Non-causal strictly stationary solutions of random recurrence equations

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Abstract

Let $(M_n, Q_n)_{n\in\mathbb{N}}$ be an i.i.d. sequence in $\mathbb{R}^2$. Much attention has been paid to causal strictly stationary solutions of the random recurrence equation $X_n = M_n X_{n-1} + Q_n$, $n \in \mathbb{N}$, i.e. to strictly stationary solutions of this equation when X_0 is assumed to be independent of $(M_n, Q_n)_{n\in\mathbb{N}}$. Goldie and Maller (2000) gave a complete characterisation when such causal solutions exist. In the present paper we shall dispose of the independence assumption of X_0 and $(M_n, Q_n)_{n\in\mathbb{N}}$ and derive necessary and sufficient conditions for a strictly stationary, not necessarily causal solution of this equation to exist.

Keywords: Causal, non-anticipative, non-causal, random recurrence equation, strictly stationary

1. Introduction

Let $(M_n, Q_n)_{n\in\mathbb{N}}$ be an i.i.d. sequence in $\mathbb{R}^2$, (M, Q) a generic copy of it, and let the real-valued process $(X_n)_{n\in\mathbb{N}_0}$ be defined recursively by

$$X_n = M_n X_{n-1} + Q_n, \quad n \in \mathbb{N}, \tag{1}$$

where X_0 is some starting random variable, defined on the same probability space. Our goal is to characterise when the starting random variable X_0 can be chosen such that the derived process $(X_n)_{n\in\mathbb{N}_0}$ is strictly stationary, meaning that for all $n \in \mathbb{N}_0, m \in \mathbb{N}$ and $h_1, \ldots, h_m \in \mathbb{N}_0$,

$$\mathcal{L}(X_{h_1}, \ldots, X_{h_m}) = \mathcal{L}(X_{h_1+n}, \ldots, X_{h_m+n})$$

where $\mathcal{L}(Y)$ denotes the law of a random vector Y. Much attention has been paid to this question when X_0 is assumed to be *independent* of $(M_n, Q_n)_{n\in\mathbb{N}}$, in which case $(X_n)_{n\in\mathbb{N}_0}$ becomes a time-homogeneous Markov process. In this case, an independent X_0 can be chosen such that the process becomes stationary if and only if the Markov process admits an invariant probability measure μ, in which case X_0 and μ are related by $\mu = \mathcal{L}(X_0)$. By the definition of the invariant measure, this is further equivalent to saying that the distributional fixed point equation

$$\mathcal{L}(X) = \mathcal{L}(Q + MX), \quad \text{with } X \text{ independent of } (M, Q),$$

has a solution. A complete solution of when such a distributional fixed point and hence a choice of an independent X_0 exists making $(X_n)_{n\in\mathbb{N}_0}$ strictly stationary has been achieved by Goldie and Maller (2000, Theorem 3.1), while necessary and sufficient conditions under some extra conditions had been obtained earlier by Vervaat (1979, Theorems 1.5 and 1.6). We also mention Brandt (1986, Theorem 1), who gave sufficient conditions when $(M_n, Q_n)_{n\in\mathbb{N}_0}$ was allowed to be stationary and ergodic rather than i.i.d., and Bougerol and Picard (1992) who consider a multivariate extension of Vervaat's result. We refer to the paper by Goldie and Maller (2000) for further references when X_0 is assumed to be independent of $(M_n, Q_n)_{n\in\mathbb{N}}$.

In time series analysis, the assumption that X_0 is independent of the sequence $(M_n, Q_n)_{n\in\mathbb{N}}$ is termed a *causality-assumption* or also a *non-anticipativity assumption*, and a corresponding solution a *causal* solution. The aim of the present paper is to dispose of this causality assumption and to characterise completely when X_0, possibly dependent on $(M_n, Q_n)_{n\in\mathbb{N}}$, can be chosen such that $(X_n)_{n\in\mathbb{N}_0}$ becomes strictly stationary. It will turn out that non-causal solutions which depend on the future may indeed exist. The present paper can then be seen as a discrete time analogue of Behme at al. (2011), who consider strictly stationary solutions of the stochastic differential equation $dV_t = V_{t-}dU_t + dL_t$ with Lévy noise. Note also that related questions for ARMA processes (with deterministic coefficients) have been dealt with in (Brockwell and Davis, 1991, Theorem 3.1.3 and Problem 4.28) for the second order stationary case, and in (Brockwell and Lindner, 2010, Theorem 1) for the strictly stationary case. A discussion of non-causal autoregressive models in economic time series can be found in Lanne and Saikkonen (2011).

2. Preliminaries

Let $(M_n, Q_n)_{n\in\mathbb{N}}$ be an $\mathbb{R}^2$-valued i.i.d. sequence defined on a probability space $(\Omega, \mathcal{F}, P)$, let X_0 be a random variable on the same probability space, and define $(X_n)_{n\in\mathbb{N}_0}$ by (1). Denote

$$\Pi_n := \prod_{i=1}^{n} M_i, \quad n \in \mathbb{N}_0,$$

with the usual convention that the empty product is 1. By successive iteration, it is easy to see that

$$X_{n+h} = \left(\prod_{i=h+1}^{n+h} M_i\right) X_h + \sum_{i=h+1}^{n+h} \left(\prod_{j=i+1}^{n+h} M_j\right) Q_i \quad \forall\, h, n \in \mathbb{N}_0. \tag{2}$$

By Theorem 3.1 (c) of Goldie and Maller (2000), if $P(M = 0) = 0$ and $P(Q + Mc = c) < 1$ for all $c \in \mathbb{R}$, then a *causal* strictly stationary solution of (1) exists if and only if $\sum_{n=1}^{\infty} \Pi_{n-1} Q_n$ converges almost surely absolutely, in which case $\mathcal{L}(\sum_{n=1}^{\infty} \Pi_{n-1} Q_n)$ is the unique invariant measure. In (Goldie and Maller, 2000, Theorem 2.1), they also give a necessary and sufficient condition for this sum to converge almost surely absolutely. It will be also an important tool for the proof of our characterisation of all (not-necessarily causal) solutions we give in Theorem 2 below. For a random variable X, we denote its distribution by P_X, and if $P(X > 0) > 0$ we denote

$$A_X(y) := E(X^+ \wedge y) = \int_0^y P(X > x)\, dx, \quad y > 0. \tag{3}$$

Then the function $(0, \infty) \to (0, 1]$, $y \mapsto \frac{A_X(y)}{y}$ is nonincreasing, cf. (Goldie and Maller, 2000, Remark 2.2). We can now state those parts of Theorem 2.1 of Goldie and Maller (2000) which are relevant for our further investigations. In the formulation below, the equivalence of (ii) and (iii) and the last assertions follow from Theorem 2.1 together with Lemma 5.5 (applied with $Z_0 := 0$) in Goldie and Maller (2000).

Theorem 1. (Goldie and Maller, 2000, Theorem 2.1)
Let $(M_n, Q_n)_{n\in\mathbb{N}}$ be an i.i.d. sequence in $\mathbb{R}^2$ with generic copy (M, Q) such that $P(Q = 0) < 1$ and $P(M = 0) = 0$. Then the following are equivalent:

(i) $\Pi_n \to 0$ *a.s. as* $n \to \infty$ *and* $\int_1^{\infty} \frac{\log q}{A_{-\log|M|}(\log q)} P_{|Q|}(dq) < \infty$.

(ii) *The infinite sum* $\sum_{n=1}^{\infty} \Pi_{n-1} Q_n$ *converges almost surely absolutely.*

(iii) $\Pi_n \to 0$ *a.s. as* $n \to \infty$ *and* $\sum_{i=1}^{n} \Pi_{i-1} Q_i$ *converges in distribution to a finite random variable as* $n \to \infty$.

If $\Pi_n \to 0$ *a.s.* $(n \to \infty)$ *but* $\int_1^\infty \frac{\log q}{A_{-\log|M|}(\log q)} P_{|Q|}(dq) = \infty$, *then* $|\sum_{i=1}^{n} \Pi_{i-1} Q_i|$ *converges in probability to* ∞ *as* $n \to \infty$. *Further, if* Π_n *does not converge almost surely to 0 as* $n \to \infty$ *and* $P(Q + Mc = c) < 1$ *for all* $c \in \mathbb{R}$, *then* $|\sum_{i=1}^{n} \Pi_{i-1} Q_i|$ *converges in probability to* ∞ *as* $n \to \infty$.

Observe that if $P(M = 0) = 0$, $P(Q = 0) = 1$ and $\Pi_n \to 0$ a.s. $(n \to \infty)$, then conditions (i), (ii), (iii) of Theorem 1 are automatically satisfied.

Conditions for the almost sure convergence of Π_n to 0 have been obtained by Kesten and Maller (1996, Lemma 1.1). To state their results, let $(M_n)_{n\in\mathbb{N}}$ be an i.i.d. sequence of real valued random variables such that $P(M_1 = 0) = 0$. Consider the random walk

$$S_n := \sum_{i=1}^{n} (-\log M_i), \quad n \in \mathbb{N}.$$

Then $\prod_{i=1}^{n} M_i$ converges almost surely to 0 if and only if S_n drifts almost surely to $+\infty$ as $n \to \infty$, and $\prod_{i=1}^{n} M_i^{-1}$ converges almost surely to 0 if and only if S_n converges almost surely to $-\infty$ as $n \to \infty$. Further, it is well known that $(S_n)_{n\in\mathbb{N}}$ either converges almost surely to $+\infty$, or converges almost surely to $-\infty$, or oscillates in the sense that $-\infty = \liminf_{n\to\infty} S_n < \limsup_{n\to\infty} S_n = +\infty$ almost surely. Then by Lemma 1.1 in Kesten and Maller (1996), we have $S_n \to \infty$ a.s. as $n \to \infty$ if and only if either $0 < E(-\log|M|) \le E\big|\log|M|\big| < \infty$, or $E(\log|M|)^- = \infty$ and $\int_1^\infty \frac{y}{A_{-\log|M|}(y)} P_{\log|M|}(dy) < \infty$ with $A_{-\log|M|}$ as defined in (3). Similarly, $S_n \to -\infty$ a.s. as $n \to \infty$ if and only if either

$$0 < E(\log|M|) \le E\big|\log|M|\big| < \infty, \tag{4}$$

or

$$E(\log|M|)^+ = \infty \quad \text{and} \quad \int_1^\infty \frac{y}{A_{\log|M|}(y)} P_{-\log|M|}(dy) < \infty. \tag{5}$$

Since $\lim_{y\to\infty} A_{\log|M|}(y)$ is finite if and only if $E(\log|M|)^+ < \infty$, we see that

$$\int_1^\infty \frac{y}{A_{\log|M|}(y)} P_{-\log|M|}(dy) < \infty \tag{6}$$

is implied by both (4) and (5), hence (6) always holds whenever $\prod_{i=1}^{n} M_i^{-1} \to 0$ a.s. $(n \to \infty)$.

Remark 1. *If* $P(M=0)=0$ *and* $E\big|\log|M|\big|<\infty$*, then* $\Pi_n \to 0$ *a.s.* $(n\to\infty)$ *if and only* $E\log|M|<0$*, in which case* $A_{-\log|M|}(x)$ *converges to* $E((-\log|M|)^+)<\infty$ *as* $x\to\infty$*. Hence, provided that* $E\big|\log|M|\big|<\infty$*, condition (i) of Theorem 1 can be replaced by*

$$E\log|M|<0 \quad \textit{and} \quad E\log^+|Q|<\infty,$$

where $\log^+(x)=\log(\max\{1,x\})$ *for* $x\in\mathbb{R}$*, cf. (Goldie and Maller, 2000, Cor. 4.1).*

3. Results

The following is our main result and characterises when X_0 can be chosen for (1) to have a strictly stationary, not necessarily causal, solution.

Theorem 2. *Let* $(M_n,Q_n)_{n\in\mathbb{N}_0}$ *be an i.i.d. sequence in* $\mathbb{R}^2$ *with generic copy* (M,Q)*. Consider the random recurrence equation (1).*
(a) Suppose that $P(M=0)>0$*. Then a random variable* X_0 *(possibly on a suitably enlarged probability space) can be chosen such that the stochastic process* $(X_n)_{n\in\mathbb{N}_0}$ *is strictly stationary. This stationary solution is unique in distribution and obtained by choosing* X_0 *independent of* $(M_n,Q_n)_{n\in\mathbb{N}}$ *with*

$$\mathcal{L}(X_0)=\mathcal{L}\left(\sum_{i=0}^{\infty}\left(\prod_{j=1}^{i}M_j\right)Q_{i+1}\right). \tag{7}$$

(b) Suppose that $P(M=0)=0$ *and that* $\prod_{i=1}^n M_i$ *converges almost surely to 0 as* $n\to\infty$*, i.e. that* $\sum_{i=1}^n\log|M_i|\to-\infty$ *a.s. as* $n\to\infty$*. Then the following are equivalent:*

- *(i) A random variable* X_0 *(possibly on a suitably enlarged probability space) can be chosen such that* $(X_n)_{n\in\mathbb{N}_0}$ *is strictly stationary.*
- *(ii) The infinite sum* $\sum_{i=0}^{\infty}\left(\prod_{j=1}^{i}M_j\right)Q_{i+1}$ *converges almost surely absolutely.*
- *(iii) With* $A_{-\log|M|}$ *as defined in (3), it holds*

$$\int_1^\infty \frac{\log q}{A_{-\log|M|}(\log q)}\,P_{|Q|}(dq)<\infty.$$

If these equivalent conditions are satisfied, then the stationary solution is unique in distribution, and it is obtained by choosing X_0 independent of $(M_n, Q_n)_{n\in\mathbb{N}}$ and with distribution $\mathcal{L}(X_0)$ given by (7).
(c) Suppose that $P(M = 0) = 0$ and that $\prod_{i=1}^n M_i^{-1}$ converges almost surely to 0 as $n \to \infty$, i.e. that $\sum_{i=1}^n \log|M_i| \to +\infty$ a.s. as $n \to \infty$. Then the following are equivalent:

(i) A random variable X_0 can be chosen such that $(X_n)_{n\in\mathbb{N}_0}$ is strictly stationary.

(ii) The infinite sum $\sum_{i=1}^\infty \left(\prod_{j=1}^i M_j^{-1}\right) Q_i$ converges almost surely absolutely.

(iii) With $A_{\log|M|}$ as defined in (3), it holds

$$\int_1^\infty \frac{\log q}{A_{\log|M|}(\log q)}\, P_{|M^{-1}Q|}(dq) < \infty.$$

If these equivalent conditions are satisfied, then the stationary solution is unique and given by

$$X_n = -\sum_{i=1}^\infty \left(\prod_{j=1}^i M_{n+j}^{-1}\right) Q_{n+i}, \quad n \in \mathbb{N}_0. \tag{8}$$

(d) Suppose that $P(M = 0) = 0$ and that neither $\prod_{i=1}^n M_i$ nor $\prod_{i=1}^n M_i^{-1}$ converges almost surely to 0 as $n \to \infty$. Then a random variable X_0 can be chosen such that $(X_n)_{n\in\mathbb{N}_0}$ is strictly stationary if and only if there is some $c \in \mathbb{R}$ such that $P(Q + Mc = c) = 1$. If this condition is satisfied, a strictly stationary solution is given by the degenerate and constant process $X_n = c$ for all $n \in \mathbb{N}_0$. If additionally $P(|M| = 1) < 1$, then $(X_n = c)_{n\in\mathbb{N}_0}$ is the only strictly stationary solution of (1).

Observe that the solution given by (8) depends on the future and is a non-causal solution.

Proof. (b) Suppose that $P(M = 0) = 0$ and that $\prod_{i=1}^n M_i \to 0$ a.s. as $n \to \infty$. The equivalence of (ii) and (iii) is clear from Theorem 1. Now assume (i) and let $(X_n)_{n\in\mathbb{N}_0}$ be a strictly stationary solution of (1). Since $\mathcal{L}(X_{n+h_1}, \ldots, X_{n+h_m}) = \mathcal{L}(X_{h_1}, \ldots, X_{h_m})$ for all $m \in \mathbb{N}$ and $h_1, \ldots, h_m \in \mathbb{N}_0$

by strict stationarity, and since $\prod_{i=1+h_k}^{n+h_k} M_i \to 0$ a.s. as $n \to \infty$ for each $k \in \{1, \ldots, m\}$, it follows from (2) and Slutsky's lemma that

$$\left(\sum_{i=1+h_1}^{n+h_1} \left(\prod_{j=i+1}^{n+h_1} M_j \right) Q_i, \ldots, \sum_{i=1+h_m}^{n+h_m} \left(\prod_{j=1+h_m}^{n+h_m} M_j \right) Q_i \right)$$

converges in distribution as $n \to \infty$ to $\mathcal{L}(X_{h_1}, \ldots, X_{h_m})$. Since this limit does not depend on X_0, we see that the stationary solution must be unique in distribution. Further, setting $m = 1$ and $h_1 = 0$, we get convergence in distribution of $\sum_{i=1}^{n} \left(\prod_{j=i+1}^{n} M_j \right) Q_i$, and since

$$\mathcal{L}\left(\sum_{i=1}^{n} \left(\prod_{j=i+1}^{n} M_j \right) Q_i \right) = \mathcal{L}\left(\sum_{i=1}^{n} \left(\prod_{j=1}^{i-1} M_j \right) Q_i \right) \tag{9}$$

as a consequence of the i.i.d. assumption on $(M_n, Q_n)_{n \in \mathbb{N}}$, we see that also $\sum_{i=1}^{n} \Pi_{i-1} Q_i$ converges in distribution to a finite random variable as $n \to \infty$. Hence (ii) follows from Theorem 1.

For the converse, assume (ii), and choose X_0 independent of $(M_n, Q_n)_{n \in \mathbb{N}}$ with distribution given by (7). Then $(X_n)_{n \in \mathbb{N}_0}$ is a time–homogeneous Markov process, and it is easy to check that $\mathcal{L}(M_1 X_0 + Q_1) = \mathcal{L}(X_0)$. Hence $\mathcal{L}(X_0)$ is an invariant probability measure and the Markov process $(X_n)_{n \in \mathbb{N}_0}$ consequently strictly stationary.

(a) If $P(M = 0) > 0$, for each $h_k \in \mathbb{N}_0$ we automatically have $\prod_{i=1+h_k}^{n+h_k} M_i \to 0$ a.s. and almost sure absolute convergence of $\sum_{i=1}^{n} \left(\prod_{j=1}^{i-1} M_j \right) Q_i$ as $n \to \infty$. The existence of a stationary solution and the uniqueness assertion is then in complete analogy to the corresponding proof in (b).

(c) Suppose that $P(M = 0) = 0$ and that $\prod_{i=1}^{n} M_i^{-1} \to 0$ almost surely as $n \to \infty$. Since

$$\sum_{i=1}^{n} \left(\prod_{j=1}^{i} M_j^{-1} \right) Q_i = \sum_{i=1}^{n} \left(\prod_{j=1}^{i-1} M_j^{-1} \right) M_i^{-1} Q_i$$

for $n \in \mathbb{N}$ and since $(M_n, M_n^{-1} Q_n)_{n \in \mathbb{N}}$ is an i.i.d. sequence, the equivalence of (ii) and (iii) follows from Theorem 1. Now assume (i) and let X_0 be chosen such that $(X_n)_{n \in \mathbb{N}_0}$ is strictly stationary. Rewriting (2) we have

$$X_h = \left(\prod_{i=h+1}^{n+h} M_i^{-1} \right) X_{n+h} - \sum_{i=h+1}^{n+h} \left(\prod_{j=h+1}^{i} M_j^{-1} \right) Q_i \tag{10}$$

for every $h \in \mathbb{N}_0$ and $n \in \mathbb{N}$. Since $\mathcal{L}(X_{h+n}) = \mathcal{L}(X_0)$ by strict stationarity, and since $\prod_{i=h+1}^{n+h} M_i^{-1}$ converges almost surely to 0 as $n \to \infty$, we conclude from Slutsky's lemma that $\left(\prod_{i=h+1}^{n+h} M_i^{-1}\right) X_{h+n}$ converges in probability to 0 as $n \to \infty$, hence $-\sum_{i=1}^n \left(\prod_{j=1}^i M_{h+j}^{-1}\right) Q_{h+i}$ must converge in probability to X_h as $n \to \infty$. This shows uniqueness of the solution and the given form, and from the discussion above and Theorem 1 we see that the convergence must be almost surely absolutely, hence we obtain (ii). Conversely, if (ii) is satisfied, define X_n by (8). Then it is easy to see that $(X_n)_{n\in\mathbb{N}_0}$ is a strictly stationary solution of (1).

(d) Suppose that $P(M = 0) = 0$ and that neither Π_n nor Π_n^{-1} converges to 0 a.s. as $n \to \infty$. Suppose that $P(Q + Mc = c) < 1$ for all $c \in \mathbb{R}$. Then $P(Q = 0) < 1$ and $|\sum_{i=1}^n \Pi_{i-1} Q_i|$ converges in probability to ∞ as $n \to \infty$ by Theorem 1, hence so does $\left|\sum_{i=1}^n \left(\prod_{j=i+1}^n M_j\right) Q_i\right|$ by (9). Assume that a stationary version $(X_n)_{n\in\mathbb{N}_0}$ exists. By (2) for $h = 0$ this implies that $|\Pi_n X_0|$ converges in probability to ∞ as $n \to \infty$, hence so does $|\Pi_n|$. By stationarity, we conclude that $\Pi_n^{-1} X_n$ converges in probability to 0 as $n \to \infty$, and hence we conclude from (10) for $h = 0$ that $\sum_{i=1}^n \Pi_{i-1}^{-1} M_i^{-1} Q_i$ converges in probability to $-X_0$. Since $P(Q + Mc = c) < 1$ for all $c \in \mathbb{R}$, we also have $P(M^{-1}Q + M^{-1}d = d) < 1$ for all $d \in \mathbb{R}$, and since Π_n^{-1} does not converge to 0 a.s. by assumption it follows again from Theorem 1 that $|\sum_{i=1}^n \Pi_{i-1}^{-1} M_i^{-1} Q_i|$ converges in probability to ∞, a contradiction. Hence no strictly stationary solution can exist unless $P(Q + Mc = c) = 1$ for some $c \in \mathbb{R}$.

Now if there is some $c \in \mathbb{R}$ such that $P(Q + Mc = c) = 1$, then $Q_n = (c - M_n c)$ a.s., and (1) is equivalent to

$$X_n - c = M_n(X_{n-1} - c), \quad n \in \mathbb{N}. \tag{11}$$

Hence $X_n = c$ for each $n \in \mathbb{N}_0$ is obviously a strictly stationary solution. To show uniqueness if $P(|M| = 1) < 1$, let $(X_n)_{n\in\mathbb{N}_0}$ be some strictly stationary solution of (1). From (11) we obtain $|X_n - c| = |\Pi_n|\,|X_0 - c|$, hence

$$\log|X_n - c| = \log|X_0 - c| + \sum_{i=1}^n \log|M_i|, \quad n \in \mathbb{N},$$

with the convention that $\log 0 = -\infty$. But as the modulus of a random walk with $P(\log|M_i| = 0) < 1$, $\left|\sum_{i=1}^n \log|M_i|\right|$ converges in probability to $+\infty$ as $n \to \infty$ (this is well known; for instance it is an immediate consequence of

Theorem III.9 in Petrov (1975)), hence $\big|\log|X_n - c|\big|$ converges in probability to ∞ as $n \to \infty$. But since $(X_n)_{n\in\mathbb{N}_0}$ is strictly stationary, this is only possible if $\big|\log|X_n - c|\big| = \infty$ a.s., i.e. if $X_n = c$ a.s. □

Remark 2. *It follows from Theorem 2 that the strictly stationary solution to (1), provided it exists, is unique in distribution unless* $P(|M| = 1) = 1$ *and* $Q = (1-M)c$ *a.s. for some* $c \in \mathbb{R}$. *If* $P(|M| = 1) = P(Q = (1-M)c) = 1$ *for some* $c \in \mathbb{R}$, *then the strictly stationary solution is indeed no longer unique in distribution, as follows from Theorem 3.1 (b) (i)–(iii) in Goldie and Maller (2000), where moreover all causal solutions in this case are characterised.*

Remark 3. *Let* $(M_n, Q_n)_{n\in\mathbb{Z}}$ *be an i.i.d. sequence in* $\mathbb{R}^2$ *with generic copy* (M, Q). *Then the same characterisation as in Theorem 2 also holds for the existence of strictly stationary solutions to the equation* $X_n = M_n X_{n-1} + Q_n$ *indexed by* $n \in \mathbb{Z}$. *The only difference is now that, in cases (a) and (b), the strictly stationary solution (if existent) is not only unique in distribution, but unique almost surely, and given by* $X_t = \sum_{i=0}^{\infty}\left(\prod_{j=0}^{i-1} M_{t-j}\right) Q_{t-i}$ *for all* $t \in \mathbb{Z}$, *with convergence almost surely absolutely. This follows from (2) by fixing* $t = n + h$ *and letting* $h \to -\infty$.

In light of part (c) of Theorem 2, in comparison with part (b) of Theorem 2, it is natural to ask for the relationship between the almost sure absolute convergence of $\sum_{i=1}^{\infty}\left(\prod_{j=1}^{i} M_j^{-1}\right) Q_i$ and that of $\sum_{i=1}^{\infty}\left(\prod_{j=1}^{i-1} M_j^{-1}\right) Q_i$, or in other words, the relationship between the convergence of the integrals $\int_1^{\infty} \frac{\log q}{A_{\log|M|}(\log q)} P_{|M^{-1}Q|}(dq)$ and $\int_1^{\infty} \frac{\log q}{A_{\log|M|}(\log q)} P_{|Q|}(dq)$. We have the following result:

Proposition 1. *Let* $(M_n, Q_n)_{n\in\mathbb{N}}$ *be an i.i.d. sequence in* $\mathbb{R}^2$ *with generic copy* (M, Q) *such that* $P(M = 0) = 0$.
(a) If $\sum_{i=1}^{\infty}\left(\prod_{j=1}^{i-1} M_j^{-1}\right) Q_i$ *converges almost surely absolutely, then so does* $\sum_{i=1}^{\infty}\left(\prod_{j=1}^{i} M_j^{-1}\right) Q_i$.
(b) Conversely, if additionally $E\big|\log|M|\big| < \infty$, *then almost sure absolute convergence of* $\sum_{i=1}^{\infty}\left(\prod_{j=1}^{i} M_j^{-1}\right) Q_i$ *implies that of* $\sum_{i=1}^{\infty}\left(\sum_{j=1}^{i-1} M_j^{-1}\right) Q_i$.
(c) If (M, Q) *are such that* $P(M > 1) = 1$, $E \log M = \infty$ *and* $Q = M$, *then* $\sum_{i=1}^{\infty}\left(\prod_{j=1}^{i} M_j^{-1}\right) Q_i$ *converges almost surely absolutely, while* $\sum_{i=1}^{\infty}\left(\prod_{j=1}^{i-1} M_j^{-1}\right) Q_i$ *does not.*

Proof. The proof of (a) and (b) is in complete analogy to the proof of Theorem 3.1 in Lindner and Maller (2005) for convergence of Lévy integrals and hence omitted. We only remark that for the proof of (a), Equation (7.1) in Lindner and Maller (2005) has to be replaced by

$$P(|M^{-1}Q| > q) \leq P(|M^{-1}| > \sqrt{q}) + P(|Q| > \sqrt{q})$$

for $q \geq 1$ and that (6) is used to show convergence of the corresponding integral involving $P(|M^{-1}| > \sqrt{q})$. The proof of (b) is similar to that in Lindner and Maller (2005), using

$$P(|Q| > q) \leq P(|M| > \sqrt{q}) + P(|M^{-1}Q| > \sqrt{q}).$$

The convergence statement in (c) is trivial from Theorem 1 since $P_{|M^{-1}Q|}$ is the Dirac measure at 1, while the divergence assertion follows as in (Lindner and Maller, 2005, Thm. 3.1 (c)). □

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