case treated there we obtain sharper conditions for existence of the moments by omitting the use of Hölder's inequality. Indeed, a comparison with Theorems 3.5 and 3.6 shows that the moment conditions in the following Theorem are sharp.

Theorem 3.3. *Let $(V_t)_{t\geq 0}$ be a non-constant strictly stationary solution of (3.1).*

(a) *Suppose that $\lim_{t\to\infty} \mathcal{E}(U)_t = 0$ a.s. and that for $\kappa > 0$ fixed*

$$E|\widetilde{U}_1|^{\max\{1,\kappa\}} < \infty, E|L_1|^{\max\{1,\kappa\}} < \infty \text{ and } E|\mathcal{E}(\widetilde{U})_1|^\kappa < e^\lambda, \tag{3.9}$$

for $\lambda = \Pi_U(\{-1\}) \geq 0$. Then $E|V_0|^\kappa < \infty$. Especially for $\kappa = 1$ and $\kappa = 2$ resp. it holds

$$E[V_0] = -\frac{E[L_1]}{E[U_1]} \tag{3.10}$$

$$\textit{and} \quad \mathrm{Var}\,(V_0) = -\frac{E\left[(U_1E[L_1] - E[U_1]L_1)^2\right]}{(E[U_1])^2(2E[U_1] + \mathrm{Var}\,(U_1))}. \tag{3.11}$$

Remark for $\kappa = 1$ that $E[U_1]$ is negative by (3.4) and (3.9) while for $\kappa = 2$ by (3.5) and (3.9) it holds $2E[U_1] + \mathrm{Var}\,(U_1) < 0$.

(b) *Suppose that $\Pi_U(\{-1\}) = 0$ and $\lim_{t\to\infty}[\mathcal{E}(U)_t]^{-1} = 0$ a.s. and that for $\kappa > 0$ fixed*

$$E|W_1|^{\max\{1,\kappa\}} < \infty, E|\eta_1|^{\max\{1,\kappa\}} < \infty \text{ and } E|\mathcal{E}(W)_1|^\kappa < 1. \tag{3.12}$$

Then it holds $E|V_0|^\kappa < \infty$. Especially for $\kappa = 1$ and $\kappa = 2$ equations (3.10) and (3.11) hold for U and L replaced by W and η, respectively.

Finally we give the autocorrelation function of the stationary processes $(V_t)_{t\geq 0}$ in the following theorem.

Theorem 3.4. *Let $(V_t)_{t\geq 0}$ be a non-constant strictly stationary solution of* (3.1).

(a) *Suppose that* $\lim_{t\to\infty} \mathcal{E}(U)_t = 0$ *a.s. and* (3.9) *holds for* $\kappa = 2$. *Then*

$$\mathrm{Cov}\,(V_s, V_t) = -e^{E[U_1](t-s)} \frac{E\left[(U_1 E[L_1] - E[U_1] L_1)^2\right]}{(E[U_1])^2 (2E[U_1] + \mathrm{Var}\,(U_1))}, \quad 0 \leq s \leq t. \quad (3.13)$$

(b) *Suppose that* $\Pi_U(\{-1\}) = 0$ *and* $\lim_{t\to\infty} [\mathcal{E}(U)_t]^{-1} = 0$. *Then if* (3.12) *holds for* $\kappa = 2$, *equation* (3.13) *is true with U and L replaced by W and η, respectively.*

It should be mentioned here that the proof of Theorem 3.4 does not make use of the stationarity assumption. In fact every solution $(V_t)_{t\geq 0}$ of (1.11) such that V_u is independent of $(U_{u+v} - U_u, L_{u+v} - L_u)_{v\geq 0}$ fulfills $\mathrm{Cov}\,(V_s, V_t) = e^{E[U_1](t-s)} \mathrm{Var}\, V_s$ given that $\mathrm{Var}\, V_s$ and $E|U_1|$ are finite. In the same way, given $\Pi_U(\{-1\}) = 0$, every $(V_t)_{t\geq 0}$ with V_u independent of $(U_v, L_v)_{0\leq v<u}$ satisfies $\mathrm{Cov}\,(V_s, V_t) = e^{E[W_1](t-s)} \mathrm{Var}\, V_t$ if $\mathrm{Var}\, V_t$ and $E|W_1|$ are finite.

3.2 Tail behaviour

In this section we study the tail behaviour of the stationary solutions of (3.1) which were given in Corollary 2.3. To analyse the non-constant stationary solutions we start with a result corresponding to case (ii) in Corollary 2.3 which is based on classical results on the tails of solutions of random recurrence equations by Kesten [38] and Goldie [22]. For the special case of generalized Ornstein-Uhlenbeck processes this result is also given in [46, Theorem 4.5] with slightly stronger conditions.

Theorem 3.5. *Let $(U_t, L_t)_{t\geq 0}$ be a bivariate Lévy process and suppose there exists $\kappa > 0$ such that*

$$E|\widetilde{U}_1|^{\max\{1,\kappa+\epsilon\}} < \infty, E|L_1|^{\max\{1,\kappa\}} < \infty \text{ and } E|\mathcal{E}(\widetilde{U})_1|^\kappa = e^\lambda \quad (3.14)$$

for some $\epsilon > 0$ and $\lambda = \Pi_U(\{-1\}) \geq 0$ is fulfilled. If U is of finite variation additionally assume that the drift of U is non-zero or that there is no $r > 0$ such that $\mathrm{supp}\,(\Pi_U) \subset \{-1 \pm e^{rz}, z \in \mathbb{Z}\}$. *Then* $\lim_{t\to\infty} \mathcal{E}(U)_t = 0$ *a.s. and there exist a strictly stationary solution $(V_t)_{t\geq 0}$ of* (3.1) *and constants $C_+, C_- \geq 0$ such that*

$$\lim_{x\to\infty} x^\kappa P(V_0 > x) = C_+ \text{ and } \lim_{x\to\infty} x^\kappa P(V_0 < -x) = C_-. \quad (3.15)$$

If $(V_t)_{t\geq 0}$ is not constant it holds $C_+ + C_- > 0$ and in the case that $\Pi_U((-\infty, -1)) > 0$ *we get $C_+ = C_-$.*

In the analogue statement for the non-causal stationary solution corresponding to case (b) in Theorem 2.1 we have to assure that $\lambda = 0$ holds, since otherwise such a solution does not exist. Apart from that the result is similar to the one before and can be stated as following.

Theorem 3.6. *Let $(U_t, L_t)_{t\geq 0}$ be a bivariate Lévy process with $\Pi_U(\{-1\}) = 0$ and suppose that there exists $\kappa > 0$ such that*

$$E|W_1|^{\max\{1,\kappa\}} < \infty, E|\eta_1|^{\max\{1,\kappa\}} < \infty \text{ and } E|\mathcal{E}(W)_1|^{\kappa} = 1. \tag{3.16}$$

If U is of finite variation additionally assume that the drift of U is non-zero or that there is no $r > 0$ such that $\operatorname{supp}(\Pi_U) \subset \{-1 \pm e^{rz}, z \in \mathbb{Z}\}$. Then $\lim_{t\to\infty} \mathcal{E}(U)_t^{-1} = 0$ a.s. and there exist a strictly stationary solution $(V_t)_{t\geq 0}$ of (3.1) and constants $C_+, C_- \geq 0$ such that

$$\lim_{x\to\infty} x^{\kappa} P(V_0 > x) = C_+ \text{ and } \lim_{x\to\infty} x^{\kappa} P(V_0 < -x) = C_-.$$

If $(V_t)_{t\geq 0}$ is not constant it holds $C_+ + C_- > 0$ and in the case that $\Pi_U((-\infty, -1)) > 0$ we get $C_+ = C_-$.

Since in the following we want to apply the results on tails of perpetuities given by Goldie and Grübel [23], we first reveal that the process V as defined in (3.1) can be interpreted as a perpetuity. This formulation will then also be used in Section 3.3. In fact it is known that the fixed point random equation

$$X \stackrel{d}{=} AX' + B \tag{3.17}$$

where X and X' are equally distributed random variables and X' is independent of the random vector (A, B) is related to the almost sure absolute convergence of the perpetuity

$$X_\infty := \sum_{k=0}^{\infty} \left(\prod_{i=0}^{k-1} A_i \right) B_k \tag{3.18}$$

where $(A_k, B_k)_{k\in\mathbb{N}_0}$ is an i.i.d sequence with the same distribution as (A, B).
More detailed Goldie and Maller [24, Theorems 2.1 and 3.1] showed the following. One direction of part (b) is already due to Vervaat [62].

Proposition 3.7.

(a) *Suppose $P(A = 0) > 0$. Then the sum in (3.18) converges almost surely to X_∞ and (3.17) has a unique solution which is given by $\mathcal{L}(X_\infty)$*

(b) *Suppose $P(A = 0) = 0$ and $P(Ac + B = c) < 1$ for all $c \in \mathbb{R}$. Then (3.17) has a solution if and only if the sum in (3.18) converges almost surely absolutely in which case $\mathcal{L}(X_\infty)$ is the unique solution of the random fixed point equation (3.17).*

From (2.15) we know, that the stationary solutions $(V_t)_{t\geq 0}$ of the SDE (1.11) satisfy the distributional fixed point equation

$$V_0 \stackrel{d}{=} V_t = A_t V_0 + B_t \tag{3.19}$$

for any $t \geq 0$ where

$$\begin{aligned} A_t &= \mathcal{E}(\widetilde{U})_t \mathbb{1}_{K(t)=0} \quad \text{and} && (3.20)\\ B_t &= \mathcal{E}(\widetilde{U})_t \int_{(0,t]} [\mathcal{E}(\widetilde{U})_{s-}]^{-1} d\widetilde{\eta}_s \mathbb{1}_{K(t)=0} && (3.21)\\ &\quad + \mathcal{E}(\widetilde{U})_{(T(t),t]} \left(\Delta L_{T(t)} + \int_{(T(t),t]} [\mathcal{E}(\widetilde{U})_{(T(t),s)}]^{-1} d\widetilde{\eta}_s \right) \mathbb{1}_{K(t)>0} \end{aligned}$$

which are independent of V_0, if the solution is causal as in Theorem 2.1(a) and Theorem 2.2. In the case of strictly non-causal solutions as in Theorem 2.1(b) we may rearrange (3.19) to get

$$V_t \stackrel{d}{=} V_0 = A_t^{-1} V_t - A_t^{-1} B_t \tag{3.22}$$

with $(A_t^{-1}, -A_t^{-1}B_t) = (\mathcal{E}(W)_t, -\int_{(0,t]} \mathcal{E}(W)_{s-} d\eta_s)$ independent of V_t.
Hence in our applications the case $P(A = 0) = 0$ coincides with the case $\lambda = \Pi_U(\{-1\}) = 0$ while $P(A = 0) > 0$ holds if and only if $\lambda > 0$ and hence only occurs in the causal case. If there exists no $k \in \mathbb{R}$ such that $kU = -L$, the resulting process is non-degenerate by Lemma 3.2. Hence convergence of the perpetuity is given in both cases under the conditions given in Corollary 2.3, since then a non-degenerate stationary solution exists as has been shown in Chapter 2.
Now that we can interpret our stationary solutions as perpetuities, we can apply the results on the tail behaviour of perpetuities in [23]. We start with the following proposition which is a direct consequence of [23, Theorem 4.1]. Remark, that we do not need any assumptions on the process L here.

Proposition 3.8. *Let $(U_t, L_t)_{t\geq 0}$ be a bivariate Lévy process and $(V_t)_{t\geq 0}$ a non-constant strictly stationary solution of* (3.1).

(a) *Assume that $\lim_{t\to\infty} \mathcal{E}(U)_t = 0$. If U is of finite variation suppose that it has strictly positive drift or that $\Pi_U(\mathbb{R}\backslash[-2, 0]) > 0$. Then the law of V_0 has at least a power-law tail, i.e.*

$$\liminf_{x\to\infty} \frac{\log(P(|V_0| \geq x))}{\log x} > -\infty. \tag{3.23}$$

(b) *Assume $\lambda = \Pi_U(\{-1\}) = 0$ and that $\lim_{t\to\infty}[\mathcal{E}(U)_t]^{-1} = 0$. If U is of finite variation suppose that it has strictly negative drift or that $\Pi_U([-2, 0]) > 0$. Then* (3.23) *holds true.*

The conditions on U formulated in the previous proposition part (a) ensure that $P(|\mathcal{E}(\widetilde{U})_t| > 1) > 0$ for all $t \geq 0$. If by contrast $|\mathcal{E}(\widetilde{U})_t|$ is bounded by 1 and not constant, then the tails of V_0 decrease at least exponentially fast under some additional condition on L as formulated in the following theorem.

Theorem 3.9. *Let $(U_t, L_t)_{t\geq 0}$ be a bivariate Lévy process and $(V_t)_{t\geq 0}$ a strictly stationary, non-constant solution of* (3.1)*.*

(a) *Assume that $\lim_{t\to\infty} \mathcal{E}(U)_t = 0$. Suppose that U is of finite variation, has non-positive drift and that $\Pi_U(\mathbb{R}\backslash[-2,0]) = 0$. Assume either the drift is non-zero or that $\Pi_U(\mathbb{R}\backslash\{-1\}) > 0$. Then, given that there exists $\kappa > 0$ such that $Ee^{\kappa|L_1|} < \infty$, the tails of $\mathcal{L}(V_0)$ decrease at least exponentially fast, i.e.*

$$\limsup_{x\to\infty} x^{-1} \log(P(|V_0| \geq x)) < 0. \tag{3.24}$$

(b) *Assume $\lambda = \Pi_U(\{-1\}) = 0$ and that $\lim_{t\to\infty}[\mathcal{E}(U)_t]^{-1} = 0$. Suppose that U is of finite variation, has non-negative drift and that $\Pi_U([-2,0]) = 0$. Assume either the drift is non-zero or that $\Pi_U(\mathbb{R}\backslash\{-1\}) > 0$. Then, given that there exists $\kappa > 0$ such that $Ee^{\kappa|\eta_1|} < \infty$,* (3.24) *holds true.*

3.3 Absolute continuity

In this section we determine necessary and sufficient conditions for the stationary solutions of (3.1) in the case $\lambda = \Pi_U(-1) > 0$ to be (absolutely) continuous. By the above exposition this corresponds to studying the law of the perpetuity (3.18). In the case that $P(A = 0) = 0$ this problem has first been treated by Grincevičius [25, 26] and later on also by Alsmeyer et al. [1]. An application of Grincevičius' results to generalized Ornstein-Uhlenbeck processes has been given by Bertoin et al. [8]. Here we concentrate on the case $P(A = 0) > 0$ and give necessary and sufficient conditions for the law of a perpetuity to be (absolutely) continuous as follows.

Theorem 3.10. *Let (A, B) be a pair of real-valued random variables with $P(A = 0) > 0$ and let X_∞ be the unique solution of the fixed point random equation* (3.17)*.*

(a) *The distribution of X_∞ is continuous if and only if the conditional distribution of B given $A = 0$ is continuous.*
(b) *The distribution of X_∞ is absolutely continuous if and only if the conditional distribution of B given $A = 0$ is absolutely continuous.*

If we apply Theorem 3.10 on the stationary solutions of (3.1), using (3.20) and (3.21) we obtain that the distribution of V_0 is (absolutely) continuous if and only if the distribution of

$$R_t := \mathcal{E}(\widetilde{U})_{(T(t),t]} \Delta L_{T(t)} + \mathcal{E}(\widetilde{U})_t \int_{(T(t),t]} [\mathcal{E}(\widetilde{U})_{s-}]^{-1} d\widetilde{\eta}_s$$

given $K(t) > 0$ is (absolutely) continuous and by the proof of Theorem 2.2 it holds for all $B \in \mathcal{B}_1$

$$P(R_t \in B | K(t) > 0) = \lambda \int_{(0,t]} P(Z_y \in B) e^{-\lambda y} dy,$$

with Z_t given by

$$Z_t = \mathcal{E}(\widetilde{U})_t \left(Y + \int_{(0,t]} [\mathcal{E}(\widetilde{U})_{s-}]^{-1} d\widetilde{\eta}_s \right), \quad t \geq 0. \tag{3.25}$$

Hence we can formulate the following corollary.

Corollary 3.11. *Suppose* $(V_t)_{t \geq 0}$ *to be a strictly stationary solution of* (3.1) *with* $\lambda := \Pi_U(\{-1\}) > 0$.

(a) $\mathcal{L}(V_0)$ *is continuous if and only if*

$$\int_{(0,1]} P(Z_y = a) e^{-\lambda y} dy = 0, \quad \forall a \in \mathbb{R}.$$

(b) $\mathcal{L}(V_0)$ *is absolutely continuous if and only if*

$$\int_{(0,1]} P(Z_y \in B) e^{-\lambda y} dy = 0, \quad \forall B \in \mathcal{B}_1 \text{ with Lebesgue measure } 0.$$

In particular we can conclude that if $\mathcal{L}(Z_t)$ is (absolutely) continuous for Lebesgue-almost every $t > 0$, then so is $\mathcal{L}(V_0)$. In the following we will discuss some examples for the behaviour of the distributions of Z_t and hence of $\mathcal{L}(V_0)$.

Examples

(1) Suppose the processes U and L to be independent. Then by (3.25) it holds almost surely $Z_t = \mathcal{E}(\widetilde{U})_t \int_{(0,t]} [\mathcal{E}(\widetilde{U})_{s-}]^{-1} d\widetilde{\eta}_s$, $t \geq 0$ such that by Lemma 2.5 it is

$$Z_t \overset{d}{=} \int_{(0,t]} \mathcal{E}(\widetilde{U})_{s-} d\widetilde{L}_s,$$

with the process $(\widetilde{L}_t)_{t \geq 0}$ defined by

$$\widetilde{L}_t = L_t - \sum_{\substack{0<s\leq t \\ \Delta U_s = -1}} \Delta L_s, \quad t \geq 0 \tag{3.26}$$

which in this setting is almost surely equal to $(L_t)_{t \geq 0}$.
Assume additionally that L_t is a standard Brownian motion, then for all $t > 0$ the conditional distribution of Z_t given $(\mathcal{E}(\widetilde{U})_s)_{0 \leq s < t}$ is normally distributed with

mean 0 and variance $\int_{(0,t]} |\mathcal{E}(\widetilde{U})_{s-}|^2 ds > 0$ a.s. Hence $P(Z_t \in B | (\mathcal{E}(\widetilde{U})_s)_{0\le s<t}) = 0$ for all $B \in \mathcal{B}_1$ with Lebesgue measure 0 and it follows $P(Z_t \in B) = E[P(Z_t \in B | (\mathcal{E}(\widetilde{U})_s)_{0\le s<t})] = 0$ for all $B \in \mathcal{B}_1$ with Lebesgue measure 0. Hence $\mathcal{L}(V_0)$ is absolutely continuous.

(2) Suppose U and L to be independent and let L_t be a compound Poisson process. Then $\mathcal{L}(V_0)$ has an atom, since $\mathcal{L}(Z_t)$ has an atom at $a = 0$ for $t \ge 0$.
If additionally the jump distribution of L is continuous, then also the distribution of Z_t given $L_t \neq 0$ is continuous such that $P(Z_t = a) = 0$ for all $t \ge 0$ and $a \neq 0$. Thus $\mathcal{L}(V_0)$ has a continuous part and an atom at 0, hence it is not a pure-type measure.

(3) Suppose that the distribution of Y is (absolutely) continuous. Then $\mathcal{L}(V_0)$ is (absolutely) continuous, too.
Indeed we get from (3.25) for $B \in \mathcal{B}_1$ with Lebesgue measure 0 in the absolute continuous case or for a single point set $B = \{b\}$ in the continuous case

$$
\begin{aligned}
P(Z_t \in B) &= P\left(Y \in [\mathcal{E}(\widetilde{U})_t]^{-1} B - \int_{(0,t]} [\mathcal{E}(\widetilde{U})_{s-}]^{-1} d\widetilde{\eta}_s\right) \\
&= \int_{\mathbb{R}^2} P\left(Y \in xB - y \,\middle|\, [\mathcal{E}(\widetilde{U})_t]^{-1} = x, \int_{(0,t]} [\mathcal{E}(\widetilde{U})_{s-}]^{-1} d\widetilde{\eta}_s = y\right) \nu(dx, dy) \\
&= \int_{\mathbb{R}^2} 0 \, \nu(dx, dy) = 0,
\end{aligned}
$$

where ν is the distribution of $([\mathcal{E}(\widetilde{U})_t]^{-1}, \int_{(0,t]} [\mathcal{E}(\widetilde{U})_{s-}]^{-1} d\widetilde{\eta}_s)$. It follows that Z_t is (absolutely) continuous for all $t > 0$. Hence $\mathcal{L}(V_0)$ is (absolutely) continuous by Corollary 3.11.

3.4 Proofs

For the proofs of the preceding results we need some auxiliary Lévy processes already introduced in Chapter 2 which we repeat here for convenience. Namely, in the case that $\lambda = \Pi_U(\{-1\}) = 0$ we will often make use of the formulation

$$
\mathcal{E}(U)_t = (-1)^{N_t} e^{-\widehat{U}_t}. \tag{3.27}
$$

where the processes $N = (N_t)_{t\ge 0}$ and $\widehat{U} = (\widehat{U}_t)_{t\ge 0}$ are defined by

$$
\begin{aligned}
N_t &:= \text{number of jumps of size } < -1 \text{ of } U \text{ in } [0,t], && (3.28) \\
\widehat{U}_t &:= -U_t + \sigma_U^2 t/2 + \sum_{0<s\le t} \left[\Delta U_s - \log|1 + \Delta U_s|\right]. && (3.29)
\end{aligned}
$$

See Chapter 2 for details on $N = (N_t)_{t\ge 0}$ and $\widehat{U} = (\widehat{U}_t)_{t\ge 0}$.

On the other hand, if $\lambda = \Pi_U(\{-1\}) > 0$, we will use the processes $\widetilde{U}$, $\widetilde{\eta}$, $\widetilde{L}$ and $\widetilde{W}$ defined by

$$\widetilde{U}_t = U_t - \sum_{\substack{0<s\leq t\\ \Delta U_s=-1}} \Delta U_s \quad \text{and} \quad \widetilde{\eta}_t = \eta_t - \sum_{\substack{0<s\leq t\\ \Delta U_s=-1}} \Delta\eta_s, \quad t\geq 0, \tag{3.30}$$

(3.26) and

$$\widetilde{W}_t = W_t - \sum_{\substack{0<s\leq t\\ \Delta U_s=-1}} \Delta W_s, \quad t\geq 0. \tag{3.31}$$

3.4.1 Proofs for Section 3.1

For the following calculations we need a short lemma on stochastic integrals with respect to Lévy processes which can be deduced from [10, Proposition 4.6.16] or, in the case that $\kappa \geq 2$, from [55, Theorem V.66]. Since the proof in [10] is not carried out completely we present an alternative proof here.
Remark that the following lemma allows to circumvent Hölder's inequality in the proof of Theorem 3.3 such that we get slightly sharper results than the corresponding ones obtained in [46] for generalized Ornstein-Uhlenbeck processes.

Lemma 3.12. *Let $(L_s)_{s\geq 0}$ be a Lévy process and $(H_s)_{s\geq 0}$ an adapted, càdlàg process. Suppose there exists $\kappa \geq 1$ such that $E|L_1|^\kappa < \infty$ and $E\sup_{0<s\leq 1}|H_s|^\kappa < \infty$. Then*

$$E\sup_{0<t\leq 1}\left|\int_{(0,t]} H_{s-}dL_s\right|^\kappa < \infty. \tag{3.32}$$

In particular, if $E|L_1| < \infty$ and $E\sup_{0<s\leq t}|H_s| < \infty$, for $t>0$ it holds

$$E\int_{(0,t]} H_{s-}dL_s = E[L_1]\int_{(0,t]} EH_{s-}ds. \tag{3.33}$$

Proof. Define the processes L^+ and L^- such that $L = L^+ + L^-$ and $EL_1 = EL_1^+$ with L^- having only jumps of size in $(-\frac{1}{2}, \frac{1}{2})$ and $L_s^+ = \sum_{i=1}^{N_s} Y_i + \gamma s$ being a compound Poisson process with parameter a, jump times T_i, $i = 1, 2, \ldots$, jump heights Y_i, $i = 1, 2, \ldots$ such that $|Y_i| \geq 1/2$ for all $i \in \mathbb{N}$ and additional drift term

γs. Then we can derive by standard calculation using Minkowski's inequality

$$\begin{aligned}
&\left(E\sup_{0<t\leq 1}\left|\int_{(0,t]}H_{s-}dL_s^+\right|^\kappa\right)^{1/\kappa}\\
\leq\; &\left(E\left(\sum_{i=1}^{N_1}|H_{T_i-}||Y_i|\right)^\kappa\right)^{1/\kappa}+\left(E\left(|\gamma|\sup_{0<t\leq 1}\left|\int_{(0,t]}H_{s-}ds\right|\right)^\kappa\right)^{1/\kappa}\\
\leq\; &\left(\sum_{j=0}^{\infty}P(N_1=j)\cdot j\cdot E|Y_1|^\kappa E\sup_{0<s\leq 1}|H_s|^\kappa\right)^{1/\kappa}+|\gamma|\left(E\sup_{0<s\leq 1}|H_s|^\kappa\right)^{1/\kappa}\\
=\; &\left(EN_1\cdot E|Y_1|^\kappa E\sup_{0<s\leq 1}|H_s|^\kappa\right)^{1/\kappa}+|\gamma|\left(E\sup_{0<s\leq 1}|H_s|^\kappa\right)^{1/\kappa}\\
<\; &\infty.
\end{aligned}$$

On the other hand using the notations as in [55] by [55, Theorem V.2] for some constant c_1 it is

$$\begin{aligned}
E\sup_{0<t\leq 1}\left|\int_{(0,t]}H_{s-}dL_s^-\right|^\kappa &= \left|\left|\int_{(0,\cdot]}\mathbb{1}_{(0,1]}(s)H_{s-}dL_s^-\right|\right|^\kappa_{\underline{\underline{S}}^\kappa}\\
&\leq c_1\left|\left|\int_{(0,\cdot]}\mathbb{1}_{(0,1]}(s)H_{s-}dL_s^-\right|\right|^\kappa_{\underline{\underline{H}}^\kappa}
\end{aligned}$$

and by equation (14) in [51] we know for some constant c_2 that

$$\left|\left|\int_{(0,\cdot]}\mathbb{1}_{(0,1]}(s)H_{s-}dL_s^-\right|\right|^\kappa_{\underline{\underline{H}}^\kappa}\leq c_2\,E\sup_{0<s\leq 1}|H_s|^\kappa\cdot||(L^-_{s\wedge 1})_{s\geq 0}||^\kappa_{BMO},$$

where $||\cdot||_{BMO}$ denotes the BMO-norm as defined e.g. in [55, p. 197]. Since L^- is a zero-mean Lévy process with bounded jumps, $(L^-_{s\wedge 1})_{s\geq 0}$ is a BMO-process. Hence the latter is finite and this yields $E\sup_{0<t\leq 1}\left|\int_{(0,t]}H_{s-}dL_s^-\right|^\kappa<\infty$ such that (3.32) follows directly.
For the second assertion, notice that since L_s-sEL_1 is a local martingale the same holds true for $M_t:=\int_{(0,t]}H_{s-}d(L_s-sE[L_1])$. By a calculus similar to above it holds $E\sup_{0<s\leq t}|M_t|<\infty$ and hence by [55, Theorem I.51] M_t is a martingale. Thus $E\int_{(0,t]}H_{s-}dL_s=E[L_1]E\int_{(0,t]}H_{s-}ds$ and using Fubini's theorem the second assertion follows. □

Proof of Proposition 3.1. (a) First remark that $|\mathcal{E}(U)_t|^\kappa=|\mathcal{E}(\widetilde{U})_t|^\kappa\mathbb{1}_{K(t)=0}$ is integrable if and only if $|\mathcal{E}(\widetilde{U})_t|^\kappa$ is. Thus it is sufficient to show the integrability condition under the assumption that $\Pi_U(\{-1\})=0$.
Because of (3.27) we have $E\left[|\mathcal{E}(U)_t|^\kappa\right]=E\left[e^{-\kappa\widehat{U}_t}\right]$ and hence by [58, Theorem 25.17] it holds $E\left[|\mathcal{E}(U)_t|^\kappa\right]<\infty$ if and only if $\int_{|x|>1}e^{-\kappa x}\Pi_{\widehat{U}}(dx)<\infty$. Using $\Pi_{\widehat{U}}=X(\Pi_U)$

for the transformation

$$X : \mathbb{R} \setminus \{-1\} \to \mathbb{R}, \quad x \mapsto X(x) = -\log|1+x| \tag{3.34}$$

as introduced in Lemma 2.8 this is equivalent to

$$\int_{\mathbb{R}\setminus([-1-e,-1-e^{-1}]\cup[-1+e^{-1},-1+e])} |1+x|^\kappa \Pi_U(dx) < \infty$$

which is fulfilled if and only if $|U_1|^\kappa$ is integrable.
To compute $E[\mathcal{E}(U)_t]$ for U with $\Pi_U(\{-1\}) \geq 0$ recall that the Doléans-Dade exponential fulfills the integral equation

$$\mathcal{E}(U)_t = 1 + \int_0^t \mathcal{E}(U)_{s-} dU_s. \tag{3.35}$$

Under the given assumptions we have $E \sup_{0<s\leq t} |\mathcal{E}(U)_s| < \infty$ by [58, Theorem 25.18] and hence using Lemma 3.12 it holds

$$E[\mathcal{E}(U)_t] = 1 + E[U_1] \int_0^t E[\mathcal{E}(U)_s] ds.$$

Thus by differentiation $dE[\mathcal{E}(U)_t]/dt = E[U_1]E[\mathcal{E}(U)_t]$ such that $E[\mathcal{E}(U)_t] = ce^{E[U_1]t}$ for some constant $c \neq 0$. But since $\mathcal{E}(U)_0 = 1$ a.s. one easily sees that $c = 1$ which gives (3.4).
To show (3.5) we derive from (3.35) via partial integration

$$\begin{aligned} E\left[(\mathcal{E}(U)_t)^2\right] &= 1 + 2E\left[\int_0^t \mathcal{E}(U)_{s-} dU_s\right] + E\left[\left[\int_0^\bullet \mathcal{E}(U)_{s-} dU_s, \int_0^\bullet \mathcal{E}(U)_{s-} dU_s\right]_t\right] \\ &\quad + 2E\left[\int_0^t \left(\int_0^s \mathcal{E}(U)_{u-} dU_u\right) d\left(\int_0^s \mathcal{E}(U)_{u-} dU_u\right)\right] \end{aligned}$$

which by a standard calculation using (3.4), (3.32) and (3.33) leads to

$$\frac{d\left(E\left[(\mathcal{E}(U)_t)^2\right]\right)}{dt} = \left(E[[U,U]_1] + 2E[U_1]\right) E\left[(\mathcal{E}(U)_t)^2\right]$$

such that

$$E\left[(\mathcal{E}(U)_t)^2\right] = e^{t(E[[U,U]_1]+2E[U_1])}$$

since $\mathcal{E}(U)_0 = 1$ a.s. Finally remark that $E[[U,U]_1] = E[U_1^2] - 2E[U_1]\int_0^1 sE[U_1]ds = \mathrm{Var}\,(U_1)$ which gives

$$E\left[(\mathcal{E}(U)_t)^2\right] = e^{t(2E[U_1]+\mathrm{Var}\,(U_1))} \tag{3.36}$$

and hence (3.5).
For proving (b) recall that $E|\mathcal{E}(U)_t|^{-\kappa} < \infty$ iff $\int_{|x|>1} e^{-\kappa x}\Pi_{\widehat{W}}(dx) < \infty$ where $\widehat{W}$ is the process corresponding to W via (3.27). Since $\Pi_{\widehat{W}} = X(\Pi_W) = X(Y(\Pi_U))$ with

$$Y : \mathbb{R} \setminus \{-1\} \to \mathbb{R} \setminus \{-1\}, \quad x \mapsto Y(x) = \frac{-x}{1+x} \tag{3.37}$$

as defined in Lemma 2.8, this is equivalent to

$$\int_{\mathbb{R}\setminus([-1-e,-1-e^{-1}]\cup[-1+e^{-1},-1+e])} |1+x|^{-\kappa}\,\Pi_U(dx) < \infty$$

and hence to (3.6). Equations (3.4) and (3.5) can then by shown by similar calculations as above while the formula for $E[W_1]$ has already been given in Lemma 2.8. The variance of W_1 is given by $\mathrm{Var}\,(W_1) = \sigma_W^2 + \int_{\mathbb{R}} x^2 \Pi_W(dx)$ (see [58, Example 25.12]). Using the transformation Y in (3.37) this directly leads to the given formula.
□

Proof of Lemma 3.2. Suppose $V_t = k$ a.s. By (1.11) we know that it holds $V_t = V_0 + \int_{(0,t]} V_{s-} dU_s + L_t$ which gives $k = k + kU_t + L_t$ and hence the desired result.
For the converse remark that $kU_t = -L_t$, $t \geq 0$ implies $\widetilde{\eta}_t = k\widetilde{W}_t$ by (2.1), (3.30), (3.3) and (3.31) and also $\Delta L_{T(t)} = k$ for all $t \geq 0$ such that we get from (3.1)

$$\begin{aligned} V_t &= \mathcal{E}(U)_t \left(k + k \int_{(0,t]} \mathcal{E}(\widetilde{W})_{s-} d\widetilde{W}_s \right) \mathbb{1}_{\{K(t)=0\}} \\ &\quad + \mathcal{E}(U)_{(T(t),t]} \left(k + k \int_{(T(t),t]} \mathcal{E}(\widetilde{W})_{(T(t),s)} d\widetilde{W}_s \right) \mathbb{1}_{\{K(t)\geq 1\}}. \end{aligned}$$

From (3.35) it follows that the Doléans-Dade exponential fulfills the integral equation $\mathcal{E}(\widetilde{W})_{(T(t),t]} = 1 + \int_{(T(t),t]} \mathcal{E}(\widetilde{W})_{(T(t),s)} d\widetilde{W}_s$ for all $t > 0$ with $K(t) > 0$ and this together with (3.35) directly gives $V_t = k$ a.s. □

Proof of Theorem 3.3. (a) Using Proposition 3.1(a) it follows from (3.9) for $k := \max\{1, \kappa\}$ that $E|\mathcal{E}(\widetilde{U})_s|^k < \infty$. By [58, Theorem 25.18] this is equivalent to $E \sup_{0<s\leq 1} |\mathcal{E}(\widetilde{U})_s|^k = E \sup_{0<s\leq 1} e^{-k\widehat{\widetilde{U}}_s} < \infty$ and hence from Lemma 3.12 it follows

$$E \sup_{0<t\leq 1} \left| \int_{(0,t]} \mathcal{E}(\widetilde{U})_{s-} d\widetilde{L}_s \right|^k < \infty. \tag{3.38}$$

Set $\alpha := \lfloor \kappa \rfloor$ the integer part of κ, then it can be shown for any $m, n \in \mathbb{N}_0$, $m < n$ exactly as it is done in the proof of [46, Proposition 4.1] that it holds (Equation (8.4) in [46])

$$\begin{aligned} & E \left| \int_{(m,n]} \mathcal{E}(\widetilde{U})_{s-} d\widetilde{L}_s \right|^\kappa \qquad (3.39) \\ \leq\; & E \left| \int_{(0,1]} \mathcal{E}(\widetilde{U})_{s-} d\widetilde{L}_s \right|^\kappa \left(\sum_{j=m}^{n-1} \exp\left(\frac{j}{\kappa} \psi_{\widehat{\widetilde{U}}}(\kappa) \right) \right)^\alpha \left(\sum_{j=m}^{n-1} \exp\left(\frac{j(\kappa-\alpha)}{\kappa} \psi_{\widehat{\widetilde{U}}}(\kappa) \right) \right) \end{aligned}$$

where the last factor can be omitted if $\kappa = \alpha$.

(i) Assume for the first, that $\lambda = \Pi_U(\{-1\}) = 0$, i.e. by Theorem 2.1(a) it holds $V_0 \stackrel{d}{=} \int_{(0,\infty)} \mathcal{E}(U)_{s-}dL_s$. Since we have $E|\mathcal{E}(U)_1|^\kappa < 1$ which is equivalent to $\psi_{\widehat{U}}(\kappa) < 0$, the sums in (3.39) converge absolutely when $n \to \infty$ such that with (3.38) it follows that $\int_{(0,t]} \mathcal{E}(U)_{s-}dL_s$ is a Cauchy sequence in L^κ and thus converges in L^κ to $\int_{(0,\infty)} \mathcal{E}(U)_{s-}dL_s$ as $t \to \infty$ such that we have $E|V_0|^\kappa < \infty$.
For the expectation we obtain using (3.33) and (3.4)

$$E[V_0] = E\left[\int_{(0,\infty)} \mathcal{E}(U)_{s-}dL_s\right] = E[L_1]\int_{(0,\infty)} e^{E[U_1]s}ds = -\frac{E[L_1]}{E[U_1]}.$$

To compute the variance we deduce using partial integration and Lemma 3.12

$$\begin{aligned} E[V_0^2] &= E\left[\left[\int_0^\bullet \mathcal{E}(U)_{s-}dL_s, \int_0^\bullet \mathcal{E}(U)_{s-}dL_s\right]_\infty\right] \\ &\quad +2E\left[\int_0^\infty \left(\int_0^t \mathcal{E}(U)_{s-}dL_s\right) d\left(\int_0^t \mathcal{E}(U)_{s-}dL_s\right)\right] \\ &= E[[L,L]_1]\int_0^\infty E\left[(\mathcal{E}(U)_{s-})^2\right]ds \\ &\quad +2E[L_1]\int_0^\infty E\left[\mathcal{E}(U)_{t-}\left(\int_0^t \mathcal{E}(U)_{s-}dL_s\right)\right]dt. \end{aligned}$$

By (3.36) it holds

$$\int_0^\infty E\left[(\mathcal{E}(U)_{s-})^2\right]ds = -\left(2E[U_1] + \mathrm{Var}\,(U_1)\right)^{-1}$$

which is strictly positive and finite since $E[\mathcal{E}(U)_1^2] < 1$ holds by assumption. For the calculation of

$$\int_0^\infty E\left[\mathcal{E}(U)_{t-}\left(\int_0^t \mathcal{E}(U)_{s-}dL_s\right)\right]dt \quad =: \quad \int_0^\infty X_t\,dt$$

again by partial integration, use of Lemma 3.12 and equations (3.4), (3.36) and (3.35) one gets

$$X_t = \frac{E[[U,L]_1] + E[L_1]}{2E[U_1] + \mathrm{Var}\,(U_1)}\left(e^{t(2E[U_1]+\mathrm{Var}\,(U_1))} - 1\right) + E[U_1]\int_0^t X_s\,ds.$$

Solving this integral equation leads to

$$E[V_0^2] = \frac{1}{2E[U_1] + \mathrm{Var}\,(U_1)}\left(\frac{2E[L_1]\left(\mathrm{Cov}\,(U_1,L_1) + E[L_1]\right)}{E[U_1]} - \mathrm{Var}\,(L_1)\right) \tag{3.40}$$

where we replaced

$$E[[U,L]_1] = E[U_1L_1] - E[L_1]\int_0^1 sE[U_1]ds - E[U_1]\int_0^1 sE[L_1]ds = \mathrm{Cov}\,(U_1,L_1).$$

Under use of the results above the variance can now be derived by standard algebra.
(ii) Suppose that $\lambda = \Pi_U(\{-1\}) > 0$ and deduce from Theorem 2.2 that

$$E|V_0|^\kappa = E|Z_\tau|^\kappa = \int_{(0,\infty)} \lambda e^{-\lambda t} E|Z_t|^\kappa dt$$

with $(Z_t)_{t\geq 0}$ defined in (3.25).
Let $\lambda' := \psi_{\widehat{\widetilde{U}}}(\kappa) = \log E|\mathcal{E}(\widetilde{U})_1|^\kappa$ then by assumption we have $\lambda' < \lambda$. Choose λ'' such that $\lambda' \leq \lambda'' < \lambda$. Then, for finiteness of $E|V_0|^\kappa$, it is needed to show that $E|Z_t|^\kappa \leq c \cdot e^{t\lambda''}$ for some constant c. First observe that we have

$$\begin{aligned} E|Z_t|^\kappa &= E\left|\mathcal{E}(\widetilde{U})_t Y + \mathcal{E}(\widetilde{U})_t \int_{(0,t]} [\mathcal{E}(\widetilde{U})_{s-}]^{-1} d\widetilde{\eta}_s\right|^\kappa \\ &\leq 2^\kappa \left(E\left|\mathcal{E}(\widetilde{U})_t Y\right|^\kappa + E\left|\int_{(0,t]} \mathcal{E}(\widetilde{U})_{s-} d\widetilde{L}_s\right|^\kappa\right), \end{aligned}$$

where we used Lemma 2.5.
Since $E|L_1|^\kappa < \infty$ implies $E|Y|^\kappa < \infty$ and since $\widetilde{U}$ and Y are independent we conclude

$$\begin{aligned} \int_0^\infty e^{-t\lambda} 2^\kappa E\left|\mathcal{E}(\widetilde{U})_t Y\right|^\kappa dt &\leq 2^\kappa \int_0^\infty e^{-t\lambda} e^{t\lambda''} E|Y|^\kappa dt \\ &< \infty. \end{aligned}$$

For the second term observe that from (3.39) and (3.38) it follows

$$\begin{aligned} E\left|\int_{(0,n]} \mathcal{E}(\widetilde{U})_{s-} d\widetilde{L}_s\right|^\kappa &\leq E\left|\int_{(0,1]} \mathcal{E}(\widetilde{U}) d\widetilde{L}_s\right|^\kappa \\ &\quad \cdot \left(\sum_{j=0}^{n-1} \exp\left(\frac{j}{\kappa}\lambda'\right)\right)^\alpha \left(\sum_{j=0}^{n-1} \exp\left(\frac{j(\kappa-\alpha)}{\kappa}\lambda'\right)\right) \\ &\leq C \cdot \exp\left(\frac{n\alpha}{\kappa}\lambda''\right) \exp\left(\frac{n(\kappa-\alpha)}{\kappa}\lambda''\right) \\ &\leq C \cdot e^{n\lambda''} \end{aligned}$$

for any $n \in \mathbb{N}$ and a suitable constant C. Finally, for arbitrary $t > 0$, we get

$$\begin{aligned} E\left|\int_{(0,t]} \mathcal{E}(\widetilde{U})_{s-} d\widetilde{L}_s\right|^\kappa &= E\left|\int_{(0,\lfloor t\rfloor]} \mathcal{E}(\widetilde{U})_{s-} d\widetilde{L}_s + \mathcal{E}(\widetilde{U})_{\lfloor t\rfloor} \int_{(\lfloor t\rfloor,t]} \mathcal{E}(\widetilde{U})_{(\lfloor t\rfloor,s)} d\widetilde{L}_s\right|^\kappa \\ &\leq 2^\kappa \left(C \cdot e^{\lfloor t\rfloor\lambda''} + e^{\lfloor t\rfloor\lambda''} E \sup_{0<t\leq 1}\left|\int_{(0,t]} \mathcal{E}(\widetilde{U}) d\widetilde{L}_s\right|^\kappa\right) \\ &\leq C' \cdot e^{t\lambda''} \end{aligned}$$

by (3.38) for some constant C'.
We conclude

$$\int_0^\infty e^{-t\lambda} 2^\kappa E\left|\int_{(0,t]} \mathcal{E}(\widetilde{U})_{s-} d\widetilde{L}_s\right|^\kappa dt < \infty$$

and altogether $\int_0^\infty e^{-t\lambda}|Z_t|^\kappa dt < \infty$ giving finiteness of $E|V_0|^\kappa$.
To compute the expectation using the result (3.4) and calculations as in part (i) above we derive under the assumption that $E[\widetilde{U}_1] \neq 0$

$$E[Z_t] = e^{E[\widetilde{U}_1]t}EY + \frac{E[\widetilde{L}_1]}{E[\widetilde{U}_1]}\left(e^{E[\widetilde{U}_1]t} - 1\right)$$

such that by integration and using $E[U_1] = E[\widetilde{U}_1] - \lambda$

$$E[V_0] = -\frac{E[\widetilde{L}_1] + \lambda EY}{E[U_1]}.$$

If on the other hand $E[\widetilde{U}_1] = 0$ it follows $E[Z_t] = EY + tE[\widetilde{L}_1]$ and then $E[V_0] = EY + \lambda^{-1}E[\widetilde{L}_1]$ which is a special case of the formula derived above. Hence since $E[L_1] = E[\widetilde{L}_1] + \lambda EY$ we have shown the formula for $E[V_0]$, provided that $E|V_0| < \infty$.
To prove the formula for $\mathrm{Var}\,(V_0)$ because of

$$E[V_0^2] = E[Z_\tau^2] = \int_{(0,\infty)} \lambda e^{-\lambda t} E[Z_t^2]dt \tag{3.41}$$

one first has to derive $E[Z_t^2]$ for which one gets by a long calculation starting from (3.25)

$$\begin{aligned} E[Z_t^2] \quad = \quad & e^{t(2E[\widetilde{U}_1]+\mathrm{Var}\,(\widetilde{U}_1))}\left(E[Y^2] + \frac{\mathrm{Var}\,(\widetilde{L}_1)}{2E[\widetilde{U}_1] + \mathrm{Var}\,(\widetilde{U}_1)}\right. \\ & \left. + \frac{\mathrm{Cov}\,(\widetilde{U}_1, \widetilde{L}_1) + E[\widetilde{L}_1]}{E[\widetilde{U}_1] + \mathrm{Var}\,(\widetilde{U}_1)}\left(2EY + \frac{2E[\widetilde{L}_1]}{2E[\widetilde{U}_1] + \mathrm{Var}\,(\widetilde{U}_1)}\right)\right) \\ & - e^{tE[\widetilde{U}_1]}\left(\frac{\mathrm{Cov}\,(\widetilde{U}_1, \widetilde{L}_1) + E[\widetilde{L}_1]}{E[\widetilde{U}_1] + \mathrm{Var}\,(\widetilde{U}_1)}\left(2EY + \frac{2E[\widetilde{L}_1]}{E[\widetilde{U}]}\right)\right) \\ & + \frac{2E[\widetilde{L}_1](\mathrm{Cov}\,(\widetilde{U}_1, \widetilde{L}_1) + E[\widetilde{L}_1]) - E[\widetilde{U}_1]\mathrm{Var}\,(\widetilde{L}_1)}{E[\widetilde{U}_1](2E[\widetilde{U}_1] + \mathrm{Var}\,(\widetilde{U}_1))}. \end{aligned}$$

By integration and standard algebra this leads to (3.40) and hence to the given formula for $\mathrm{Var}\,(V_0)$ where we used the following relationships: (all sums are meant over the jumps of U of size -1 during the time interval $[0, 1]$)

$$\begin{aligned} E[U_1^2] \quad &= \quad E[\widetilde{U}_1^2] + 2E[\widetilde{U}_1]E[\Sigma\Delta U] + E[(\Sigma\Delta U)^2] \\ &= \quad E[\widetilde{U}_1^2] - 2\lambda E[\widetilde{U}_1] + \lambda + \lambda^2, \\ & \qquad \text{since } \Sigma\Delta U \text{ is Poisson distributed, such that} \\ \mathrm{Var}\,(\widetilde{U}_1) \quad &= \quad \mathrm{Var}\,(U_1) - \lambda. \end{aligned}$$

On the other hand for L and $\widetilde{L}$ we have

$$\begin{aligned} E[L_1^2] &= E[\widetilde{L}_1^2] + 2\lambda EY E[\widetilde{L}_1] + \mathrm{Var}\,(\Sigma\Delta L) + (E[\Sigma\Delta L])^2 \\ &\quad \text{and since } \Sigma\Delta L \text{ is compound Poisson distributed this gives} \\ E[L_1^2] &= E[\widetilde{L}_1^2] + 2\lambda EY E[\widetilde{L}_1] + \lambda E[Y^2] + \lambda^2 (EY)^2 \text{ and hence} \\ \mathrm{Var}\,(\widetilde{L}_1) &= \mathrm{Var}\,(L_1) - \lambda E[Y^2], \end{aligned}$$

while for the covariance one deduces $\mathrm{Cov}\,(\widetilde{U}_1, \widetilde{L}_1) = \mathrm{Cov}\,(U_1, L_1) + \lambda EY$.
The proof of (b) can be carried out as the proof of (a) in the case $\lambda = 0$. We omit it here. □

Proof of Theorem 3.4. (a) For $s < t$ take $A_{s,t}$ and $B_{s,t}$ as defined in (2.14) and recall (2.15). Since by Corollary 2.3 the stationary solution $(V_t)_{t\geq 0}$ is unique in law, we may and shall assume that V_0 is independent of $(U_t, L_t)_{t\geq 0}$. Observe that due to the independence it follows by Proposition 3.1 and Theorem 3.3 that $A_{s,t}\mathbb{1}_{K(s)=K(t)}V_s$ has finite expectation. Hence for $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ denoting the natural filtration induced by $(U_t, L_t)_{t\geq 0}$ we get from (2.15)

$$\begin{aligned} E[V_t|\mathcal{F}_s] &= E\left[(A_{s,t}V_s + B_{s,t})\,\mathbb{1}_{K(s)=K(t)} + \left(A_{T(t),t}\Delta\eta_{T(t)} + B_{T(t),t}\right)\mathbb{1}_{K(s)<K(t)}|\mathcal{F}_s\right] \\ &= V_s E\left[A_{s,t}\mathbb{1}_{K(s)=K(t)}\right] \\ &\quad + E\left[A_{T(t),t}\Delta\eta_{T(t)}\mathbb{1}_{K(s)<K(t)} + B_{s,t}\mathbb{1}_{K(s)=K(t)} + B_{T(t),t}\mathbb{1}_{K(s)<K(t)}\right]. \end{aligned}$$

On the other hand since V_s is independent of $A_{s,t}\mathbb{1}_{K(s)=K(t)}$ it holds

$$\begin{aligned} E[V_t] &= E\left[V_s\right] E\left[A_{s,t}\mathbb{1}_{K(s)=K(t)}\right] \\ &\quad + E\left[A_{T(t),t}\Delta\eta_{T(t)}\mathbb{1}_{K(s)<K(t)} + B_{s,t}\mathbb{1}_{K(s)=K(t)} + B_{T(t),t}\mathbb{1}_{K(s)<K(t)}\right] \end{aligned}$$

such that

$$E[V_t|\mathcal{F}_s] = E[V_t] + E[A_{s,t}\mathbb{1}_{K(s)=K(t)}](V_s - E[V_s]).$$

Finally since

$$E\left[A_{s,t}\mathbb{1}_{K(s)=K(t)}\right] = E\left[\mathcal{E}(\widetilde{U})_{(s,t]}\right] E\left[\mathbb{1}_{K(s)=K(t)}\right] = e^{E[\widetilde{U}_1](t-s)}e^{-\lambda(t-s)} = e^{E[U_1](t-s)}$$

we get

$$\mathrm{Cov}\,(V_s, V_t) = E\left[V_s E[V_t|\mathcal{F}_s]\right] - E[V_s]E[V_t] = e^{E[U_1](t-s)}\mathrm{Var}\,V_s$$

as had to be shown.
(b) By Theorem 2.1(b) the stationary solution is non-causal and from (2.15) we obtain $V_s = A_{s,t}^{-1}V_t - A_{s,t}^{-1}B_{s,t}$ for $s < t$. Defining $\mathcal{G}_t = \sigma((U_{u+t} - U_t, L_{u+t} - L_t)_{u\geq 0})$ we can then compute $E[V_s|\mathcal{G}_t]$ for $s < t$ and finally $\mathrm{Cov}\,(V_s, V_t)$ similar as in (a). □

3.4.2 Proofs for Section 3.2

Proof of Theorem 3.5. In the case that $\lambda = 0$ we first conclude from $E|\mathcal{E}(U)_1|^\kappa = 1$, which is then equivalent to $\psi_{\widehat{U}}(\kappa) = 0$, that $\lim_{t\to\infty} \mathcal{E}(U)_t = 0$ as in the proof of [46, Proposition 4.1]. Remark that $E|U_1|^{\max\{1,\kappa\}} < \infty$ implies $E|\mathcal{E}(U)_1|^{\max\{1,\kappa\}} < \infty$ by Proposition 3.1(a). Since further $E\log^+|L_1| < \infty$ by (3.14), it follows from Theorem 2.10 that $\int_{(0,\infty)} \mathcal{E}(U)_{s-} dL_s$ converges a.s. Hence a strictly stationary solution V_t exists by Corollary 2.3(ii).
If $\lambda > 0$ it is clear that $\lim_{t\to\infty} \mathcal{E}(U)_t = 0$ holds and the existence of a stationary solution is again guaranteed by Corollary 2.3(ii).
We know by (3.19) that for all $t \geq 0$ the stationary solution fulfills $V_0 \stackrel{d}{=} A_t V_0 + B_t$ for A_t and B_t defined by (3.20) and (3.21), respectively. Thus we have for any $t > 0$ fixed

$$E|A_t|^\kappa = E\left|\mathcal{E}(\widetilde{U})_t \mathbb{1}_{K(t)=0}\right|^\kappa = E\left|\mathcal{E}(\widetilde{U})_t\right|^\kappa P(K(t)=0) = e^{t\lambda}e^{-t\lambda} = 1$$

by assumption and with $\mathcal{E}(\widetilde{U})_t = (-1)^{N_t} e^{-\widehat{\widetilde{U}}_t}$ it holds

$$\begin{aligned} E|A_t|^\kappa \log^+|A_t| &= E\left[\mathbb{1}_{K(t)=0} e^{-\kappa\widehat{\widetilde{U}}_t} \log^+(e^{-\kappa\widehat{\widetilde{U}}_t})\right] = e^{-t\lambda} E\left[e^{-\kappa\widehat{\widetilde{U}}_t} \log^+(e^{-\kappa\widehat{\widetilde{U}}_t})\right] \\ &< \infty, \end{aligned}$$

since $E\left[e^{-(\kappa+\epsilon)\widehat{\widetilde{U}}_t}\right] < \infty$ by (3.14) and Proposition 3.1. Additionally for $k := \max\{1,\kappa\}$ by Minkowski's inequality it is

$$\begin{aligned} 0 < \left(E|B_t|^k\right)^{1/k} &\leq \left(E\left|\mathcal{E}(\widetilde{U})_{(T(t),t]} \mathbb{1}_{K(t)>0} \Delta L_{T(t)}\right|^k\right)^{1/k} \\ &\quad + \left(E\left|\mathcal{E}(\widetilde{U})_{(T(t),t]} \int_{(T(t),t]} \left[\mathcal{E}(\widetilde{U})_{(T(t),s)}\right]^{-1} d\widetilde{\eta}_s\right|^k\right)^{1/k} \end{aligned} \tag{3.42}$$

where we set $T(t) := 0$ if $K(t) = 0$. The first term in the latter sum vanishes if $\lambda = 0$ and in any case is less or equal to

$$\begin{aligned} \left(E\left[\left|\mathcal{E}(\widetilde{U})_{(T(t),t]}\right|^k \left|\Delta L_{T(t)}\right|^k\right]\right)^{1/k} &= \left(E\left|\mathcal{E}(\widetilde{U})_{(T(t),t]}\right|^k \cdot E\left|\Delta L_{T(t)}\right|^k\right)^{1/k} \\ &= \left(E e^{-k(\widehat{\widetilde{U}}_t - \widehat{\widetilde{U}}_{T(t)})} \cdot E\left|\Delta L_{T(t)}\right|^k\right)^{1/k} \end{aligned}$$

which is finite by assumption.
On the other hand observe that by conditioning on $T(t)$, which is independent of $(\widetilde{U}_t, \widetilde{L}_t)_{t\geq 0}$, it follows from Lemma 2.5 that

$$\mathcal{E}(\widetilde{U})_{(T(t),t]} \int_{(T(t),t]} [\mathcal{E}(\widetilde{U})_{(T(t),s)}]^{-1} d\widetilde{\eta}_s \stackrel{d}{=} \int_{(T(t),t]} \mathcal{E}(\widetilde{U})_{(T(t),s)} d\widetilde{L}_t.$$

Hence the second term of (3.42) is finite by Lemma 3.12 since $E\sup_{0<s\leq 1}|\mathcal{E}(\widetilde{U})_s|^k < \infty$ by [58, Theorem 25.18] is equivalent to $E|\mathcal{E}(\widetilde{U})_1|^k < \infty$ which is given by assumption. Altogether we obtain $0 < E|B_t|^\kappa \leq E|B_t|^k < \infty$.
From (3.27) it is clear that $\widehat{\widehat{U}}$ has infinite variation if and only if $\widetilde{U}$ has and thus if and only if U has infinite variation. Hence by [58, Corollary 24.6] in this case $\widehat{\widehat{U}}_t$ has non-arithmetic law for each $t > 0$, i.e. there exists no $r > 0$ such that the law of $\widehat{\widehat{U}}_t$ has support in $r\mathbb{Z}$. Otherwise, if we assume $\widehat{\widehat{U}}$ to be deterministic, then from (3.14) it follows $\kappa\widehat{\widehat{U}}_1 = \lambda > 0$ which contradicts to $\lim_{t\to\infty}\mathcal{E}(U)_t = 0$, such that this case cannot occur. Thus, if U (and hence $\widehat{\widehat{U}}$) is of finite variation by [58, Corollary 24.6] it suffices to ensure that either the drift $\gamma^0_{\widehat{\widehat{U}}}$ of $\widehat{\widehat{U}}$ is non-zero or that there is no $r > 0$ with $\operatorname{supp}(\Pi_{\widehat{\widehat{U}}}) \subset \operatorname{supp}(\Pi_{\widehat{U}}) \subset r\mathbb{Z}$, to guarantee that $\widehat{\widehat{U}}_t$ has non-arithmetic law for t from a dense subset of $(0,\infty)$. Via the relations between U and $\widehat{U}$ known from Lemma 2.8, we have $\gamma^0_{\widehat{\widehat{U}}} = -\gamma^0_U$ and $\operatorname{supp}(\Pi_{\widehat{U}}) = X(\operatorname{supp}(\Pi_U))$ with X as defined in (3.34), such that this holds by assumption. Hence $\mathcal{L}(\log|A_t|\,|A_t \neq 0) = \mathcal{L}(\widehat{\widehat{U}}_t)$ is non-arithmetic for t from a dense subset of $(0,\infty)$.
Now by [22, Theorem 4.1] it follows that there exists a unique law of V_0 fulfilling $V_0 \overset{d}{=} A_tV_0 + B_t$ and by uniqueness in law of the stationary solution this law is equal to $\mathcal{L}(V_t)$ for all $t \geq 0$. Hence [22, Theorem 4.1] shows the existence of $C_+, C_- \geq 0$ such that (3.15) holds as well as the fact that $C_+ = C_-$ if $\Pi_U((-\infty,-1)) > 0$.
Finally, fix a sequence t_n tending to infinity such that $\mathcal{L}(\widehat{\widehat{U}}_{t_n})$ is non-arithmetic for all $n \in \mathbb{N}$. Now from [22, Theorem 4.1] it follows additionally that if $C_+ + C_- = 0$ it holds $B_{t_n} = (1 - A_{t_n})c_n$ for some real constants c_n. But letting n tend to infinity we observe that $A_{t_n} \to 0$ a.s., $n \to \infty$, and hence $B_{t_n} \overset{d}{\to} V_0$ by (3.19). This implies $c_n \overset{d}{\to} V_0$, $n \to \infty$, such that V_0 and hence V_t, $t \geq 0$ is constant. □

The **Proof of Theorem 3.6** can be carried out analogously to that of Theorem 3.5 simplified to the case $\lambda = 0$. Observe that by defining a process $\widehat{W}$ similar as $\widehat{U}$, it holds $\widehat{W} = -\widehat{U}$, such that $\widehat{W}$ has non-arithmetic law if and only if $\widehat{U}$ has.

Proof of Proposition 3.8. Part (a) follows from [23, Theorem 4.1] once it is shown that $P(|A_t| > 1) > 0$ holds for A_t, $t > 0$ defined in (3.20). This is equivalent to $P(\widehat{\widehat{U}}_t < 0) > 0$ such that we have to ensure that $\widehat{\widehat{U}}$ is not a subordinator which is equivalent to the assumptions stated in Proposition 3.8(a) using the relations between the processes U and $\widehat{\widehat{U}}$.
Due to (3.22) and arguments as above we have to ensure that $\widehat{W}$ is not a subordinator to prove (b). Since $\widehat{W} = -\widehat{U}$ this is equivalent to the given assumptions. □

Proof of Theorem 3.9. We know that V_t fulfills (3.19) for all $t \geq 0$. Remark that due to the relations between the processes U and $\widehat{\widehat{U}}$ the assumptions on U given in

the theorem are equivalent to stating that $\widehat{\widehat{U}}$ is a subordinator with $P(\widehat{\widehat{U}}_t > 0) > 0$ for all $t > 0$. By (3.20) this implies $P(|A_t| \leq 1) = P(|\mathcal{E}(\widetilde{U})_t| \mathbb{1}_{K(t)=0} \leq 1) = 1$ and $P(|A_t| < 1) > 0$ for all $t > 0$. It remains to show that the moment generating function (m.g.f.) of $|B_t|$ for $t > 0$ fixed is finite in some neighbourhood $(-\epsilon, \epsilon)$ of 0, then the result follows directly from Theorem 2.1 of [23].
By (3.21) for $t > 0$ fixed and $\epsilon > 0$ it holds

$$\begin{aligned} e^{\epsilon|B_t|} &= \exp\left(\epsilon \left| \mathcal{E}(\widetilde{U})_{(T(t),t]} \int_{(T(t),t]} [\mathcal{E}(\widetilde{U})_{(T(t),s)}]^{-1} d\widetilde{\eta}_s + \mathcal{E}(\widetilde{U})_{(T(t),t]} \Delta L_{T(t)} \mathbb{1}_{K(t)>0} \right| \right) \\ &\leq \exp\left(\epsilon \left| \mathcal{E}(\widetilde{U})_{(T(t),t]} \int_{(T(t),t]} [\mathcal{E}(\widetilde{U})_{(T(t),s)}]^{-1} d\widetilde{\eta}_s \right| \right) \cdot \exp\left(\epsilon |\Delta L_{T(t)}| \mathbb{1}_{K(t)>0}\right) \end{aligned}$$

where we set $T(t) = 0$ if $K(t) = 0$ and hence by Hölder's inequality it is enough to show that both factors have finite expectation in some neighbourhood of 0. Due to our assumption on L this holds true for the second factor, while for the first factor we obtain with Lemma 2.5

$$\begin{aligned} &E \exp\left(\epsilon \left| \mathcal{E}(\widetilde{U})_{(T(t),t]} \int_{(T(t),t]} [\mathcal{E}(\widetilde{U})_{(T(t),s)}]^{-1} d\widetilde{\eta}_s \right| \right) \\ &= E\left[E\left[\exp\left(\epsilon \left| \mathcal{E}(\widetilde{U})_{t-T(t)} \int_{(0,t-T(t)]} [\mathcal{E}(\widetilde{U})_{s-}]^{-1} d\widetilde{\eta}_s \right| \right) \middle| T(t) \right] \right] \\ &= \int_{[0,t]} E\left[\exp\left(\epsilon \left| \int_{(0,t-w]} \mathcal{E}(\widetilde{U})_{s-} d\widetilde{L}_s \right| \right) \right] dP_{T(t)}(w). \end{aligned}$$

Recall the process $\widetilde{L}$ and define $\widetilde{L}^+$ and $\widetilde{L}^-$ similar to L^+ and L^- in the proof of Lemma 3.12. Then

$$\begin{aligned} &E\left[\exp\left(\epsilon \left| \int_{(0,t-w]} \mathcal{E}(\widetilde{U})_{s-} d\widetilde{L}_s \right| \right) \right] \qquad (3.43) \\ &\leq \left(E\left[\exp\left(2\epsilon \left| \int_{(0,t-w]} \mathcal{E}(\widetilde{U})_{s-} d\widetilde{L}_s^+ \right| \right) \right] \right)^{\frac{1}{2}} \left(E\left[\exp\left(2\epsilon \left| \int_{(0,t-w]} \mathcal{E}(\widetilde{U})_{s-} d\widetilde{L}_s^- \right| \right) \right] \right)^{\frac{1}{2}} \end{aligned}$$

by the Cauchy-Schwarz inequality. Denoting the total variation of $\widetilde{L}^+$ on $(0, t]$ by $||\widetilde{L}^+||_t$ we get

$$\left| \int_{(0,t-w]} \mathcal{E}(\widetilde{U})_{s-} d\widetilde{L}_s^+ \right| \leq \int_{(0,t-w]} \sup_{0<u\leq t-w} |\mathcal{E}(\widetilde{U})_{u-}| \, |d\widetilde{L}_s^+| \leq \int_{(0,t-w]} |d\widetilde{L}_s^+| \leq ||\widetilde{L}^+||_{t-w}.$$

Since $\widetilde{L}^+$ is by definition a finite variation process and L has finite m.g.f. in some neighbourhood of 0, so has $||\widetilde{L}^+||$. Thus the first term on the RHS of (3.43) is finite for ϵ small enough.
For fixed $t > 0$ set $M_t := \int_{(0,t]} \mathcal{E}(\widetilde{U})_{s-} d\widetilde{L}_s^-$, then $(M_s)_{0<s\leq t}$ is a square integrable martingale by [55, Lemma to Theorem IV.27] since $E[\int_{(0,t)} \mathcal{E}(\widetilde{U})_{s-}^2 d[\widetilde{L}^-, \widetilde{L}^-]_s] \leq$

$E[\sigma_L^2 t + \sum_{0<s\le t}(\Delta \widetilde{L}_s^-)^2] < \infty$. Additionally it holds $|\Delta M_t| = |\mathcal{E}(\widetilde{U})_{t-}\Delta\widetilde{L}_t^-| \le 1/2$ and $[M,M]_t \le \sigma_L^2 t + \sum_{0<s\le t}(\Delta\widetilde{L}_s^-)^2$, where the latter is a Lévy process with bounded jumps having finite exponential moments by [58, Theorem 25.17]. Hence by [56, Theorem 6.1] $(\mathcal{E}(M)_s)_{0<s\le t}$ is a martingale.
By the definition of the Doléans-Dade exponential we have

$$\exp(\epsilon M_s) = \mathcal{E}(M)_s^\epsilon \cdot \exp\left(\frac{1}{2}\epsilon\sigma_M^2 s\right)\cdot\left(\prod_{0<u\le s}(1+\Delta M_u)^{-1}e^{\Delta M_u}\right)^\epsilon$$

where the first two factors on the RHS have bounded expectation uniformly in $s \in [0,t]$ and sufficiently small $\epsilon > 0$ and for the last factor observe that

$$\sum_{0<u\le s}(\Delta M_u - \log(1+\Delta M_u)) \le \sum_{0<u\le s}(\Delta M_u)^2 \le \sum_{0<u\le s}(\Delta\widetilde{L}_u^-)^2,$$

since $|\Delta M_u| < 1/2$, such that $\sup_{0\le s\le t} E\left(\prod_{0<u\le s}(1+\Delta M_u)^{-1}e^{\Delta M_u}\right)^\epsilon$ is finite for sufficiently small ϵ. An application of Hölder's inequality hence gives $E[\exp(\epsilon M_s)] \le C_1$ for all $s \le t$, some constant $C_1 = C_1(t)$ and sufficiently small $\epsilon > 0$.
Remark that

$$E\left[e^{\epsilon|M_t|}\right] = E\left[e^{\epsilon M_t}\mathbb{1}_{M_t>0}\right] + E\left[e^{-\epsilon M_t}\mathbb{1}_{M_t<0}\right] \le E\left[e^{\epsilon M_t}\right] + E\left[e^{-\epsilon M_t}\right]$$

such that $E[\exp(\epsilon|M_t|)] \le 2C_1$ since the above calculations also hold true for $-M_t = \int_{(0,t]}\mathcal{E}(\widetilde{U})_{s-}d(-\widetilde{L}^-)_s$. Hence the second term on the RHS of (3.43) is bounded and we conclude that

$$E\left[\exp\left(\epsilon\left|\int_{(0,t-w]}\mathcal{E}(\widetilde{U})_{s-}d\widetilde{L}_s\right|\right)\right] \le C_2$$

for some constant $C_2 = C_2(t)$ and sufficiently small $\epsilon > 0$ uniformly in $w \in [0,t]$. Thus

$$\int_{[0,t]} E\left[\exp\left(\epsilon\left|\int_{(0,t-w]}\mathcal{E}(\widetilde{U})_{s-}d\widetilde{L}_s\right|\right)\right] dP_{T(t)}(w) < \infty$$

such that $|B_t|$ is shown to have finite m.g.f. in some neighbourhood of 0.
To prove (b) due to (3.22) we have to show for $t > 0$ that $P(|A_t^{-1}| \le 1) = P(|\mathcal{E}(W)_t| \le 1) = 1$, $P(|A_t| < 1) > 0$ and that the m.g.f. of $|A_t^{-1}B_t|$, i.e. of $|\int_{[0,t)}\mathcal{E}(W)_{s-}d\eta_s|$, $t > 0$, is finite in some neighbourhood $(-\epsilon,\epsilon)$ of 0. This can be done as in (a) and the result follows again from Theorem 2.1 of [23]. □

3.4.3 Proofs for Section 3.3

The following lemma will be needed to prove our main theorem on absolute continuity. An analogous result for almost surely non-zero M_t has been shown in Bertoin et al. [8, Lemma 2.1].

Lemma 3.13. *For $t \in \mathbb{N}_0$, let ψ_t, Q_t and M_t be random variables such that it holds $P(M_t = 0) > 0$, ψ_t is independent of (Q_t, M_t) and for large enough $t \in \mathbb{N}$ the conditional distribution of Q_t given $M_t = 0$ is continuous. Suppose ψ to be a random variable satisfying $\psi = Q_t + M_t\psi_t$ and $\psi \overset{d}{=} \psi_t$ for all $t \geq 0$ and such that $Q_t \overset{P}{\to} \psi$ as $t \to \infty$. Then ψ has an atom if and only if it is a constant random variable.*

Proof. Suppose that ψ has an atom at $a \in \mathbb{R}$, i.e. $P(\psi = a) =: \beta > 0$. Then for all $\epsilon \in (0, \beta)$ there exists $\delta > 0$ such that $P(|\psi - a| < 2\delta) < \beta + \epsilon$. Additionally $Q_t \overset{P}{\to} \psi$ implies the existence of some $t' = t'(\epsilon)$ such that $P(|\psi - Q_t| \geq \delta) = P(|M_t\psi_t| \geq \delta) < \epsilon$ for all $t \geq t'$.
Following the lines of the proof of [8, Lemma 2.1] one can now show, that for all $t \geq t'$ there exists $s_t \in \mathbb{R}$ such that $\beta_t := P(Q_t + M_t s_t = a, |M_t s_t| < \delta) \geq \beta - \epsilon$ and it holds

$$\begin{aligned} &P(|\psi - Q_t| \geq \delta) + P(|Q_t - a| < \delta) \\ &\quad\geq\; P(\psi = a) + P(Q_t + M_t s_t = a, |M_t s_t| < \delta) \\ &\qquad -P(Q_t + M_t s_t = a, |M_t s_t| < \delta, \psi = a). \end{aligned} \tag{3.44}$$

Since $P(M_t = 0) > 0$ we have

$$\begin{aligned} &\{Q_t + M_t s_t = a, |M_t s_t| < \delta, \psi = a\} \\ &\quad= \{Q_t + M_t s_t = a, |M_t s_t| < \delta, \psi = a, M_t = 0\} \\ &\qquad \cup\{Q_t + M_t s_t = a, |M_t s_t| < \delta, \psi = a, M_t \neq 0\} \\ &\quad\subset \{Q_t = a, M_t = 0\} \cup \{Q_t + M_t s_t = a, |M_t s_t| < \delta, M_t \neq 0, \psi_t = s_t\} \\ &\quad\subset \{Q_t = a, M_t = 0\} \cup (\{Q_t + M_t s_t = a, |M_t s_t| < \delta\} \cap \{\psi_t = s_t\}) \end{aligned}$$

and by the continuity assumption on Q_t given $M_t = 0$, we obtain

$$P(Q_t + M_t s_t = a, |M_t s_t| < \delta, \psi = a) \leq 0 + \beta_t P(\psi_t = s_t),$$

such that we can conclude from (3.44) that it holds

$$P(|\psi - Q_t| \geq \delta) + P(|Q_t - a| < \delta) \geq \beta + \beta_t - \beta_t\, P(\psi_t = s_t).$$

From here again following directly the proof of [8, Lemma 2.1] we get the assumption $P(\psi = a) = 1$. □

Now we can prove the conditions for the distribution of the perpetuity X_∞ to be (absolutely) continuous in the case $P(A = 0) > 0$ as stated in Theorem 3.10.

Proof of Theorem 3.10. To show (a), first suppose that the conditional distribution of B given $A = 0$ is continuous. Let $(A_k, B_k)_{k\in\mathbb{N}_0}$ be an i.i.d. sequence of random variables such that (A_0, B_0) has the same distribution as (A, B). Define

$$\psi := X_\infty = \sum_{k=0}^{\infty} \left(\prod_{i=0}^{k-1} A_i\right) B_k, \qquad M_t = \prod_{i=0}^{t-1} A_i,$$

$$\psi_t = \sum_{k=t}^{\infty} \left(\prod_{i=t}^{k-1} A_i\right) B_k, \qquad Q_t = \sum_{k=0}^{t-1} \left(\prod_{i=0}^{k-1} A_i\right) B_k.$$

Then it follows from Lemma 3.13 that ψ is continuous or a Dirac measure if we can show that the conditional distribution of Q_t given $M_t = 0$ is continuous for all $t \in \mathbb{N}$. We do this by induction.
For $t = 1$ the claim is true by assumption. Now suppose that the conditional distribution of Q_t given $M_t = 0$ is continuous, so that $P(Q_t = a, M_t = 0) = 0$ for each $a \in \mathbb{R}$. We then have, since $Q_{t+1} = Q_t + M_t B_t$, for each $a \in \mathbb{R}$,

$$\begin{aligned} & P(Q_{t+1} = a, M_{t+1} = 0) \\ &= P(Q_t + M_t B_t = a, M_t A_t = 0) \\ &= P(Q_t = a, M_t = 0) + P(Q_t + M_t B_t = a, M_t \neq 0, A_t = 0) \\ &= P(B_t = M_t^{-1}(a - Q_t), M_t \neq 0, A_t = 0) \end{aligned}$$

by induction hypothesis. Now using regular conditional probabilities, we obtain further

$$\begin{aligned} & P(B_t = M_t^{-1}(a - Q_t), M_t \neq 0, A_t = 0) \\ &= \int_{(\mathbb{R}\setminus\{0\})\times\mathbb{R}} P(B_t = u^{-1}(a - v), A_t = 0 | M_t = u, Q_t = v)\, dP_{(M_t,Q_t)}(u, v) \\ &= \int_{(\mathbb{R}\setminus\{0\})\times\mathbb{R}} P(B_t = u^{-1}(a - v), A_t = 0)\, dP_{(M_t,Q_t)}(u, v) \\ & \quad \text{(since } (A_t, B_t) \text{ is independent of } (M_t, Q_t)) \\ &= \int_{(\mathbb{R}\setminus\{0\})\times\mathbb{R}} 0\, dP_{(M_t,Q_t)}(u, v) = 0, \end{aligned}$$

the latter following from the fact that the conditional distribution of B_t given $A_t = 0$ is continuous. Hence we see that the conditional distribution of Q_{t+1} given $M_{t+1} = 0$ is continuous, too, completing the induction step. Lemma 3.13 hence shows that X_∞ is continuous or degenerate to a Dirac measure. But X_∞ cannot be degenerate to a Dirac measure, since $X_\infty \stackrel{d}{=} B + AX' = B$ on $A = 0$, where $P(A = 0) > 0$ and $\mathcal{L}(B|A = 0)$ is continuous.
To see the converse, suppose that the conditional distribution of B given $A = 0$ is not continuous. Then there is $a \in \mathbb{R}$ such that $P(B = a|A = 0) = \beta > 0$. Since $\mathcal{L}(X_\infty)$ satisfies the fixed point equation (3.17), we have

$$P(X_\infty = a) = P(AX' + B = a) \geq P(B = a, A = 0) > 0.$$

Hence $\mathcal{L}(X_\infty)$ has an atom.
For (b) we will first show, that $\mathcal{L}(X_\infty)$ is either absolutely continuous or a Dirac measure, given that the conditional distribution of B given $A = 0$ is absolutely continuous. Then it follows from part (a), that $\mathcal{L}(X_\infty)$ is absolutely continuous. In doing so we follow the arguments of Grincevičius [25] who considered the case $P(A = 0) = 0$.
Assume that $\mathcal{L}(X_\infty)$ is not singular and denote its characteristic function by

$$\begin{aligned} f(x) &= Ee^{ixX_\infty} = E\left[E\left[e^{iBx}e^{iAxX'}|A, B\right]\right] \\ &= E\left[e^{iBx} f(Ax)\right]. \end{aligned}$$

Then by the Lebesgue decomposition theorem we may write $f(x) = \alpha_1 f_1(x) + \alpha_2 f_2(x)$, where $\alpha_1 > 0$, $\alpha_2 \geq 0$, $\alpha_1 + \alpha_2 = 1$ and $f_1(x)$ and $f_2(x)$ are the characteristic functions, respectively, of an absolutely continuous and a singular probability distribution. Hence

$$\alpha_1 f_1(x) + \alpha_2 f_2(x) = \alpha_1 E[e^{iBx} f_1(Ax)] + \alpha_2 E[e^{iBx} f_2(Ax)].$$

Let Y be a random variable independent of (A, B), having characteristic function f_1 and set $Z := AY + B$, then for $C \in \mathcal{B}_1$ with Lebesgue measure 0 it holds

$$\begin{aligned} P(Z \in C) &= P(AY + B \in C) \\ &= P(B \in C, A = 0) + P(AY + B \in C, A \neq 0) \\ &= 0 + \int_{(\mathbb{R}\setminus\{0\})\times\mathbb{R}} P(Y \in u^{-1}(C - v)) dP_{A,B}(u, v) \\ &= 0. \end{aligned}$$

It follows that Z is absolutely continuous and its characteristic function $x \mapsto E(e^{iBx} f_1(Ax))$ is the characteristic function of an absolutely continuous function.
Applying the Lebesgue decomposition to the distribution having characteristic function $x \mapsto Ee^{iBx} f_2(Ax)$, we can write $Ee^{iBx} f_2(Ax) = \alpha_3 f_3(x) + \alpha_4 f_4(x)$ with $\alpha_3, \alpha_4 \geq 0$, $\alpha_3 + \alpha_4 = 1$, and f_3 and f_4 are, respectively, the characteristic functions of an absolutely continuous and a singular distribution. By the uniqueness of the Lebesgue decomposition if follows

$$\alpha_1 f_1(x) = \alpha_1 E\left[e^{iBx} f_1(Ax)\right] + \alpha_2 \alpha_3 f_3(x),$$

which for $x = 0$ yields $\alpha_2 \alpha_3 = 0$. Hence $f_1(x) = E\left[e^{iBx} f_1(Ax)\right]$. Since also $f(x) = E\left[e^{iBx} f(Ax)\right]$ it follows that $f(x) = f_1(x)$ by an easy extension of Proposition 1 of [25]. Hence we conclude that $\mathcal{L}(X_\infty)$ is absolutely continuous.
It remains to show that if the conditional distribution of B given $A = 0$ is not absolutely continuous, then $\mathcal{L}(X_\infty)$ cannot be absolutely continuous. For in that case, there is $C \in \mathcal{B}_1$ with Lebesgue measure zero but $P(B \in C | A = 0) > 0$. We conclude $P(B \in C, A = 0) > 0$ and hence (for $X' \stackrel{d}{=} X_\infty$ being independent of (A, B))

$$P(X_\infty \in C) = P(AX' + B \in C) \geq P(B \in C, A = 0) > 0,$$

so that $\mathcal{L}(X_\infty)$ cannot be absolutely continuous. $\square$

Chapter 4: Distributions of Exponential Integrals related to Generalized Gamma Convolutions[1]

Let $(\xi, L) = (\xi_t, L_t)_{t\geq 0}$ be a bivariate càdlàg independent increment process. In most cases in this chapter, (ξ, L) is assumed to be a bivariate Lévy process, but we will also treat more general cases where ξ or L is a compound sum process, which is not necessarily a Lévy process but another typical independent increment process.
Our concern in this chapter is to examine distributional properties of the exponential integral

$$V_\infty := \int_{(0,\infty)} e^{-\xi_{t-}} dL_t, \tag{4.1}$$

provided that this integral converges almost surely. As we saw in the previous chapters, integrals of the form (4.1) occur as stationary solutions of generalized Ornstein-Uhlenbeck processes. This motivates the question of when $\mathcal{L}(V_\infty)$ is in a given class of distributions. More precisely, in the following we will investigate when $\mathcal{L}(V_\infty)$ is selfdecomposable or moreover is a generalized gamma convolution.
We say that a probability distribution μ on $\mathbb{R}$ (resp. an $\mathbb{R}$-valued random variable X) is selfdecomposable, if for any $b > 1$, there exists a probability distribution μ_b (resp. a random variable Y_b independent of X) such that

$$\mu = D_{b^{-1}}(\mu) * \mu_b, \quad (\text{resp. } X \overset{d}{=} b^{-1}X + Y_b),$$

where $D_a(\mu)$ means the distribution induced by $D_a(\mu)(aB) := \mu(B)$ for $B \in \mathcal{B}_1$ and $*$ is the convolution operator. Every selfdecomposable distribution is infinitely divisible. Some well-known distributional properties of non-trivial selfdecomposable distributions are absolute continuity and unimodality, (see [58] p.181 and p.404).

First we review existing results on $\mathcal{L}(V_\infty)$.

[1]Based on [6]: A. Behme, M. Maejima, M. Muneya and N. Sakuma (2010):
Distributions of exponential integrals of independent increment processes related to generalized gamma convolutions,
submitted.

Bertoin et al. [8] (in the case when $L = (L_t)_{t\geq 0}$ is a one-dimensional Lévy process) and Kondo et al. [42] (in the case when L is a multi-dimensional Lévy process) showed that $\mathcal{L}(V_\infty)$ is selfdecomposable if $\xi = (\xi_t)_{t\geq 0}$ is a spectrally negative Lévy process satisfying $\lim_{t\to\infty} \xi_t = +\infty$ a.s. and if the integral (4.1) converges a.s., or equivalently, if $\int_{\mathbb{R}} \log^+ |y| \Pi_L(dy) < \infty$.
On the other hand, there is an example of non-infinitely divisible $\mathcal{L}(V_\infty)$, which is due to Samorodnitsky, (see [40]). In fact if $(\xi_t, L_t) = (N_t + at, t)$, where $(N_t)_{t\geq 0}$ is a subordinator and $a > 0$, then the support of $\mathcal{L}(V_\infty)$ is bounded so that $\mathcal{L}(V_\infty)$ cannot be infinitely divisible.
Recently, Lindner and Sato [47] considered the exponential integral

$$\int_{(0,\infty)} \exp\left(-(\log c)N_{t-}\right) dY_t = \int_{(0,\infty)} c^{-N_{t-}} dY_t, \quad c > 0,$$

where $(N_t, Y_t)_{t\geq 0}$ is a bivariate compound Poisson process whose Lévy measure is concentrated on $(1,0)$, $(0,1)$ and $(1,1)$. They showed a necessary and sufficient condition for the infinite divisibility of $\mathcal{L}(V_\infty)$ and pointed out that $\mathcal{L}(V_\infty)$ is always c^{-1}-decomposable, namely there exists a probability distribution ρ such that $\mu = D_{c^{-1}}(\mu) * \rho$. Note here, that a c^{-1}-decomposable distribution is not necessarily infinitely divisible, unless ρ is infinitely divisible.
In their second paper [48], Lindner and Sato also gave a condition under which $\mathcal{L}(V_\infty)$, generated by a bivariate compound Poisson process $(N_t, Y_t)_{t\geq 0}$ whose Lévy measure is concentrated on $(1,0)$, $(0,1)$ and $(1, c^{-1})$, is infinitely divisible.

In this chapter we focus on "Generalized Gamma Convolutions" (GGCs, for short,) to get finer distributional informations on V than selfdecomposability.
We say that for $r > 0$ and $\lambda > 0$ a random variable $\gamma_{r,\lambda}$ has a gamma(r,λ) distribution if its probability density function f on $(0,\infty)$ is

$$f(x) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x},$$

where $\Gamma(r)$ denotes the Gamma function.
A gamma$(1,\lambda)$ distribution simply is an exponential distribution with parameter $\lambda > 0$. When we do not have to emphasize the parameters (r,λ), we just write γ for a gamma random variable.
The class of GGCs is defined to be the smallest class of distributions on the positive half line that contains all gamma distributions and is closed under convolution and weak convergence. By including gamma distributions on the negative real axis, we obtain the class of distributions on $\mathbb{R}$ which will be called "Extended Generalized Gamma Convolutions" (EGGCs, for short). We refer to [12] and [61] for many properties of GGCs and EGGCs with relations among other subclasses of infinitely divisible distributions.

One well-known concrete example of the law of an exponential integral originally due to Dufresne [15] is the following. When $(\xi_t, L_t) = (B_t + at, t)$ with a standard

Brownian motion $(B_t)_{t\geq 0}$ and a drift $a > 0$, the distribution of (4.1) is $\mathcal{L}(1/(2\gamma))$ which is a GGC (and thus is selfdecomposable). This motivates the question if there are other exponential integrals in the class of GGCs, which will be investigated in the following.

This chapter is organized as follows. In Section 4.1 we give some preliminaries. In Section 4.2 we consider exponential integrals for two independent Lévy processes ξ and L such that either ξ or L is a compound Poisson process, and construct concrete examples related to our question. In the special case that both ξ and L are compound Poisson processes, we also treat a model allowing dependence between the two components of (ξ, L). In Section 4.3, we consider exponential integrals for independent increment processes such that ξ and L are independent and one is a compound sum process (which is not necessarily a Lévy process) while the other is a Lévy process.

4.1 Preliminaries

We denote the class of all infinitely divisible distributions on $\mathbb{R}$ (resp. $\mathbb{R}_+$) by $I(\mathbb{R})$ (resp. $I(\mathbb{R}_+)$). The class of selfdecomposable distributions on $\mathbb{R}$ (resp. $\mathbb{R}_+$) will be denoted by $L(\mathbb{R})$ (resp. $L(\mathbb{R}_+)$) while the class of EGGCs on $\mathbb{R}$ (resp. GGCs on $\mathbb{R}_+$) is noted as $T(\mathbb{R})$ (resp. $T(\mathbb{R}_+)$). The moment generating function of a random variable X and of a distribution μ are written as $\mathbb{L}_X$ and $\mathbb{L}_\mu$, respectively. If X is positive and μ has support in $\mathbb{R}_+$, $\mathbb{L}_X$ and $\mathbb{L}_\mu$ coincide with the Laplace transforms.

We are especially interested in distributions on $\mathbb{R}_+$. The class $T(\mathbb{R}_+)$ can be characterized by the Laplace transform as follows: A probability distribution μ is GGC if and only if there exist $a \geq 0$ and a measure U satisfying

$$\int_{(0,1]} |\log x| U(dx) < \infty \quad \text{and} \quad \int_{(1,\infty)} x^{-1} U(dx) < \infty, \tag{4.2}$$

such that the Laplace transform $\mathbb{L}_\mu(u)$ can be uniquely represented as

$$\mathbb{L}_\mu(u) = \int_{[0,\infty)} \mathrm{e}^{-ux} \mu(dx) = \exp\left\{-au + \int_{(0,\infty)} \log\left(\frac{x}{x+u}\right) U(dx)\right\}. \tag{4.3}$$

Another class of distributions which we are interested in is the class of distributions on $\mathbb{R}_+$ whose densities are hyperbolically completely monotone (HCM, for short). Here we say that a function $f(x)$ on $(0,\infty)$ with values in $\mathbb{R}_+$ is HCM if for every $u > 0$ the function $v \mapsto f(uv) \cdot f(u/v)$, $v > 0$, is completely monotone with respect to the variable $w = v + v^{-1}$. Examples of HCM functions are x^β ($\beta \in \mathbb{R}$), e^{-cx} ($c > 0$) and $(1 + cx)^{-\alpha}$ ($c > 0$, $\alpha > 0$). The class of all distributions on $\mathbb{R}_+$ whose probability densities are HCM is denoted by $H(\mathbb{R}_+)$. Note that $H(\mathbb{R}_+) \subset T(\mathbb{R}_+) \subset L(\mathbb{R}_+) \subset I(\mathbb{R}_+)$.

For illustration we give some examples. Log-normal distributions are in $H(\mathbb{R}_+)$ [12, Example 5.2.1]. So these are also GGCs. Positive strictly stable distributions with Laplace transform $\mathbb{L}(u) = e^{-u^\alpha}$ for $\alpha \in \{1/2, 1/3, \ldots\}$ are in $H(\mathbb{R}_+)$ [12, Example 5.6.2] while they are GGCs for all $\alpha \in (0, 1]$ [61, Proposition 5.7]. Let $Y = e^{\gamma_{r,\lambda}} - 1$. If $r \geq 1$, then $\mathcal{L}(Y)$ is in $H(\mathbb{R}_+)$, but if $r < 1$, $\mathcal{L}(Y)$ is not in $H(\mathbb{R}_+)$ [12, p. 88]. But $\mathcal{L}(Y)$ or equivalently $\mathcal{L}(e^{\gamma_{r,\lambda}})$ is always in $T(\mathbb{R}_+)$, independent of the value of r [12, Theorem 6.2.3].
Remark that by treating $H(\mathbb{R}_+)$ we cannot replace $e^{\gamma_{r,\lambda}} - 1$ by $e^{\gamma_{r,\lambda}}$. Namely, set $r = 1$ and observe that the probability density function $\lambda(x+1)^{-\lambda-1}\mathbb{1}_{[0,\infty)}(x)$ is HCM, but the probability density function $\lambda x^{-\lambda-1}\mathbb{1}_{[1,\infty)}(x)$ is not HCM. It follows from this that $\mathcal{L}(e^{\gamma_{1,\lambda}} - 1)$ is in $H(\mathbb{R}_+)$ but $\mathcal{L}(e^{\gamma_{1,\lambda}})$ is not in $H(\mathbb{R}_+)$.

In addition, we also investigate the extended HCM class denoted by $\widetilde{H}(\mathbb{R})$, which gives some interesting examples of $\mathcal{L}(V_\infty)$ on $\mathbb{R}$. The class $\widetilde{H}(\mathbb{R})$ is characterized to be the class of distributions of random variables $\sqrt{X}Z$, where X and Z are independent, $\mathcal{L}(X) \in H(\mathbb{R}_+)$ or degenerate and Z is a standard normal random variable. By the definition, any distribution in $\widetilde{H}(\mathbb{R}_+)$ is a so-called type-G distribution, i.e. it is the distribution of the variance mixture of a standard normal random variable. Note that $H(\mathbb{R}_+) \not\subseteq \widetilde{H}(\mathbb{R})$ and $\widetilde{H}(\mathbb{R}) \subset T(\mathbb{R})$. As will be seen in Proposition 4.1, there are nice relations between $\widetilde{H}(\mathbb{R})$ and $T(\mathbb{R})$ in common with those of $H(\mathbb{R}_+)$ and $T(\mathbb{R}_+)$.

Here we state some known facts that we will use later ([61, Theorem VI.5.24 and Propositions VI.5.27 and VI.5.19] and [12, Theorems 7.3.3, 7.3.5 and 7.3.6]).

Proposition 4.1.

(a) *A continuous function $\mathbb{L}(u)$, $u > 0$, with $\mathbb{L}(0+) = 1$ is HCM if and only if it is the Laplace transform of a GGC.*
(b) *If $\mathcal{L}(X) \in H(\mathbb{R}_+)$, $\mathcal{L}(Y) \in T(\mathbb{R}_+)$ and X and Y are independent, then $\mathcal{L}(XY) \in T(\mathbb{R}_+)$.*
(c) *If $\mathcal{L}(X) \in H(\mathbb{R}_+)$, $\mathcal{L}(Y) \in H(\mathbb{R}_+)$ and X and Y are independent, then $\mathcal{L}(XY) \in H(\mathbb{R}_+)$ and $\mathcal{L}(X/Y) \in H(\mathbb{R}_+)$.*
(d) *If $\mathcal{L}(X) \in H(\mathbb{R}_+)$ and $|q| \geq 1$, then $\mathcal{L}(X^q) \in H(\mathbb{R}_+)$.*
(e) *Suppose that $\mathcal{L}(X) \in H(\mathbb{R}_+)$, $\mathcal{L}(Y) \in T(\mathbb{R})$ and that X and Y are independent. If $\mathcal{L}(Y)$ is symmetric, then $\mathcal{L}(\sqrt{X}Y) \in T(\mathbb{R})$.*
(f) *Suppose that $\mathcal{L}(X) \in \widetilde{H}(\mathbb{R})$ and $\mathcal{L}(Y) \in T(\mathbb{R})$ and that X and Y are independent. If $\mathcal{L}(Y)$ is symmetric, then $\mathcal{L}(XY) \in T(\mathbb{R})$.*
(g) *If $\mathcal{L}(X) \in \widetilde{H}(\mathbb{R})$, then $\mathcal{L}(|X|^q) \in H(\mathbb{R}_+)$ for all $|q| \geq 2$, $q \in \mathbb{R}$. Furthermore, $\mathcal{L}(|X|^q \mathrm{sign}(X)) \in \widetilde{H}(\mathbb{R})$ for all $q \in \mathbb{N}$, $q \neq 2$, but not always for $q = 2$.*

Remark 4.2. *Notice that the distribution of a sum of independent random variables with distributions in $H(\mathbb{R}_+)$ does not necessarily belong to $H(\mathbb{R}_+)$ [12, p. 101].*

Some distributional properties of GGCs are stated in the following proposition [12, Theorems 4.1.1. and 4.1.3.].

Proposition 4.3.

(a) *The probability density function of a GGC without Gaussian part satisfying* $0 < \int_{(0,\infty)} U(du) = \beta < \infty$ *with the measure* U *as in* (4.3)*, admits the representation* $x^{\beta-1}h(x)$*, where* $h(x)$ *is some completely monotone function.*

(b) *Let* f *be the probability density function of a GGC distribution without Gaussian part satisfying* $1 < \int_{(0,\infty)} U(du) = \beta \leq \infty$*. Let* $k \geq 0$ *be an integer such that* $k < \beta - 1$*. Then* f *is continuously differentiable any times on* $(0,\infty)$*, and at* 0 *at least* k *times differentiable with* $f^{(j)}(0) = 0$ *for* $j \leq k$*.*

Examples of GGCs and the explicit calculation of their Lévy measure are found in [12] and [32].

4.2 Exponential Integrals of Compound Poisson Processes

In this section we study exponential integrals of the form (4.1), where either ξ or L is a compound Poisson process and the other is an arbitrary Lévy process. First we assume the two processes to be independent, later we also investigate the case that (ξ, L) is a bivariate compound Poisson processes with dependent components.

4.2.1 Independent component case

We start with a general lemma which gives a sufficient condition for distributions of perpetuities to be GGCs.

Lemma 4.4. *Suppose* A *and* B *are two independent random variables such that* $\mathcal{L}(A) \in H(\mathbb{R}_+)$ *and* $\mathcal{L}(B) \in T(\mathbb{R}_+)$*. Let* $(A_j, B_j), j = 0, 1, 2, \ldots$ *be i.i.d. copies of* (A, B)*. Then, given its a.s. convergence, the distribution of the perpetuity* $Z := \sum_{k=0}^{\infty} \left(\prod_{i=0}^{k-1} A_i\right) B_k$ *belongs to* $T(\mathbb{R}_+)$*. Similarly, if* $\mathcal{L}(A) \in \widetilde{H}(\mathbb{R})$*,* $\mathcal{L}(B) \in T(\mathbb{R})$ *and* $\mathcal{L}(B)$ *is symmetric, then* $\mathcal{L}(Z) \in T(\mathbb{R})$*.*

Proof. If we put

$$Z_n := \sum_{k=0}^{n} \left(\prod_{i=0}^{k-1} A_i\right) B_k,$$

then we can rewrite

$$Z_n = B_0 + A_0(B_1 + A_1(B_2 + \cdots + A_{n-2}(B_{n-1} + A_{n-1}B_n) \cdots)).$$

Since A_{n-1} and B_n are independent, $\mathcal{L}(A_{n-1}) \in H(\mathbb{R}_+)$ and $\mathcal{L}(B_n) \in T(\mathbb{R}_+)$, we get $\mathcal{L}(A_{n-1}B_n) \in T(\mathbb{R}_+)$ by Proposition 4.1(b). Further B_{n-1} and $A_{n-1}B_n$ are independent and both distributions belong to $T(\mathbb{R}_+)$ such that $\mathcal{L}(B_{n-1}+A_{n-1}B_n) \in T(\mathbb{R}_+)$. By the same argument we can now conclude that $\mathcal{L}(B_{n-2}+A_{n-2}(B_{n-1}+A_{n-1}B_n)) \in T(\mathbb{R}_+)$ and by induction we obtain that $\mathcal{L}(Z_n) \in T(\mathbb{R}_+)$. Since the class $T(\mathbb{R}_+)$ is closed under weak convergence, the first part follows immediately. A similar argument using Proposition 4.1(f) gives the second part. □

Case 1: The process ξ is a compound Poisson process

Proposition 4.5. *Suppose that the processes ξ and L are independent Lévy processes where $\xi_t = \sum_{i=1}^{N_t} X_i$ is a compound Poisson process with i.i.d. jump heights $X_i,\ i = 1, 2, \ldots$ such that $0 < E[X_1] < \infty$, $\mathcal{L}(e^{-X_1}) \in H(\mathbb{R}_+)$ and L has finite log-moment $E\log^+|L_1|$ and it holds $\mathcal{L}(L_\tau) \in T(\mathbb{R}_+)$ for an exponential random variable τ independent of L with the same distribution as the waiting times of N. Then the integral* (4.1) *converges a.s. and*

$$\mathcal{L}\left(\int_{(0,\infty)} e^{-\xi_{t-}}\,dL_t\right) \in T(\mathbb{R}_+).$$

Similarly, if $\mathcal{L}(e^{-X_1}) \in \widetilde{H}(\mathbb{R})$, $\mathcal{L}(L_\tau) \in T(\mathbb{R})$ and $\mathcal{L}(L_\tau)$ is symmetric, then $\mathcal{L}(V_\infty) \in T(\mathbb{R})$.

Proof. Convergence of the integral follows directly from (1.14).
Set $T_0 = 0$ and let $T_j, j = 1, 2, \ldots$ be the time of the j-th jump of $(N_t)_{t\geq 0}$. Then we can write

$$\begin{aligned}\int_{(0,\infty)} e^{-\xi_{t-}}\,dL_t &= \sum_{j=0}^{\infty}\int_{(T_j,T_{j+1}]} e^{-\sum_{i=1}^{j} X_i}\,dL_t = \sum_{j=0}^{\infty} e^{-\sum_{i=1}^{j} X_i}\int_{(T_j,T_{j+1}]} dL_t\\ &=: \sum_{j=0}^{\infty}\left(\prod_{i=1}^{j} A_i\right) B_j,\end{aligned}$$

where $A_i = e^{-X_i}$ and $B_j = \int_{(T_j,T_{j+1}]} dL_t \overset{d}{=} L_{T_{j+1}-T_j}$. Now Lemma 4.4 yields the conclusion. □

In the following we give some examples for choices of ξ and L fulfilling the assumptions of Proposition 4.5.

Example 4.6 (X_1 is a normal random variable with positive mean)**.** *We immediately see that $\mathcal{L}(e^{-X_1})$ is log-normal and hence is in $H(\mathbb{R}_+)$.*

Example 4.7 (X_1 is the logarithm of the power of a gamma random variable)**.** *Let $Y = \gamma_{r,\lambda}$ and set $X_1 = c\log Y$ for $|c| \geq 1$. By [12, Example 5.6.3] we know that $\mathcal{L}(e^{-X_1}) = \mathcal{L}(\gamma_{r,\lambda}^{-c}) \in H(\mathbb{R}_+)$. Note that it follows from [12, Example 7.2.3] that $E[X_1] = c\psi(\lambda)$, where $\psi(x)$ denotes the derivative of $\log\Gamma(x)$. If we take $c \in \mathbb{R}$ such that $c\psi(\lambda) > 0$ the integrability condition on ξ in Proposition 4.5 is fulfilled.*

Example 4.8 (X_1 is logarithm of a positive strictly stable random variable)**.** *Let $X_1 = \log Y$ be a random variable, where Y is a positive strictly stable random variable with parameter $0 < \alpha < 1$. Then X_1 is in the class of EGGC when $\alpha = 1/n$, $n = 2, 3, \dots$ (see Example 7.2.5 of [12]) and*

$$E[e^{uX_1}] = E[Y^u] = \frac{\Gamma(1-u/\alpha)}{\Gamma(1-u)}.$$

It follows that $E[X_1] = -\frac{1}{\alpha}\psi(1)+\psi(1) = (1-1/\alpha)\psi(1) > 0$ and $\mathcal{L}(e^{-X_1}) = \mathcal{L}(Y^{-1}) \in H(\mathbb{R}_+)$ by [12, Example 5.6.2].

Example 4.9 (X_1 is the logarithm of the ratio of two exponential random variables)**.** *Let $X_1 = \log(Y_1/Y_2)$, where $Y_j, j = 1, 2$, are independent exponential random variables with $\lambda = 1$. The density function of X_1 is given by [12, Example 7.2.4]*

$$f(x) = \frac{1}{B(1,1)}\frac{e^{-x}}{(1+e^{-x})^2}, \quad x \in \mathbb{R},$$

where $B(\cdot,\cdot)$ denotes the Beta-function. Now if $E[X_1] > 0$, then we can set X_1 to be a jump distribution of ξ. By Proposition 4.1(c) it is easy to see that $\mathcal{L}(e^{-X_1}) = \mathcal{L}(Y_2/Y_1) \in H(\mathbb{R}_+)$ since $\mathcal{L}(Y_j) \in H(\mathbb{R}_+)$.

Example 4.10 (L is nonrandom)**.** *When $L_t = t$, it holds that $\mathcal{L}(L_\tau) = \mathcal{L}(\tau) \in T(\mathbb{R}_+)$. Hence for a suitable ξ, we have $\mathcal{L}(V_\infty) \in T(\mathbb{R}_+)$.*

Example 4.11 (L is a stable subordinator)**.** *Consider L to be a stable subordinator without drift. Then the Laplace transform of L_1 is given by $\mathbb{L}_{L_1}(u) = \exp\{-u^\alpha\}$. The Laplace transform of $B := L_\tau$ with $\tau = \gamma_{1,\lambda}$ is given by (see e.g. [61, p.10])*

$$\mathbb{L}_B(u) = \frac{\lambda}{\lambda - \log \mathbb{L}_{L_1}(u)}. \tag{4.4}$$

such that we obtain

$$\mathbb{L}_B(u) = \frac{\lambda}{\lambda + u^\alpha}.$$

This function is HCM, since $\frac{\lambda}{\lambda+u}$ is HCM by the definition and due to the fact that the composition of an HCM function and x^α, $|\alpha| \leq 1$, is also HCM. Thus the Laplace transform of B is HCM and by Proposition 4.1(a) we conclude that $\mathcal{L}(L_\tau)$ is GGC.

Remark that if L admits an additional drift term, the distribution $\mathcal{L}(L_\tau)$ in the setting of the last example is not GGC. This result was pointed out in [43].

Example 4.12 (L is an inverse Gaussian subordinator)**.** *Now we suppose L to be an inverse Gaussian subordinator with parameters $\beta > 0$ and $\delta > 0$. The Laplace transform of L_t is*

$$\mathbb{L}_{L_t}(u) = \exp\left(-\delta t(\sqrt{\beta^2 + 2u} - \beta)\right).$$

Now by choosing the parameters satisfying $\lambda = \delta\beta$, we have, for $B = L_\tau$,

$$\mathbb{L}_B(u) = \frac{\beta}{\sqrt{\beta^2 + 2u}}.$$

This function is HCM from the definition and again by using Proposition 4.1(a) we see that $\mathcal{L}(L_\tau)$ is GGC.

Case 2: The process L is a compound Poisson process

In the following we now assume the integrator L to be a compound Poisson process, while ξ is an arbitrary Lévy process, independent of L. We can argue similarly as above to obtain the following result.

Proposition 4.13. *Let ξ and L be independent and assume $L_t = \sum_{i=1}^{N_t} Y_i$ to be a compound Poisson process with i.i.d. jump heights Y_i, $i = 1, 2, \ldots$. Suppose that $E[\xi_1] > 0$, $E \log^+ |L_1| < \infty$, $\mathcal{L}(Y_1) \in T(\mathbb{R}_+)$ and $\mathcal{L}(e^{-\xi_\tau}) \in H(\mathbb{R}_+)$ for an exponentially distributed random variable τ independent of ξ with the same distribution as the waiting times of N. Then the integral (4.1) converges a.s. and it holds that*

$$\mathcal{L}\left(\int_{(0,\infty)} e^{-\xi_{t-}} dL_t\right) \in T(\mathbb{R}_+).$$

Similarly, if $\mathcal{L}(e^{-\xi_\tau}) \in \widetilde{H}(\mathbb{R})$, $\mathcal{L}(Y_1) \in T(\mathbb{R})$ and $\mathcal{L}(Y_1)$ is symmetric, then $\mathcal{L}(V_\infty) \in T(\mathbb{R})$.

Proof. Convergence of the integral is again guaranteed by (1.14).
Now set $T_0 = 0$ and let $T_j, j = 1, 2, \ldots$ be the jump times of $(N_t)_{t\geq 0}$. Then we have

$$\begin{aligned}\int_{(0,\infty)} e^{-\xi_{t-}} dL_t &= \sum_{j=1}^{\infty} e^{-\xi_{T_j}} Y_j = \sum_{j=1}^{\infty} e^{-(\xi_{T_j} - \xi_{T_{j-1}})} \cdots e^{-(\xi_{T_1} - \xi_{T_0})} Y_j \\ &= \sum_{j=1}^{\infty} \left(\prod_{i=1}^{j} e^{-(\xi_{T_i} - \xi_{T_{i-1}})}\right) Y_j =: \sum_{j=1}^{\infty} \left(\prod_{i=1}^{j} A_i\right) B_j,\end{aligned}$$

where $A_i = e^{-(\xi_{T_i} - \xi_{T_{i-1}})} \stackrel{d}{=} e^{-\xi_{T_i - T_{i-1}}}$ and $B_j = Y_j$. Remark that the proof of Lemma 4.4 remains valid even if the summation starts from $j = 1$. Hence the assumption follows from Lemma 4.4. □

4.2.2 Dependent component case

In this subsection, we generalize a model of [47] and study which class $\mathcal{L}(V_\infty)$ belongs to.

Let $0 < p < 1$. Suppose that $(\xi_t, L_t)_{t\geq 0}$ is a bivariate compound Poisson process with parameter $\lambda > 0$ and normalized Lévy measure

$$\nu_{\xi,L}(dx, dy) = p\delta_0(dx)\rho_0(dy) + (1-p)\delta_1(dx)\rho_1(dy),$$

where ρ_0 and ρ_1 are probability measures on $(0, \infty)$ and $[0, \infty)$, respectively, such that

$$\int_{(1,\infty)} \log y \, d\rho_0(y) < \infty \quad \text{and} \quad \int_{(1,\infty)} \log y \, d\rho_1(y) < \infty. \tag{4.5}$$

For the bivariate compound Poisson process (ξ, L) we have the following representation (see [58] p.18):

$$(\xi_t, L_t) = \sum_{k=0}^{N_t} S_k = \left(\sum_{k=0}^{N_t} S_k^{(1)}, \sum_{k=0}^{N_t} S_k^{(2)} \right),$$

where $S_0^{(1)} = S_0^{(2)} = 0$ and $\left\{ S_k = \left(S_k^{(1)}, S_k^{(2)} \right) \right\}_{k=1}^{\infty}$ is a sequence of i.i.d. random variables with values in $\mathbb{R}^2 \backslash \{0\}$. It implies that the projections of the compound Poisson process on $\mathbb{R}^2$ are also compound Poisson processes. Precisely in the given model, since $P(S_1^{(1)} = 0) = p$ and $P(S_1^{(1)} = 1) = 1 - p$, the marginal process ξ is a Poisson process with parameter $(1-p)\lambda > 0$. Note that $S_k^{(1)}$ and $S_k^{(2)}$ may be dependent for any $k \in \mathbb{N}$. In this case, $\rho_i(B)$ is equal to $\mathbb{P}(S_k^{(2)} \in B | S_k^{(1)} = i)$ for $i = 0, 1$ and $B \in \mathcal{B}_1$.

Example 4.14. *In [47] the authors considered the bivariate compound Poisson process with parameter $u + v + w$, $u, v, w \geq 0$ and normalized Lévy measure*

$$\nu_{\xi,L}(dx, dy) = \frac{v}{u+v+w}\delta_0(dx)\delta_1(dy) + \frac{u+w}{u+v+w}\delta_1(dx)\left(\frac{u}{u+w}\delta_0(dy) + \frac{w}{u+w}\delta_1(dy) \right).$$

So, choosing $p = \frac{v}{u+v+w}$, $\rho_0 = \delta_1$ and $\rho_1 = \frac{u}{u+w}\delta_0 + \frac{w}{u+w}\delta_1$ in the above setting results in the setting of [47].

In the following theorem, we give a sufficient condition for $\mathcal{L}(V_\infty)$, given by (4.1) with (ξ, L) as described above, to be GGC.

Theorem 4.15. *If the function $\frac{(1-p)\mathbb{L}_{\rho_1}(u)}{1-p\mathbb{L}_{\rho_0}(u)}$ is HCM, then $\mathcal{L}(V_\infty)$ is GGC.*

Proof. Convergence of the integral V_∞ follows from Theorem 1.11 due to (4.5). Define T_ξ and M to be the first jump time of the Poisson process ξ and the number of the jumps of the bivariate compound Poisson process (ξ, L) before T_ξ, respectively.

Due to the strong Markov property of the Lévy process (ξ, L), we have

$$\begin{aligned}\int_{(0,\infty)} e^{-\xi_{s-}} dL_s &= \int_{(0,T_\xi]} e^{-\xi_{s-}} dL_s + \int_{(T_\xi,\infty]} e^{-\xi_{s-}} dL_s \\ &= L_{T_\xi} + \int_{(0,\infty)} e^{-\xi_{T_\xi+s-}} dL_{s+T_\xi} \\ &= L_{T_\xi} + e^{-\xi_{T_\xi}} \int_{(0,\infty)} e^{-(\xi_{T_\xi+s-} - \xi_{T_\xi})} d((L_{(s+T_\xi)} - L_{T_\xi}) + L_{T_\xi}) \\ &\overset{d}{=} L_{T_\xi} + e^{-1} \int_{(0,\infty)} e^{-\widetilde{\xi}_{s-}} d\widetilde{L}_s,\end{aligned}$$

where the process $(\widetilde{\xi}, \widetilde{L})$ is an independent, identically distributed copy of (ξ, L). Therefore, we have

$$\mathbb{L}_\mu(u) = \mathbb{L}_\mu(e^{-1}u)\mathbb{L}_\rho(u), \tag{4.6}$$

with $\mu = \mathcal{L}(V_\infty)$ and ρ denoting the distribution of L_{T_ξ}. Thus μ is e^{-1}-decomposable and it follows that

$$\mathbb{L}_\mu(u) = \prod_{n=0}^{\infty} \mathbb{L}_\rho(e^{-n}u).$$

In the given setting, we have

$$L_{T_\xi} = \left(\sum_{k=0}^{M} S_k^{(2)}\right) + S_{M+1}^{(2)},$$

where M is geometrically distributed with parameter p, namely,

$$\mathbb{P}(M = k) = (1-p)p^k \quad \text{for any } k \in \mathbb{N}_0.$$

Hence we obtain

$$\begin{aligned}\mathbb{L}_\rho(u) &= \mathbb{E}\left[\exp(-uL_{T_\xi})\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[\exp\left(-u\left(\sum_{k=0}^{M} S_k^{(2)} + S_{M+1}^{(2)}\right)\right)\Big| M\right]\right] \\ &= \mathbb{E}\left[(\mathbb{L}_{\rho_0}(u))^M \mathbb{L}_{\rho_1}(u)\right] \\ &= (1-p)\mathbb{L}_{\rho_1}(u)\sum_{k=0}^{\infty}(p\mathbb{L}_{\rho_0}(u))^k \\ &= \frac{(1-p)\mathbb{L}_{\rho_1}(u)}{1-p\mathbb{L}_{\rho_0}(u)}.\end{aligned}$$

The class of HCM functions is closed under scale transformation, multiplication and pointwise limit. Therefore $\mathbb{L}_\mu(u)$ is HCM if $\mathbb{L}_\rho(u)$ is HCM and hence μ is GGC if ρ is GGC by Proposition 4.1(a). As a result, μ is GGC if $\frac{(1-p)\mathbb{L}_{\rho_1}(u)}{1-p\mathbb{L}_{\rho_0}(u)}$ is HCM. □

A distribution with Laplace transform of the form $\frac{(1-p)}{1-p\mathbb{L}_{\rho_0}(u)}$ is called a compound geometric distribution. It is compound Poisson, because every geometric distribution is compound Poisson with Lévy measure given by

$$\Pi(\{k\}) = -\frac{1}{\log(1-p)}\frac{1}{k+1}p^{k+1}, k \in \mathbb{N}$$

(see p. 147 in [61]). Since HCM functions are closed under multiplication the following is an immediate consequence of Theorem 4.15.

Corollary 4.16. *If ρ_1 and the compound geometric distribution of ρ_0 are GGCs, then $\mathcal{L}(V_\infty)$ is a GGC.*

In addition, we make the following observations.

Corollary 4.17.

(a) *For any constant $c > 1$, the distribution*

$$\mu_c = \mathcal{L}\left(\int_{(0,\infty)} c^{-\xi_{s-}} dL_s\right) \tag{4.7}$$

is c^{-1}-selfdecomposable. Thus in the non-degenerate case it is absolutely continuous or continuous singular ([64]) and Theorem 4.15 holds true also for μ_c instead of μ.

(b) *Let $B(\mathbb{R}_+)$ be the Goldie-Steutel-Bondesson class, which is the smallest class that contains all mixtures of exponential distributions and is closed under convolution and weak convergence.*
If $\frac{(1-p)\mathbb{L}_{\rho_1}(u)}{1-p\mathbb{L}_{\rho_0}(u)}$ is the Laplace transform of a distribution in $B(\mathbb{R}_+)$, then μ_c is in $B(\mathbb{R}_+)$. Moreover μ_c will be a c^{-1}-semi-selfdecomposable distribution.

About the definition and basic properties of semi-selfdecomposable distributions see [58]. The proof of (a) is obvious. The proof of (b) follows from the characterization of the class $B(\mathbb{R}_+)$ in Chapter 9 of [12] and our proof of Theorem 4.15.

Remark that a distribution with Laplace transform $\frac{(1-p)}{1-p\mathbb{L}_{\rho_0}(u)}$ is always infinitely divisible since it is a compound Poisson distribution. Hence only ρ_1 has influence on that property of $\mathcal{L}(V_\infty)$ or μ_c as defined in (4.7), which yields the following interesting conclusion of Theorem 4.15.

Corollary 4.18. *If ρ_1 is infinitely divisible, then μ_c is also infinitely divisible.*

To finish this section we treat an example.

Example 4.19. *Let ρ_1 be a GGC, i.e. $\mathbb{L}_{\rho_1}(u)$ is HCM. Then if $\frac{1}{1-p\mathbb{L}_{\rho_0}(u)}$ is HCM, μ is found to be GGC. For example, if ρ_0 is an exponential random variable with* *density* $f(x) = be^{-bx}$, $b > 0$, *then it fulfills* (4.5) *and* $\frac{1}{1-p\mathbb{L}_{\rho_0}(u)}$ *is HCM. To see this, for* $u > 0$, $v > 0$, *write*

$$\begin{aligned}\frac{1}{1-p\mathbb{L}_{\rho_0}(uv)}\frac{1}{1-p\mathbb{L}_{\rho_0}(u/v)} &= \frac{1+\frac{u}{b}(v+v^{-1})+\frac{u^2}{b^2}}{(1-p)^2+\frac{u}{b}(v+v^{-1})(1-p)+\frac{u^2}{b^2}} \\ &= \frac{1}{1-p}+\frac{p+\left(1-\frac{1}{1-p}\right)\frac{u^2}{b^2}}{(1-p)^2+\frac{u}{b}(v+v^{-1})(1-p)+\frac{u^2}{b^2}}.\end{aligned}$$

This is nonnegative and completely monotone as a function of $v + v^{-1}$.

4.3 Exponential Integrals of Independent Increment Processes

We say that a process $X = (X_t)_{t\geq 0}$ with $X_t = \sum_{i=1}^{M_t} X_i, t \geq 0$ is a compound sum process, if the $\{X_i\}_{i\in\mathbb{N}}$ are i.i.d. random variables, $(M_t)_{t\geq 0}$ is a renewal (or counting) process and they are independent. When $(M_t)_{t\geq 0}$ is a Poisson process, X is nothing but a compound Poisson process and is a Lévy process. Unless $(M_t)_{t\geq 0}$ is a Poisson process, X is no Lévy process. In this section we consider exponential integrals of the form (4.1) in the case when either ξ or L is a compound sum process and the other is an arbitrary Lévy process and the both are independent. Although (ξ, L) is not a Lévy process, the exponential integral (4.1) can be defined and its distribution can be infinitely divisible and/or GGC in many cases as we will show in the following.

Case 1: The process ξ is a compound sum process

First we give a condition for the convergence of the exponential integral (4.1) in the given setting when (ξ, L) is not a Lévy process.

Proposition 4.20. *Suppose that* $(\xi_t, L_t)_{t\geq 0}$ *is a stochastic process where* ξ *and* L *are independent,* L *is a Lévy process and* $\xi_t = \sum_{i=1}^{M_t} X_i$ *is a compound sum process with i.i.d. jump heights* X_i, $i = 1, 2, \dots$ *and i.i.d. waiting times* W_i. *Then* (4.1) *converges in probability to a finite random variable if*

$$\xi_t \to \infty \text{ a.s. and } \int_{(1,\infty)} \left(\frac{\log q}{A_\xi(\log q)}\right) P(|L_{W_1}| \in dq) < \infty \tag{4.8}$$

for $A_\xi(x) = \int_{(0,x)} P(X_1 > u)du$.

Proof. As argued in the proof of Proposition 4.5, we can rewrite the exponential integral as a perpetuity

$$\int_{(0,\infty)} e^{-\xi_{t-}} dL_t = \sum_{j=0}^{\infty}\left(\prod_{i=1}^{j} A_i\right) B_j,$$

where $A_i = e^{-X_i}$ and $B_j \stackrel{d}{=} L_{W_j}$. By Theorem 2.1 of [24] the above converges a.s. to a finite random variable if and only if $\prod_{i=1}^n A_i \to 0$ a.s. and

$$\int_{(1,\infty)} \left(\frac{\log q}{A(\log q)} \right) P(|B_1| \in dq) < \infty$$

for $A(x) = \int_{(0,x)} P(-\log A_1 > u)du$. Using the given expressions for A_1 and B_1 in our setting we observe that this is equivalent to (4.8). It remains to show that a.s. convergence of the perpetuity implies convergence in probability of (4.1). Therefore remark that

$$\int_{(0,t]} e^{-\xi_{s-}} dL_s = \int_{(0,T_{M_t}]} e^{-\xi_{s-}} dL_s + e^{-\xi_{T_{M_t}}} (L_t - L_{T_{M_t}})$$

where the first term converges to a finite random variable while the second converges in probability to 0 since $\sup_{t \in [T_{M_t}, T_{M_{t+1}})} |L_t - L_{M_t}| \stackrel{d}{=} \sup_{t \in [0, W_1)} |L_t|$. □

Now we can extend Proposition 4.5 in this new setting as follows.

Proposition 4.21. *Suppose $(\xi_t, L_t)_{t \geq 0}$ is a stochastic process where ξ and L are independent, $\xi_t = \sum_{i=1}^{M_t} X_i$ is a compound sum process with i.i.d. jump heights X_i, $i = 1, 2, \dots$ and i.i.d. waiting times W_i, $i = 1, 2, \dots$ such that (4.8) is fulfilled and L is a Lévy process. Further, suppose that $\mathcal{L}(e^{-X_1}) \in H(\mathbb{R}_+)$ and $\mathcal{L}(L_\tau) \in T(\mathbb{R}_+)$ for τ being a random variable with the same distribution as W_1 and independent of L. Then*

$$\mathcal{L} \left(\int_{(0,\infty)} e^{-\xi_{t-}} dL_t \right) \in T(\mathbb{R}_+).$$

Similarly, if $\mathcal{L}(e^{-X_1}) \in \widetilde{H}(\mathbb{R})$, $\mathcal{L}(L_\tau) \in T(\mathbb{R})$ and $\mathcal{L}(L_\tau)$ is symmetric, then $\mathcal{L}(V_\infty) \in T(\mathbb{R})$.

In the following we give some examples fulfilling the assertions of Proposition 4.21.

Example 4.22 (L is non-random)**.** *For the case $L_t = t$, $\mathcal{L}(L_\tau)$ belongs to $T(\mathbb{R}_+)$ if and only if $\mathcal{L}(\tau)$ does. Hence for all waiting times which are GGC and have finite log-moment, a suitable choice of jump heights of ξ yields that $\mathcal{L}(V_\infty) \in T(\mathbb{R}_+)$.*

Example 4.23 (L is a stable subordinator and $\mathcal{L}(W_1)$ is GGC with finite log-moment)**.** *Consider L to be a stable subordinator having Laplace transform $\mathbb{L}_L(u) = \exp\{-u^\alpha\}$ with $0 < \alpha < 1$. Then the Laplace transform of $B := L_\tau$ is given by $\mathbb{L}_B(u) = \mathbb{L}_\tau(u^\alpha)$. This function is HCM if τ is GGC, since by Proposition 4.1(a), $\mathbb{L}_\tau$ is HCM and hence also its composition with x^α. Thus whenever $\mathcal{L}(\tau) = \mathcal{L}(W_1)$ is GGC, $\mathcal{L}(L_\tau)$ is GGC, too. Finally, we observe using [61, Proposition A.3.2] that in our setting $E \log^+ |\tau| < \infty$ is equivalent to $E \log^+ |L_\tau| < \infty$ such that the assumptions of Proposition 4.21 are fulfilled if $\mathcal{L}(W_1)$ is a GGC with finite log-moment.*

Example 4.24 (L is a standard Brownian motion and $\mathcal{L}(W_1)$ is GGC with finite log-moment)**.** *Given that L is a standard Brownian motion, L_1 has characteristic function $Ee^{izL_1} = \exp(-z^2/2)$, which yields $\mathbb{L}_B(u) = \mathbb{L}_\tau(u^2/2)$. We can not see $\mathcal{L}(B) \in T(\mathbb{R}_+)$ from this, and in fact $\mathcal{L}(B)$ is in $T(\mathbb{R})$ but not in $T(\mathbb{R}_+)$ [12, p. 117]). Again by [61, Proposition A.3.2] we obtain that $E\log^+|\tau| < \infty$ is equivalent to $E\log^+|L_\tau| < \infty$. Finally, due to the symmetric property of L, we can apply Proposition 4.1(f) and conclude that $\mathcal{L}(V_\infty) \in T(\mathbb{R})$ for suitable jump heights of ξ.*

Example 4.25 (L is a Lévy subordinator and $\mathcal{L}(W_1)$ is a half normal distribution)**.** *The $1/2$-stable subordinator L and the standard half normal random variable τ have densities, respectively, given by,*

$$f_{L_t}(x) = \frac{t}{2\sqrt{\pi}} x^{-3/2} e^{-t^2/2x} \quad \text{and} \quad f_\tau(x) = \sqrt{\frac{2}{\pi}} e^{-x^2/2},\ x > 0.$$

These yield the density function of L_τ as

$$f_{L_\tau}(x) = \int f_{L_y}(x) f_\tau(y) dy = \frac{1}{\pi} \frac{x^{-1/2}}{1+x}.$$

Interestingly, this is an F distribution (see [58, p.46]) and since a random variable with an F distribution is constricted to be the quotient of two independent gamma random variables, we have that $\mathcal{L}(L_\tau) \in T(\mathbb{R}_+)$.

Case 2: The process L_t is a compound sum process

Again we start with a condition for the convergence of the exponential integral (4.1). It can be shown similar to Proposition 4.20.

Proposition 4.26. *Suppose $(\xi_t, L_t)_{t\geq 0}$ is a stochastic process where ξ and L are independent, ξ is a Lévy process and $L_t = \sum_{i=1}^{M_t} Y_i$ is a compound sum process with i.i.d. jump heights $Y_i, i = 1, 2, \ldots$ and i.i.d. waiting times $U_i, i = 1, 2, \ldots$. Then* (4.1) *converges a.s. to a finite random variable if and only if*

$$\xi_t \to \infty \text{ a.s. and } \int_{(1,\infty)} \left(\frac{\log q}{A_L(\log q)} \right) P(|Y_1| \in dq) < \infty \qquad (4.9)$$

for $A_L(x) = \int_{(0,x)} P(\xi_{U_1} > u) du$.

In the same manner as before we can now extend Proposition 4.13 to the new setting and obtain the following result.

Proposition 4.27. *Let ξ and L be independent and assume $L_t = \sum_{i=1}^{M_t} Y_i$ to be a compound renewal process with i.i.d. jump heights $Y_i,\ i = 1, 2, \ldots$ and i.i.d. waiting times U_i such that* (4.9) *holds. Suppose that $\mathcal{L}(Y_1) \in T(\mathbb{R}_+)$ and $\mathcal{L}(e^{-\xi_\tau}) \in H(\mathbb{R}_+)$*

for a random variable τ having the same distribution as U_1 and being independent of ξ. Then

$$\mathcal{L}\left(\int_{(0,\infty)} e^{-\xi_{t-}} dL_t\right) \in T(\mathbb{R}_+).$$

Furthermore, if $\mathcal{L}(e^{-\xi_\tau}) \in \widetilde{H}(\mathbb{R})$, $\mathcal{L}(Y_1) \in T(\mathbb{R})$ and $\mathcal{L}(Y_1)$ is symmetric, then $\mathcal{L}(V_\infty) \in T(\mathbb{R})$.

We end up this chapter with a simple example fulfilling the assumptions in Proposition 4.27.

Example 4.28 (L is a random walk and ξ is a standard Brownian motion with drift)**.** *Suppose $\xi_t = B_t + at$ is a standard Brownian motion with drift $a > 0$ and U_1 is degenerated at 1. Then $\mathcal{L}(e^{-\xi_\tau}) = \mathcal{L}(e^{-a}e^{-B_1})$ is a log-normal distribution and hence in $H(\mathbb{R}_+)$. So for all GGC jump heights $\mathcal{L}(Y_1)$ with finite log-moment, the exponential integral is GGC.*

Chapter 5: Multivariate Generalized Ornstein-Uhlenbeck Processes

According to Definition 1.9 for any Lévy process $(\eta_t)_{t\geq 0}$ the Ornstein-Uhlenbeck type process driven by η with starting random variable V_0 and parameter $\lambda \neq 0$ is given via

$$V_t = e^{-\lambda t}\left(V_0 + \int_{(0,t]} e^{\lambda s} d\eta_s\right), \quad t \geq 0.$$

The associated stochastic differential equation (SDE) is $dV_t = -\lambda V_{t-}dt + d\eta_t$, $t \geq 0$. Since for every $h > 0$ the Ornstein-Uhlenbeck type process fulfills the random recurrence equation $V_{nh} = e^{-\lambda h} V_{(n-1)h} + Z_{n,h}$, $n \in \mathbb{N}$, with i.i.d. noise $(Z_{n,h})_{n\in\mathbb{N}}$ such that $\mathcal{L}(Z_{1,h}) = \mathcal{L}(\int_{(0,h]} e^{-\lambda(h-s)} d\eta_s)$, it can be seen as a natural generalization in continuous time of an AR(1) time series. As already mentioned in Section 1.2, by embedding the more general random sequence $Y_{nh} = A_{n,h}Y_{(n-1)h} + B_{n,h}$, $n \in \mathbb{N}$, $h > 0$, with $(A_{n,h}, B_{n,h})_{n\in\mathbb{N}}$ i.i.d., $A_{1,h} > 0$ a.s., into a continuous time setting, De Haan and Karandikar [27] introduced the generalized Ornstein-Uhlenbeck process

$$V_t = e^{-\xi_t}\left(V_0 + \int_{(0,t]} e^{\xi_{s-}} d\eta_s\right), \quad t \geq 0, \tag{5.1}$$

driven by a bivariate Lévy process $(\xi_t, \eta_t)_{t\geq 0}$ with starting random variable V_0. Provided that V_0 is independent of $(\xi_t, \eta_t)_{t\geq 0}$, they also showed that it is the unique solution of the SDE $dV_t = V_{t-}dU_t + dL_t$, $t \geq 0$, where the bivariate Lévy process $(U_t, L_t)_{t\geq 0}$ is given by (1.12).

In this chapter we extend the setting of De Haan and Karandikar [27] to random matrices with real valued entries, i.e. we aim to construct a process

$$(V_t)_{t\geq 0}, \text{ with } V_t = (V_t^{(i,j)})_{\substack{0<i\leq m \\ 0<j\leq n}} \in \mathbb{R}^{m\times n}$$

in continuous time which fulfills the random recurrence equation

$$V_t = A_{s,t}V_s + B_{s,t} \quad \text{a.s., } 0 \leq s \leq t, \tag{5.2}$$

for random functionals $(A_{s,t})_{0\le s\le t}$, $(B_{s,t})_{0\le s\le t}$ such that $A_{s,t} \in \mathbb{R}^{m\times m}$ and $B_{s,t} \in \mathbb{R}^{m\times n}$, the $A_{s,t}$ are supposed to be non-singular and $(A_{(n-1)h,nh}, B_{(n-1)h,nh})$, $n \in \mathbb{N}$, are i.i.d. for all $h > 0$. Observe that the question of when a solution of (5.2) exists can be treated seperately for each column of $(V_t)_{t\ge0}$. Thus, if not stated otherwise, for simplicity we set $n = 1$ throughout this chapter, hence V_t and $B_{s,t}$ are elements in $\mathbb{R}^m$.

A crucial point for the investigations in this chapter which is covered in Section 5.2 is the fact that the stochastic process $A_t := A_{0,t}$ in the autoregressive model above has to be a multiplicative *right Lévy process in the general linear group* $\mathrm{GL}(\mathbb{R}, m)$ *of order* m, i.e. a stochastic process, having càdlàg paths and stationary and independent increments, which have to be multiplied from the left side, starting a.s. in I, the identity matrix. As by an observation due to Skorokhod [59] every right Lévy process in $\mathrm{GL}(\mathbb{R}, m)$ is a *right stochastic exponential* as defined in Definition 5.4 this classifies all possible choices for the random functional $(A_{s,t})_{0\le s\le t}$ and leads to the general form of the processes $(V_t)_{t\ge0}$ described by the model (5.2) given in (5.20). The resulting processes $(V_t)_{t\ge0}$, will be called *multivariate generalized Ornstein-Uhlenbeck processes (MGOU)* and we show in Proposition 5.9 that they solve the SDE $dV_t = dU_tV_{t-} + dL_t$ for some $\mathbb{R}^{m\times m} \times \mathbb{R}^m$-valued Lévy process (U_t, L_t) which can be uniquely determined from $(A_{0,t}, B_{0,t})_{t\ge0}$.

A new aspect compared to the one-dimensional GOU process is the possibility of the existence of affine subspaces H of $\mathbb{R}^m$ which are *invariant* under the model (5.2) in the sense that $V_0 \in H$ implies $V_t \in H$ for all $t \ge 0$. In Section 5.3.2 we give necessary and sufficient conditions for the existence of an invariant affine subspace of the model (5.2) and show that given the existence of a d-dimensional invariant affine subspace H, after an appropriate orthogonal transformation of the underlying space, the MGOU process with $V_0 \in H$ consists of an $(m-d)$-dimensional constant process and an $\mathbb{R}^d$-valued MGOU process. Subsequently in Section 5.3.3 strictly stationary solutions of MGOU processes are treated. Under some extra conditions on the limit behaviour of the stochastic exponential of U we give necessary and sufficient conditions for their existence and determine their form in terms of U and L. Finally, Section 5.4 contains the proofs for the results of Section 5.3.3 as well as several auxiliary results about multivariate stochastic exponentials.

5.1 Setting

Following the lines of De Haan and Karandikar [27] observe that the condition of (5.2) to hold for all $0 \le s \le t$ yields

$$V_t = A_{s,t}V_s + B_{s,t} = A_{s,t}A_{u,s}V_u + A_{s,t}B_{u,s} + B_{s,t}, \quad 0 \le u \le s \le t.$$

Assuming that $(A_{s,t}, B_{s,t})_{0\le s\le t}$ is unique now leads to Assumption 5.1(a) given below while extending the i.i.d. property of $(A_{(n-1)h,nh}, B_{(n-1)h,nh})$, $n \in \mathbb{N}$, for all $h > 0$ into the continuous time setting yields the requirements 5.1(b) and (c). Finally, it is natural to impose that $(A_{0,t})_{t\ge0}$ and $(B_{0,t})_{t\ge0}$ are continuous in probability at 0 since this, together with 5.1(a),(b) and (c), implies the existence of càdlàg modifications of the processes

$$(A_t)_{t\geq 0} := (A_{0,t})_{t\geq 0} \quad \text{and} \quad (B_t)_{t\geq 0} := (B_{0,t})_{t\geq 0}$$

as will be shown later. This motivates Assumption 5.1(d) below.

Assumption 5.1. *Suppose the* $\mathrm{GL}(\mathbb{R}, m) \times \mathbb{R}^m$*-valued random functional* $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ *satisfies the following four conditions.*

(a) *For all* $0 \leq u \leq s \leq t$ *almost surely*

$$A_{u,t} = A_{s,t}A_{u,s} \quad \textit{and} \quad B_{u,t} = A_{s,t}B_{u,s} + B_{s,t}. \tag{5.3}$$

(b) *For* $0 \leq a \leq b \leq c \leq d$ *the families of random matrices* $\{(A_{s,t}, B_{s,t}), a \leq s \leq t \leq b\}$ *and* $\{(A_{s,t}, B_{s,t}), c \leq s \leq t \leq d\}$ *are independent.*

(c) *For all* $0 \leq s \leq t$ *it holds*

$$(A_{s,t}, B_{s,t}) \stackrel{d}{=} (A_{0,t-s}, B_{0,t-s}). \tag{5.4}$$

(d) *It holds*

$$P - \lim_{t\downarrow 0} A_{0,t} = A_{0,0} = I \quad \textit{and} \quad P - \lim_{t\downarrow 0} B_{0,t} = B_{0,t} = 0, \tag{5.5}$$

where I *denotes the identity matrix and* 0 *the vector (or matrix) only having zero entries.*

Remark that it follows directly from these assumptions, that any process $(V_t)_{t\geq 0}$ satisfying (5.2) and with starting random variable V_0 independent of $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ is a time-homogeneous Markov process. We will also see that Assumption 5.1 implies that the process $(A_t)_{t\geq 0} = (A_{0,t})_{t\geq 0}$ is a multiplicative Lévy process in the general linear group $\mathrm{GL}(\mathbb{R}, m)$, $m \geq 1$. Therefore, first remark that for a stochastic process $(X_t)_{t\geq 0}$ in $\mathrm{GL}(\mathbb{R}, m)$ the group structure of $\mathrm{GL}(\mathbb{R}, m)$ allows us to construct *left increments* $X_tX_s^{-1}$ and *right increments* $X_s^{-1}X_t$ for $0 \leq s \leq t < \infty$. We say that the process $(X_t)_{t\geq 0}$ in $\mathrm{GL}(\mathbb{R}, m)$ has *independent left increments* if for any $n \in \mathbb{N}$, $0 < t_1 < \ldots < t_n$, the random variables $X_0, X_{t_1}X_0^{-1}, \ldots, X_{t_n}X_{t_{n-1}}^{-1}$ are independent. The process has *stationary left increments* if $X_tX_s^{-1} \stackrel{d}{=} X_{t-s}X_0^{-1}$ holds for all $s < t$. Stationarity and independence of right increments is understood analogously. Following the notations in the book of Liao [45] multiplicative Lévy processes in $\mathrm{GL}(\mathbb{R}, m)$ can now be defined as follows.

Definition 5.2. *A càdlàg process* $(X_t)_{t\geq 0}$ *in* $\mathrm{GL}(\mathbb{R}, m)$, $m \geq 1$, *with* $X_0 = I$ *a.s. is called a* left Lévy process, *if it has independent and stationary right increments. Similarly, a càdlàg process* $(X_t)_{t\geq 0}$ *in* $\mathrm{GL}(\mathbb{R}, m)$, $m \geq 1$, *with* $X_0 = I$ *a.s. is called a* right Lévy process, *if it has independent and stationary left increments.*
Given a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, *a left Lévy process* $(X_t)_{t\geq 0}$ *in* $\mathrm{GL}(\mathbb{R}, m)$ *is called a* left $\mathbb{F}$-Lévy process, *if it is adapted to* $\mathbb{F}$ *and for any* $s < t$ *the increment* $X_s^{-1}X_t$ *is independent of* $\mathcal{F}_s$. *A* right $\mathbb{F}$-Lévy process *is defined similarly.*

Now we can state the following Lemma.

Lemma 5.3. *For any* $(A_{s,t})_{0\le s\le t}$ *fulfilling Assumption 5.1 the process* $(A_t)_{t\ge 0} = (A_{0,t})_{t\ge 0}$ *has a càdlàg modification which is a right Lévy process in* $\mathrm{GL}(\mathbb{R}, m)$*. In the following we will only refer to this version.*
Conversely, if $(A_t)_{t\ge 0}$ *is a right Lévy process in* $\mathrm{GL}(\mathbb{R}, m)$*, then* $(A_{s,t})_{0\le s\le t}$ *defined by* $A_{s,t} = A_t A_s^{-1}$ *fulfills Assumption 5.1.*

Proof. Since by Assumption 5.1(a) we have $A_t A_s^{-1} = A_{s,t}$ it follows directly from Assumption 5.1(b) and (c), that $(A_t)_{t\ge 0}$ is a stochastic process in $\mathrm{GL}(\mathbb{R}, m)$ with stationary and independent left (multiplicative) increments. It is everywhere continuous in probability from the right since by 5.1(a), (c) and (d)

$$P - \lim_{h\downarrow 0} A_{t+h} = P - \lim_{h\downarrow 0} A_{t,t+h} A_t = A_t, \quad t \ge 0.$$

Similarly due to

$$P - \lim_{h\downarrow 0} A_{t-h} = P - \lim_{h\downarrow 0} A_t A_{t-h,t}^{-1} = A_t \cdot P - \lim_{h\downarrow 0} A_h^{-1} = A_t, \quad t \ge 0,$$

it is also continuous in probability from the left such that by [60, Theorem V.3] a càdlàg modification exists which is a right Lévy process in $\mathrm{GL}(\mathbb{R}, m)$ as specified in Definition 5.2.
The converse is true by the definition of right Lévy processes. □

The process $(B_t)_{t\ge 0} = (B_{0,t})_{t\ge 0}$ is a stochastic process on $\mathbb{R}^m$ whose paths are due to $B_{t+h} = A_{t+h} A_t^{-1} B_t + B_{t,t+h}$ by Assumption 5.1(c) and (d) everywhere continuous in probability from the right and similarly from the left. It admits a càdlàg modification which can be shown by a simple extension of the proof in the one-dimensional case given in [27, Lemma 2.1]. In the following we will only refer to this càdlàg version of $(B_t)_{t\ge 0}$.

For any matrix $M \in \mathbb{R}^{m\times n}$ we write $M^{\perp}$ for its transpose and let $M^{(i,j)}$ denote the component in the ith row and jth column of M.
Since the matrix multiplication in general is non-commutative, we will use two different integral operators for matrices. Namely, for a semimartingale M in $\mathbb{R}^{m\times n}$, i.e. a matrix-valued stochastic process whose single components are semimartingales, and a locally bounded predictable process H in $\mathbb{R}^{l\times m}$ the $\mathbb{R}^{l\times n}$-valued stochastic integral $I = \int H dM$ is given by $I^{(i,j)} = \sum_{k=1}^m \int H^{(i,k)} dM^{(k,j)}$ and in the same way for $M \in \mathbb{R}^{l\times m}$, $H \in \mathbb{R}^{m\times n}$ we define the $\mathbb{R}^{l\times n}$-valued integral $J = \int dMH$ by $J^{(i,j)} = \sum_{k=1}^m \int H^{(k,i)} dM^{(j,k)}$.
Given two semimartingales M and N in $\mathbb{R}^{l\times m}$ and $\mathbb{R}^{m\times n}$ define the quadratic variation $[M, N]$ in $\mathbb{R}^{l\times n}$ by its components via $[M, N]^{(i,j)} = \sum_{k=1}^m [M^{(i,k)}, N^{(k,j)}]$. Similarly its continuous part $[M, N]^c$ is given by $([M, N]^c)^{(i,j)} = \sum_{k=1}^m [M^{(i,k)}, N^{(k,j)}]^c$.

With these notations, for two semimartingales M and N in $\mathbb{R}^{m\times m}$ and two locally bounded predictable processes G and H in $\mathbb{R}^{m\times m}$ we have the following a.s. equalities as stated e.g. in [37]

$$\left[\int_{(0,\cdot]} G_s dM_s, \int_{(0,\cdot]} dN_s H_s\right]_t = \int_{(0,t]} G_s d[M,N]_s H_s, \quad t \geq 0, \tag{5.6}$$

$$\left[M, \int_{(0,\cdot]} G_s dN_s\right]_t = \left[\int_{(0,\cdot]} dM_s G_s, N\right]_t, \quad t \geq 0, \tag{5.7}$$

and the integration by parts formula (1.8) takes the form

$$(MN)_t = \int_{(0,t]} M_{s-} dN_s + \int_{(0,t]} dM_s\, N_{s-} + [M,N]_t, \quad t \geq 0. \tag{5.8}$$

5.2 The Multivariate Stochastic Exponential

The two integral operators defined above lead to two different options to define a multivariate stochastic exponential. Hence we need the following generalization of Definition 1.8. The SDE of the stochastic exponential has been studied on arbitrary Lie groups by Estrade [20].

Definition 5.4. *Let $(X_t)_{t\geq 0}$ be a semimartingale in $(\mathbb{R}^{m\times m}, +)$. Then its* left stochastic exponential $\overleftarrow{\mathcal{E}}(X)_t$ *is defined as the unique $\mathbb{R}^{m\times m}$-valued, adapted, càdlàg solution of the SDE*

$$Z_t = I + \int_{(0,t]} Z_{s-} dX_s, \quad t \geq 0, \tag{5.9}$$

while the unique adapted, càdlàg solution of the SDE

$$Z_t = I + \int_{(0,t]} dX_s\, Z_{s-}, \quad t \geq 0, \tag{5.10}$$

will be called right stochastic exponential *and denoted by $\overrightarrow{\mathcal{E}}(X)_t$. In the case that there is no need to distinguish between left and right exponentials, we simply write $\mathcal{E}(X)$.*

Remark that replacing Z and X by their transposes in (5.9) leads to the SDE (5.10) and vice versa. Hence we have

$$\overleftarrow{\mathcal{E}}(X)^{\perp} = \overrightarrow{\mathcal{E}}(X^{\perp}). \tag{5.11}$$

As has been observed by Karandikar [37] a necessary and sufficient condition for non-singularity of the left stochastic exponential of an $\mathbb{R}^{m\times m}$-valued process X at time t, is to claim that $(I + \Delta X_s)$ is invertible for all $0 < s \leq t$. Due to the above stated relationship between left and right exponential this result holds true also for right exponentials and hence any stochastic exponential is invertible for all $t \geq 0$ if and only if

$$\det(I + \Delta X_t) \neq 0 \,\text{for all}\, t \geq 0. \tag{5.12}$$

The following one-to-one relation between multiplicative Lévy processes and stochastic exponentials of additive Lévy processes is a key result for the investigations in this chapter.

Proposition 5.5. *Let $(Z_t)_{t\geq 0}$ be a left (resp. right) $\mathbb{F}$-Lévy process in $(\mathrm{GL}(\mathbb{R}, m), \cdot)$ for some filtration $\mathbb{F}$ satisfying the usual hypotheses. Then there exists an $\mathbb{F}$-Lévy process $(X_t)_{t\geq 0}$ in $(\mathbb{R}^{m\times m}, +)$, i.e. a Lévy process which is adapted to $\mathbb{F}$ and such that for any $s < t$ the increment $X_t - X_s$ is independent of $\mathcal{F}_s$, such that* (5.12) *holds and $Z_t = \overleftarrow{\mathcal{E}}(X)_t$ (resp. $Z_t = \overrightarrow{\mathcal{E}}(X)_t$). The process $(X_t)_{t\geq 0}$ is the unique solution of the SDE*

$$X_t := \int_{(0,t]} Z_{u-}^{-1} dZ_u, \quad t \geq 0, \tag{5.13}$$

(resp. of $X_t := \int_{(0,t]} dZ_u Z_{u-}^{-1}$, $t \geq 0$).
Conversely, for every $\mathbb{F}$-Lévy process $(X_t)_{t\geq 0}$ in $(\mathbb{R}^{m\times m}, +)$ fulfilling (5.12)*, the exponential $Z_t = \overleftarrow{\mathcal{E}}(X)_t$ (resp. $Z_t = \overrightarrow{\mathcal{E}}(X)_t$) is a left (resp. right) $\mathbb{F}$-Lévy process in $(\mathrm{GL}(\mathbb{R}, m), \cdot)$.*

The fact that every left Lévy process in $(\mathrm{GL}(\mathbb{R}, m), \cdot)$ fulfills the SDE (5.9) for some stochastic process $(X_t)_{t\geq 0}$ with stationary and independent additive increments has already been noted by Holevo [28] as a conclusion of results by Skorokhod [59]. Nevertheless, a complete proof of Proposition 5.5 is given in Section 5.4.1.

The above proposition shows in particular, that the following definition is unambiguous.

Definition 5.6. *Let $(Z_t)_{t\geq 0}$ be a left Lévy process in $(\mathrm{GL}(\mathbb{R}, m), \cdot)$. Then the Lévy process $(X_t)_{t\geq 0}$ in $(\mathbb{R}^{m\times m}, +)$ with $Z_t = \overleftarrow{\mathcal{E}}(X)_t$ will be called* left stochastic logarithm *of Z and we write $X_t = \overleftarrow{\mathrm{Log}}\,(Z_t)$. Conversely, if $(Z_t)_{t\geq 0}$ is a right Lévy process in $(\mathrm{GL}(\mathbb{R}, m), \cdot)$ its* right stochastic logarithm *$\overrightarrow{\mathrm{Log}}\,(Z_t)$ is the Lévy process $(X_t)_{t\geq 0}$ in $(\mathbb{R}^{m\times m}, +)$ with $Z_t = \overrightarrow{\mathcal{E}}(X)_t$.*

Since the inverse and the transpose of a left Lévy process in $\mathrm{GL}(\mathbb{R}, m)$ are right Lévy processes and vice versa, for any additive Lévy process $(X_t)_{t\geq 0}$ fulfilling (5.12) the process $([\overleftarrow{\mathcal{E}}(X)_t]^{-1})_{t\geq 0}$ is a right Lévy process and hence by the above theorem it is the right stochastic exponential of another Lévy process $(U_t)_{t\geq 0}$. In fact, the following holds ([37, Theorem 1]).

Theorem 5.7. *Let $(X_t)_{t\geq 0}$ be a semimartingale such that* (5.12) *is fulfilled. Then*

$$[\overleftarrow{\mathcal{E}}(X)_t]^{-1} = [\overleftarrow{\mathcal{E}}(U^{\perp})_t]^{\perp} = \overrightarrow{\mathcal{E}}(U)_t,\ t \geq 0$$

with

$$U_t := -X_t + [X,X]_t^c + \sum_{0<s\leq t} \left((I + \Delta X_s)^{-1} - I + \Delta X_s\right),\ t \geq 0. \tag{5.14}$$

Remark that it follows from (5.14) by standard calculations that the processes U and X fulfill the relation

$$U_t = -X_t - [X,U]_t, \quad t \geq 0, \tag{5.15}$$

and that if X is a Lévy process, then so is U and vice versa.

5.3 Multivariate Generalized Ornstein-Uhlenbeck Processes

In this section we will show that every process $(V_t)_{t\geq 0}$ satisfying the random recurrence equation (5.2) with $(A_{s,t}, B_{s,t})$ fulfilling Assumption (5.1) is a solution of the SDE

$$dV_t = dU_t V_{t-} + dL_t \tag{5.16}$$

for a bivariate Lévy process $(U_t, L_t)_{t\geq 0}$ such that U fulfills (5.12) and vice versa. In particular, the explicit formula of the solution in terms of U and L will be given. Secondly, we will investigate degenerate MGOU processes which are carried by affine subspaces of $\mathbb{R}^m$ and finally we treat strictly stationary solutions of the MGOU process $(V_t)_{t\geq 0}$ and present, under extra conditions on the limit behaviour of $\overleftarrow{\mathcal{E}}(U)$, necessary and sufficient conditions for their existence.

Throughout this section the SDE (5.16) is understood with respect to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, which we define to be the smallest filtration satisfying the usual hypotheses such that $(U_t, L_t)_{t\geq 0}$ is adapted and such that V_0 is $\mathcal{F}_0$ measurable. The natural assumption of V_0 being independent of $(U_t, L_t)_{t\geq 0}$ ensures by Corollary 1 of Theorem VI.11 in [55] that $(U_t, L_t)_{t\geq 0}$ is a semimartingale with respect to $\mathbb{F}$.

5.3.1 The Stochastic Differential Equation and its Solution

We start with a proposition showing one direction of the equivalence between the recursion equality (5.2) and the SDE (5.16).

Proposition 5.8. *Suppose* $(A_{s,t}, B_{s,t})_{0\le s\le t}$ *is a process satisfying Assumption 5.1 and* $(V_t)_{t\ge0}$ *with starting random variable* V_0 *independent of* $(A_{s,t}, B_{s,t})_{0\le s\le t}$ *is a process which fulfills* (5.2). *Define* $(U_t, L_t)_{t\ge0}$ *in* $\mathbb{R}^{m\times m}\times\mathbb{R}^m$ *by*

$$\begin{pmatrix} U_t \\ L_t \end{pmatrix} = \begin{pmatrix} \overrightarrow{\operatorname{Log}} A_t \\ B_t - \int_{(0,t]} dA_s\, A_{s-}^{-1} B_{s-} \end{pmatrix}, \quad t\ge 0. \tag{5.17}$$

Then $(U_t, L_t)_{t\ge0}$ *is a Lévy process and* $(V_t)_{t\ge0}$ *is a solution of the SDE* (5.16) *with respect to* $\mathbb{F}$.

Proof. Let $\mathbb{H}$ be the augmented natural filtration of $(A_t, B_t)_{t\ge0}$. Then by Lemma 5.3 it follows directly from Assumption 5.1 that $(A_t)_{t\ge0}$ is a right $\mathbb{H}$-Lévy process and thus as argued in the proof of Proposition 5.5 it is semimartingale with respect to $\mathbb{H}$. Hence $(U_t, L_t)_{t\ge0}$ is well defined and by Proposition 5.5 $(U_t)_{t\ge0}$ fulfills $dA_t = dU_t A_{t-}$. Since $V_t = A_t V_0 + B_t$ by (5.2) we obtain

$$\begin{aligned} dV_t &= dA_t V_0 + dB_t = dU_t A_{t-} V_0 + dB_t \\ &= dU_t(A_{t-}V_0 + B_{t-}) + dB_t - dU_t B_{t-} = dU_t V_{t-} + dB_t - dA_t A_{t-}^{-1} B_{t-} \\ &= dU_t V_{t-} + dL_t. \end{aligned}$$

It remains to show that $(U_t, L_t)_{t\ge0}$ given by (5.17) is a Lévy process. By computations similar to those in the proof of [27, Theorem 2.2] one can show that

$$\begin{pmatrix} U_t - U_s \\ L_t - L_s \end{pmatrix} = \begin{pmatrix} \int_{(s,t]} dA_{s,u}\, A_{s,u-}^{-1} \\ B_{s,t} - \int_{(s,t]} dA_{s,u}\, A_{s,u-}^{-1} B_{s,u-} \end{pmatrix}, \quad 0\le s\le t. \tag{5.18}$$

By Assumption 5.1(b) and (c) we observe that $(A_{s,s+u}, B_{s,s+u})_{u\ge0} \stackrel{d}{=} (A_{0,u}, B_{0,u})_{u\ge0}$ are equal in law and thus we obtain from (5.18) that (U, L) has stationary and independent increments. We also know that $(U_0, L_0) = 0$ a.s. and that the paths of (U, L) are càdlàg since that held true for $(A_t, B_t)_{t\ge0}$. Hence $(U_t, L_t)_{t\ge0}$ is a Lévy process as had to be shown. □

The following proposition establishes a choice of random functionals $(A_{s,t}, B_{s,t})_{0\le s\le t}$ fulfilling Assumption (5.1) and determines the corresponding process (V_t) which, by the above proposition, also solves the SDE (5.16).

Proposition 5.9. *Suppose* $(X_t, Y_t)_{t\ge0}$ *to be a Lévy process in* $(\mathbb{R}^{m\times m}\times\mathbb{R}^m, +)$ *such that* $(X_t)_{t\ge0}$ *fulfills* (5.12). *For* $0\le s\le t$ *define*

$$\begin{pmatrix} A_{s,t} \\ B_{s,t} \end{pmatrix} := \begin{pmatrix} \overleftarrow{\mathcal{E}}(X)_t^{-1}\overleftarrow{\mathcal{E}}(X)_s \\ \overleftarrow{\mathcal{E}}(X)_t^{-1}\int_{(s,t]} \overleftarrow{\mathcal{E}}(X)_{u-} dY_u \end{pmatrix}. \tag{5.19}$$

Then $(A_{s,t}, B_{s,t})_{0\le s\le t}$ *satisfies Assumption 5.1 and for any starting random variable* V_0 *independent of* $(X_t, Y_t)_{t\ge0}$ *the process*

$$V_t := \overleftarrow{\mathcal{E}}(X)_t^{-1}\left(V_0 + \int_{(0,t]} \overleftarrow{\mathcal{E}}(X)_{s-} dY_s\right) \tag{5.20}$$

satisfies (5.2) *and is the unique solution of the SDE* (5.16) *for the Lévy process* $(U_t, L_t)_{t\geq 0}$ *in* $(\mathbb{R}^{m\times m} \times \mathbb{R}^m, +)$ *with* U *as defined in* (5.14) *and* L *given by*

$$L_t = Y_t + \sum_{0<s\leq t} \big((I + \Delta X_s)^{-1} - I\big)\, \Delta Y_s - [X, Y]^c_t, \quad t \geq 0. \tag{5.21}$$

Proof. Let $\mathbb{H}$ be the augmented natural filtration of $(X_t, Y_t)_{t\geq 0}$, then $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ is well defined with respect to $\mathbb{H}$ and we know by Proposition 5.5 that the right stochastic exponential $\overleftarrow{\mathcal{E}}(X)_t^{-1}$ is a right $\mathbb{H}$-Lévy process whose left increments are given by $A_{s,t}$. Thus we have that for all $0 \leq s \leq u \leq t$ almost surely $A_{s,t} = A_{u,t}A_{s,u}$ holds. Also it follows directly from the definitions of $A_{s,t}$ and $B_{s,t}$ that $B_{s,t} = A_{u,t}B_{s,u} + B_{u,t}$ a.s. such that Assumption 5.1(a) is fulfilled.
For the common process $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ observe that for $0 \leq s \leq t$ we have

$$\begin{pmatrix} A_{s,t} \\ B_{s,t} \end{pmatrix} = \begin{pmatrix} \overleftarrow{\mathcal{E}}(X)_t^{-1}\overleftarrow{\mathcal{E}}(X)_s \\ \overleftarrow{\mathcal{E}}(X)_t^{-1}\overleftarrow{\mathcal{E}}(X)_s \int_{(s,t]} \overleftarrow{\mathcal{E}}(X)_s^{-1}\overleftarrow{\mathcal{E}}(X)_{u-}dY_u \end{pmatrix}.$$

Since $(A_t)_{t\geq 0}$ is a right $\mathbb{H}$-Lévy process the common increments $(\overleftarrow{\mathcal{E}}(X)_t^{-1}\overleftarrow{\mathcal{E}}(X)_s, Y_t - Y_s)_{t\geq s}$ are independent of $(X_u, Y_u)_{0\leq u\leq s}$. Hence it follows that $\{(A_{s,t}, B_{s,t}), a \leq s \leq t \leq b\}$ and $\{(A_{s,t}, B_{s,t}), c \leq s \leq t \leq d\}$ with $b \leq c$ are independent. Similarly we conclude that

$$\begin{pmatrix} A_{s,t} \\ B_{s,t} \end{pmatrix} \overset{d}{=} \begin{pmatrix} \overleftarrow{\mathcal{E}}(X)_{t-s}^{-1}\overleftarrow{\mathcal{E}}(X)_0 \\ \overleftarrow{\mathcal{E}}(X)_{t-s}^{-1}\overleftarrow{\mathcal{E}}(X)_0 \int_{(0,t-s]} \overleftarrow{\mathcal{E}}(X)_0^{-1}\overleftarrow{\mathcal{E}}(X)_{u-}dY_u \end{pmatrix} = \begin{pmatrix} A_{0,t-s} \\ B_{0,t-s} \end{pmatrix}$$

which yields Assumption 5.1(c).
The continuity in probability at 0 of $A_t = A_{0,t}$ is clear, while for $B_t = B_{0,t}$ it follows from that of A_t and Y_t and the continuity of the integral.
Now a simple calculation shows that $(V_t)_{t\geq 0}$ as defined in (5.20) fulfills the random recurrence equation (5.2) for all $0 < s \leq t$ and hence by Proposition 5.8 it is a solution of the SDE (5.16) for the bivariate Lévy process $(U_t, L_t)_{t\geq 0}$ defined in (5.17). Thus for U we have $\overrightarrow{\mathcal{E}}(U)_t = A_t = \overleftarrow{\mathcal{E}}(X)_t^{-1}$ which yields (5.14) while for L by the definition of A_t and B_t we obtain using the partial integration formula (5.8) and (5.7)

$$\begin{aligned} L_t &= B_t - \int_{(0,t]} dA_s\, A_{s-}^{-1} B_{s-} \\ &= \overleftarrow{\mathcal{E}}(X)_t^{-1} \int_{(0,t]} \overleftarrow{\mathcal{E}}(X)_{s-} dY_s - \int_{(0,t]} d(\overleftarrow{\mathcal{E}}(X)_s^{-1}) \int_{(0,s)} \overleftarrow{\mathcal{E}}(X)_{u-} dY_u \\ &= \overleftarrow{\mathcal{E}}(X)_t^{-1} \int_{(0,t]} \overleftarrow{\mathcal{E}}(X)_{s-} dY_s - \overleftarrow{\mathcal{E}}(X)_t^{-1} \int_{(0,t]} \overleftarrow{\mathcal{E}}(X)_{s-} dY_s \\ &\quad + \int_{(0,t]} \overleftarrow{\mathcal{E}}(X)_{s-}^{-1} d\left(\int_{(0,s]} \overleftarrow{\mathcal{E}}(X)_{u-} dY_u\right) + \left[\overleftarrow{\mathcal{E}}(X)^{-1}, \int_{(0,\cdot]} \overleftarrow{\mathcal{E}}(X)_{u-} dY_u\right]_t \end{aligned}$$

$$\begin{aligned}
&= \int_{(0,t]} \overleftarrow{\mathcal{E}}(X)_{s-}^{-1} \overleftarrow{\mathcal{E}}(X)_{s-} dY_u + \left[\int_{(0,\cdot]} d(\overleftarrow{\mathcal{E}}(X)_s^{-1}) \overleftarrow{\mathcal{E}}(X)_{s-}, Y \right]_t \\
&= Y_t + \left[\int_{(0,\cdot]} dA_s\, A_{s-}^{-1}, Y \right]_t \\
&= Y_t + [U, Y]_t. \qquad (5.22)
\end{aligned}$$

By the definition of U in (5.14) we have

$$\begin{aligned}
[U, Y]_t &= -[X, Y]_t + [[X, X]^c, Y]_t + \left[\sum_{0<s\leq \cdot} \left((I + \Delta X_s)^{-1} - I + \Delta X_s \right), Y \right]_t \\
&= -[X, Y]_t^c + \sum_{0<s\leq t} \left((I + \Delta X_s)^{-1} - I \right) \Delta Y_s.
\end{aligned}$$

Together with (5.22) this gives (5.21) and we are done. □

Propositions 5.8 and 5.9 together yield the following theorem.

Theorem 5.10. *Suppose $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ to be a process in $\mathrm{GL}(RR, m) \times \mathbb{R}^m$ satisfying Assumption 5.1 and suppose $(V_t)_{t\geq 0}$ fulfills (5.2). Then there exists a Lévy process $(X_t, Y_t, U_t, L_t)_{t\geq 0}$ in $(\mathbb{R}^{m\times m} \times \mathbb{R}^m \times \mathbb{R}^{m\times m} \times \mathbb{R}^m, +)$ defined by (5.17) and*

$$\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \begin{pmatrix} \overleftarrow{\mathrm{Log}}\,[\overrightarrow{\mathcal{E}}(U)_t]^{-1} \\ L_t + [\overleftarrow{\mathrm{Log}}\,[\overrightarrow{\mathcal{E}}(U)]^{-1}, L]_t \end{pmatrix}, \quad t \geq 0, \qquad (5.23)$$

such that $(A_{s,t}, B_{s,t})$ is given by (5.19). Further, provided that V_0 is independent of $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$, the process $(V_t)_{t\geq 0}$ is the unique solution of the SDE (5.16) with respect to $\mathbb{F}$.
Conversely let $(U_t, L_t)_{t\geq 0}$ be a Lévy process in $(\mathbb{R}^{m\times m} \times \mathbb{R}^m, +)$ such that $(U_t)_{t\geq 0}$ fulfills (5.12) and define $(X_t, Y_t)_{t\geq 0}$ by (5.23). Then $(X_t, Y_t, U_t, L_t)_{t\geq 0}$ is a Lévy process in $(\mathbb{R}^{m\times m} \times \mathbb{R}^m \times \mathbb{R}^{m\times m} \times \mathbb{R}^m, +)$ and the functional $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ defined by (5.19) satisfies Assumption 5.1. Furthermore, for any starting random variable V_0 independent of (U, L), the process $(V_t)_{t\geq 0}$ which solves the SDE (5.16) is the only càdlàg process fulfilling (5.2) and is given by (5.20).

Proof. Let $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ be a process in $\mathbb{R}^{m\times m} \times \mathbb{R}^m$ satisfying Assumption 5.1. Then Proposition 5.8 provides the existence of $(U_t, L_t)_{t\geq 0}$ and shows that $(V_t)_{t\geq 0}$ is a solution of the SDE (5.16). Hence we can define $(X_t, Y_t)_{t\geq 0}$ by (5.23). Since the left logarithm is a Lévy process by definition, we obtain that $(X_t, Y_t, U_t, L_t)_{t\geq 0}$ is a Lévy process. Observe that the given definition of $(X_t)_{t\geq 0}$ is equivalent to (5.14) and that from the definition of $(Y_t)_{t\geq 0}$ we deduce

$$\begin{aligned} Y_t &= L_t + [X, L]_t \\ &= L_t + \sum_{0<s\leq t} (\Delta X_s \Delta L_s) + [X, L]_t^c \\ &= L_t - \sum_{0<s\leq t} \left((I + \Delta X_s)^{-1} - I\right)(I + \Delta X_s)\Delta L_s + [X, L]_t^c \\ &= L_t - \sum_{0<s\leq t} \left((I + \Delta X_s)^{-1} - I\right)\Delta Y_s + [X, Y]_t^c, \quad t \geq 0, \end{aligned} \tag{5.24}$$

which is equivalent to (5.21). Thus by Proposition 5.9, the process $(V_t')_{t\geq 0}$ defined by

$$V_t' := \overleftarrow{\mathcal{E}}(X)_t^{-1}\left(V_0 + \int_{(0,t]} \overleftarrow{\mathcal{E}}(X)_{s-} dY_s\right)$$

solves the SDE (5.16), too. By the uniqueness of the solution we obtain $(V_t)_{t\geq 0} = (V_t')_{t\geq 0}$. In particular, by setting $V_0 = 0$ the validity of (5.2) together with (5.20) yields $B_t = \overleftarrow{\mathcal{E}}(X)_t^{-1}\int_{(0,t]} \overleftarrow{\mathcal{E}}(X)_{s-} dY_s$. On the other hand by (5.14) and (5.17) we have $A_t = \overleftarrow{\mathcal{E}}(X)_t^{-1}$ and thus (5.19) holds.
For the converse remark that as argued above $(X_t, Y_t, U_t, L_t)_{t\geq 0}$ is a Lévy process. Hence by Proposition 5.9 the functional $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ satisfies Assumption 5.1 and the process $(V_t)_{t\geq 0}$ defined by (5.20) solves the SDE (5.16) since (5.23) is equivalent to (5.14) and (5.21). As the solution of the SDE (5.16) is unique, uniqueness of $(V_t)_{t\geq 0}$ is clear. □

The results in this section motivate the following definition.

Definition 5.11. *Let $(X_t, Y_t)_{t\geq 0}$ be a Lévy process in $(\mathbb{R}^{m\times m} \times \mathbb{R}^m, +)$ such that $(X_t)_{t\geq 0}$ fulfills* (5.12) *and let V_0 be a random variable in $\mathbb{R}^m$. Then the process $(V_t)_{t\geq 0}$ in $\mathbb{R}^m$ as given in* (5.20) *will be called* multivariate generalized Ornstein-Uhlenbeck (MGOU) process *driven by $(X_t, Y_t)_{t\geq 0}$.*

Remark that even for $m = 1$ Definition 5.11 is generalizing Definition 1.9 since we do not assume a priori that V_0 is independent of $(X_t, Y_t)_{t\geq 0}$ and also the condition of $\mathcal{E}(X)_t^{-1}$ to be strictly positive is dropped. Nevertheless in view of the results in this thesis it seems natural to include these cases in the class of generalized Ornstein-Uhlenbeck processes.

5.3.2 MGOU Processes Carried by Affine Subspaces

To classify degenerate cases of MGOU processes, we introduce the notation of irreducibility which we mainly adopt from Bougerol and Picard [13] who studied generalized autoregressive models in discrete time.

Definition 5.12. *Suppose $(X_t, Y_t)_{t\geq 0}$ is a Lévy process in $(\mathbb{R}^{m\times m}\times\mathbb{R}^m, +)$ such that $(X_t)_{t\geq 0}$ fulfills* (5.12) *and let $(V_t)_{t\geq 0}$ be the MGOU driven by $(X_t, Y_t)_{t\geq 0}$ satisfying the autoregressive model* (5.2) *with $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ as defined in* (5.19)*. An affine subspace H of $\mathbb{R}^m$, $m \geq 1$, is called* invariant *under the model* (5.2) *if $V_0 \in H$ a.s. implies $V_t \in H$ a.s. for all $t \geq 0$. If $\mathbb{R}^m$ is the only invariant affine subspace, the model* (5.2) *is called* irreducible.

Remark that the given definition of invariant subspaces is more restrictive than the one in [13], since e.g. setting $Y_t = B_t = 0$ and letting A_t be a rotation operator with angle $2\pi t$ implies that in the discrete time model $V_n = A_{n-1,n}V_{n-1} + B_{n-1,n}$, $n \in \mathbb{N}$, every point is a zero-dimensional invariant affine subspace, while only the rotation axis is invariant for all $t \geq 0$.
Accordingly, irreducibility of the continuous time model does not directly imply that for all $h > 0$ the discrete time model $V_{nh} = A_{(n-1)h,nh}V_{(n-1)h} + B_{(n-1)h,nh}$, $n \in \mathbb{N}$, is irreducible in the sense of [13]. But we can show the following proposition whose rather technical proof can be found at the end of Section 5.4.2.

Proposition 5.13. *Suppose $m \leq 3$ and that the autoregressive model* (5.2) *with $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ fulfilling Assumption 5.1 is irreducible. Then there exists $h > 0$ for which the time-discrete autoregressive model*

$$V_{nh} = A_{(n-1)h,nh}V_{(n-1)h} + B_{(n-1)h,nh},\ n \in \mathbb{N}, \tag{5.25}$$

is irreducible in the sense that there exists no affine subspace H of $\mathbb{R}^m$, $H \neq \mathbb{R}^m$ such that $V_0 \in H$ implies $V_h \in H$.

It is very likely that the above proposition holds for arbitrary dimensions. Nevertheless an extension of the proof to larger values of m seemed to go beyond the scope of this work and thus was omitted.

The next theorem treats MGOU processes where the corresponding autoregressive model admits a d-dimensional invariant affine subspace H. It turns out that in this case we can split up the process carried by H in a constant part and an $\mathbb{R}^{m-d}$-valued MGOU process. For convenience we assume that H is parallel to the axes.

Theorem 5.14. *Suppose $(V_t)_{t\geq 0}$ is a MGOU process with starting random variable V_0, driven by the Lévy process $(X_t, Y_t)_{t\geq 0}$ in $(\mathbb{R}^{m\times m}\times\mathbb{R}^m, +)$ such that $(X_t)_{t\geq 0}$ fulfills* (5.12)*. Assume $(V_t)_{t\geq 0}$ is satisfying the autoregressive model* (5.2) *with $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ as defined in* (5.19)*.*

(a) *Assume that $H = \{(k_1, \ldots, k_d, h_{d+1}, \ldots, h_m)^\perp, h_{d+1}, \ldots, h_m \in \mathbb{R}\}$ with $0 < d \leq m$ and constants $k_1, \ldots, k_d \in \mathbb{R}$ is an invariant, affine subspace of $\mathbb{R}^m$ with respect to the model* (5.2)*. Then, given that $V_0 \in H$ a.s., it holds $V_t = \begin{pmatrix} K \\ \mathcal{V}_t \end{pmatrix}$*

with $K = (k_1, \ldots, k_d)^\perp$ *and* $\mathcal{V}_t \in \mathbb{R}^{m-d}$ *and further it holds for all* $t \geq 0$

$$X_t = \begin{pmatrix} \mathcal{X}_t^1 & 0 \\ \mathcal{X}_t^2 & \mathcal{X}_t^3 \end{pmatrix} \quad \text{a.s. where} \quad \mathcal{X}_t^1 \in \mathbb{R}^{d\times d} \quad \text{and} \tag{5.26}$$

$$Y_t = \begin{pmatrix} \mathcal{Y}_t^1 \\ \mathcal{Y}_t^2 \end{pmatrix} = \begin{pmatrix} \mathcal{X}_t^1 K \\ \mathcal{Y}_t^2 \end{pmatrix} \quad \text{a.s. where} \quad \mathcal{Y}_t^1 \in \mathbb{R}^d. \tag{5.27}$$

The process $(\mathcal{V}_t)_{t\geq 0}$ *is a MGOU process driven by the Lévy process*

$$\big(\mathcal{X}_t^3, \mathcal{Y}_t^2 - \mathcal{X}_t^2 K\big)_{t\geq 0} \tag{5.28}$$

in $(\mathbb{R}^{(m-d)\times(m-d)} \times \mathbb{R}^{m-d}, +)$*. Further, if* $(V_t)_{t\geq 0}$ *solves the SDE* (5.16) *with respect to* $\mathbb{F}$*, then we have a.s. for* $t \geq 0$

$$U_t = \begin{pmatrix} \mathcal{U}_t^1 & 0 \\ \mathcal{U}_t^2 & \mathcal{U}_t^3 \end{pmatrix} \quad \text{and} \quad L_t = \begin{pmatrix} \mathcal{L}_t^1 \\ \mathcal{L}_t^2 \end{pmatrix} \quad \text{with} \quad \mathcal{U}_t^1 \in \mathbb{R}^{d\times d}, \mathcal{L}_t^1 \in \mathbb{R}^d, \tag{5.29}$$

where $\mathcal{L}^1 = -\mathcal{U}^1 K$ *a.s. and* $(\mathcal{V}_t)_{t\geq 0}$ *solves the SDE*

$$d\mathcal{V}_t = d\mathcal{U}_t^3 \mathcal{V}_{t-} + d(\mathcal{L}_t^2 + \mathcal{U}_t^2 K), \quad t \geq 0, \tag{5.30}$$

with respect to $\mathbb{F}$*.*

(b) *Conversely, if* (5.26) *and* (5.27) *hold for* $K = (k_1, \ldots, k_d)^\perp \in \mathbb{R}$ *constant, then the affine subspace* $H = \{(k_1, \ldots, k_d, h_{d+1}, \ldots, h_m)^\perp, h_{d+1}, \ldots, h_m \in \mathbb{R}\}$ *of* $\mathbb{R}^m$ *is invariant with respect to the model* (5.2) *and for any starting random variable* $V_0 \in H$ *the MGOU process defined by* (5.20) *can be written as* $V_t = \begin{pmatrix} K \\ \mathcal{V}_t \end{pmatrix}$ *a.s. where* $(\mathcal{V}_t)_{t\geq 0}$ *is a MGOU process driven by the Lévy process* (5.28)*.*

If in the setting of Theorem 5.14 the invariant affine subspace H is not parallel to the axes, then there exists an orthogonal transformation matrix O, such that OH fulfills the assumptions in Theorem 5.14 for the transformed MGOU process $V' = OV$. The process $(V_t')_{t\geq 0}$ fulfills the random recurrence equation $V_t' = A_{s,t}' V_s' + B_{s,t}'$ for $0 \leq s \leq t$ where $A_{s,t}' = OA_{s,t}O^{-1}$ and $B_{s,t}' = OB_{s,t}$ and hence is a MGOU process driven by $(OX_tO^{-1}, OY_t)_{t\geq 0}$. Thus the study of arbitrary invariant affine subspaces reduces to the case treated in Theorem 5.14.

This observation and Theorem 5.14 together imply the following characterization of irreducibility of the model (5.2).

Corollary 5.15. *Suppose* $(X_t, Y_t)_{t\geq 0}$ *in* $(\mathbb{R}^{m\times m} \times \mathbb{R}^m, +)$ *is a Lévy process such that* $(X_t)_{t\geq 0}$ *fulfills* (5.12)*. Then the autoregressive model* (5.2) *with* $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ *as defined in* (5.19) *is irreducible if and only if there exists no pair* (O, K) *of an orthogonal transformation* $O \in \mathbb{R}^{m\times m}$ *and a constant* $K = (k_1, \ldots, k_d)^\perp \in \mathbb{R}^d$, $1 \leq d \leq m$*, such that a.s.*

$$OX_tO^{-1} = \begin{pmatrix} \mathcal{X}_t^1 & 0 \\ \mathcal{X}_t^2 & \mathcal{X}_t^3 \end{pmatrix} \quad \text{and} \quad OY_t = \begin{pmatrix} \mathcal{X}_t^1 K \\ \mathcal{Y}_t^2 \end{pmatrix} \quad \text{where} \quad \mathcal{X}_t^1 \in \mathbb{R}^{d\times d}, \quad t \geq 0. \tag{5.31}$$

Proof of Theorem 5.14. (a) We start by verifying (5.26) and (5.27). Since H is an invariant affine subspace we deduce from (5.2) that for any $t \geq 0$ and all $h_{d+1}, \dots, h_m \in \mathbb{R}$ the equation

$$A_t(k_1, \dots, k_d, h_{d+1}, \dots, h_m)^\perp + B_t = (k_1, \dots, k_d, g_{d+1}, \dots, g_m)^\perp \quad \text{a.s.}$$

has to admit a solution $g_{d+1}, \dots, g_m \in \mathbb{R}$. This is equivalent to

$$\begin{aligned} \sum_{j=1}^d k_j A_t^{(i,j)} + \sum_{j=d+1}^m h_j A_t^{(i,j)} + b_i &= k_i, \quad \forall i = 1, \dots, d \\ \sum_{j=1}^d k_j A_t^{(i,j)} + \sum_{j=d+1}^m h_j A_t^{(i,j)} + b_i &= g_i, \quad \forall i = d+1, \dots, m. \end{aligned}$$

Thus we can conclude that $A_t^{(i,j)} = 0$ holds a.s. for $i \leq d, j > d$. Observe by simple algebraic calculations that if two matrices M and N in $\mathbb{R}^{m \times m}$ have a $d \times (m-d)$ block of zero entries in the upper right corner, then so do M^{-1} and MN. More detailed we have for

$$M = \begin{pmatrix} \mathcal{M}_1 & 0 \\ \mathcal{M}_2 & \mathcal{M}_3 \end{pmatrix} \in \mathrm{GL}(\mathbb{R}, m) \quad \text{and} \quad N = \begin{pmatrix} \mathcal{N}_1 & 0 \\ \mathcal{N}_2 & \mathcal{N}_3 \end{pmatrix} \in \mathbb{R}^{m \times m},\ \mathcal{M}_1, \mathcal{N}_1 \in \mathbb{R}^{d \times d}$$

that $\mathcal{M}_1$ and $\mathcal{M}_3$ are non-singular and it holds

$$M^{-1} = \begin{pmatrix} \mathcal{M}_1^{-1} & 0 \\ -\mathcal{M}_3^{-1}\mathcal{M}_2\mathcal{M}_1^{-1} & \mathcal{M}_3^{-1} \end{pmatrix} \quad \text{and} \quad MN = \begin{pmatrix} \mathcal{M}_1\mathcal{N}_1 & 0 \\ \mathcal{M}_2\mathcal{N}_1 + \mathcal{M}_3\mathcal{N}_2 & \mathcal{M}_3\mathcal{N}_2 \end{pmatrix}.$$

Now recall that $A_t = \overleftarrow{\mathcal{E}}(X)_t^{-1}$ and thus we know that $\overleftarrow{\mathcal{E}}(X)_t$ and $\overleftarrow{\mathcal{E}}(X)_t^{-1}$ a.s. admit a $d \times (m-d)$ zero block for all $t \geq 0$. Hence it follows from (5.13) that also X_t a.s. has such a zero block which is (5.26). Thus we deduce from (5.9) that

$$\overleftarrow{\mathcal{E}}(X)_t := \begin{pmatrix} \mathcal{E}_t^1 & 0 \\ \mathcal{E}_t^2 & \mathcal{E}_t^3 \end{pmatrix} = \begin{pmatrix} I + \int_0^t \mathcal{E}_{s-}^1 d\mathcal{X}_s^1 & 0 \\ \int_0^t \mathcal{E}_{s-}^2 d\mathcal{X}_s^1 + \int_0^t \mathcal{E}_{s-}^3 d\mathcal{X}_s^2 & I + \int_0^t \mathcal{E}_{s-}^3 d\mathcal{X}_s^3 \end{pmatrix}, \quad t \geq 0, \tag{5.32}$$

and observe in particular that $\mathcal{E}_t^1 = \overleftarrow{\mathcal{E}}(\mathcal{X}^1)_t$ and $\mathcal{E}_t^3 = \overleftarrow{\mathcal{E}}(\mathcal{X}^3)_t$ hold for $t \geq 0$. Inserting the previous results in (5.20) yields for all $t \geq 0$ a.s.

$$\begin{aligned} V_t = \begin{pmatrix} K \\ \mathcal{V}_t \end{pmatrix} &= \begin{pmatrix} (\mathcal{E}_t^1)^{-1} & 0 \\ -(\mathcal{E}_t^3)^{-1}\mathcal{E}_t^2(\mathcal{E}_t^1)^{-1} & (\mathcal{E}_t^3)^{-1} \end{pmatrix} \left[\begin{pmatrix} K \\ \mathcal{V}_0 \end{pmatrix} + \int_{(0,t]} \begin{pmatrix} \mathcal{E}_{s-}^1 & 0 \\ \mathcal{E}_{s-}^2 & \mathcal{E}_{s-}^3 \end{pmatrix} d \begin{pmatrix} \mathcal{Y}_s^1 \\ \mathcal{Y}_s^2 \end{pmatrix} \right] \\ &= \begin{pmatrix} (\mathcal{E}_t^1)^{-1} K \\ -(\mathcal{E}_t^3)^{-1}\mathcal{E}_t^2(\mathcal{E}_t^1)^{-1} K + (\mathcal{E}_t^3)^{-1}\mathcal{V}_0 \end{pmatrix} \\ &\quad + \begin{pmatrix} (\mathcal{E}_t^1)^{-1} \int_0^t \mathcal{E}_{s-}^1 d\mathcal{Y}_s^1 \\ -(\mathcal{E}_t^3)^{-1}\mathcal{E}_t^2(\mathcal{E}_t^1)^{-1} \int_0^t \mathcal{E}_{s-}^1 d\mathcal{Y}_s^1 + (\mathcal{E}_t^3)^{-1}[\int_0^t \mathcal{E}_{s-}^3 d\mathcal{Y}_s^2 + \int_0^t \mathcal{E}_{s-}^2 d\mathcal{Y}_s^1] \end{pmatrix}. \end{aligned} \tag{5.33}$$

The first line of (5.33) is equivalent to

$$K + \int_{(0,t]} \overleftarrow{\mathcal{E}}(\mathcal{X}^1)_{s-} d\mathcal{Y}^1_s = \overleftarrow{\mathcal{E}}(\mathcal{X}^1)_t K = K + \int_{(0,t]} \overleftarrow{\mathcal{E}}(\mathcal{X}^1)_{s-} d\mathcal{X}^1_s K \quad \text{a.s., } t \geq 0,$$

from where we deduce (5.27). From the second line of (5.33) we derive under use of (5.27), (5.9) and (5.32)

$$\begin{aligned}
&\mathcal{V}_t - \overleftarrow{\mathcal{E}}(\mathcal{X}^3)_t^{-1}\mathcal{V}_0 \\
&= \overleftarrow{\mathcal{E}}(\mathcal{X}^3)_t^{-1} \left(\int_{(0,t]} \mathcal{E}^3_{s-} d\mathcal{Y}^2_s + \int_{(0,t]} \mathcal{E}^2_{s-} d\mathcal{X}^1_s K - \mathcal{E}^2_t (\mathcal{E}^1_t)^{-1} \left(I + \int_{(0,t]} \mathcal{E}^1_{s-} d\mathcal{X}^1_s \right) K \right) \\
&= \overleftarrow{\mathcal{E}}(\mathcal{X}^3)_t^{-1} \left(\int_{(0,t]} \mathcal{E}^3_{s-} d\mathcal{Y}^2_s + \int_{(0,t]} \mathcal{E}^2_{s-} d\mathcal{X}^1_s K - \mathcal{E}^2_t \overleftarrow{\mathcal{E}}(\mathcal{X}^1)_t^{-1} \overleftarrow{\mathcal{E}}(\mathcal{X}^1)_t K \right) \\
&= \overleftarrow{\mathcal{E}}(\mathcal{X}^3)_t^{-1} \left(\int_{(0,t]} \mathcal{E}^3_{s-} d\mathcal{Y}^2_s + \int_{(0,t]} \mathcal{E}^2_{s-} d\mathcal{X}^1_s K - \left(\int_{0,t]} \mathcal{E}^2_{s-} d\mathcal{X}^1_s + \int_{(0,t]} \mathcal{E}^3_{s-} d\mathcal{X}^2_s \right) K \right) \\
&= \overleftarrow{\mathcal{E}}(\mathcal{X}^3)_t^{-1} \int_{(0,t]} \overleftarrow{\mathcal{E}}(\mathcal{X}^3)_{s-} d(\mathcal{Y}^2_s - \mathcal{X}^2_s K) \quad \text{a.s., } t \geq 0,
\end{aligned}$$

such that (5.28) is shown.
Finally let $(U_t, L_t)_{t\geq 0}$ be the Lévy process defined in (5.14) and (5.21) and assume that $(V_t)_{t\geq 0}$ solves the SDE (5.16) with respect to $\mathbb{F}$. Observe that by the same argumentation as for X or alternatively by (5.14) we deduce that for all $t \geq 0$ it holds $U_t^{(i,j)} = 0$ a.s. for $i \leq d, j > d$. By inserting U and L as given in (5.29) in the SDE (5.16) we obtain $\mathcal{L}^1 = -\mathcal{U}^1 K$ in the first and (5.30) in the second component. This completes the proof.
(b) Inserting (5.26) and (5.27) in (5.20) directly gives the assumption by calculations similar as under (a). □

5.3.3 Stationary Solutions of MGOU Processes

In the following we investigate conditions for the existence of stationary solutions of multivariate generalized Ornstein-Uhlenbeck processes. To maintain the flow we will defer the proofs of the results in this section to Section 5.4.2.
Given some extra information on the limit behaviour of $\overleftarrow{\mathcal{E}}(U)$ and $\overleftarrow{\mathcal{E}}(X)$ our first theorem provides necessary and sufficient conditions for the existence of stationary solutions of MGOU processes.

Theorem 5.16. *Suppose $(V_t)_{t\geq 0}$ is a MGOU process driven by the Lévy process $(X_t, Y_t)_{t\geq 0}$ in $(\mathbb{R}^{m\times m} \times \mathbb{R}^m, +)$ where $(X_t)_{t\geq 0}$ fulfills* (5.12). *Let $(U_t, L_t)_{t\geq 0}$ be the Lévy process defined in* (5.14) *and* (5.21).

(a) *Suppose $\lim_{t\to\infty} \overleftarrow{\mathcal{E}}(U)_t = 0$ in probability. Then a finite random variable V_0 can be chosen such that $(V_t)_{t\geq 0}$ is strictly stationary if and only if the integral*

$\int_{(0,t]} \overleftarrow{\mathcal{E}}(U)_{s-} dL_s$ *converges in distribution for* $t \to \infty$ *to a finite random variable. In this case, the distribution of the strictly stationary process* $(V_t)_{t\geq 0}$ *is uniquely determined and is obtained by choosing* V_0 *independent of* $(X_t, Y_t)_{t\geq 0}$ *as the distributional limit of* $\int_{(0,t]} \overleftarrow{\mathcal{E}}(U)_{s-} dL_s$ *as* $t \to \infty$.

(b) *Suppose* $\lim_{t\to\infty} \overleftarrow{\mathcal{E}}(X)_t = 0$ *in probability. Then a finite random variable* V_0 *can be chosen such that* $(V_t)_{t\geq 0}$ *is strictly stationary if and only if the integral* $\int_{(0,t]} \overleftarrow{\mathcal{E}}(X)_{s-} dY_s$ *converges in probability to a finite random variable as* $t \to \infty$. *In this case the strictly stationary solution is unique and given by*

$$V_t = -\overleftarrow{\mathcal{E}}(X)_t^{-1} \int_{(t,\infty)} \overleftarrow{\mathcal{E}}(X)_{s-} dY_s \quad \text{a.s. for all} \quad t \geq 0.$$

By adding the assumption of irreducibility of the underlying model the above theorem can be sharpened as follows.

Theorem 5.17. *Suppose* $(X_t, Y_t)_{t\geq 0}$ *in* $(\mathbb{R}^{m\times m} \times \mathbb{R}^m, +)$ *with* $m \leq 3$ *is a Lévy process where* $(X_t)_{t\geq 0}$ *fulfills* (5.12) *and such that the corresponding autoregressive model* (5.2) *with* $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ *as defined in* (5.19) *is irreducible. If* $m > 3$ *assume that there exists* $h > 0$ *for which the model* $V_{nh} = A_{(n-1)h,nh} V_{(n-1)h} + B_{(n-1)h,nh}$, $n \in \mathbb{N}$, *is irreducible in the sense that there exists no affine subspace* $H \subsetneq \mathbb{R}^m$ *such that* $V_0 \in H$ *implies* $V_h \in H$ *a.s. Let* $(V_t)_{t\geq 0}$ *be the MGOU process driven by* $(X_t, Y_t)_{t\geq 0}$ *and let* $(U_t, L_t)_{t\geq 0}$ *be the Lévy process defined in* (5.14) *and* (5.21).

(a) *A finite random variable* V_0, *independent of* $(X_t, Y_t)_{t\geq 0}$, *can be chosen such that* $(V_t)_{t\geq 0}$ *is strictly stationary if and only if* $\lim_{t\to\infty} \overleftarrow{\mathcal{E}}(U)_t = 0$ *in probability and the integral* $\int_{(0,t]} \overleftarrow{\mathcal{E}}(U)_{s-} dL_s$ *converges in distribution for* $t \to \infty$ *to a finite random variable.*

(b) *A finite random variable* V_0 *can be chosen such that* $(V_t)_{t\geq 0}$ *is strictly stationary and strictly non-causal in the sense that* V_t *is independent of* $(X_s, Y_s)_{0\leq s<t}$ *for* $t \geq 0$ *if and only if* $\lim_{t\to\infty} \overleftarrow{\mathcal{E}}(X)_t = 0$ *in probability and the integral* $\int_{(0,t]} \overleftarrow{\mathcal{E}}(X)_{s-} dY_s$ *converges in probability as* $t \to \infty$.

In the case that the underlying model is not irreducible, $P - \lim_{t\to\infty} \overleftarrow{\mathcal{E}}(U)_t = 0$ is not necessary for the existence of a causal strictly stationary solution as shown in the following corollary of Theorems 5.14 and 5.16.

Corollary 5.18. *Suppose* $(X_t, Y_t)_{t\geq 0}$ *is a Lévy process in* $(\mathbb{R}^{m\times m} \times \mathbb{R}^m, +)$ *such that* $(X_t)_{t\geq 0}$ *fulfills* (5.12) *and let* $(V_t)_{t\geq 0}$ *be the MGOU driven by* $(X_t, Y_t)_{t\geq 0}$ *satisfying the autoregressive model* (5.2) *with* $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ *as defined in* (5.19). *Choose an* *orthogonal transformation* $O \in \mathbb{R}^{m\times m}$ *such that* (5.31) *holds for* $K = (k_1, \ldots, k_d)^\perp$, $0 \le d \le m$. *Define* $(U_t, L_t)_{t\ge0}$ *via* (5.14) *and* (5.21), *then we have*

$$OU_tO^{-1} = \begin{pmatrix} \mathcal{U}_t^1 & 0 \\ \mathcal{U}_t^2 & \mathcal{U}_t^3 \end{pmatrix} \quad \text{and} \quad OL_t = \begin{pmatrix} -\mathcal{U}_t^1 K \\ \mathcal{L}_t^2 \end{pmatrix} \quad \text{with} \quad \mathcal{U}_t^1 \in \mathbb{R}^{d\times d} \tag{5.34}$$

and we obtain sufficient conditions for the existence of causal or strictly non-causal strictly stationary solutions as follows.

(a) *A finite random variable* V_0 *can be chosen such that* $(V_t)_{t\ge0}$ *is strictly stationary and causal if* $P-\lim_{t\to\infty} \overleftarrow{\mathcal{E}}(\mathcal{U}^3)_t = 0$ *and* $\int_{(0,t]} \overleftarrow{\mathcal{E}}(\mathcal{U}^3)_{s-} d(\mathcal{L}_s^2 + \mathcal{U}_s^2 K)$ *converges in distribution to a finite random variable as* $t \to \infty$. *A strictly stationary solution can be obtained by choosing* V_0 *independent of* $(X_t, Y_t)_{t\ge0}$ *as the distributional limit as* $t \to \infty$ *of*

$$O^{-1}\begin{pmatrix} K \\ \int_{(0,t]} \overleftarrow{\mathcal{E}}(\mathcal{U}^3)_{s-} d(\mathcal{L}_s^2 + \mathcal{U}_s^2 K) \end{pmatrix}.$$

(b) *A finite random variable* V_0 *can be chosen such that* $(V_t)_{t\ge0}$ *is strictly stationary and strictly non-causal if* $P-\lim_{t\to\infty} \overleftarrow{\mathcal{E}}(\mathcal{X}^3)_t = 0$ *and* $\int_{(0,t]} \overleftarrow{\mathcal{E}}(\mathcal{X}^3)_{s-} d(\mathcal{Y}_s^2 - \mathcal{X}_s^2 K)$ *converges in probability to a finite random variable as* $t \to \infty$. *A strictly stationary solution is given by*

$$V_t = O^{-1}\begin{pmatrix} K \\ -\overleftarrow{\mathcal{E}}(\mathcal{X}^3)_t^{-1} \int_{(t,\infty)} \overleftarrow{\mathcal{E}}(\mathcal{X}^3)_{s-} d(\mathcal{Y}_s^2 - \mathcal{X}_s^2 K) \end{pmatrix} \quad \text{a.s. for all} \quad t \ge 0.$$

Remark 5.19. *The results in Section 5.3 remain valid if we treat a MGOU process* $(V_t)_{t\ge0}$ *with* $V_t \in \mathbb{R}^{m\times n}$ *and drop the condition of* $n = 1$. *As the value of* n *has no influence on the proofs, apart from Theorem 5.14, we can simply replace the vector valued processes* $(Y_t)_{t\ge0}$ *and* $(L_t)_{t\ge0}$ *by* $\mathbb{R}^{m\times n}$*-valued processes. Theorem 5.14 may be applied column-by-column or, alternatively, it is possible to interpret the MGOU process* $(V_t)_{t\ge0}$ *in* $\mathbb{R}^{m\times n}$ *driven by* $(X_t, Y_t)_{t\ge0}$, $X_t \in \mathbb{R}^{m\times m}$, $Y_t = (Y_t^1, \ldots, Y_t^n) \in \mathbb{R}^{m\times n}$ *as an MGOU process in* $\mathbb{R}^{mn}$ *driven by the Lévy process*

$$\left(\begin{pmatrix} X_t & & 0 \\ & \ddots & \\ 0 & & X_t \end{pmatrix}, \begin{pmatrix} Y_t^1 \\ \vdots \\ Y_t^n \end{pmatrix}\right)_{t\ge0} \quad \text{in} \quad \mathbb{R}^{mn\times mn} \times \mathbb{R}^{mn}.$$

5.4 Auxiliary Results and Proofs

5.4.1 Properties of the Stochastic Exponential

We start this section by proving the one-to-one relationship between the stochastic exponential and multiplicative Lévy processes stated in Proposition 5.5.

Proof of Proposition 5.5. Due to similarity only the relation between left Lévy processes and left stochastic exponentials will be shown.
Since it seems more constructive to us, we start with the second part of the proposition, so first suppose $(X_t)_{t\geq 0}$ to be an additive Lévy process with respect to $\mathbb{F}$ and define $Z_t = \overleftarrow{\mathcal{E}}(X)_t$ by (5.9). Then Z_t, $t \geq 0$, is non-singular. Hence we can deduce for the right-increment $Z_{s,t}$, $0 \leq s \leq t$ of Z

$$\begin{aligned} Z_{s,t} := Z_s^{-1} Z_t &= \left(I + \int_{(0,s]} Z_{u-} dX_u\right)^{-1} \left(I + \int_{(0,s]} Z_{u-} dX_u + \int_{(s,t]} Z_{u-} dX_u\right) \\ &= I + \left(I + \int_{(0,s]} Z_{u-} dX_u\right)^{-1} \int_{(s,t]} Z_{u-} dX_u \\ &= I + Z_s^{-1} \int_{(0,t-s]} Z_{s+u-} dX_{s+u} \\ &= I + \int_{(0,t-s]} Z_{s,s+u-} dX_{s+u}. \end{aligned}$$

This integral equation yields that $(Z_{s,s+r})_{r\geq 0} = \overleftarrow{\mathcal{E}}(X_{s+\cdot} - X_s)_{r\geq 0}$ and hence by the stationarity of the increments of X we see directly that for all $0 \leq s \leq t$ it holds $Z_{s,t} \overset{d}{=} Z_{0,t-s}$ such that the right increments of Z are shown to be stationary. On the other hand for $s \geq 0$ it follows from the Markov property of X that $(Z_{s,s+r})_{r\geq 0} = \overleftarrow{\mathcal{E}}(X_{s+\cdot} - X_s)_{r\geq 0}$ is independent of $\mathcal{F}_s$. Hence for any $0 \leq s < t \leq u < v$ the increments $Z_{s,t}$ and $Z_{u,v}$ are independent and by an inductive argumentation this yields that Z has independent right increments. Finally, remark that the integration preserves the càdlàg-paths such that with X also the paths of Z are càdlàg and hence Z is shown to be a left $\mathbb{F}$-Lévy process.
Conversely, suppose $(Z_t)_{t\geq 0}$ to be a left $\mathbb{F}$-Lévy process in $(\mathrm{GL}(\mathbb{R}, m), \cdot)$. Since every multiplicative Lévy process admits an Itô decomposition, i.e. an integral representation, as shown in [28], it is a semimartingale with respect to its augmented natural filtration and we can define $(X_t)_{t\geq 0}$ by (5.13). Then it holds for all $0 \leq s < t$

$$\begin{aligned} X_t - X_s &= \int_{(s,t]} Z_{u-}^{-1} dZ_u = \int_{(0,t-s]} Z_{(s+u)-}^{-1} dZ_{s+u} \\ &= \int_{(0,t-s]} Z_{(s+u)-}^{-1} Z_s d(Z_s^{-1} Z_{s+u}) \end{aligned}$$

$$= \int_{(0,t-s]} (Z_s^{-1} Z_{(s+u)-})^{-1} d(Z_s^{-1} Z_{s+u}) \tag{5.35}$$
$$\stackrel{d}{=} \int_{(0,t-s]} Z_{u-}^{-1} dZ_u$$

where the last equality follows from the fact that the processes $(Z_s^{-1} Z_{s+u})_{u \geq 0}$ and $(Z_u)_{u \geq 0}$ are equal in distribution (see e.g. [45, Proposition 1.1]). Thus we have that $X_t - X_s \stackrel{d}{=} X_{t-s}$ for all $0 \leq s < t$ and hence X has stationary additive increments. Additionally, since Z is a multiplicative Lévy process, for $0 \leq s < t \leq u < v$ the increment $X_v - X_u$ is independent of $\mathcal{F}_u$ and hence of $X_t - X_s$ as can be seen from (5.35). Again, an inductive argument yields the property of independent increments for X. As integration preserves the càdlàg paths, X is shown to be an additive Lévy process with respect to $\mathbb{F}$. Observe that (5.13) is equivalent to $dX_t = Z_{t-}^{-1} dZ_t$ with $X_0 = 0$ and hence to (5.9). Thus it holds $\overleftarrow{\mathcal{E}}(X)_t = Z_t$, the exponential Z_t is an $\mathbb{F}$-semimartingale and since Z_t is non-singular for all $t \geq 0$ Equation (5.12) holds. □

Next we introduce an approximation of the stochastic exponential which will be a useful tool throughout this section. Namely, the following result is due to Emery [17].

Lemma 5.20. *Let $\sigma = (t_0 = 0, t_1, \ldots, t_j, \ldots)$ with $t_j \to \infty$ and $|\sigma| := \sup_{j \in \mathbb{N}} |t_j - t_{j-1}| < \infty$ be a subdivision of the positive real line. Let X be a Lévy process. Then the processes $\overleftarrow{\mathcal{E}}(X)^\sigma$ given by $\overleftarrow{\mathcal{E}}(X)^\sigma_0 = I$ and*

$$\overleftarrow{\mathcal{E}}(X)^\sigma_t = (I + X_{t_1})(I + X_{t_2} - X_{t_1}) \cdots (I + X_{t_j} - X_{t_{j-1}})(I + X_t - X_{t_j}) \tag{5.36}$$

for $t_j < t \leq t_{j+1}$ converge to $\overleftarrow{\mathcal{E}}(X)$ uniformly on compacts in probability when $|\sigma|$ tends to 0. Similarly, by (5.11) it follows that the approximating processes $\overrightarrow{\mathcal{E}}(X)^\sigma_0 = I$ and for $t_j < t \leq t_{j+1}$

$$\overrightarrow{\mathcal{E}}(X)^\sigma_t = (I + X_t - X_{t_j})(I + X_{t_j} - X_{t_{j-1}}) \cdots (I + X_{t_2} - X_{t_1})(I + X_{t_1}) \tag{5.37}$$

converge to $\overrightarrow{\mathcal{E}}(X)$ uniformly on compacts in probability when $|\sigma|$ tends to 0.

Now we can easily show the following.

Lemma 5.21. *Let $(X_t)_{t \geq 0}$ be a Lévy process in $(\mathbb{R}^{m \times m}, +)$. Then for any $t \geq 0$ fixed we have that*

$$\overleftarrow{\mathcal{E}}(X)_t \stackrel{d}{=} \overrightarrow{\mathcal{E}}(X)_t.$$

In particular this implies

$$P - \lim_{t \to \infty} \overleftarrow{\mathcal{E}}(X)_t = 0 \quad \Leftrightarrow \quad P - \lim_{t \to \infty} \overrightarrow{\mathcal{E}}(X)_t = 0. \tag{5.38}$$

Proof. Fix $t > 0$ and for $n \in \mathbb{N}$ let $\sigma = (0, t/n, 2t/n, \ldots)$ be a subdivision of the positive real line. Then the approximations of the left and right stochastic exponential as defined in (5.36) and (5.37) are given by

$$\begin{aligned}\overleftarrow{\mathcal{E}}(X)_t^\sigma &= (I + X_{t/n})(I + X_{2t/n} - X_{t/n})\cdots \\ &\quad \cdots(I + X_{(n-1)t/n} - X_{(n-2)t/n})(I + X_t - X_{(n-1)t/n}) \quad \text{and} \\ \overrightarrow{\mathcal{E}}(X)_t^\sigma &= (I + X_t - X_{(n-1)t/n})(I + X_{(n-1)t/n} - X_{(n-2)t/n})\cdots \\ &\quad \cdots(I + X_{2t/n} - X_{t/n})(I + X_{t/n}).\end{aligned}$$

Since X is a Lévy process it has stationary and independent increments, such that $\overleftarrow{\mathcal{E}}(X)_t^\sigma \overset{d}{=} \overrightarrow{\mathcal{E}}(X)_t^\sigma$. Letting n tend to infinity yields the assumption by Lemma 5.20. □

Apart from transposition and inversion, another connection between left and right Lévy processes on $\mathrm{GL}(\mathbb{R}, m)$ is given via time reversal. Recall that for a fixed time $t > 0$, the time reversal of an additive Lévy process $(X_s)_{0 \leq s \leq t}$ is defined by

$$\tilde{X}_s := \begin{cases} 0, & s = 0 \\ X_{(t-s)-} - X_{t-}, & 0 < s < t \\ -X_{t-}, & s = t, \end{cases} \tag{5.39}$$

and it has the same law as the process $-X$ with respect to its natural filtration. Then we can observe the following.

Lemma 5.22. *Let $t > 0$ be fixed and suppose $(X_s)_{s \geq 0}$ is a Lévy process in $(\mathbb{R}^{m \times m}, +)$ which fulfills* (5.12)*. Then it holds*

$$\overrightarrow{\mathcal{E}}(X)_t \overrightarrow{\mathcal{E}}(X)_{(t-s)-}^{-1} = \overleftarrow{\mathcal{E}}(-\tilde{X})_s \quad \text{a.s. for all} \quad 0 \leq s \leq t, \tag{5.40}$$

while

$$\overrightarrow{\mathcal{E}}(X)_{(t+s)-}^{-1} \overrightarrow{\mathcal{E}}(X)_t = \overrightarrow{\mathcal{E}}(X_{t+\cdot} - X_t)_{s-} \quad \text{a.s. for all} \quad s > 0. \tag{5.41}$$

Proof. Due to similarity we only prove (5.40). For notational simplicity assume $t = 1$. Let $\sigma = (s_0 = 0, s_1 = 1/n, s_2 = 2/n, \ldots)$, $n \in \mathbb{N}$, be a partition of the positive real line. Then for any $i = 0, \ldots, n$ we have a.s. by (5.36)

$$\begin{aligned}\overleftarrow{\mathcal{E}}(-\tilde{X})_{1-s_i}^\sigma &= (I - \tilde{X}_{s_1})\cdots(I - \tilde{X}_{s_{n-i}} + \tilde{X}_{s_{n-i-1}}) \\ &= (I - \tilde{X}_{s_1-})\cdots(I - \tilde{X}_{s_{n-i}-} + \tilde{X}_{s_{n-i-1}-}) \\ &= (I + X_{s_n} - X_{s_{n-1}})\cdots(I + X_{s_{i+1}} - X_{s_i}) \\ &= (I + X_{s_n} - X_{s_{n-1}})\cdots(I + X_{s_{i+1}} - X_{s_i}) \\ &\quad (I + X_{s_i} - X_{s_{i-1}})\cdots(I + X_{s_1})(I + X_{s_1})^{-1}\cdots(I + X_{s_i} - X_{s_{i-1}})^{-1} \\ &= \overrightarrow{\mathcal{E}}(X)_1^\sigma(\overrightarrow{\mathcal{E}}(X)_{s_i}^\sigma)^{-1}\end{aligned}$$

where we have used the fact that at fixed time s_i the process X and thus $\tilde{X}$ a.s. does not jump. Hence we have established that $\overleftarrow{\mathcal{E}}(-\tilde{X})^{\sigma}_{1-s} = \overrightarrow{\mathcal{E}}(X)^{\sigma}_1(\overrightarrow{\mathcal{E}}(X)^{\sigma}_s)^{-1}$ holds for all $s \in \{0, 1/n, 2/n, \ldots\}$. Setting $|\sigma| = 2^k$ and taking the limit for $k \to \infty$ gives us $\overleftarrow{\mathcal{E}}(-\tilde{X})_{1-s} = \overrightarrow{\mathcal{E}}(X)_1(\overrightarrow{\mathcal{E}}(X)_{s-})^{-1}$ a.s. for all $s \in \mathbb{Q} \cap [0,1]$. Finally the fact that left and right exponential as multiplicative Lévy processes have càdlàg paths yields the assumption. □

5.4.2 Proofs

Proofs for Section 5.3.3

The next is a proposition which will be needed to prove Theorem 5.16. It generalizes Proposition 2.3 in [46] and its extension in Lemma 2.5 to a multivariate setting. Remark the switch of direction of the exponential in the distributional equality which results from a time change.

Proposition 5.23. *Suppose $(X_t, Y_t)_{t\geq 0}$ is a Lévy process in $(\mathbb{R}^{m\times m} \times \mathbb{R}^m, +)$ such that $(X_t)_{t\geq 0}$ fulfills (5.12) and define the process $(U_t, L_t)_{t\geq 0}$ by (5.14) and (5.21). Then it holds*

$$\overrightarrow{\mathcal{E}}(U)_t \int_{(0,t]} \overrightarrow{\mathcal{E}}(U)^{-1}_{s-} dY_s \overset{d}{=} \int_{(0,t]} \overleftarrow{\mathcal{E}}(U)_{s-} dY_s + \left[\overleftarrow{\mathcal{E}}(U), Y\right]_t \overset{a.s.}{=} \int_{(0,t]} \overleftarrow{\mathcal{E}}(U)_{s-} dL_s \quad (5.42)$$

and analogously

$$\overrightarrow{\mathcal{E}}(X)_t \int_{(0,t]} \overrightarrow{\mathcal{E}}(X)^{-1}_{s-} dL_s \overset{d}{=} \int_{(0,t]} \overleftarrow{\mathcal{E}}(X)_{s-} dL_s + \left[\overleftarrow{\mathcal{E}}(X), L\right]_t \overset{a.s.}{=} \int_{(0,t]} \overleftarrow{\mathcal{E}}(X)_{s-} dY_s. \quad (5.43)$$

Proof. The almost sure equalities in (5.42) and (5.43) follow directly from (5.22) and (5.24), respectively, under use of (5.6) and (5.9), while the distributional equalities can be shown following the proof of [55, Theorem VI.22]. Due to similarity we restrict on showing (5.42). Fix $t > 0$ and define for $0 \leq s \leq t$

$$\hat{U}_s := U_t - U_{(t-s)-} \quad \text{and} \quad \hat{Y}_s := Y_t - Y_{(t-s)-}.$$

For $n \in \mathbb{N}$ let $\sigma = (0, t/n, 2t/n, \ldots)$ be a partition of the positive real line, set

$$H_s := \overleftarrow{\mathcal{E}}(\hat{U})_s \quad \text{and} \quad G_s := \hat{Y}_s$$

and define the additional random variables

$$\begin{aligned} A^{\sigma} &:= \sum_{i=0}^{n-1} H_{t(i+1)/n}(G_{t(i+1)/n} - G_{ti/n}) \\ &= \sum_{i=0}^{n-1} H_{ti/n}(G_{t(i+1)/n} - G_{ti/n}) + \sum_{i=0}^{n-1} (H_{t(i+1)/n} - H_{ti/n})(G_{t(i+1)/n} - G_{ti/n}) \end{aligned}$$

$$B^\sigma \quad := \quad -\sum_{i=0}^{n-1} H_{t(i+1)/n-}(G_{t(i+1)/n-} - G_{ti/n-}),$$

where we set $G_{0-} := 0$. Since integral and quadratic variation are defined component-by-component, letting $|\sigma|$ tend to zero, we obtain by [55, Theorems II.21 and II.23]

$$\begin{aligned} A^\sigma \quad &\xrightarrow{P} \int_{(0,t]} H_{s-}dG_s + [H,G]_t = \int_{(0,t]} \overleftarrow{\mathcal{E}}(\hat{U})_{s-}d\hat{Y}_s + [\overleftarrow{\mathcal{E}}(\hat{U}), \hat{Y}]_t \\ &\stackrel{d}{=} \int_{(0,t]} \overleftarrow{\mathcal{E}}(U)_{s-}dY_s + [\overleftarrow{\mathcal{E}}(U), Y]_t \end{aligned}$$

where the last equality follows from the fact that $(\hat{U}_s, \hat{Y}_s)_{0\le s\le t} \stackrel{d}{=} (U_s, Y_s)_{0\le s\le t}$ which yields $(\overleftarrow{\mathcal{E}}(\hat{U})_s, \hat{Y}_s)_{0\le s\le t} \stackrel{d}{=} (\overleftarrow{\mathcal{E}}(U)_s, Y_s)_{0\le s\le t}$. On the other hand remark that by definition $G_{t(i+1)/n-} - G_{ti/n-} = Y_{t(n-i)/n} - Y_{t(n-i-1)/n}$ and since by (5.40) we have for $0 \le s \le t$ that $H_{s-} = \overleftarrow{\mathcal{E}}(-\tilde{U})_{s-} = \overrightarrow{\mathcal{E}}(U)_t \overrightarrow{\mathcal{E}}(U)^{-1}_{t-s}$, it holds

$$\begin{aligned} B^\sigma \quad &= \quad -\sum_{i=0}^{n-1} \overrightarrow{\mathcal{E}}(U)_t \overrightarrow{\mathcal{E}}(U)^{-1}_{t(n-i-1)/n}(Y_{t(n-i)/n} - Y_{t(n-i-1)/n}) \\ &= \quad -\overrightarrow{\mathcal{E}}(U)_t \sum_{i=1}^{n} \overrightarrow{\mathcal{E}}(U)^{-1}_{t(i-1)/n}(Y_{t(i/n)} - Y_{t(i-1)/n}) \\ &\xrightarrow{P} \quad -\overrightarrow{\mathcal{E}}(U)_t \int_{(0,t]} \overrightarrow{\mathcal{E}}(U)^{-1}_{s-}dY_s, \quad |\sigma| \to 0. \end{aligned}$$

A combination of A^σ and B^σ gives

$$\begin{aligned} &A^\sigma + B^\sigma \\ &= \sum_{i=0}^{n-1} H_{t(i+1)/n}(\Delta G_{t(i+1)/n} - \Delta G_{ti/n}) + \sum_{i=0}^{n-1} \Delta H_{t(i+1)/n}(G_{t(i+1)/n-} - G_{ti/n-}) \\ &= 0 \quad \text{a.s.} \end{aligned}$$

since at fixed times G and H a.s. do not jump. Hence the limits of A^σ and B^σ add to zero which gives the assumption. □

With the above proposition at hand we can now prove the conditions for strict stationarity of MGOU processes stated in Theorem 5.16.

Proof of Theorem 5.16. Assume that $\lim_{t\to\infty} \overleftarrow{\mathcal{E}}(U)_t = 0$ in probability and suppose that $(V_t)_{t\ge 0}$ is strictly stationary. Then by (5.38) we have that $\lim_{t\to\infty} \overrightarrow{\mathcal{E}}(U)_t = 0$ in probability and obtain

$$V_0 = \text{d-}\lim_{t\to\infty} V_t = \text{d-}\lim_{t\to\infty} \left(\overrightarrow{\mathcal{E}}(U)_t V_0 + \overrightarrow{\mathcal{E}}(U)_t \int_{(0,t]} \overrightarrow{\mathcal{E}}(U)^{-1}_{s-}dY_s \right).$$

Thus by (5.42) we conclude that $\int_{(0,t]} \overleftarrow{\mathcal{E}}(U)_{s-} dL_s \overset{d}{=} \overrightarrow{\mathcal{E}}(U)_t \int_{(0,t]} \overrightarrow{\mathcal{E}}(U)_{s-}^{-1} dY_s$ tends to V_0 in distribution as stated in (a).
Conversely, assume that $\lim_{t\to\infty} \overrightarrow{\mathcal{E}}(U)_t = 0$ in probability and $\int_{(0,\infty)} \overleftarrow{\mathcal{E}}(U)_{s-} dL_s$ converges in distribution and set V_0 independent of $(U_t, L_t)_{t\geq 0}$ such that $\mathcal{L}(V_0) = \mathcal{L}(\int_{(0,\infty)} \overleftarrow{\mathcal{E}}(U)_{s-} dL_s)$. Then by (5.42), letting t tend to infinity, V_t converges in distribution to V_0. Since $(V_t)_{t\geq 0}$ satisfies (5.2) with $(A_{t,t+h}, B_{t,t+h})$ independent of V_t this yields for all $h > 0$

$$V_0 = \text{d-}\lim_{t\to\infty} V_{t+h} = \text{d-}\lim_{t\to\infty} A_{t,t+h} V_t + B_{t,t+h} \overset{d}{=} A_{0,h} V_0 + B_{0,h} = V_h$$

such that $(V_t)_{t\geq 0}$ is strictly stationary, since it is a time homogeneous Markov process.
For (b) suppose that $\lim_{t\to\infty} \overleftarrow{\mathcal{E}}(X)_t = 0$ in probability and that $(V_t)_{t\geq 0}$ is strictly stationary. Then we have that $\overleftarrow{\mathcal{E}}(X)_t V_t = V_0 + \int_{(0,t]} \overleftarrow{\mathcal{E}}(X)_{s-} dY_s \to 0$ in probability as t tends to infinity. Hence $V_0 = \text{P-}\lim_{t\to\infty}(-\int_{(0,t]} \overleftarrow{\mathcal{E}}(X)_{s-} dY_s)$ showing one direction of (b).
Conversely, setting $V_0 = -\int_{(0,\infty)} \overleftarrow{\mathcal{E}}(X)_{s-} dY_s$ yields directly that

$$V_t = -\overleftarrow{\mathcal{E}}(X)_t^{-1} \int_{(t,\infty)} \overleftarrow{\mathcal{E}}(X)_{s-} dY_s = -\int_{(t,\infty)} \overrightarrow{\mathcal{E}}(U)_t \overrightarrow{\mathcal{E}}(U)_{s-}^{-1} dY_s$$

and hence by applying the inverse of (5.41) we observe that for any $t \geq 0$ it holds

$$V_t = -\int_{(0,\infty)} \overrightarrow{\mathcal{E}}(U_{t+\cdot} - U_t)_{s-}^{-1} d(Y_{t+s} - Y_t) \overset{d}{=} -\int_{[0,\infty)} \overrightarrow{\mathcal{E}}(U)_{s-}^{-1} dY_s = V_0.$$

Thus for any $t \geq 0$, $n \in \mathbb{N}$ and $0 \leq h_1 \leq \ldots \leq h_n$ we obtain from (5.2) with $(A_{t,t+h}^{-1}, A_{t,t+h}^{-1} B_{t,t+h})$ independent of V_{t+h} that

$$\begin{aligned}
&(V_t, V_{t+h_1}, \ldots, V_{t+h_n}) \\
&= (A_{t,t+h_n}^{-1}(V_{t+h_n} - B_{t,t+h_n}), A_{t+h_1,t+h_n}^{-1}(V_{t+h_n} - B_{t+h_1,t+h_n}), \ldots, V_{t+h_n}) \\
&\overset{d}{=} (A_{0,h_n}^{-1}(V_{h_n} - B_{0,h_n}), A_{h_1,h_n}^{-1}(V_{h_n} - B_{h_1,h_n}), \ldots, V_{h_n}) \\
&= (V_0, V_{h_1}, \ldots, V_{h_n})
\end{aligned}$$

such that $(V_t)_{t\geq 0}$ is strictly stationary. □

Proof of Theorem 5.17. (a) In view of Theorem 5.16 and Proposition 5.13 it remains to show that given the irreducibility of the underlying discrete model for $h > 0$ fixed, if $(V_t)_{t\geq 0}$ is strictly stationary and causal, then $\overleftarrow{\mathcal{E}}(U)_t$ tends to 0 in probability as $t \to \infty$. Therefore for $n \in \mathbb{N}$ set $\bar{A}_{n,h} := A_{(n-1)h,nh} = \overrightarrow{\mathcal{E}}(U)_{nh} \overrightarrow{\mathcal{E}}(U)_{(n-1)h}^{-1}$ and $\bar{B}_{n,h} := B_{(n-1)h,nh}$ such that we have $V_{nh} = \bar{A}_{n,h} V_{(n-1)h} + \bar{B}_{n,h}$.

Since $(\bar{A}_{n,h}, \bar{B}_{n,h}, V_{(n-1)h})_{n\in\mathbb{N}}$ is strictly stationary, we can extend it to a new stationary process $(\bar{A}_{n,h}, \bar{B}_{n,h}, V_{(n-1)h})_{n\in\mathbb{Z}}$ and observe that $(V_{nh})_{n\in\mathbb{Z}}$ is a strictly stationary, causal solution of the irreducible autoregressive model $V_{nh} = \bar{A}_{n,h}V_{(n-1)h} + \bar{B}_{n,h}$, $n \in \mathbb{Z}$. Thus by [13, Theorem 2.4] we have that a.s. the product $\bar{A}_{0,h}\bar{A}_{-1,h}\cdots\bar{A}_{-k,h}$ converges to 0 as $k \to \infty$. By the stationarity of $(\bar{A}_{n,h})_{n\in\mathbb{Z}}$ this yields that the product $\bar{A}_{k,h}\bar{A}_{k-1,h}\cdots\bar{A}_{1,h}$ tends to 0 in probability as $k \to \infty$ which is equivalent to $P - \lim_{n\to\infty} \overrightarrow{\mathcal{E}}(U)_{nh} = 0$ and by (5.38) also to $P - \lim_{n\to\infty} \overleftarrow{\mathcal{E}}(U)_{nh} = 0$, $h > 0$. For any $t \geq 0$, $\epsilon > 0$ we now obtain

$$P(||\overleftarrow{\mathcal{E}}(U)_t|| > \epsilon) \leq P\left(||\overleftarrow{\mathcal{E}}(U)_{\lfloor t/h\rfloor h}|| \cdot \sup_{\lfloor t/h\rfloor h \leq s < (\lfloor t/h\rfloor +1)h} ||\overleftarrow{\mathcal{E}}(U)^{-1}_{\lfloor t/h\rfloor h}\overleftarrow{\mathcal{E}}(U)_s|| > \epsilon \right),$$

where $||\cdot||$ denotes some submultiplicative matrix norm. As

$$\sup_{\lfloor t/h\rfloor h \leq s < (\lfloor t/h\rfloor +1)h} ||\overleftarrow{\mathcal{E}}(U)^{-1}_{\lfloor t/h\rfloor h}\overleftarrow{\mathcal{E}}(U)_s|| \stackrel{d}{=} \sup_{s\in[0,h)} ||\overleftarrow{\mathcal{E}}(U)_s||$$

is finite, we have $\lim_{t\to\infty} P(||\overleftarrow{\mathcal{E}}(U)_t|| > \epsilon) = 0$ as had to be shown.
(b) We need to prove $P - \lim_{t\to\infty} \overleftarrow{\mathcal{E}}(X)_t = 0$ given the irreducibility of the underlying discrete model for $h > 0$ fixed and provided that $(V_t)_{t\geq 0}$ is strictly stationary and strictly non-causal. Using the notations as introduced under (a) it can easily be seen that $(\bar{A}_{n,h}, \bar{B}_{n,h}, V_{(n-1)h})_{n\in\mathbb{N}}$ is strictly stationary and thus can again be extended to a strictly stationary process $(\bar{A}_{n,h}, \bar{B}_{n,h}, V_{(n-1)h})_{n\in\mathbb{Z}}$ where by the provided strict non-causality $V_{n,h}$ is independent of $(\bar{A}_{k,h}, \bar{B}_{k,h})_{k\leq n}$. Defining the process $(C_{n,h}, D_{n,h}, W_{(n-1)h})_{n\in\mathbb{Z}}$ by $C_{n,h} := \bar{A}^{-1}_{-n,h}$, $D_{n,h} := -\bar{A}^{-1}_{-n,h}\bar{B}_{-n,h}$ and $W_{nh} := V_{-nh}$ we see that it is strictly stationary and obtain that W_{nh} fulfills the autoregressive model

$$W_{(n+1)h} = C_{n,h}W_{nh} + D_{n,h} \tag{5.44}$$

where W_{nh} is independent of $(C_{k,h}, D_{k,h})_{k\geq n}$ and hence it is non-anticipative. The model (5.44) is irreducible since any invariant affine subspace of the model (5.44) is also an invariant affine subspace of the initial discrete model. Namely, suppose there exists an invariant affine subspace H of (5.44) then we have a.s. $\bar{A}^{-1}_{-n,h}H - \bar{A}^{-1}_{-n,h}\bar{B}_{-n,h} = H$ since the mapping $x \mapsto \bar{A}^{-1}_{-n,h}x - \bar{A}^{-1}_{-n,h}\bar{B}_{-n,h}$ is bijective. Thus it follows $\bar{A}_{-n,h}H + \bar{B}_{-n,h} = H$ such that H is invariant under the initial model. Hence we can again apply [13, Theorem 2.4] and an argumentation as under (a) yields the result. □

Proof of Corollary 5.18. Define $(U'_t, L'_t)_{t\geq 0}$ as the process corresponding to $(OX_tO^{-1}, OY_t)_{t\geq 0}$ via (5.14) and (5.21). Then $(U'_t)_{t\geq 0}$ and $(L'_t)_{t\geq 0}$ are of the form given in (5.29) as shown in Theorem 5.14. In particular we deduce from (5.14) that

$$\begin{aligned} U'_t &= -OX_tO^{-1} + [OXO^{-1}, OXO^{-1}]^c_t \\ &\quad + \sum_{0<s\leq t} \big((I+\Delta(OX_sO^{-1}))^{-1} - I + \Delta(OX_sO^{-1})\big) \\ &= -OX_tO^{-1} + O[X,X]^c_tO^{-1} + \sum_{0<s\leq t} O\left((I+\Delta X_s)^{-1} - I + \Delta X_s\right)O^{-1} \\ &= OU_tO^{-1}, \quad t \geq 0, \end{aligned}$$

and together with this, (5.22) yields

$$L'_t = OY_t + [OUO^{-1}, OY]_t = OY_t + O[U,Y]_t = OL_t, \quad t \geq 0,$$

such that (5.34) is shown.
Now Theorem 5.16 provides the sufficient conditions for the existence of strictly stationary solutions as stated in (a) and (b) as well as the given form of the solutions.
□

Proof of Proposition 5.13

Finally, to show Proposition 5.13 we need to make a short excursion in affine geometry. Taking up the notations as in [52] we say that the vectors $v_0, v_1, \ldots, v_n \in \mathbb{R}^m$ are *affine-independent* if

$$\lambda_0 v_0 + \lambda_1 v_1 + \ldots + \lambda_n v_n = 0 \quad \text{and} \quad \lambda_0 + \lambda_1 + \ldots + \lambda_n = 0, \quad \lambda_i \in \mathbb{R}$$

implies $\lambda_0 = \lambda_1 = \ldots = \lambda_n = 0$. The *affine span* of $(n+1)$ vectors $v_0, v_1, \ldots, v_n \in \mathbb{R}^m$ is the set of vectors

$$\{\lambda_0 v_0 + \lambda_1 v_1 + \ldots + \lambda_n v_n,\ \lambda_0 + \ldots + \lambda_n = 1,\ \lambda_i \in \mathbb{R}\}$$

and a subset of $\mathbb{R}^m$ is called an *affine n-flat* if it is the *affine span* of $(n+1)$ affine-independent vectors. Thus the affine n-flats are simply all sets $a+W$ where $a \in \mathbb{R}^m$ and W is an n-dimensional subspace of $\mathbb{R}^m$.
Given K affine n_k-flats H_k each of which is the affine span of the vectors $v_0^{n_k}, \ldots, v_{n_k}^{n_k}$, we will shortly call the affine span of the vectors $v_0^1, \ldots, v_{n_1}^1, v_0^2, \ldots, v_{n_K}^{n_K}$ the affine span of $H_1, \ldots, H_K$.

In the following we investigate a classification of affine flats which are invariant under a given invertible affine transformation $f : \mathbb{R}^m \to \mathbb{R}^m, x \mapsto Ax + B$. Therefore we will write $\mathcal{H}^d$ for the set of all affine d-flats H, $0 \leq d < m$, which are f-invariant in the sense that we have $f(H) \subset H$ (which by the bijectivity implies $f(H) = H$). A first observation is the following Lemma which is a direct consequence of [52, Corollary 12.3].

Lemma 5.24. *Suppose the affine transformation* $f : \mathbb{R}^m \to \mathbb{R}^m, x \mapsto Ax+B$ *admits* $m+1$ *affine independent fixed points. Then* f *is the identity.*

The above lemma can also be concluded from the following one.

Lemma 5.25. *Suppose we have* $H_1 \in \mathcal{H}^{d_1}, H_2 \in \mathcal{H}^{d_2}$, $0 \leq d_1 \leq d_2 < m$ *where* H_1 *and* H_2 *are parallel in the sense that there exists* $a \in \mathbb{R}^m$ *such that* $a + H_1 \subset H_2$. *Then all affine* d_2*-flats which are parallel to* H_1 *and* H_2 *and lie in the affine span of* H_1 *and* H_2 *are in* $\mathcal{H}^{d_2}$.

Proof. We can write

$$\begin{aligned} H_1 &= \{a + \lambda_1 h_1 + \ldots + \lambda_{d_1} h_{d_1}, \lambda_i \in \mathbb{R}\} \quad \text{and} \\ H_2 &= \{b + \lambda_1 h_1 + \ldots + \lambda_{d_1} h_{d_1} + \lambda_{d_1+1} h_{d_1+1} + \ldots + \lambda_{d_2} h_{d_2}, \lambda_i \in \mathbb{R}\}. \end{aligned}$$

As H_1 and H_2 are f-invariant, this yields that

$$\begin{aligned} A(a + \lambda_1 h_1 + \ldots + \lambda_{d_1} h_{d_1}) + B &= a + \mu_1 h_1 + \ldots + \mu_{d_1} h_{d_1} \\ A(b + \lambda'_1 h_1 + \ldots + \lambda'_{d_2} h_{d_2}) + B &= b + \mu'_1 h_1 + \ldots + \mu'_{d_2} h_{d_2} \end{aligned}$$

admits a solution for all $\lambda_1, \ldots, \lambda_{d_1}, \lambda'_1, \ldots, \lambda'_{d_2}$. By substraction and setting $\lambda_i = \lambda'_i = 0$ we obtain that

$$A(a - b) = (a - b) + \nu_1 h_1 + \ldots + \nu_{d_2} h_{d_2}.$$

Let H_3 be an affine d_2-flat, parallel to H_2, in the affine span of H_1 and H_2. Then we can represent H_3 as

$$H_3 = \{b + \alpha(a - b) + \lambda_1 h_1 + \ldots + \lambda_{d_2} h_{d_2}, \lambda_i \in \mathbb{R}\}.$$

Now observe that

$$\begin{aligned} & A\left(b + \alpha(a - b) + \lambda_1 h_1 + \ldots + \lambda_{d_2} h_{d_2}\right) + B \\ &= A\left(b + \lambda_1 h_1 + \ldots + \lambda_{d_2} h_{d_2}\right) + B + \alpha A(a - b) \\ &= b + \mu_1 h_1 + \ldots + \mu_{d_2} h_{d_2} + \alpha(a - b) + \alpha\nu_1 h_1 + \ldots + \alpha\nu_{d_2} h_{d_2} \end{aligned}$$

such that $f(H_3) \subset H_3$ as had to be shown. □

An immediate consequence of the last lemma is the fact that $\mathcal{H}^0$ is either empty, a single-point set or consists of all points in an affine d-flat, $1 \leq d \leq m$.

To treat invariant affine d-flats, $d \geq 1$, which do not consist of fixed points, we make the following additional observations.

Lemma 5.26. *Let* $f : \mathbb{R}^m \to \mathbb{R}^m, x \mapsto Ax+B$ *be an invertible affine transformation on* $\mathbb{R}^m$ *and suppose we have* $H_1 \in \mathcal{H}^{d_1}, H_2 \in \mathcal{H}^{d_2}$.

(a) *If H_1 and H_2 intersect in an affine d_3-flat, $0 \leq d_3 \leq d_1, d_2$, then the intersection is an f-invariant d_3-flat and thus lies in $\mathcal{H}^{d_3}$.*

(b) *Suppose H_1 and H_2 are not parallel. Let d_4, $0 \leq d_4 < d_1, d_2$, be the dimension of the intersection of the subspaces of $\mathbb{R}^m$ which are parallel to H_1 and H_2. Then all affine $(d_1 + d_2 - d_4)$-flats which are parallel to H_1 and H_2 and lie in the affine span of H_1 and H_2 are in $\mathcal{H}^{d_1+d_2-d_4}$.*

Proof. (a) For $x \in H_1 \cap H_2$ we obviously have $f(x) \in H_1$ and $f(x) \in H_2$ such that the intersection of H_1 and H_2 is shown to be f-invariant.
(b) The general representations of H_1 and H_2 are

$$\begin{aligned} H_1 &= \{a + \lambda_1 h_1 + \ldots + \lambda_{d_4} h_{d_4} + \lambda_{d_4+1} h_{d_4+1} + \ldots + \lambda_{d_1} h_{d_1}, \lambda_i \in \mathbb{R}\} \quad \text{and} \\ H_2 &= \{b + \lambda_1 h_1 + \ldots + \lambda_{d_4} h_{d_4} + \lambda_{d_4+1} g_{d_4+1} + \ldots + \lambda_{d_2} g_{d_2}, \lambda_i \in \mathbb{R}\}, \end{aligned}$$

where $0 \leq d_4 < d_1, d_2$. The f-invariance of H_1 and H_2 implies that

$$\begin{aligned} & A(a - b + (\lambda_1 - \lambda_1')h_1 + \ldots + (\lambda_{d_4} - \lambda_{d_4}')h_{d_4} \\ & \qquad + \lambda_{d_4+1} h_{d_4+1} + \ldots + \lambda_{d_1} h_{d_1} - \lambda'_{d_4+1} g_{d_4+1} - \ldots - \lambda'_{d_2} g_{d_2}) \\ & = \quad a - b + \mu_1 h_1 + \ldots + \mu_{d_1} h_{d_1} - \mu'_{d_4+1} g_{d_4+1} - \ldots - \mu'_{d_2} g_{d_2}. \end{aligned}$$

Now let H_3 be the affine $(d_1 + d_2 - d_4)$-flat which contains H_1 and is parallel to H_2, i.e.

$$H_3 = \{a + \lambda_1 h_1 + \ldots + \lambda_{d_1} h_{d_1} + \lambda'_{d_4+1} g_{d_4+1} + \ldots + \lambda'_{d_2} g_{d_2}, \lambda_i, \lambda_i' \in \mathbb{R}\},$$

and we have

$$\begin{aligned} & A\left(a + \lambda_1 h_1 + \ldots + \lambda_{d_1} h_{d_1} + \lambda'_{d_4+1} g_{d_4+1} + \ldots + \lambda'_{d_2} g_{d_2}\right) + B \\ & = \quad A\left(a - b + \lambda_1 h_1 + \ldots + \lambda_{d_1} h_{d_1} + \lambda'_{d_4+1} g_{d_4+1} + \ldots + \lambda'_{d_2} g_{d_2}\right) + A(b) + B \\ & = \quad a - b + \mu_1 h_1 + \ldots + \mu_{d_1} h_{d_1} - \mu'_{d_4+1} g_{d_4+1} - \ldots - \mu'_{d_2} g_{d_2} \\ & \qquad + b + \nu_{d_4+1} g_{d_4+1} + \ldots + \nu_{d_2} g_{d_2} \end{aligned}$$

such that $f(H_3) \subset H_3$. Finally Lemma 5.25 gives the assumption. □

With the derived lemmata we can now prove Proposition 5.13.

Proof of Proposition 5.13. Let $f_n : \mathbb{R}^m \to \mathbb{R}^m, x \mapsto A_{0,h}x + B_{0,h}$, $n \in \mathbb{N}_0$, be the affine transformation on $\mathbb{R}^m$ corresponding to the model (5.25) for $h = 2^{-n}$. Define $\mathcal{H}_n^d$ to be the set of all affine d-flats H, $0 \leq d < m$, which are f_n-invariant. Observe that obviously any affine subspace which is invariant under the model (5.25) for $h > 0$ also is invariant for all kh, $k \in \mathbb{N}$. Thus in particular we have $\mathcal{H}_{n+1}^d \subset \mathcal{H}_n^d$ for all $n \in \mathbb{N}_0$, $0 \leq d < m$ such that $\mathcal{H}_\infty^d := \lim_{n\to\infty} \mathcal{H}_n^d = \bigcap_{n=0}^\infty \mathcal{H}_n^d$ can be defined for $0 \leq d < m$.

We will prove the proposition by contradiction, hence we assume that for all $h > 0$ there exists an invariant subspace of the model (5.25) i.e. there exists no $h > 0$

such that (5.25) is irreducible. It will be shown below that this assumption implies $\mathcal{H}_\infty = \bigcup_{d=0}^{m-1} \mathcal{H}_\infty^d \neq \emptyset$, i.e. there exists an affine subspace H of $\mathbb{R}^m$ which is invariant under f_n for all $n \in \mathbb{N}$. Thus it is also invariant under the model (5.25) for all $h = k2^{-n}$, $k \in \mathbb{N}$, $n \in \mathbb{N}$ and hence for all $h > 0$, $h \in \mathbb{Q}$. It then remains to show that $A_{0,t}H + B_{0,t} \subset H$ holds for all $t > 0$, i.e. that H is invariant under (5.2). Therefore let $t > 0$ be non-rational and define a monotonely decreasing sequence $(t_n)_{n\geq 0}$ with $\lim_{n\to\infty} t_n = t$ and $t_n \in \mathbb{Q}$ for all $n \in \mathbb{N}$. Denote the underlying probability space by $(\Omega, \mathcal{F}, P)$. Then we have that for all $n \in \mathbb{N}$ there exists Ω_n with $P(\Omega_n) = 1$ such that $A_{0,t_n}(\omega)H + B_{0,t_n}(\omega) \subset H$ for all $\omega \in \Omega_n$. Taking Ω_∞ the infinite intersection of the Ω_n, we have $P(\Omega_\infty) = 1$ and

$$A_{0,t_n}(\omega)H + B_{0,t_n}(\omega) \subset H, \quad \forall n \in \mathbb{N}, \forall \omega \in \Omega_\infty.$$

Since $A_{0,t}(\omega)$ and $B_{0,t}$ a.s. have càdlàg paths, $A_{0,t_n}(\omega)$ and $B_{0,t_n}(\omega)$ converge to $A_{0,t}(\omega)$ and $B_{0,t}(\omega)$ for all $\omega \in C$ where $P(C) = 1$. Thus for all $x \in H$ and $\omega \in \Omega_\infty \cap C$ it holds

$$\begin{aligned} A_{0,t}(\omega)x + B_{0,t}(\omega) &= \lim_{n\to\infty} A_{0,t_n}(\omega)x + \lim_{n\to\infty} B_{0,t_n}(\omega) \\ &= \lim_{n\to\infty} (A_{0,t_n}(\omega)x + B_{0,t_n}(\omega)) \\ &\in H \end{aligned}$$

since H is closed. Hence the continuous time model (5.2) admits an invariant affine subspace as had to be shown.

It remains to show that $\mathcal{H}_\infty = \bigcup_{d=0}^{m-1} \mathcal{H}_\infty^d = \emptyset$ implies the existence of some N such that for all $n > N$ we have $\bigcup_{d=0}^{m-1} \mathcal{H}_n^d = \emptyset$ for $m = 1, 2$ and 3.
We have characterized above that $\mathcal{H}_n^0$ is of one of the following types.
(1) $\mathcal{H}_n^0 = \emptyset$
(2) $\mathcal{H}_n^0 = \{x, x \in H\}$ where H is an affine d-flat, $0 \leq d \leq m$.
Thus, recalling again that $\mathcal{H}_{n+1}^0 \subset \mathcal{H}_n^0$, the assumption $\mathcal{H}_\infty^0 = \bigcap_{n=0}^\infty \mathcal{H}_n^0 = \emptyset$ implies the existence of some constant N_0 such that for all $n \geq N_0$ we have $\mathcal{H}_n^0 = \emptyset$. Hence for $m = 1$ the result is shown and in treating higher dimensions we can assume without loss of generality that no fixed point exists.

Suppose $m = 2$. Then using $\mathcal{H}_n^0 = \emptyset$ and Lemmata 5.25 and 5.26(a) we can classify $\mathcal{H}_n^1$ as follows.
(1) $\mathcal{H}_n^1 = \emptyset$
(2) $\mathcal{H}_n^1 = \{g\}$ where g is an affine 1-flat.
(3) $\mathcal{H}_n^1 = \{h, h||g\}$ where g and h are affine 1-flats and $||$ denotes parallelism.
Hence $\mathcal{H}_\infty^1 = \bigcap_{n=0}^\infty \mathcal{H}_n^1 = \emptyset$ implies the existence of some constant $N_1 \geq N_0$ such that for all $n \geq N_1$ we have $\mathcal{H}_n^1 = \emptyset$ which yields the result for $m = 2$.

In the case of $m = 3$ the assumption of $\mathcal{H}_n^0 = \emptyset$ and Lemma 5.25 lead to the following classification of $\mathcal{H}_n^1$.

(1) $\mathcal{H}_n^1 = \emptyset$
(2) $\mathcal{H}_n^1 = \{g\}$ where g is an affine 1-flat.
(3) $\mathcal{H}_n^1 = \{h, h||g, h \in H\}$ where g and h are affine 1-flats and H is an affine 2 or 3-flat such that $g \in H$.
(4) $\mathcal{H}_n^1 \supset \{g, h\}$ where g and h are two skew affine 1-flats, i.e. they do not intersect and are not parallel.

The case (4) implies by Lemma 5.26(b) that $\mathcal{H}_n^2 \supset \{H, H||G\}$ where G and H are affine 2-flats. In fact we have $\mathcal{H}_n^2 = \{H, H||G\}$ since given the existence of another affine 2-flat K in $\mathcal{H}_n^2$ this one would intersect at least one of the 1-flats g and h in a single point, i.e. in a fixed point in contradiction to our assumptions. This observation allows us to argue as above such that $\mathcal{H}_\infty^1 = \bigcap_{n=0}^\infty \mathcal{H}_n^1 = \emptyset$ and $\mathcal{H}_\infty^2 = \bigcap_{n=0}^\infty \mathcal{H}_n^2 = \emptyset$ imply the existence of some constant $N_1 \geq N_0$ such that for all $n \geq N_1$ we have $\mathcal{H}_n^1 = \emptyset$.

Finally given that $\mathcal{H}_n^0 = \mathcal{H}_n^1 = \emptyset$ for $\mathcal{H}_n^2$ only the following possibilities remain.

(1) $\mathcal{H}_n^2 = \emptyset$
(2) $\mathcal{H}_n^2 = \{G\}$ where G is an affine 2-flat.
(3) $\mathcal{H}_n^2 = \{H, H||G\}$ where G and H are affine 2-flats.

Hence $\mathcal{H}_\infty^2 = \bigcap_{n=0}^\infty \mathcal{H}_n^2 = \emptyset$ implies that there exists $N_2 \geq N_1$ such that for all $n \geq N_2$ we have $\mathcal{H}_n = \bigcup_{d=1}^{m-1} \mathcal{H}_n^d = \emptyset$ and we are done. □

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Bibliography

[1] G. Alsmeyer, A. Iksanov and U. Rösler (2009) On distributional properties of perpetuities. *J. Theoret. Probab.* **22**, 666–682.

[2] D. Applebaum (2004) *Lévy Processes and Stochastic Calculus.* Cambridge University Press, Cambridge.

[3] O. E. Barndorff-Nielsen, M. Maejima and K. Sato (2006) Some classes of infinitely divisible distributions admitting stochastic integral representations. *Bernoulli* **12**, 1–33.

[4] A. Behme (2011) Distributional properties of solutions of $dV_t = V_{t-}dU_t + dL_t$ with Lévy noise. *Adv. Appl. Probab.* **43.3**.

[5] A. Behme, A. Lindner and R. Maller (2011) Stationary solutions of the stochastic differential equation $dV_t = V_{t-}dU_t + dL_t$ with Lévy noise. *Stoch. Proc. Appl.* **121**, 91–108.

[6] A. Behme, M. Maejima, M. Matsui and N. Sakuma (2010) Distributions of exponential integrals of independent increment processes related to generalized gamma convolutions. submitted

[7] J. Bertoin (1996) *Lévy Processes.* Cambridge University Press, Cambridge.

[8] J. Bertoin, A. Lindner and R. Maller (2008) On continuity properties of the law of integrals of Lévy processes. In: C. Donati-Martin, M. Émery, A. Rouault, C. Stricker (eds.): *Séminaire de Probabilités XLI, Lecture Notes in Mathematics* **1934**, 137–159, Springer, Berlin.

[9] J. Bertoin and M. Yor (2005) Exponential integrals of Lévy processes. *Probab. Surveys* **2**, 191–212.

[10] K. Bichteler (2002) *Stochastic Integration with Jumps.* Cambridge University Press, Cambridge.

[11] T. Bollerslev (1986) Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics* **31**, 307–327.

[12] L. Bondesson (1992) *Generalized Gamma Convolutions and Related Classes of Distributions and Densities.* Lecture Notes in Statistics, Springer, **76**.

[13] P. Bougerol and N. Picard (1992) Strict stationarity of generalized autoregressive processes. *Ann. Probab.* **20**, 1714–1730

[14] R. Doney and R. Maller (2002) Stability and attraction to normality for Lévy processes at zero and at infinity. *J. Theoret. Probab.* **15**, 751–792.

[15] D. Dufresne (1990) The distribution of a perpetuity with application to risk theory and pension funding. *Scand. Actuarial J.* **1-2**, 39–79

[16] A. Einstein (1905) Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen. *Annalen der Physik* **322**, 549–560.

[17] M. Emery (1978) Stabilité des solutions des équations différentielles stochastiques, applications aux intégrales multiplicatives stochastiques. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, **41**, 241–262.

[18] R. F. Engle (1982) Autoregressive conditional heteroskedasticity with estimates of the variance of United Kingdom inflation. *Econometrica* **50**, 987–1008.

[19] K. B. Erickson and R. A. Maller (2005) Generalised Ornstein-Uhlenbeck processes and the convergence of Lévy integrals. In: M. Émery, M. Ledoux, M. Yor (eds.): *Séminaire de Probabilités XXXVIII, Lecture Notes in Mathematics* **1857**, 70–94, Springer, Berlin.

[20] A. Estrade (1992) Exponentielle stochastique et intégrale multiplicative discontinues. *Ann. Inst. Henri Poincaré* **28**, 107–129.

[21] H. K. Gjessing and J. Paulsen (1997) Present value distributions with applications to ruin theory and stochastic equations. *Stoch. Proc. Appl.* **71**, 123–144.

[22] C. M. Goldie (1991) Implicit renewal theory and tails of solutions of random equations. *Ann. Appl. Probab.* **1**, 126–166.

[23] C. M. Goldie and R. Grübel (1996) Perpetuities with thin tails. *Adv. Appl. Probab.* **28**, 463-480.

[24] C. M. Goldie and R. Maller (2000) Stability of perpetuities. *Ann. Probab.* **28**, 1195-1218.

[25] A. K. Grincevičius (1974) On the continuity of the distribution of a sum of dependent variables connected with independent walks on lines. *Theory Probab. Appl.* **19**, 163-168.

[26] A. K. Grincevičius (1980) Products of random affine transformations. *Lithuanian Math. J.* **20** no. 4, 279-282. Translated from *Litosvk. Mat. Sb.* **20** no. 4, 49–53.

[27] L. De Haan and R. L. Karandikar (1989) Embedding a stochastic difference equation into a continuous-time process. *Stoch. Proc. Appl.*, **32**, 225–235.

[28] A. S. Holevo (1990) An analog of the Itô decomposition for multiplicative processes with values in a Lie group. *Quantum probability and applications, V (Heidelberg, 1988), Lecture Notes in Math.* **1442**, 211–215, Springer, Berlin.

[29] J. Jacod (1982) Equations differentielles stochastiques lineares: La methode de variation des constantes. *Séminaire de Probabilités XVI, Lecture Notes in Math.* **920**, 442–446, Springer, Berlin.

[30] J. Jacod (1985) Grossissement de filtration et processus d'Ornstein Uhlenbeck généralisé. In: Th. Jeulin, M. Yor (eds.): *Grossissements de filtrations: exemples et applications, Lecture Notes in Mathematics* **1118**, 36–44, Springer, Berlin.

[31] J. Jacod and A. N. Shiryaev (2003) *Limit Theorems for Stochastic Processes.* Second Edition. Springer, Berlin.

[32] L. F. James, B. Roynette and M. Yor (2008) Generalized gamma convolutions, Dirichlet means, Thorin measures, with explicit examples. *Probab. Surveys* **5**, 346–415.

[33] S. Jaschke (2003) A note on the inhomogeneous linear stochastic differential equation. *Insurance: Mathematics and Economics* **32**, 461–464.

[34] Z. J. Jurek and W. Vervaat (1983) An integral representation for self-decomposable Banach space valued random variables. *Z. Wahrsch. Verw. Gebiete* **62**, 247–262.

[35] O. Kallenberg (2002) *Foundations of Modern Probability.* 2nd edition. Probability and Applications, Springer-Verlag, New York and Berlin.

[36] R. L. Karandikar (1981) A.s. approximation results for multiplicative stochastic integrals. *Séminaire de Probabilités XVI. Lecture Notes in Math.* **920** 384–391, Springer, Berlin

[37] R. L. Karandikar (1991) Multiplicative decomposition of nonsingular matrix valued semimartingales. *Séminaire de Probabilités XXV, Lecture Notes in Math.* **1485**, 262–269, Springer, Berlin.

[38] H. Kesten (1973) Random difference equations and renewal theory for products of random matrices. *Acta. Math.* **131**, 207–248.

[39] C. Klüppelberg, A. Lindner and R. Maller (2004) A continuous time GARCH process driven by a Lévy process: stationarity and second order behaviour. *J. Appl. Probab.* **41**, 601–622.

[40] C. Klüppelberg, A. Lindner and R. Maller (2006) Continuous time volatility modelling: COGARCH versus Ornstein-Uhlenbeck mmodels. In Y. Kabanov, R. Liptser and J. Stoyanov (eds.): *The Shiryaev Festschrift: From Stochastic Calculus to Mathematical Finance*, Springer, Berlin, 393–419.

[41] C. Klüppelberg, R. Maller and A. Szimayer (2010) The COGARCH: a review, with news on option pricing and statistical inference. Available at SSRN: http://ssrn.com/abstract=1538115.

[42] H. Kondo, M. Maejima and K. Sato (2006) Some properties of exponential integrals of Lévy processes and examples. *Elect. Comm. Probab.* **11**, 291–303,

[43] T. J. Kozubowski (2005) A note on self-decomposability of stable process subordinated to self-decomposable subordinator. *Stat. Prob. Lett.* **74**, 89–91.

[44] A. E. Kyprianou (2006) *Introductory Lectures on Fluctuations of Lévy Processes with Applications.* Springer, Berlin.

[45] M. Liao (2004) *Lévy Processes in Lie Groups.* Cambridge University Press, Cambridge.

[46] A. Lindner and R. Maller (2005) Lévy integrals and the stationarity of generalised Ornstein-Uhlenbeck processes. *Stoch. Proc. Appl.* **115**, 1701–1722

[47] A. Lindner and K. Sato (2009) Continuity properties and infinite divisibility of stationary distributions of some generalised Ornstein-Uhlenbeck processes. *Ann. Probab.* **37**, 250–274.

[48] A. Lindner and K. Sato (2010) Properties of stationary distributions of a sequence of generalized Ornstein-Uhlenbeck processes. *Math. Nach.*, to appear.

[49] R. Maller, G. Müller and A. Szimayer (2009) Ornstein–Uhlenbeck processes and extensions. In: T. Andersen, R.A. Davis, J.-P. Kreiß, T. Mikosch (eds.): *Handbook of Financial Time Series*, 421–437. Springer, Berlin.

[50] K. Maulik and B. Zwart (2006) Tail asymptotics for exponential integrals of Lévy processes. *Stoch. Proc. Appl.* **116**, 156–177.

[51] P. A. Meyer (1978) Inégalités de normes pour les intégrales stochastiques. In A. Dold, B. Eckmann (eds.): *Séminaire de Probabilités XII, Lecture Notes in Mathematics 649*, 757–762, Springer, Berlin.

[52] P. M. Neumann, G. A. Stoy and E. C. Thompson (1994) *Groups and Geometry.* Oxford Science Publications, Oxford.

[53] L. Ornstein and G. Uhlenbeck (1930) On the theory of the Brownian motion. *Phys. Rev.* **36**, 823–841.

[54] J. Paulsen (1993) Risk theory in a stochastic economic environment. *Stoch. Proc. Appl.* **46**, 327–361.

[55] P. E. Protter (2005) *Stochastic Integration and Differential Equations.* Second Edition, Version 2.1, Springer, Berlin.

[56] P. E. Protter and K. Shimbo (2008) No arbitrage and general semimartingales. *Markov processes and related topics: a Festschrift for Thomas G. Kurtz*, 267–283, Inst. Math. Stat. Collect., **4**, Inst. Math. Statist., Beachwood, OH, 2008.

[57] V. Rivero (2010) Tail asymptotics for exponential integrals of Lévy processes: the convolution equivalent case. Preprint, arXiv:0905.2401.

[58] K. Sato (1999) *Lévy Processes and Infinitely Divisible Distributions.* Cambridge University Press, Cambridge.

[59] A. V. Skorokhod (1982) Operator stochastic differential equations and stochastic semigroups. *Uspekhi Mat. Nauk*, **37:6**, 157–183.

[60] A. V. Skorokhod (1991) *Random Processes with Independent Increments.* Kluwer Academic Publishers Group, Dordrecht.

[61] F. W. Steutel and K. Van Harn (2003) *Infinite Divisibility of Probability Distributions on the Real Line.* Marcel Dekker Inc.

[62] W. Vervaat (1979) On a stochastic difference equation and a representation of nonnegative infinitely divisible random variables. *Adv. in Appl. Probab.* **11**, 750–783.

[63] T. Watanabe (2009) Private communication.

[64] S. J. Wolfe (1983): Continuity properties of decomposable probability measure on Euclidean spaces. *J. Multivariate Anal.* **13**, 534–538.

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