Finite Variation of Fractional Lévy Processes

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Abstract

Various characterizations for fractional Lévy process to be of finite variation are obtained, one of which is in terms of the characteristic tiplet of the driving Lévy process, while others are in terms of differentiability properties of the sample paths. A zero-one law and a formula for the expected total variation is also given.

1 Introduction

Recently there has been increased interest in fractional Lévy processes, which are generalizations of fractional Brownian motion. Benassi et al. [2, 3] and Marquardt [7] introduced real harmonizable fractional Lévy processes, well-balanced (moving average) fractional Lévy processes N_d and non-anticipative (moving average) fractional Lévy processes M_d. Apart from a normalizing constant, these arise by replacing the Brownian motion in the corresponding representation of fractional Brownian motion by a centered square-integrable Lévy process, and the precise definitions of M_d and N_d are given below in (1.1) and (1.3), respectively. Note that although the different representations all give fractional Brownian motion if the driving process is Brownian motion, in general the corresponding definitions lead to different processes for arbitrary driving Lévy processes. However, all the mentioned processes have the same second order structure as fractional Brownian motion. Other properties, such as self-similarity, are not necessarily shared with fractional Brownian motion (cf. [3, 7]), and in this paper we shall concentrate on the semimartingale property and on finite variation of the paths. While it is well known that a fractional Brownian motion with Hurst parameter $H \in (0, 1)$ cannot be a semimartingale unless $H = 1/2$ (e.g. Mishura [8], p. 71), in particular cannot be of finite variation on compacts almost surely for any $H \in (0, 1)$, this is not the case for M_d and N_d. Marquardt [7], Theorems 4.6, 4.7, has shown that M_d will be of finite variation

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if the driving Lévy process is a compensated compound Poisson process, and has given examples when M_d is not a semimartingale. The aim of this paper is to provide a complete characterization for the non-anticipative fractional Lévy process M_d and for the well-balanced fractional Lévy process N_d to be of finite variation on compacts, equivalently for them to be semimartingales, in the long memory case, i.e. when $H := d + 1/2 \in (1/2, 1)$. This subject and the obtained results are closely related to a recent paper by Basse and Pedersen [1], who characterised the semimartingale property of general one-sided Lévy driven moving average processes and applied their results to obtain a necessary condition for the non-anticipative fractional Lévy process M_d to be of finite variation (see Remark 2.3 below). This condition is expressed in terms of the absence of a Brownian motion component and an integrability condition on the Lévy measure at zero. We shall show that the condition obtained by Basse and Pedersen [1] is also sufficient and give a totally different proof for the necessity assertion, which is based on the stationary increments property of fractional Lévy processes. We also obtain further characterisations based on differentiability properties of M_d, show that the total variation is finite if and only if its expectation is finite, and obtain a zero-one law for the property of being of finite variation. Note that when M_d is a semimartingale it may be used as a driving process for various stochastic differential equations, and hence allows to incorporate the long memory property into various classes of processes.

To set notation, fix a complete probability space $(\Omega, \mathcal{F}, P)$ on which a real valued, two sided Lévy process $L = (L(t))_{t\in\mathbb{R}}$ is defined, i.e. a process with independent and stationary increments having càdlàg paths satisfying $L_0 = 0$. We shall further assume throughout that $L(1)$ has finite variance and mean zero. Recall that a two sided Lévy processes L indexed by $\mathbb{R}$ can be easily constructed from a one-sided Lévy processes L_1, indexed by $[0,\infty)$, by letting $L(t) = L_1(t)$ for $t \geq 0$ and $L(t) = -L_2(-t-)$ for $t < 0$, where L_2 is an independent copy of L_1. We shall use the Lévy–Khintchine representation of L in the form

$$\mathbb{E}e^{izL(1)} = \exp\left\{-\frac{1}{2}z^2\sigma + iz\gamma + \int_{\mathbb{R}} \left[e^{izx} - 1 - izx\mathbb{1}_{[-1,1]}(x)(1-|x|)\right]\,\nu(\mathrm{d}x)\right\}$$

for $z \in \mathbb{R}$, with $\gamma \in \mathbb{R}$, $\sigma \geq 0$ and ν being the Lévy measure of L, and refer to (σ, ν, γ) as the characteristic triplet of L. See Sato [10] for further information regarding Lévy processes.

Let $d \in (0, 1/2)$ which corresponds to a Hurst index $H := d + 1/2 \in (1/2, 1)$ and hence to the long memory situation, to which we limit ourselves in this paper. As in Marquardt [7], define the *non-anticipative fractional Lévy process* $M_d = (M_d(t))_{t\in\mathbb{R}}$ by

$$M_d(t) \;:=\; \frac{1}{\Gamma(d+1)}\int_{\mathbb{R}} \left[(t-s)_+^d - (-s)_+^d\right] L(\mathrm{d}s), \quad t \in \mathbb{R}, \tag{1.1}$$

where for $\alpha \in \mathbb{R}$ we put $x_+^\alpha := 0$ for $x \le 0$ and $x_+^\alpha := x^\alpha$ for $x > 0$. The integral in (1.1) converges in the L^2 sense (e.g. [7], Proposition 2.1). As shown in [7], Theorem 3.4, M_d admits a modification with continuous sample paths, which is given by the following improper Riemann integral representation

$$M_d(t) = \frac{1}{\Gamma(d)} \int_{\mathbb{R}} \left[(t-s)_+^{d-1} - (-s)_+^{d-1} \right] L(s)\,\mathrm{d}s, \quad t \in \mathbb{R}. \tag{1.2}$$

We shall always assume that we are working with the continuous modification given by (1.2).

Following Taqqu and Samorodnitsky [9] the process defined in (1.1) is called non-anticipative, since for positive t the value $M_d(t)$ depends on $\{L(s); s \le t\}$ only. This is in contrast to the *well-balanced fractional Lévy process* defined by

$$N_d(t) := \frac{1}{\Gamma(d+1)} \int_{\mathbb{R}} \left(|t-s|^d - |s|^d \right) \, L(\mathrm{d}s), \quad t \in \mathbb{R}, \tag{1.3}$$

where $N_d(t)$ depends also on the future behaviour of L. It is easy to see that both M_d and N_d have stationary increments, which will be a crucial tool in the proofs presented.

The structure of the paper is as follows: Section 2 contains the main results for the non-anticipative fractional Lévy process M_d, giving various characterisations for it to be of finite variation, one of which is the already mentioned condition in terms of the characteristic triplet of L, while some others are given in terms of differentiability properties of M_d. A formula for the derivative of M_d at zero and for the expected total variation is further obtained. The results of Section 2 are proved in Section 3. Section 4 is concerned with the well-balanced case N_d and shows that the same characterisations hold for N_d. Finally, in Section 5, the connection between fractional Lévy processes and the fractional Riemann-Liouville integral of a Lévy process is investigated, which establishes a direct link between the results obtained in this article and those of Basse and Pedersen [1] for this situation.

2 Main results

The results in this section completely characterise when fractional Lévy processes are semimartingales. One characterisation is given by integrability conditions on the Lévy measure of the driving Lévy process, a second by finiteness of the expected total variation, others are provided via various differentiability conditions on the sample paths of M_d. The characterisation also contains a 0-1 law for the finite variation property of the sample paths.

Theorem 2.1. *Let L be a Lévy process with finite variance and $\mathbb{E}L(1) = 0$ and characteristic triplet (σ, ν, γ), let $d \in (0, 1/2)$ and $[a, b]$ be a non-empty non-degenerate*

compact interval in $\mathbb{R}$. The following are equivalent:
(a) M_d is almost surely of finite variation on $[a, b]$.
(a') M_d is of finite variation on $[a, b]$ with positive probability.
(b) With probability one the sample paths of M_d are Lebesgue almost everywhere differentiable on $[a, b]$.
(b') With positive probability the sample paths of M_d are Lebesgue almost everywhere differentiable on $[a, b]$.
(c) M_d is almost surely differentiable in 0.
(c') M_d is differentiable in 0 with positive probability.
(d) M_d is almost surely differentiable from the left in 0.
(e) The Brownian motion part of L is zero (i.e. $\sigma = 0$) and

$$\int_{-1}^{1} |x|^{\frac{1}{1-d}} \nu(\mathrm{d}x) < \infty.$$

(f) The expected total variation of M_d on compacts is finite, i.e. $\mathbb{E}\left[\mathrm{TV}\left(M_d|_{[a,b]}\right)\right] < \infty$.
(g) M_d is a semimartingale on $[0, \infty]$ with respect to any filtration (satisfying the usual hypothesis) it is adapted to.
(h) M_d is a semimartingale on $[0, \infty]$ with respect to some filtration (satisfying the usual hypothesis) it is adapted to.

Remark 2.2. By Théorème III in Bretagnolle [4] property (e) in the above theorem is equivalent to the driving Lévy process L being of finite $1/(1-d)$-variation.

Remark 2.3. Basse and Pedersen [1] showed that property (a) implies (e) in the above Theorem. This is stated explicitly in Corollary 5.4 of [1] when σ is assumed a priori to be zero, but that also $\sigma = 0$ is necessary can be seen immediately from Corollary 3.3 and Lemma 5.2 in [1] along with a symmetrization argument. Observe that in Theorem 2.1 above, (e) is deduced from the weaker conditions (b), (c), or (d), and that (e) is shown to imply (a) and even the stronger condition (f).

For the derivative of M_d at zero and for the expected total variation, we have the following explicit representation:

Theorem 2.4. *If the assumptions as well as one of the equivalent conditions of Theorem 2.1 hold, then the derivative of M_d at zero can be expressed by*

$$\frac{\mathrm{d}}{\mathrm{d}t} M_d(0) = \frac{1}{\Gamma(d)} \int_{-\infty}^{0} (-s)^{d-1} L(\mathrm{d}s) = \frac{1}{\Gamma(d-1)} \int_{-\infty}^{0} (-s)^{d-2} L(s)\, \mathrm{d}s, \tag{2.1}$$

where the integrals exist as improper integrals at zero in the sense of almost sure and $L^1(P)$ convergence. Moreover, for every $a < b \in \mathbb{R}$,

$$\mathbb{E}\left[\mathrm{TV}\left(M_d|_{[a,b]}\right)\right] = \frac{(b-a)}{\Gamma(d)} \mathbb{E}\left[\left|\int_{-\infty}^{0} (-s)^{d-1} L(\mathrm{d}s)\right|\right].$$

3 The proofs

We first provide the proof of Theorem 2.1 by showing the following implications

$$(a) \Rightarrow (a') \Rightarrow (b') \Rightarrow (c') \Rightarrow (c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (f) \Rightarrow (g) \Rightarrow (h) \Rightarrow (a)$$

and

$$(a) \Rightarrow (b) \Rightarrow (b').$$

Note that

$$(a) \Rightarrow (a') \Rightarrow (b'),\ (c) \Rightarrow (d),\ (f) \Rightarrow (g) \Rightarrow (h),\ (a) \Rightarrow (b) \Rightarrow (b')$$

are obvious and $(h) \Rightarrow (a)$ is a consequence of the zero quadratic variation property and the continuous paths of M_d, see Theorem 4.7 in Marquardt [7] for details.

Proof of Theorem 2.1, (b') $\Longrightarrow$ (c'). By the stationary increments property, we can assume without loss of generality that $[a,b] = [-1,1]$. Denote

$$B := \{(t_0,\omega) \in (-1,1) \times \Omega : \frac{\mathrm{d}}{\mathrm{d}t} M_d(t_0,\omega) \text{ exists}\}.$$

Note that the sample paths of M_d are continuous. Hence, $M_d : \mathbb{R} \times \Omega \to \mathbb{R}$ is $(\mathcal{B} \otimes \mathcal{F}) - \mathcal{B}$-measurable, and so is $h^{-1}(M_d(t+h) - M_d(t))$ for every $h \neq 0$, where $\mathcal{B}$ denotes the Borel σ-algebra in $\mathbb{R}$. Thus, thanks to the continuous paths of M_d, we observe that

$$\begin{aligned} B \;=\; & \Bigg\{(t,\omega) \in (-1,1) \times \Omega :\ -\infty < \liminf_{h \in \mathbb{Q},\, h \to 0} \frac{M_d(t+h) - M_d(t)}{h} \\ & \qquad = \limsup_{h \in \mathbb{Q},\, h \to 0} \frac{M_d(t+h) - M_d(t)}{h} < \infty \Bigg\} \in \mathcal{B} \otimes \mathcal{F}. \end{aligned}$$

Now, define the cuts

$$B_{t_0} := \{\omega \in \Omega : (t_0,\omega) \in B\}, \quad B^{\omega} := \{t_0 \in (-1,1) : (t_0,\omega) \in B\},$$

which are $\mathcal{F}$- and $\mathcal{B}$-measurable, respectively. By assumption we have $P(\{\omega \in \Omega : \lambda_1(B^{\omega}) = 1\}) > 0$, where λ_1 denotes the Lebesgue measure on $[-1,1]$. Further, by stationarity, $P(B_t) = P(B_0)$ for all $t \in (-1,1)$, and consequently Fubini's theorem gives

$$2P(B_0) = \int_{-1}^{1} P(B_t)\, \lambda_1(\mathrm{d}t) = (\lambda_1 \otimes P)\ (B) = \int_{\Omega} \lambda_1(B^{\omega})\, dP > 0.$$

□

Proof of Theorem 2.1, (c') $\Longrightarrow$ (c). With the definition above we have

$$B_0 = \{\omega \in \Omega : \frac{\mathrm{d}}{\mathrm{d}t} M_d(0,\omega) \text{ exists}\}.$$

For fixed $r < 0$ and $t \in (r, |r|)$, (1.2) yields

$$\Gamma(d) M_d(t) = \int_{-\infty}^{r} \left[(t-s)^{d-1} - (-s)^{d-1}\right] L(s)\,\mathrm{d}s + \int_{r}^{0\vee t} \left[(t-s)_+^{d-1} - (-s)_+^{d-1}\right] L(s)\,\mathrm{d}s.$$

Note that $M_d(0) = 0$ and

$$\limsup_{s\to-\infty} \frac{|L(s)|}{(2|s|\log\log|s|)^{1/2}} < \infty$$

(cf. Sato [10], Proposition 48.9). Moreover, L is pathwise bounded on compacts. Therefore Lebesgue's dominated convergence theorem ensures the existence of

$$\lim_{t\to 0} t^{-1} \int_{-\infty}^{r} \left[(t-s)^{d-1} - (-s)^{d-1}\right] L(s)\,\mathrm{d}s = (d-1)\int_{-\infty}^{r} (-s)^{d-2} L(s)\,\mathrm{d}s.$$

It follows that for each fixed $r < 0$, the set B_0 is measurable with respect to $(L(s))_{r\le s\le |r|}$. Letting $r \uparrow 0$, we conclude from the Blumenthal zero-one law, e.g. as given in Proposition 40.4 in Sato [10], that $P(B_0) \in \{0,1\}$. Since $P(B_0) > 0$ by assumption, we have $P(B_0) = 1$. $\square$

Proof of Theorem 2.1, (d) $\Longrightarrow$ (e). (i) Suppose first that L is symmetric. Denote by D the left derivative of $\Gamma(d)M_d$ in 0. Then D is infinitely divisible as a limit of infinitely divisible distributions. Since D is also symmetric, its characteristic triplet with respect to the continuous cut-off function $\beta(x) = (1-|x|)\mathbf{1}_{[-1,1]}(x)$ is given by $(A_D, \nu_D, 0)$ with Gaussian variance A_D and Lévy measure ν_D, i.e. we have

$$\mathbb{E}e^{iDz} = \exp\left[-\frac{1}{2}A_D z^2 + \int_{\mathbb{R}} \left(e^{izx} - 1 - izx\beta(x)\right)\,\nu_D(\mathrm{d}x)\right], \quad z\in\mathbb{R}.$$

For any $r < 0$ we have (as shown in the proof of "(c') $\Longrightarrow$ (c)") that

$$D = (d-1)\int_{-\infty}^{r} (-s)^{d-2} L(s)\,\mathrm{d}s + \lim_{t\uparrow 0} t^{-1} \int_{r}^{0} \left[(t-s)_+^{d-1} - (-s)_+^{d-1}\right] L(s)\,\mathrm{d}s.$$

Integrating by parts, we obtain

$$\begin{aligned}
(d-1)\int_{-\infty}^{r} (-s)^{d-2} L(s)\,\mathrm{d}s &= \int_{-\infty}^{r} (-s)^{d-1} L(\mathrm{d}s) - |r|^{d-1} L(r), \\
t^{-1}\int_{r}^{0} \left[(t-s)_+^{d-1} - (-s)_+^{d-1}\right] L(s)\,\mathrm{d}s &= (td)^{-1}\int_{r}^{0} \left[(t-s)_+^{d} - (-s)_+^{d}\right] L(\mathrm{d}s) \\
&\quad + L(r)\frac{(|r|+t)^d - |r|^d}{td}.
\end{aligned}$$

Hence, for every $r < 0$ we have,

$$D = Y_r + Z_r$$

where

$$\begin{aligned}
Y_r &:= \int_{-\infty}^{r} (-s)^{d-1}\, L(\mathrm{d}s), \\
Z_r &:= \lim_{t\uparrow 0} (td)^{-1} \int_{r}^{0} \left[(t-s)_+^{d} - (-s)_+^{d}\right] L(\mathrm{d}s).
\end{aligned}$$

Then Y_r and Z_r are independent for each r, and also infinitely divisible. Denote the characteristic triplets of Y_r and Z_r with respect to β by $(A_r, \nu_r, 0)$ and $(A_{Z_r}, \nu_{Z_r}, 0)$, respectively, and observe that

$$A_r + A_{Z_r} = A_D \text{ and } \nu_r + \nu_{Z_r} = \nu_D$$

by independence. Observe further that A_r is a monotone increasing sequence of real numbers bounded by A_D and $(\nu_r)_{r<0}$ is an increasing sequence of Lévy measures as $r \uparrow 0$, bounded by ν_D. Denote

$$\nu_0(\Lambda) = \lim_{r \uparrow 0} \nu_r(\Lambda)$$

for each Borel set Λ. Then ν_0 is a measure (e.g. Kallenberg [6], Corollary 1.16), and it is a Lévy measure, since it is bounded by ν_D. Further,

$$\int_{[-\varepsilon,\varepsilon]} x^2 \nu_r(\mathrm{d}x) \leq \int_{[-\varepsilon,\varepsilon]} x^2 \, \nu_D(\mathrm{d}x), \quad \varepsilon > 0, \quad r < 0,$$

so that

$$\lim_{\varepsilon \downarrow 0} \limsup_{r \uparrow 0} \int_{[-\varepsilon,\varepsilon]} x^2 \, \nu_r(\mathrm{d}x) = 0.$$

We conclude that Y_r converges in distribution to an infinitely divisible random variable, Y_0 say, with characteristic triplet $(A_0, \nu_0, 0)$ with respect to β, cf. Sato [10], Theorem 8.7. By the independent increments property of $r \mapsto Y_r$, this convergence is even almost surely (e.g. Kallenberg [6], Theorem 4.18). Hence,

$$Y_{-1,0} := \lim_{r \uparrow 0} \int_{-1}^{r} (-s)^{d-1} \, L(\mathrm{d}s)$$

exists as an almost sure limit. We now apply Proposition 5.3 parts (ii) and (i) of Sato [11] to $f(s) = (-s)^{d-1}$ and $\alpha = 1 - \frac{d}{1-d} \in (0,1)$ to find that the existence of $\lim_{r \uparrow 0} Y_{-1,r}$ in probability is equivalent to L being purely non-Gaussian with a Lévy measure ν fulfilling $\int_{[-1,1]} |x|^{1/(1-d)} \, \nu(\mathrm{d}x) < \infty$.

(ii) In the general case denote by $\widetilde{L}$ an independent copy of L, and write $L^* = L - \widetilde{L}$ for the symmetrization of L. Suppose $M_d L$ (the fractional Lévy process driven by L) is almost surely left-differentiable at 0. Then $M_d \widetilde{L}$ and $M_d L^*$ are almost surely left-differentiable at 0 as well. Further, L^* has a Gaussian part if and only if L has, and the Lévy measure ν^* of L^* is given by $\nu^*(\Lambda) = \nu(\Lambda) + \nu(-\Lambda)$ for every Borel set Λ. It follows from (i) that $\int_{-1}^{1} |x|^{1/(1-d)} \nu^*(\mathrm{d}x) < \infty$, which is clearly equivalent to $\int_{-1}^{1} |x|^{1/(1-d)} \, \nu(\mathrm{d}x) < \infty$. □

We now prepare the proof of the remaining implication "(e)⇒(f)" of Theorem 2.1 and, at the same time, the proof of Theorem 2.4 by the following lemmas:

Lemma 3.1. *A symmetric and infinitely divisible random variable X without Gaussian part and with Lévy measure ν fulfills the following inequality for $\varepsilon > 0$:*

$$\mathbb{E}|X| \quad \leq \quad \varepsilon + \frac{4}{\varepsilon}\int_0^\varepsilon x\nu([x,\infty))\mathrm{d}x + 2\int_\varepsilon^\infty \nu\left([x,\infty)\right)\,\mathrm{d}x.$$

Proof. Writing $X = X^{(1)} + X^{(2)}$, where $X^{(1)}$ has characteristic triplet $(0, \nu\big|_{(-\varepsilon,\varepsilon)}, 0)$ and $X^{(2)}$ has characteristic triplet $(0, \nu\big|_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)}, 0)$ we find for all $\varepsilon > 0$ that $\mathbb{E}|X| \leq \mathbb{E}|X^{(1)}| + \mathbb{E}|X^{(2)}|$. By Chebyshev's inequality, symmetry, and integration by parts we get

$$\begin{aligned}\mathbb{E}|X^{(1)}| \quad &= \quad \int_0^\infty P(|X^{(1)}| > t)\,\mathrm{d}t \leq \varepsilon + \int_\varepsilon^\infty \frac{\mathrm{Var}\,(X^{(1)})}{t^2}\,\mathrm{d}t \\ &= \quad \varepsilon + \frac{2}{\varepsilon}\int_{(0,\varepsilon)} x^2\,\nu(\mathrm{d}x) = \varepsilon + \frac{4}{\varepsilon}\int_0^\varepsilon x\,\nu([x,\infty))\mathrm{d}x - 2\varepsilon\nu([\epsilon,\infty)). \quad (3.1)\end{aligned}$$

Moreover, denoting by $X^{(3)}$ the compound Poisson distribution with Lévy measure $\nu\big|_{[\varepsilon,\infty)}$, we clearly have

$$\mathbb{E}|X^{(2)}| \leq 2\mathbb{E}\,X^{(3)} = 2\int_{[\varepsilon,\infty)} x\,\nu(\mathrm{d}x) = 2\varepsilon\nu([\varepsilon,\infty)) + 2\int_{(\varepsilon,\infty)} \nu([x,\infty))\,\mathrm{d}x. \quad (3.2)$$

Merging equations (3.1) and (3.2) gives the assertion. □

Lemma 3.2. *Suppose condition (e) of Theorem 2.1 holds true. Then*

$$\lim_{t\downarrow 0}\mathbb{E}\left(\left|\frac{1}{t}\int_0^t (t-s)^d\,L(\mathrm{d}s)\right|\right) = 0.$$

Proof. (i) Let L be symmetric and denote by ν_t the Lévy measure of $\frac{1}{t}\int_0^t(t-s)^d\,L(\mathrm{d}s)$. Then for every $\varepsilon > 0$ we obtain from Lemma 3.1 that

$$\mathbb{E}\left(\left|\frac{1}{t}\int_0^t (t-s)^d\,L(\mathrm{d}s)\right|\right) \quad \leq \quad \varepsilon + \frac{4}{\varepsilon}\int_0^\varepsilon u\nu_t([u,\infty))\,\mathrm{d}u + 2\int_\varepsilon^\infty \nu_t([u,\infty))\,\mathrm{d}u$$

Hence it is sufficient to show that, for every $\varepsilon > 0$,

$$\text{a)}\ \lim_{t\downarrow 0}\int_0^\varepsilon u\nu_t([u,\infty))\,\mathrm{d}u = 0, \quad \text{and} \quad \text{b)}\ \lim_{t\downarrow 0}\int_\varepsilon^\infty \nu_t([u,\infty))\,\mathrm{d}u = 0.$$

To this end note that, for $u > 0$,

$$\begin{aligned}\nu_t([u,\infty)) \quad &= \quad \int_0^t\int_{\mathbb{R}\setminus\{0\}} \mathbb{1}_{[u,\infty)}\left(t^{-1}(t-s)^d x\right)\nu(\mathrm{d}x)\mathrm{d}s \\ &\leq \quad \int_0^t\int_{\{x>0\}} \mathbb{1}_{[u,\infty)}\left(t^{d-1}x\right)\nu(\mathrm{d}x)\mathrm{d}s = t\nu([ut^{1-d},\infty)),\end{aligned}$$

where the first identity is due to Theorem 3.10 in [11]. Hence,

$$0 \leq \nu_t([u,\infty)) \leq t\nu([ut^{1-d},\infty)) = \int_{ut^{1-d}}^\infty t\cdot x^{\frac{1}{1-d}}\cdot x^{\frac{-1}{1-d}}\,\nu(\mathrm{d}x) \longrightarrow 0 \quad (3.3)$$

by dominated convergence as $t \downarrow 0$, since

$$\mathbb{1}_{\{x \geq ut^{1-d}\}} \, t \cdot x^{\frac{1}{1-d}} \cdot x^{\frac{-1}{1-d}} \leq x^{\frac{1}{1-d}} u^{\frac{-1}{1-d}}.$$

By (3.3) the integrand in a) converges to zero. So a) can be obtained by dominated convergence, because

$$u\nu_t([u,\infty)) \leq u^{1-\frac{1}{1-d}} \int_0^\infty x^{\frac{1}{1-d}} \, \nu(\mathrm{d}x), \quad u > 0,$$

and $u^{1-\frac{1}{1-d}}$ is integrable at zero since $d < \frac{1}{2}$.
To get b) note that

$$\int_\varepsilon^\infty \nu_t([u,\infty)) \, \mathrm{d}u \leq t \int_\varepsilon^\infty \nu\left([ut^{1-d},\infty)\right) \, \mathrm{d}u = \frac{1}{t^{-d}} \int_{\varepsilon t^{1-d}}^\infty \nu([u,\infty)) \, \mathrm{d}u \longrightarrow 0$$

as $t \downarrow 0$ by L'Hopital's rule, since

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\varepsilon t^{1-d}}^\infty \nu([u,\infty)) \, \mathrm{d}u = \varepsilon(d-1)t^{-d}\nu([\varepsilon t^{1-d},\infty)),$$

and hence, by (3.3),

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} \int_{\varepsilon t^{1-d}}^\infty \nu([u,\infty)) \, \mathrm{d}u\right) \Big/ \left(\frac{\mathrm{d}}{\mathrm{d}t} t^{-d}\right) = \varepsilon \frac{1-d}{d} t\nu([\varepsilon t^{1-d},\infty)) \to 0.$$

(ii) If L is not symmetric, then denote by $\tilde{L}$ an independent copy of L and by $L^* = L - \tilde{L}$ the symmetrization of L. Further define

$$C_t = \frac{1}{t} \int_0^t (t-s)^d \, L(\mathrm{d}s)$$

and by $\tilde{C}_t$, C_t^* analogous expressions for $\tilde{L}$ and L^*, respectively. As it holds

$$\mathbb{E}\left[\tilde{C}_t \middle| C_t\right] = \mathbb{E}\left[\tilde{C}_t\right] = 0$$

by independence, we get

$$\mathbb{E}\left[|C_t|\right] \;=\; \mathbb{E}\left[\left|C_t - \mathbb{E}\left[\tilde{C}_t \middle| C_t\right]\right|\right] = \mathbb{E}\left[|\mathbb{E}\left[C_t^* | C_t\right]|\right] \leq \mathbb{E}\left[|C_t^*|\right],$$

and so the first case applies. □

Lemma 3.3. *Suppose condition (e) in Theorem 2.1 holds true. For $r < 0 < t$ let $\nu_{r,t}$ denote the Lévy measure of*

$$B_{r,t} := \frac{1}{t} \int_r^0 \left[(t-s)^d - (-s)^d\right] \, L(\mathrm{d}s).$$

Then for any $a > 0$ it holds that

$$\int_0^a u \cdot \nu_{r,t}([u,\infty)) \, \mathrm{d}u \leq \frac{a^2(1-d)}{1-2d} \left[|r|\nu\left(\left[\frac{a}{d}|r|^{1-d},\infty\right)\right) + \left(\frac{d}{a}\right)^{\frac{1}{1-d}} \int_0^{\frac{a}{d}|r|^{1-d}} x^{\frac{1}{1-d}} \, \nu(\mathrm{d}x)\right] \tag{3.4}$$

and

$$\int_a^\infty \nu_{r,t}\left([u,\infty)\right)\,\mathrm{d}u \le |r|^d \int_{\frac{a}{d}|r|^{1-d}}^\infty x\,\nu(\mathrm{d}x) + (1-d)\left(\frac{d}{a}\right)^{\frac{d}{1-d}} \int_0^{\frac{a}{d}|r|^{1-d}} x^{\frac{1}{1-d}}\,\nu(\mathrm{d}x). \tag{3.5}$$

Proof. From Theorem 3.10 in [11], it follows that for any $u > 0$,

$$\begin{aligned}
\nu_{r,t}\left([u,\infty)\right) &= \int_0^{|r|}\int_0^\infty \mathbb{1}_{[u,\infty)}\left(\frac{1}{t}\left[(t+s)^d - s^d\right]x\right)\nu(\mathrm{d}x)\,\mathrm{d}s \\
&\le \int_0^{|r|}\int_0^\infty \mathbb{1}_{[u,\infty)}(xs^{d-1}d)\,\nu(\mathrm{d}x)\,\mathrm{d}s \\
&\le \int_0^{|r|} \nu\left(\left[\frac{u}{d}s^{1-d},\infty\right)\right)\,\mathrm{d}s \qquad (3.6)\\
&= \frac{1}{1-d}\cdot\left(\frac{d}{u}\right)^{\frac{1}{1-d}} \int_0^{\frac{u}{d}|r|^{1-d}} x^{\frac{d}{1-d}}\nu\left([x,\infty)\right)\,\mathrm{d}x. \qquad (3.7)
\end{aligned}$$

Hence, we obtain for any $a > 0$ that

$$\begin{aligned}
\int_0^a u\cdot\nu_{r,t}\left([u,\infty)\right)\,\mathrm{d}u &\le \frac{1}{1-d}d^{\frac{1}{1-d}}\int_0^a\int_0^{\frac{u}{d}|r|^{1-d}} u^{\frac{d}{d-1}}x^{\frac{d}{1-d}}\nu\left([x,\infty)\right)\,\mathrm{d}x\,\mathrm{d}u \\
&= \frac{1}{1-d}d^{\frac{1}{1-d}}\int_0^{\frac{a}{d}|r|^{1-d}}\int_{x|r|^{d-1}d}^a u^{\frac{d}{d-1}}x^{\frac{d}{1-d}}\nu\left([x,\infty)\right)\,\mathrm{d}u\,\mathrm{d}x \\
&\le \frac{1}{1-2d}d^{\frac{1}{1-d}}a^{1+\frac{d}{d-1}}\int_0^{\frac{a}{d}|r|^{1-d}} x^{\frac{d}{1-d}}\nu\left([x,\infty)\right)\,\mathrm{d}x,
\end{aligned}$$

from which (3.4) follows using integration by parts, observing that the boundary term vanishes at 0 by (3.3). Moreover, by (3.7), we have

$$\begin{aligned}
&\int_a^\infty \nu_{r,t}\left([u,\infty)\right)\,\mathrm{d}u \\
&\le \frac{d^{\frac{1}{1-d}}}{1-d}\int_a^\infty u^{-\frac{1}{1-d}}\int_0^{\frac{u}{d}|r|^{1-d}} x^{\frac{d}{1-d}}\nu\left([x,\infty)\right)\,\mathrm{d}x\,\mathrm{d}u \\
&= \frac{d^{\frac{1}{1-d}}}{1-d}\int_0^\infty\int_{\max\{a,xd|r|^{d-1}\}}^\infty u^{\frac{1}{d-1}}\,\mathrm{d}u\,\nu\left([x,\infty)\right)x^{\frac{d}{1-d}}\,\mathrm{d}x \\
&= d^{\frac{d}{1-d}}\int_0^\infty \max\{a,xd|r|^{d-1}\}^{1-\frac{1}{1-d}}\,\nu\left([x,\infty)\right)x^{\frac{d}{1-d}}\,\mathrm{d}x \\
&= \left(\frac{d}{a}\right)^{\frac{d}{1-d}}\int_0^{\frac{a}{d}|r|^{1-d}} x^{\frac{d}{1-d}}\nu\left([x,\infty)\right)\,\mathrm{d}x + |r|^d\int_{\frac{a}{d}|r|^{1-d}}^\infty \nu\left([x,\infty)\right)\,\mathrm{d}x \\
&= (1-d)\left(\frac{d}{a}\right)^{\frac{d}{1-d}}\int_0^{\frac{a}{d}|r|^{1-d}} x^{\frac{1}{1-d}}\,\nu(\mathrm{d}x) + |r|^d\int_{\frac{a}{d}|r|^{1-d}}^\infty x\,\nu(\mathrm{d}x) \\
&\quad -a|r|\nu\left(\left[\frac{a}{d}|r|^{1-d},\infty\right)\right),
\end{aligned}$$

giving (3.5). □

Lemma 3.4. *Suppose condition (e) of Theorem 2.1 holds true.*
(a) Then $B_{r,t}$, as defined in Lemma 3.3, satisfies

$$\lim_{r\uparrow 0}\sup_{t>0} E\left[|B_{r,t}|\right] = 0.$$

(b) Moreover, the improper integral

$$\int_{-\infty}^{0} (-s)^{d-1}\, L(\mathrm{d}s) := \lim_{r\uparrow 0}\int_{-\infty}^{r} (-s)^{d-1}\, L(\mathrm{d}s)$$

exists as an L^1-limit and as an almost sure limit.

Proof. (a) Note that, analogously to part (ii) in the proof of Lemma 3.2, we may and do assume symmetry of L. For fixed $\varepsilon > 0$ we conclude from Lemmas 3.1 and 3.3 that there is a constant $c_{d,\varepsilon}$ depending on d and ε only, such that

$$\begin{aligned}\sup_{t>0} E[|B_{r,t}|] \;\; \leq \;\; & \frac{\varepsilon}{2} + c_{d,\varepsilon}\left(|r|\nu\left([\varepsilon|r|^{1-d}/(2d),\infty)\right) + \int_0^{|r|^{1-d}\varepsilon/(2d)} x^{\frac{1}{1-d}}\nu(\mathrm{d}x) \right.\\ & \left. +|r|^d \int_{|r|^{1-d}\varepsilon/(2d)}^{\infty} x\nu(\mathrm{d}x)\right).\end{aligned}$$

The right-hand side is bounded by ε for $r < 0$ sufficiently close to 0, because

$$\lim_{r\uparrow 0}\int_0^{|r|^{1-d}\varepsilon/(2d)} x^{1/(1-d)}\,\nu(\mathrm{d}x) = 0$$

$$\lim_{r\uparrow 0}|r|^d\int_{|r|^{1-d}\varepsilon/(2d)} x\nu(\mathrm{d}x) = 0 \tag{3.8}$$

$$\lim_{r\uparrow 0}|r|\nu([|r|^{1-d},\infty)) = 0 \tag{3.9}$$

by the assumption $\int_0^1 x^{\frac{1}{1-d}}\,\nu(\mathrm{d}x) < \infty$ and dominated convergence. For (3.8) the use of dominated convergence is justified by

$$\begin{aligned}\mathbb{1}_{\{|r|^{1-d}\varepsilon/(2d)\leq x\leq 1\}}|r|^d x \;\; & \leq \;\; \mathbb{1}_{\{|r|^{1-d}\varepsilon/(2d)\leq x\leq 1\}}|x|^{\frac{1}{1-d}}(\varepsilon/(2d))^{-d/(1-d)}\\ & \leq \;\; \mathbb{1}_{\{0<x\leq 1\}}|x|^{\frac{1}{1-d}}(\varepsilon/(2d))^{-\frac{d}{1-d}},\end{aligned}$$

Eq. (3.9) is already shown in (3.3).
(b) The almost sure convergence follows from the L^1-convergence by the independent increments property of $\int_{-\infty}^{r}(-s)^{d-1}\,L(\mathrm{d}s)$, $r < 0$, (see Theorem 4.18 in [6]). For proving the L^1-convergence, as above we may and do assume that L is symmetric. Denote by $\tilde{\nu}_{q,r}$ the Lévy measure of

$$d\int_r^q (-s)^{d-1}\,L(\mathrm{d}s), \qquad -\infty < r < q < 0.$$

Then, for $u > 0$,

$$\tilde{\nu}_{q,r}\left([u,\infty)\right) = \int_{|q|}^{|r|}\int_0^\infty \mathbb{1}_{[u,\infty)}(dxs^{d-1})\,\nu(\mathrm{d}x)\,\mathrm{d}s \leq \int_0^{|r|} \nu\left([\frac{u}{d}s^{1-d},\infty)\right)\,\mathrm{d}s,$$

which is the same upper bound as the one obtained for $\nu_{r,t}$ in (3.6) in the proof of Lemma 3.3. Therefore exactly the same estimates as in part (a) can be applied to conclude that, given $\varepsilon > 0$, $\mathbb{E}\left[\left|\int_r^q d(-s)^{d-1}\,L(\mathrm{d}s)\right|\right] \leq \varepsilon$ if $|r|, |q|$ are sufficiently small. Hence $\left(\int_{-\infty}^r (-s)^{d-1}\,L(\mathrm{d}s)\right)_{r\uparrow 0}$ is a Cauchy sequence and therefore convergent in L^1.

□

Lemma 3.5. *Under condition (e) of Theorem 2.1, we have*

$$L^1 - \lim_{t\downarrow 0}\frac{1}{t}M_d(t) = \frac{1}{\Gamma(d)}\int_{-\infty}^0 (-s)^{d-1}\,L(\mathrm{d}s).$$

Proof. By Lemma 3.4 part b) the candidate limit is an L^1-random variable. Moreover, for every $r < 0$ and $t > 0$,

$$\begin{aligned}
&\mathbb{E}\left[\left|\frac{1}{t}\int_{-\infty}^t \left[(t-s)_+^d - (-s)_+^d\right]\,L(\mathrm{d}s) - d\int_{-\infty}^0 (-s)^{d-1}\,L(\mathrm{d}s)\right|\right]\\
&\leq \mathbb{E}\left[\left|\frac{1}{t}\int_{-\infty}^r \left[(t-s)^d - (-s)^d\right]\,L(\mathrm{d}s) - d\int_{-\infty}^r (-s)^{d-1}\,L(\mathrm{d}s)\right|\right]\\
&+ \mathbb{E}\left[\left|\frac{1}{t}\int_0^t (t-s)^d\,L(\mathrm{d}s)\right|\right] + \sup_{t>0}\mathbb{E}\left[\left|\frac{1}{t}\int_r^0 \left[(t-s)^d - (-s)^d\right]\,L(\mathrm{d}s)\right|\right]\\
&+ \mathbb{E}\left[\left|d\int_r^0 (-s)^{d-1}\,L(\mathrm{d}s)\right|\right] =: (1) + (2) + (3) + (4), \quad \text{say.}
\end{aligned}$$

Using Itô's isometry we get for (1)

$$\begin{aligned}
(1) &\leq \mathbb{E}\left\{\left|\int_{-\infty}^r \left\{\frac{1}{t}\left[(t-s)^d - (-s)^d\right] - d(-s)^{d-1}\right\}\,L(\mathrm{d}s)\right|^2\right\}^{1/2}\\
&= \mathbb{E}\left(L(1)^2\right)^{1/2}\left\{\int_{-\infty}^r \left\{\frac{1}{t}\left[(t-s)^d - (-s)^d\right] - d(-s)^{d-1}\right\}^2\mathrm{d}s\right\}^{1/2}
\end{aligned}$$

and by dominated convergence the latter integral converges to zero as $t \downarrow 0$. For $t \downarrow 0$ (2) tends to zero as shown in Lemma 3.2. Term (3) tends to zero as $r \uparrow 0$ by Lemma 3.4(a), and so does term (4) by Lemma 3.4(b). Hence, the assertion follows by letting $t \downarrow 0$ and then $r \uparrow 0$. □

Remark 3.6. *Under condition (e) of Theorem 2.1, one can also show that*

$$L^1 - \lim_{t\uparrow 0}\frac{1}{t}M_d(t) = \frac{1}{\Gamma(d)}\int_{-\infty}^0 (-s)^{d-1}\,L(\mathrm{d}s).$$

To this end one decomposes, for $r < t < 0$,

$$\begin{aligned}
&\mathbb{E}\left[\left|\frac{1}{t}\int_{-\infty}^{0}\left[(t-s)_+^d-(-s)_+^d\right]\,L(\mathrm{d}s)-d\int_{-\infty}^{0}(-s)^{d-1}\,L(\mathrm{d}s)\right|\right]\\
&\leq\ \mathbb{E}\left[\left|\frac{1}{t}\int_{-\infty}^{r}\left[(t-s)^d-(-s)^d\right]\,L(\mathrm{d}s)-d\int_{-\infty}^{r}(-s)^{d-1}\,L(\mathrm{d}s)\right|\right]\\
&+\ \mathbb{E}\left[\left|\frac{1}{t}\int_{t}^{0}(-s)^d\,L(\mathrm{d}s)\right|\right]+\sup_{t<0}\mathbb{E}\left[\left|\frac{1}{t}\int_{r}^{t}\left[(t-s)^d-(-s)^d\right]\,L(\mathrm{d}s)\right|\right]\\
&+\ \mathbb{E}\left[\left|d\int_{r}^{0}(-s)^{d-1}\,L(\mathrm{d}s)\right|\right]
\end{aligned}$$

and shows convergence of these four terms analogously to the situation in Lemma 3.5 letting $t \uparrow 0$ and then $r \uparrow 0$.

Proof of Theorem 2.1 '(e) $\Rightarrow$ (f)', and Theorem 2.4. We assume that condition (e) in Theorem 2.1 is satisfied. In order to prove (f) we fix $a < b \in \mathbb{R}$. Due to the continuous paths of M_d the total variation can be calculated along dyadic partitions, i.e.

$$\mathrm{TV}\left(M_d|_{[a,b]}\right)=\lim_{n\to\infty}\sum_{i=1}^{2^n}|M_d(t_i)-M_d(t_{i-1})|,$$

where $t_i = a + (b-a)i2^{-n}$. By the stationary increments of M_d, monotone convergence and Lemma 3.5, we obtain,

$$\begin{aligned}
\mathbb{E}\left[\mathrm{TV}\left(M_d|_{[a,b]}\right)\right] &= (b-a)\lim_{n\to\infty}\frac{2^n}{b-a}\mathbb{E}[|M_d((b-a)2^{-n})|]\\
&= \frac{(b-a)}{\Gamma(d)}\mathbb{E}\left[\left|\int_{-\infty}^{0}(-s)^{d-1}L(\mathrm{d}s)\right|\right].
\end{aligned}$$

In view of Lemma 3.4, the expectation on the right-hand side is finite. Hence, (f) follows and the proof of Theorem 2.1 is complete. Moreover, the explicit expression for the expected total variation in Theorem 2.4 is derived under condition (e) and, thus, under every of the equivalent conditions in Theorem 2.1.

Now suppose that one of the equivalent conditions in Theorem 2.1 holds true. Then, properties (c) and (e) are valid. By (c), the limit $M_d(t)/t$ exists almost surely as t goes to zero. Thanks to (e) and Lemma 3.5 the limit $M_d(t)/t$ exists in L^1 as $t \downarrow 0$. Then both limits must coincide, and consequently,

$$\frac{\mathrm{d}}{\mathrm{d}t}M_d(0)=\frac{1}{\Gamma(d)}\int_{-\infty}^{0}(-s)^{d-1}L(\mathrm{d}s).$$

Note that by Lemma 3.4 the improper integral on the right hand side converges in L^1 and almost surely. The alternative expression in (2.1) can be derived as follows. Integration by parts yields, for $r < 0$,

$$\int_{-\infty}^{r}(-s)^{d-1}L(\mathrm{d}s)-(d-1)\int_{-\infty}^{r}L(s)(-s)^{d-2}\mathrm{d}s=L(r)|r|^{d-1}.$$

So, in view of Lemma 3.4, it is sufficient to show that the right-hand side converges to zero in L^1 and almost surely for $r \uparrow 0$. L^1-convergence is analogous to the proof of Lemma 3.2 thanks to the integrability condition (e) for the Lévy measure, because the Lévy measure $\tilde{\nu}_r$ of $L(r)|r|^{d-1}$ is given by $\tilde{\nu}_r(dx) = |r|\nu(r^{1-d}dx)$. Condition (e) also guarantees almost sure convergence to zero in view of Theorem 2.1 in Bertoin, Doney and Maller [5]. □

4 Well-balanced fractional Lévy processes

In this section we discuss the semimartingale property of the closely related well-balanced fractional Lévy process as defined in (1.3). Integration by parts yields the following representation of a well-balanced fractional Lévy process as improper Riemann integral

$$N_d(t) = \frac{1}{\Gamma(d)} \int_{-\infty}^{+\infty} \left(\operatorname{sign}(t-s)|t-s|^{d-1} + \operatorname{sign}(s)|s|^{d-1} \right) L(s)\mathrm{d}s. \tag{4.1}$$

For well-balanced fractional Lévy processes the equivalences of Theorem 2.1 do also hold true.

Theorem 4.1. *Under the assumption of Theorem 2.1 conditions (a)-(h) given there are equivalent for the well-balanced fractional Lévy process N_d. If one of the conditions holds true, then*

$$\frac{\mathrm{d}}{\mathrm{d}t} N_d(0) = \frac{-1}{\Gamma(d)} \int_{-\infty}^{\infty} \operatorname{sign}(s)|s|^{d-1} L(\mathrm{d}s) = \frac{1}{\Gamma(d-1)} \int_{-\infty}^{\infty} |s|^{d-2} L(s)\,\mathrm{d}s,$$

where the integrals exist as improper integrals at zero in the sense of almost sure and $L^1(P)$ convergence. Moreover, for every $a < b \in \mathbb{R}$,

$$\mathbb{E}\left[\mathrm{TV}\left(N_d|_{[a,b]}\right)\right] = \frac{(b-a)}{\Gamma(d)} \mathbb{E}\left[\left| \int_{-\infty}^{\infty} \operatorname{sign}(s)|s|^{d-1} L(\mathrm{d}s) \right|\right].$$

The proof of Theorem 4.1 follows similar lines as the proof of Theorem 2.1. The following decomposition of well-balanced fractional Lévy processes into the sum of two non-anticipative fractional Lévy processes $M_d^{(1)}$ and $M_d^{(2)}$ turns out to be useful:

$$\begin{aligned} \Gamma(d+1)N_d(t) &= \int_{\mathbb{R}} \left[(t-s)_+^d - (-s)_+^d + (t-s)_-^d - (-s)_-^d \right] L(\mathrm{d}s) \\ = &\int_{\mathbb{R}} \left[(t-s)_+^d - (-s)_+^d \right] L(\mathrm{d}s) + \int_{\mathbb{R}} \left[(-t-u)_+^d - (-u)_+^d \right] L(-\mathrm{d}u). \\ =: &\ \Gamma(d+1)(M_d^{(1)}(t) + M_d^{(2)}(-t)) \end{aligned} \tag{4.2}$$

Proof of Theorem 4.1. The implications

$$(a) \Rightarrow (a') \Rightarrow (b'),\ (c) \Rightarrow (d),\ (f) \Rightarrow (g) \Rightarrow (h),\ (a) \Rightarrow (b) \Rightarrow (b')$$

are again obvious.

"(h) $\Longrightarrow$ (a)": In view of (4.2) N_d inherits the continuous paths and the zero quadratic variation of $M_d^{(i)}$, $i = 1, 2$ and, hence, is a semimartingale only if its paths are of bounded variation with probability one.

"(b') $\Longrightarrow$ (c')": Analogously to Theorem 2.1 using stationarity of N_d.

"(c') $\Longrightarrow$ (c)": Denote

$$\tilde{B}_0 = \{\omega \in \Omega : \frac{\mathrm{d}}{\mathrm{d}t} N_d(0, \omega) \text{ exists}\}$$

Due to (4.1) we obtain, for $r > 0$,

$$\begin{aligned}
\Gamma(d) \cdot N_d(t) &= \int_{-\infty}^{-r} \left(\operatorname{sign}(t-s)|t-s|^{d-1} + \operatorname{sign}(s)|s|^{d-1}\right) L(s)\mathrm{d}s \\
&\quad + \int_{-r}^{r} \left(\operatorname{sign}(t-s)|t-s|^{d-1} + \operatorname{sign}(s)|s|^{d-1}\right) L(s)\mathrm{d}s \\
&\quad + \int_{r}^{\infty} \left(\operatorname{sign}(t-s)|t-s|^{d-1} + \operatorname{sign}(s)|s|^{d-1}\right) L(s)\mathrm{d}s.
\end{aligned}$$

With the reasoning of the one-sided case it follows that the first and third limit in the corresponding decomposition of

$$\lim_{t \to 0} \frac{1}{t} \Gamma(d) N_d(t)$$

exist for every fixed $r > 0$. Then the Blumenthal 0-1 law yields $P(\tilde{B}_0) = 1$.

"(d) $\Longrightarrow$ (e)": Denoting the left derivative of $\Gamma(d+1)N_d$ in 0 by $\tilde{D}$ and using the representation above we find for $r < t < 0$ analogously to the non-anticipating case

$$\begin{aligned}
\tilde{D} &= \int_{-\infty}^{r} |s|^{d-1} L(\mathrm{d}s) \\
&\quad + \lim_{t \uparrow 0} (td)^{-1} \int_{r}^{|r|} \left(|t-s|^d - |s|^d\right) L(\mathrm{d}s) - \int_{|r|}^{+\infty} |s|^{d-1} L(\mathrm{d}s) \\
&=: Y_r^- + Z_r - Y_r^+.
\end{aligned}$$

Then we apply the same reasoning to Y_r^- as in the one-sided case for Y_r.

"(e) $\Longrightarrow$ (f)": Fix $a < b \in \mathbb{R}$. By the decomposition (4.2) and the triangle inequality we have

$$\mathbb{E}\left[\mathrm{TV}\left(N_d|_{[a,b]}\right)\right] \leq \mathbb{E}\left[\mathrm{TV}\left(M_d^{(1)}|_{[a,b]}\right)\right] + \mathbb{E}\left[\mathrm{TV}\left(M_d^{(2)}|_{[-b,-a]}\right)\right] < \infty$$

thanks to Theorem 2.1, (e) $\Longrightarrow$ (f).

Representation for the derivative and expected total variation: Applying Lemma 3.5 and Remark 3.6 to $M_d^{(1)}$ and $M_d^{(2)}$ we obtain

$$\begin{aligned}
L^1 - \lim_{t \to 0} \frac{1}{t} N_d(t) &= L^1 - \lim_{t \to 0} \frac{1}{t} M_d^{(1)}(t) + L^1 - \lim_{t \to 0} \frac{1}{t} M_d^{(2)}(-t) \\
&= \frac{1}{\Gamma(d)} \int_{-\infty}^{0} (-s)^{d-1} L(\mathrm{d}s) - \frac{1}{\Gamma(d)} \int_{-\infty}^{0} (-s)^{d-1} L(-\mathrm{d}s) \\
&= \frac{-1}{\Gamma(d)} \int_{-\infty}^{\infty} \operatorname{sign}(s)|s|^{d-1} L(\mathrm{d}s).
\end{aligned}$$

Then an analogous reasoning as in the proof of Theorem 2.4 applies. $\square$

5 Fractionally integrated Lévy processes

Suppose L is a Lévy process with zero expectation and finite variance. By (1.2), the fractional Lévy process M_d of order $0 < d < 1/2$ driven by L can be split into the sum

$$M_d(t) = \frac{1}{\Gamma(d)} \int_{-\infty}^{0} \left[(t-s)^{d-1} - (-s)^{d-1}\right] L(s)\mathrm{d}s + (\mathcal{I}^d L)(t), \quad t \geq 0.$$

Here $(\mathcal{I}^d f)(t)$ is the well-known fractional Riemann-Liouville integral of order $d > 0$ defined by

$$\mathcal{I}^d f(t) = \frac{1}{\Gamma(d)} \int_0^t (t-s)^{d-1} f(s)\,\mathrm{d}s$$

for sufficiently integrable functions f.

In order to transfer our results from fractional Lévy processes to Riemann-Liouville integrals of Lévy processes we first study the process

$$F_d(t) := \frac{1}{\Gamma(d)} \int_{-\infty}^{0} \left[(t-s)^{d-1} - (-s)^{d-1}\right] L(s)\mathrm{d}s, \quad t \geq 0.$$

Proposition 5.1. *Suppose L is a Lévy process with zero expectation and finite variance and $0 < d < 1/2$. Then the expected total variation of F_d on compact intervals $[0,b]$, $b > 0$ is finite. In particular, $\mathcal{I}^d L$ is a.s. of finite variation on compact intervals (resp. has finite expected total variation on compact intervals) if and only if M_d has the respective property.*

Proof. Let $0 = t_0 < t_1 < \ldots < t_n = b$ be a partition of $[0,b]$. Then we have for $s < 0 \leq t_{i-1} < t_i$ that $(t_i - s)^{d-1} < (t_{i-1} - s)^{d-1}$ and we conclude

$$\begin{aligned}
&\Gamma(d)\,|F_d(t_i) - F_d(t_{i-1})| \\
&\quad= \left| \int_{-\infty}^{0} \left[(t_i - s)^{d-1} - (t_{i-1} - s)^{d-1}\right] L(s)\,\mathrm{d}s \right| \\
&\quad\leq \int_{-\infty}^{0} \left[(t_{i-1} - s)^{d-1} - (t_i - s)^{d-1}\right] |L(s)|\,\mathrm{d}s.
\end{aligned}$$

Taking in the following the supremum over all finite partitions of $[0,b]$, the total variation $\mathrm{TV}\left(F_d|_{[0,b]}\right)$ of F_d over $[0,b]$ can be estimated by

$$\begin{aligned}
\mathrm{TV}\left(F_d|_{[0,b]}\right) &= \sup_{t_0,\ldots,t_n} \sum_{i=1}^{n} |F_d(t_i) - F_d(t_{i-1})| \\
&\leq \frac{1}{\Gamma(d)} \sup_{t_0,\ldots,t_n} \int_{-\infty}^{0} \sum_{i=1}^{n} \left((t_{i-1} - s)^{d-1} - (t_i - s)^{d-1}\right) |L(s)|\,\mathrm{d}s \\
&= \frac{1}{\Gamma(d)} \int_{-\infty}^{0} \left[(-s)^{d-1} - (b-s)^{d-1}\right] |L(s)|\,\mathrm{d}s.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathbb{E}\left[\mathrm{TV}\left(F_d|_{[0,b]}\right)\right] &\leq \frac{1}{\Gamma(d)}\int_{-\infty}^{0}\left[(-s)^{d-1}-(b-s)^{d-1}\right]E[|L(s)|^2]^{1/2}\,\mathrm{d}s \\
&= \frac{E[|L(1)|^2]^{1/2}}{\Gamma(d)}\int_{-\infty}^{0}\left[(-s)^{d-1}-(b-s)^{d-1}\right]s^{1/2}\,\mathrm{d}s
\end{aligned}$$

The latter integral is finite because $0 < d < 1/2$ and

$$(-s)^{d-1}-(b-s)^{d-1} \sim b|d-1||s|^{d-2} \quad \text{as} \quad s \to -\infty.$$

Hence we see that F_d has finite expected total variation on compacts. The other assertions are immediate consequences of the decomposition

$$M_d(t) = F_d(t) + (\mathcal{I}^d L)(t), \quad t \geq 0.$$

□

Note that the fractional Riemann-Liouville integral $\mathcal{I}^d L$ can be defined for any Lévy process L (without any extra requirements on the expectation and variance). In this general setting, the combination of Corollaries 3.3 and 3.5(3) in Basse and Pedersen [1] states that, for $0 < d < 1/2$, $\mathcal{I}^d L$ is a.s. of finite variation on compacts, if and only if L has no Gaussian component and $|x|^{1/(1-d)}$ is integrable with respect to the Lévy measure ν around the origin (condition (e) in Theorem 2.1 above). Hence, on the one hand, Proposition 5.1 and the results of Basse and Pedersen can be combined to provide an alternative proof for the equivalence of properties (a) and (e) in Theorem 2.1. This observation complements Corollary 5.4 of [1], which only states that the condition (e) on the Lévy measure is necessary for the finite variation property of the paths of the fractional Lévy process M_d.

On the other hand, we can combine Theorem 2.1 and Proposition 5.1 to provide an alternative proof of Basse and Pedersen's Corollary 3.5(3) and, additionally, include a 0-1 law and a statement about the expected total variation, which is done in the following result.

Theorem 5.2. *Let L be a Lévy process with characteristic triplet (σ, ν, γ), but without any moment assumptions. Let $0 < d < 1/2$ and $b > 0$. Then the following statements are equivalent:*
(a) $\mathcal{I}^d L$ is a.s. of finite variation on $[0, b]$.
(b) $\mathcal{I}^d L$ is of finite variation on $[0, b]$ with positive probability.
(c) The Brownian motion part of L is zero (i.e. $\sigma = 0$) and

$$\int_{-1}^{1}|x|^{\frac{1}{1-d}}\,\nu(\mathrm{d}x) < \infty.$$

Moreover, if one of the conditions holds true and $E[|L(1)|] < \infty$, then the expected total variation of $\mathcal{I}^d L$ on $[0, b]$ is finite.

Proof. We decompose L into a sum $L = L^{(1)} + L^{(2)}$, where the Lévy process $L^{(1)}$ contains the Gaussian part of L and the compensated small jumps of L corresponding to $\nu|_{(-1,1)}$. Then $L^{(1)}$ has zero expectation and finite variance, and so Theorem 2.1 and Proposition 5.1 can be applied to this process. Observe that

$$\mathcal{I}^d L = \mathcal{I}^d L^{(1)} + \mathcal{I}^d L^{(2)} = M_d^{(1)} - F_d^{(1)} + \mathcal{I}^d L^{(2)}.$$

In view of Theorem 2.1 and Proposition 5.1 it remains to show that $\mathcal{I}^d L^{(2)}$ is a.s. of finite variation on $[0, b]$, and that its expected total variation is finite provided $E[|L(1)|] < \infty$. Note, that the Lévy process $L^{(2)}$ only contains large jumps and a drift component and, thus, is of finite variation. We decompose $L^{(2)} = L^{(2,+)} - L^{(2,-)}$ into the difference of two increasing Lévy processes (subordinators) $L^{(2,+)}$ and $L^{(2,-)}$. Then,

$$\mathcal{I}^d L^{(2,\pm)}(t) = \frac{1}{\Gamma(d)} \int_0^t s^{d-1}\, L^{(2,\pm)}(t-s)\, \mathrm{d}s$$

is increasing, because the mapping $t \mapsto \mathbf{1}_{[0,t]}(s) s^{d-1}\, L^{(2,\pm)}(t-s)$ is increasing for fixed s. Then, we obtain, for every $b > 0$,

$$\mathrm{TV}\left(\mathcal{I}^d L^{(2)}|_{[0,b]}\right) \le \mathcal{I}^d L^{(2,+)}(b) + \mathcal{I}^d L^{(2,-)}(b) < \infty.$$

Note that the Lévy measures of $L^{(2,\pm)}$ are $\nu|_{[1,\infty)}$ and $\nu|_{(-\infty,-1]}$. Hence, $E[|L(1)|] < \infty$ implies that $E[|L^{(2,+)}(t)| + |L^{(2,-)}(t)|] < \infty$ for every $t > 0$. Thus,

$$E\left[\mathrm{TV}\left(\mathcal{I}^d L^{(2)}|_{[0,b]}\right)\right] \le \frac{1}{\Gamma(d)} \int_0^b s^{d-1}\, E[|L^{(2,+)}(b)| + |L^{(2,-)}(b)|]\, \mathrm{d}s < \infty.$$

□

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