Strictly Stationary Solutions of Autoregressive Moving Average Equations

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Abstract

Necessary and sufficient conditions for the existence of a strictly stationary solution of the equations defining an ARMA(p, q) process driven by an independent and identically distributed noise sequence are determined. No moment assumptions on the driving noise sequence are made.

Keywords: ARMA process, heavy tails, infinite variance, strict stationarity

1 Introduction

A strict ARMA(p, q) process (autoregressive moving average process of order (p, q)) is defined as a complex-valued strictly stationary solution $Y = (Y_t)_{t\in\mathbb{Z}}$ of the difference equations

$$\Phi(B)Y_t = \Theta(B)Z_t, \quad t \in \mathbb{Z}, \tag{1.1}$$

where B denotes the backward shift operator, $(Z_t)_{t\in\mathbb{Z}}$ is an independent and identically distributed (i.i.d.) complex-valued sequence and $\Phi(z)$ and $\Theta(z)$ are the polynomials,

$$\Phi(z) := 1 - \phi_1 z - \ldots - \phi_p z^p, \quad \Theta(z) := 1 + \theta_1 z + \ldots + \theta_q z^q, \quad z \in \mathbb{C},$$

with $\phi_1, \ldots, \phi_p \in \mathbb{C}$, $\theta_1, \ldots, \theta_q \in \mathbb{C}$, $\phi_0 := 1, \theta_0 := 1$, $\phi_p \neq 0$ and $\theta_q \neq 0$.

A great deal of attention has been devoted to *weak* ARMA(p, q) processes which are defined as weakly stationary solutions of the difference equations (1.1) with $(Z_t)_{t\in\mathbb{Z}}$ assumed only to be white noise, i.e an uncorrelated sequence of random variables with constant

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mean μ and variance $\sigma^2 > 0$. It is easy to show, see e.g. Brockwell and Davis [1], that if $(Y_t)_{t\in\mathbb{Z}}$ is a weakly stationary solution, then it has a spectral density f_Y which satisfies,

$$|\Phi(e^{-i\omega})|^2 f_Y(\omega) = |\Theta(e^{-i\omega})|^2 \sigma^2/(2\pi), \ \omega \in [-\pi, \pi]. \tag{1.2}$$

Since f_Y must be integrable on $[-\pi, \pi]$, this shows that any zero of $\Phi(\cdot)$ on the unit circle must be removable, i.e. if $|\lambda| = 1$ and λ is a zero of order m of $\Phi(\cdot)$, then λ must also be a zero of $\Theta(\cdot)$ of order at least m.

Conversely if all the zeroes of $\Phi(z)$ on the unit circle are removable and $(Z_t)_{t\in\mathbb{Z}}$ is white noise then there exists a weakly stationary solution of (1.1), namely the unique weakly stationary solution of the equation obtained from (1.1) by cancelling the factors of $\Phi(B)$ corresponding to the zeroes on the unit circle with the corresponding factors of $\Theta(B)$. If $\Phi(z)$ has no zeroes on the unit circle then the weakly stationary solution is unique. Provided the probability space on which $(Z_t)_{t\in\mathbb{Z}}$ is defined is rich enough to support a random variable which is uniformly distributed on $[0, 1]$ and uncorrelated with $(Z_t)_{t\in\mathbb{Z}}$, then there is a unique weakly stationary solution of (1.1) only if $\Phi(z)$ has no zeroes on the unit circle. (The last statement can be proved using a slight modification of the argument in Lemma 2 below.)

In recent years the study of heavy-tailed and asymmetric time series models has become of particular importance, especially in mathematical finance, where series of daily log returns on assets (the daily log return at time t is defined as $\log P_t - \log P_{t-1}$, where P_t is the closing price of the asset on day t) show clear evidence of dependence, heavy tails and asymmetry. Daily realized volatility series also exhibit these features. The interest in ARMA processes as defined in the first paragraph has increased accordingly, especially those driven by i.i.d. noise with possible asymmetry and heavy tails.

Although sufficient conditions for the existence of strictly stationary solutions of (1.1) under the assumption of i.i.d. $(Z_t)_{t\in\mathbb{Z}}$ with not necessarily finite variance have been given in the past (see e.g. Cline and Brockwell [3], where it is assumed that the tails of the distribution of Z_t are regularly varying and that neither Φ nor Θ has a zero on the unit circle), necessary conditions are much more difficult to prescribe since the simple argument based on the spectral density in (1.2) does not apply. Yohai and Maronna [7], in their study of estimation for heavy-tailed autoregressive processes, make the assumption that

$$E \log^+ |Z_t| < \infty \tag{1.3}$$

to ensure stationarity and remark that "it seems difficult to relax it". More recently Mikosch et al. [6] and Davis [4] have investigated parameter estimation for ARMA processes for which Z_t is in the domain of attraction of a stable law. In spite of the considerable interest in heavy-tailed ARMA processes there appears however to have been no systematic investigation of precise conditions under which the equation (1.1) has a strictly stationary solution and conditions under which such a solution is unique.

The purposes of this note are to establish necessary and sufficient conditions on both the i.i.d. noise and the zeroes of the defining polynomials in (1.1) under which a strictly stationary solution $(Y_t)_{t\in\mathbb{Z}}$ of the equations (1.1) exists, to specify a solution when these conditions are satisfied and to give necessary and sufficient conditions for its uniqueness.

Such conditions are given in Theorem 2.1 which shows in particular that the condition (1.3) cannot be relaxed except in the trivial case when all the singularities of $\Theta(z)/\Phi(z)$ are removable. Furthermore, if $(Z_t)_{t\in\mathbb{Z}}$ is non-deterministic, a strictly stationary solution of (1.1) cannot exist unless all zeroes of Φ on the unit circle are also zeroes of Θ of at least the same multiplicity. Under either of these conditions a strictly stationary solution is given by the expression (familiar from the finite variance case),

$$Y_t = \sum_{k=-\infty}^{\infty} \psi_k Z_{t-k}, \tag{1.4}$$

where ψ_k is the coefficient of z^k in the Laurent expansion of $\Theta(z)/\Phi(z)$.

If $\Phi(z) \neq 0$ for all $z \in \mathbb{C}$ such that $|z| = 1$ the solution (1.4) is the *unique* strictly stationary solution of (1.1). If $\Phi(z) = 0$ for some $z \in \mathbb{C}$ such that $|z| = 1$ and there exists a strictly stationary solution of (1.1) then we show in Lemma 2 that it is not the only one (assuming that the probability space on which $(Z_t)_{t\in\mathbb{Z}}$ is defined is rich enough to support a random variable uniformly distributed on $[0, 1]$ and independent of $(Z_t)_{t\in\mathbb{Z}}$).

Lemma 2.2, which justifies the cancellation of common factors of $\Phi(z)$ and $\Theta(z)$ under appropriate conditions, plays a key role in the proof of Theorem 2.1. A version with less restrictive conditions is established as Theorem 3.2.

If $(Z_t)_{t\in\mathbb{Z}}$ is deterministic, i.e. if there exists a constant $c \in \mathbb{C}$ such that $P(Z_t = c) = 1$ for all $t \in \mathbb{Z}$, the conditions are slightly different. They are specified in Theorem 3.

2 Existence of a strictly stationary solution

The following theorem, which is the main result of the paper, gives necessary and sufficient conditions for the existence of a strictly stationary solution of (1.1) with non-deterministic i.i.d. noise $(Z_t)_{t\in\mathbb{Z}}$.

Theorem 1. *Suppose that $(Z_t)_{t\in\mathbb{Z}}$ is a non-deterministic i.i.d. sequence. Then the ARMA equation (1.1) admits a strictly stationary solution $(Y_t)_{t\in\mathbb{Z}}$ if and only if*

(i) all singularities of $\Theta(z)/\Phi(z)$ on the unit circle are removable and $E\log^+|Z_1| < \infty$,

or

(ii) all singularities of $\Theta(z)/\Phi(z)$ in $\mathbb{C}$ are removable.

If (i) or (ii) above holds, then a strictly stationary solution of (1.1) is given by

$$Y_t = \sum_{k=-\infty}^{\infty} \psi_k Z_{t-k}, \quad t \in \mathbb{Z}, \tag{2.1}$$

where

$$\sum_{k=-\infty}^{\infty} \psi_k z^k = \frac{\Theta(z)}{\Phi(z)}, \quad 1-\delta < |z| < 1+\delta \ \textit{ for some } \delta \in (0,1),$$

is the Laurent expansion of $\Theta(z)/\Phi(z)$. The sum in (2.1) converges absolutely almost surely.

If Φ does not have a zero on the unit circle, then (2.1) *is the unique strictly stationary solution of (1.1).*

Before proving Theorem 1 we need to establish conditions under which common factors of $\Phi(z)$ and $\Theta(z)$ can be cancelled. This is done in the proof of the following lemma.

Lemma 1. [Reduction Lemma] *Suppose that $Y = (Y_t)_{t\in\mathbb{Z}}$ is a strict ARMA(p,q) process satisfying (1.1). Suppose that $\lambda_1 \in \mathbb{C}$ is such that $\Phi(\lambda_1) = \Theta(\lambda_1) = 0$, and define*

$$\Phi_1(z) := \frac{\Phi(z)}{1-\lambda_1^{-1}z}, \quad \Theta_1(z) := \frac{\Theta(z)}{1-\lambda_1^{-1}z}, \quad z \in \mathbb{C}.$$

If $|\lambda_1| = 1$ suppose further that all finite dimensional distributions of Y are symmetric, that Z_0 is symmetric and that $\Phi_1(\lambda_1) = 0$, i.e. the multiplicity of the zero λ_1 of Φ is at least 2. Then Y is a strict ARMA$(p-1,q-1)$ process with autoregressive polynomial Φ_1 and moving average polynomial Θ_1, i.e.

$$\Phi_1(B)Y_t = \Theta_1(B)Z_t, \quad t \in \mathbb{Z}. \tag{2.2}$$

(As will be shown in Theorem 2 below, the symmetry conditions imposed in the case $|\lambda_1| = 1$ can be eliminated.)

Proof of Lemma 1. Define

$$W_t := \Phi_1(B)Y_t, \quad t \in \mathbb{Z}. \tag{2.3}$$

Then $(W_t)_{t\in\mathbb{Z}}$ is strictly stationary and since $\Phi(z) = (1-\lambda_1^{-1}z)\Phi_1(z)$ we have

$$W_t - \lambda_1^{-1}W_{t-1} = \Theta(B)Z_t, \quad t \in \mathbb{Z}.$$

Iterating gives for $n \in \mathbb{N}_0$

$$W_t = \lambda_1^{-n}W_{t-n} + \sum_{j=0}^{n-1} \lambda_1^{-j}\Theta(B)Z_{t-j}, \quad t \in \mathbb{Z}.$$

Writing $\Theta(B)Z_t = \sum_{k=0}^{q} \theta_k Z_{t-k}$ (with $\theta_0 := 1$), it follows that for $n \geq q$

$$\begin{aligned}
&W_t - \lambda_1^{-n} W_{t-n} \\
&\quad = \sum_{j=0}^{n-1} \lambda_1^{-j} \sum_{k=0}^{q} \theta_k Z_{t-k-j} \\
&\quad = \sum_{j=0}^{q-1} \lambda_1^{-j} \left(\sum_{k=0}^{j} \theta_k \lambda_1^k \right) Z_{t-j} + \sum_{j=q}^{n-1} \lambda_1^{-j} \left(\sum_{k=0}^{q} \theta_k \lambda_1^k \right) Z_{t-j} \\
&\qquad + \sum_{j=0}^{q-1} \lambda_1^{-n-j} \left(\sum_{k=j+1}^{q} \theta_k \lambda_1^k \right) Z_{t-n-j}.
\end{aligned} \tag{2.4}$$

Introducing the assumption that $\Theta(\lambda_1) = 0$, we easily find that

$$-\sum_{j=0}^{q-1} \lambda_1^{-j} \left(\sum_{k=j+1}^{q} \theta_k \lambda_1^k \right) z^j = \sum_{j=0}^{q-1} \lambda_1^{-j} \left(\sum_{k=0}^{j} \theta_k \lambda_1^k \right) z^j = \Theta_1(z), \quad z \in \mathbb{C},$$

so that (2.4) can be written in the form

$$W_t - \lambda_1^{-n} W_{t-n} = \Theta_1(B) Z_t - \lambda_1^{-n} \Theta_1(B) Z_{t-n}, \quad n \geq q,\ t \in \mathbb{Z}. \tag{2.5}$$

Now if $|\lambda_1| > 1$, then the strict stationarity of $(W_t)_{t\in\mathbb{Z}}$ and an application of Slutsky's lemma show that $\lambda_1^{-n} W_{t-n}$ converges in probability to 0 as $n \to \infty$, and similarly for $\lambda_1^{-n}\Theta_1(B)Z_t$, so that for $|\lambda_1| > 1$ we have $\Phi_1(B)Y_t = W_t = \Theta_1(B)Z_t$, $t \in \mathbb{Z}$, which is (2.2). If $|\lambda_1| < 1$, multiplying (2.5) by $-\lambda_1^n$ and substituting $u = t - n$ gives

$$W_u - \lambda_1^n W_{u+n} = \Theta_1(B) Z_u - \lambda_1^n \Theta_1(B) Z_{u+n}, \quad n \geq q,\ u \in \mathbb{Z}.$$

Letting $n \to \infty$ shows again that $W_u = \Theta_1(B)Z_u$, so that (2.2) is true also in this case.

Now let $|\lambda_1| = 1$ and assume that $\Phi_1(\lambda_1) = 0$. Define

$$\Phi_2(z) := \frac{\Phi_1(z)}{1 - \lambda_1^{-1} z} = \frac{\Phi(z)}{(1 - \lambda_1^{-1} z)^2}, \quad z \in \mathbb{C},$$

and

$$X_t := \Phi_2(B) Y_t, \quad t \in \mathbb{Z}.$$

Then $(X_t)_{t\in\mathbb{Z}}$ is strictly stationary, and

$$X_t - \lambda_1^{-1} X_{t-1} = \Phi_1(B) Y_t = W_t. \tag{2.6}$$

Defining

$$C_t := W_t - \Theta_1(B) Z_t, \quad t \in \mathbb{Z}, \tag{2.7}$$

it follows from (2.5) that

$$C_t = \lambda_1^{-n} C_{t-n} = \lambda_1^{-n} W_{t-n} - \lambda_1^{-n} \Theta_1(B) Z_{t-n}, \quad n \geq q, \ t \in \mathbb{Z}. \tag{2.8}$$

Summing this over n from q to N for $N \geq q$ and inserting (2.6) gives

$$(N - q + 1) C_t = \sum_{n=q}^{N} \lambda_1^{-n} (X_{t-n} - \lambda_1^{-1} X_{t-n-1}) - \sum_{n=q}^{N} \lambda_1^{-n} \Theta_1(B) Z_{t-n},$$

so that for $t \in \mathbb{Z}$ and $N \geq q$ we get

$$C_t - (N-q+1)^{-1} \left(\lambda_1^{-q} X_{t-q} - \lambda_1^{-(N+1)} X_{t-N-1} \right) = -(N-q+1)^{-1} \sum_{n=q}^{N} \lambda_1^{-n} \Theta_1(B) Z_{t-n}. \tag{2.9}$$

But since $(X_t)_{t \in \mathbb{Z}}$ is strictly stationary and since $|\lambda_1| = 1$, Slutsky's lemma shows that the left hand side of (2.9) converges in probability to C_t as $N \to \infty$. Hence the right hand side of (2.9) converges also in probability to C_t. But it is easy to see that the probability limit as $N \to \infty$ of the right hand side must be measurable with respect to the tail-σ-algebra $\bigcap_{n \in \mathbb{N}} \sigma \left(\bigcup_{k \geq n} \sigma(Z_{t-k}) \right)$, which by Kolmogorov's zero-one law is P-trivial. Hence C_t is independent of itself, so that C_t must be deterministic. Using the assumed symmetry of Y and Z_0, it follows that W_t must be symmetric for each $t \in \mathbb{Z}$, as is $W_t - C_t = \Theta_1(B) Z_t$. But this is only possible if the deterministic C_t is zero. Equation (2.7) then shows that

$$\Phi_1(B) Y_t = W_t = \Theta_1(B) Z_t, \quad t \in \mathbb{Z},$$

which is (2.2). $\square$

Proof of Theorem 1. Sufficiency of condition (ii) can be established by writing equation (1.1) as

$$\Phi(B) Y_t = \Phi(B) \Theta^*(B) Z_t,$$

where $\Theta^*(z)$ is a polynomial. It is then clear that $Y_t := \Theta^*(B) Z_t$, $t \in \mathbb{Z}$, is a strictly stationary solution of (1.1) regardless of the distribution of Z_0. To establish the sufficiency of condition (i), define $(Y_t)_{t \in \mathbb{Z}}$ as in (2.1). To see that the defining sum converges absolutely with probability one, observe that there are constants $c, d > 0$ such that $|\psi_k| \leq d e^{-c|k|}$ for all $k \in \mathbb{Z}$. For $c' \in (0, c)$ this gives

$$\begin{aligned} \sum_{k \in \mathbb{Z}} P \left(|\psi_k Z_{t-k}| > e^{-c'|k|} \right) &\leq \sum_{k \in \mathbb{Z}} P(\log^+(d|Z_{t-k}|) > (c - c')|k|) \\ &= \sum_{k \in \mathbb{Z}} P(\log^+(d|Z_0|) > (c - c')|k|) < \infty, \end{aligned}$$

the last inequality being due to the condition $E \log^+ |Z_0| < \infty$. The Borel-Cantelli lemma then shows that the event $\{|\psi_k Z_{t-k}| > e^{-c'|k|}$ infinitely often$\}$ has probability zero, giving the almost sure absolute convergence of the series. It is clear that $(Y_t)_{t\in\mathbb{Z}}$ as defined by (2.1) is strictly stationary and, from the absolute convergence of the series, that it satisfies (1.1). The uniqueness of the strictly stationary solution under either of the conditions (i) or (ii) when Φ has no zeroes on the unit circle is guaranteed by Lemma 3.1 of Brockwell and Lindner [2].

To show that the conditions are necessary, let $(Y_t)_{t\in\mathbb{Z}}$ be a strictly stationary solution of $\Phi(B)Y_t = \Theta(B)Z_t$. We shall mimic the proofs of Proposition 2.5 and Theorem 4.2 in [2]. Let λ_1 be a zero of Φ of multiplicity $m_\Phi \geq 1$, and denote by $m_\Theta \in \mathbb{N}_0$ the multiplicity of λ_1 as a zero of Θ. We shall assume that the singularity of Θ/Φ at λ_1 is not removable, i.e. that $m_\Phi > m_\Theta$, and show that this implies $E\log^+|Z_0| < \infty$ in each of the cases (a) $|\lambda_1| > 1$ and (b) $|\lambda_1| < 1$, while we derive a contradiction in the case (c) $|\lambda_1| = 1$.

(a) Let $|\lambda_1| > 1$. By Lemma 1, we may assume without loss of generality that $m_\Theta = 0$ and $m_\Phi \geq 1$. Define $W_t = \Phi_1(B)Y_t$ as in (2.3) with $\Phi_1(z) = \Phi(z)/(1-\lambda_1^{-1}z)$. Then it follows from (2.4) that

$$\begin{aligned} W_t - \lambda_1^{-n}W_{t-n} - \sum_{j=0}^{q-1}\lambda_1^{-j}\left(\sum_{k=0}^{j}\theta_k\lambda_1^k\right)Z_{t-j} - \lambda_1^{-n}\sum_{j=0}^{q-1}\lambda_1^{-j}\left(\sum_{k=j+1}^{q}\theta_k\lambda_1^k\right)Z_{t-n-j} \\ = \Theta(\lambda_1)\sum_{j=q}^{n-1}\lambda_1^{-j}Z_{t-j}. \qquad (2.10) \end{aligned}$$

Due to the stationarity of $(W_t)_{t\in\mathbb{Z}}$ and of $(Z_t)_{t\in\mathbb{Z}}$, the left hand side of (2.10) converges in probability as $n\to\infty$. Hence so does the right hand side. But since $m_\Theta = 0$ we have $\Theta(\lambda_1)\neq 0$, so that $\sum_{j=q}^{\infty}\lambda_1^{-j}Z_{t-j}$ converges in probability and hence almost surely as a sum with independent summands. Since $(Z_t)_{t\in\mathbb{Z}}$ is i.i.d., it follows from the Borel-Cantelli-Lemma that

$$\begin{aligned} \sum_{n=0}^{\infty}P(\log^+|Z_0| > n\log|\lambda_1|) &= \sum_{n=0}^{\infty}P(|Z_0| > |\lambda_1|^n) \\ &= \sum_{n=0}^{\infty}P(|\lambda_1^{-n}Z_{-n}| > 1) < \infty, \end{aligned}$$

so that $E\log^+|Z_0| < \infty$ as claimed.

(b) The case $|\lambda_1| < 1$ can be treated similarly, multiplying (2.10) by λ_1^n, substituting $u = t-n$, and letting $n\to\infty$ with u fixed.

(c) Let $|\lambda_1| = 1$. We shall show that the assumption $m_\Phi > m_\Theta$ leads to a contradiction. By taking an independent copy $(Y_t', Z_t')_{t\in\mathbb{Z}}$ of $(Y_t, Z_t)_{t\in\mathbb{Z}}$, we see that $\Phi(B)Y_t' = \Theta(B)Z_t'$

and hence that the symmetrized versions $Y_t^{\mathrm{sym}} := Y_t - Y_t'$ and $Z_t^{\mathrm{sym}} := Z_t - Z_t'$ satisfy

$$\Phi(B)Y_t^{\mathrm{sym}} = \Theta(B)Z_t^{\mathrm{sym}}, \quad t \in \mathbb{Z}.$$

Hence we may assume without loss of generality that all finite dimensional distributions of Y are symmetric and that Z_0 is symmetric, with $Z_0 \neq 0$ due to the assumption that Z_0 is not deterministic. By Lemma 1 we may hence assume that $m_\Phi \geq 1$ and $m_\Theta = 0$, so that $\Theta(\lambda_1) \neq 0$. Defining W_t as in (2.3), it follows from the stationarity of $(W_t)_{t\in\mathbb{Z}}$ and of $(Z_t)_{t\in\mathbb{Z}}$ and from $|\lambda_1| = 1$ that there exists a constant $K > 0$ such that the probability that the modulus of the left hand side of (2.10) is less than K is greater than or equal to $1/2$ for all $n \geq q$, so that by (2.10) also

$$P\left(\left|\Theta(\lambda_1)\sum_{j=q}^{n-1}\lambda_1^{-j}Z_{-j}\right| < K\right) \geq \frac{1}{2} \quad \forall\, n \geq q.$$

In particular, $\left|\Theta(\lambda_1)\sum_{j=q}^{n-1}\lambda_1^{-j}Z_{-j}\right|$ does not converge in probability to $+\infty$ as $n \to \infty$. Since $\Theta(\lambda_1) = 0$ and since $\sum_{j=q}^{n-1}\lambda_1^{-j}Z_{-j}$ is a sum of independent symmetric terms, this implies that $\sum_{j=0}^{\infty}\lambda_1^{-j}Z_{-j}$ converges almost surely (see Kallenberg [5], Theorem 4.17). The Borel-Cantelli lemma then shows that

$$\sum_{j=0}^{\infty} P(|Z_0| > r) = \sum_{j=0}^{\infty} P\left(\left|\lambda_1^{-j}Z_{-j}\right| > r\right) < \infty.$$

Hence it follows that $P(|Z_0| > r) = 0$ for each $r > 0$, so that $Z_0 = 0$ a.s., which is impossible since Z_0 (and hence its symmetrisation) was assumed to be non-deterministic. □

3 Uniqueness, order reduction and deterministic Z

In Theorem 2.1 we gave necessary and sufficient conditions for the existence of a strictly stationary solution of (1.1) and a sufficient condition, namely $\Phi(z) \neq 0$ for all $z \in \mathbb{C}$ such that $|z| = 1$, for uniqueness of the strictly stationary solution. In the following lemma we show that the latter condition is also necessary for uniqueness if the probability space $(\Omega, \mathcal{F}, P)$ on which $(Z_t)_{t\in\mathbb{Z}}$ is defined is sufficiently rich.

Lemma 2. *If equation (1.1) has a strictly stationary solution, if $\Phi(z)$ has a zero λ_1 on the unit circle and if there exists a random variable U on the probability space $(\Omega, \mathcal{F}, P)$ which is uniformly distributed on the interval $[0, 1]$ and independent of $(Z_t)_{t\in\mathbb{Z}}$, then the strictly stationary solution is not unique.*

Proof. Define

$$\tilde{Y}_t := \lambda_1^{-t} \exp(2\pi i U), \; t \in \mathbb{Z}.$$

It is clear that $(\tilde{Y}_t)_{t\in\mathbb{Z}}$ is strictly stationary since $|\lambda_1| = 1$ and $\tilde{Y}_t - \lambda_1^{-1}\tilde{Y}_{t-1} = 0$ so that $\Phi(B)\tilde{Y}_t = 0$. Consequently, if $(Y_t)_{t\in\mathbb{Z}}$ denotes the solution of (1.1) given by (2.1), then $(Y_t + \tilde{Y}_t)_{t\in\mathbb{Z}}$ is another solution of (1.1) and is strictly stationary since it is the sum of two independent strictly stationary processes. □

The following theorem is a strengthened version of the Reduction Lemma (Lemma 2.2) which was used in the proof of Theorem 2.1. The proof makes use of Theorem 2.1 to extend the lemma so as to allow cancellation of factors corresponding to autoregressive and moving average roots on the unit circle under less restrictive conditions.

Theorem 2. [Order Reduction] *Let Λ denote the set of distinct zeroes of the autoregressive polynomial $\Phi(z)$ in (1.1). For each $\lambda \in \Lambda$ denote by $m_\Phi(\lambda) \in \mathbb{N}$ the multiplicity of λ as a zero of Φ, and by $m_\Theta(\lambda) \in \mathbb{N}_0$ the multiplicity of λ as a zero of $\Theta(z)$ (with $m_\Theta(\lambda) := 0$ if $\Theta(\lambda) \neq 0$). Let $(Z_t)_{t\in\mathbb{Z}}$ be a non-deterministic i.i.d. sequence and $Y = (Y_t)_{t\in\mathbb{Z}}$ be a strictly stationary solution of the ARMA(p,q)-equations (1.1). Then (by Theorem 2.1) $m_\Theta(\lambda) \geq m_\Phi(\lambda)$ for all $\lambda \in \Lambda$ such that $|\lambda| = 1$. If we define the reduced polynomials*

$$\Phi_{\rm red}(z) := \Phi(z) \prod_{\lambda\in\Lambda:|\lambda|\neq 1} (1-\lambda^{-1}z)^{-\min(m_\Phi(\lambda), m_\Theta(\lambda))} \prod_{\lambda\in\Lambda:|\lambda|=1} (1-\lambda^{-1}z)^{1-m_\Phi(\lambda)}, \quad z \in \mathbb{C},$$

$$\Theta_{\rm red}(z) := \Theta(z) \prod_{\lambda\in\Lambda:|\lambda|\neq 1} (1-\lambda^{-1}z)^{-\min(m_\Phi(\lambda), m_\Theta(\lambda))} \prod_{\lambda\in\Lambda:|\lambda|=1} (1-\lambda^{-1}z)^{1-m_\Phi(\lambda)}, \quad z \in \mathbb{C},$$

then Y is also a strictly stationary solution of the reduced ARMA equations

$$\Phi_{\rm red}(B)Y_t = \Theta_{\rm red}(B)Z_t, \quad t \in \mathbb{Z}. \tag{3.1}$$

Conversely, if $m_\Phi(\lambda) \leq m_\Theta(\lambda)$ for all $\lambda \in \Lambda$ such that $|\lambda| = 1$ and Y is a strictly stationary solution of (3.1), then Y is also a strictly stationary solution of (1.1).

Proof. Let $Y = (Y_t)_{t\in\mathbb{Z}}$ be a strictly stationary solution of (1.1). Then $m_\Theta(\lambda) \geq m_\Phi(\lambda)$ for all $\lambda \in \Lambda$ such that $|\lambda| = 1$ by Theorem 1, so that $\Theta_{\rm red}$ is indeed a polynomial. To show that Y satisfies (3.1), it is sufficient to show that Lemma 1 is true without the symmetry assumptions on Y and Z_0 if $|\lambda_1| = 1$, since (3.1) will then follow by induction (removing one factor at a time).

Suppose therefore that $\lambda_1 \in \Lambda$ with $|\lambda_1| = 1$ and $m_\Theta(\lambda_1) \geq m_\Phi(\lambda_1) \geq 2$. With W_t and C_t as defined in (2.3) and (2.7), respectively, it follows from the proof of Lemma 1 that C_t is deterministic for every $t \in \mathbb{Z}$ (this conclusion did not use the symmetry). From (2.8) we have $C_t = \lambda_1^{-n} C_{t-n}$ for each $t \in \mathbb{Z}$ and $n \geq q$, from which it follows that

$$C_t = \lambda_1^{-t} C_0, \quad t \in \mathbb{Z},$$

for some complex number C_0, and the proof of (2.2) and hence of (3.1) is finished once C_0 is established to be 0.

To show that $C_0 = 0$, observe from (2.7) that

$$W_t = \Theta_1(B)Z_t + \lambda_1^{-t} C_0,$$

where $(W_t)_{t\in\mathbb{Z}}$ and $(\Theta_1(B)Z_t)_{t\in\mathbb{Z}}$ are both strictly stationary. Hence $\lambda_1^{-t}C_0$ must be independent of t, which is possible only if $\lambda_1 = 1$ or $C_0 = 0$.

To show that $C_0 = 0$ in either case, suppose that $\lambda_1 = 1$, in which case (2.3) and (2.7) give

$$Y_t - Y_{t-1} = \Theta_1(B)Z_t + C_0, \quad t \in \mathbb{Z}.$$

But since $m_\Theta(1) \geq m_\Phi(1) \geq 2$, we can define the polynomial $\Theta_2(z) := (1-z)^{-2}\Theta(z)$ and $V_t := \Theta_2(B)Z_t$, $t \in \mathbb{Z}$. Then $(V_t)_{t\in\mathbb{Z}}$ is strictly stationary, and

$$Y_t - Y_{t-1} = \Theta_1(B)Z_t + C_0 = V_t - V_{t-1} + C_0, \quad t \in \mathbb{Z}.$$

This implies that

$$Y_t - Y_{t-n} = V_t - V_{t-n} + C_0 n, \quad n \in \mathbb{N}.$$

If we divide this equation by n and let n tend to infinity, we see that the left hand side converges in probability to 0, and the right hand side to C_0. Consequently C_0 must be zero regardless of the value of λ_1.

This completes the proof of (2.2) and hence (3.1). The converse, that strictly stationary solutions of (3.1) are strictly stationary solutions of (1.1), is clear. $\square$

Theorem 2 shows that for the determination of stationary solutions of (1.1), common zeroes λ of Φ and Θ can be factored out without restrictions if $|\lambda| \neq 1$, and if $m_\Theta(\lambda) \geq m_\Phi(\lambda) \geq 2$ and $|\lambda| = 1$, then $(1-\lambda^{-1}z)^{m_\Phi(\lambda)-1}$ can be factored out from Φ and Θ. That in general one cannot factor out $(1-\lambda^{-1}z)^{m_\Phi(\lambda_1)}$, i.e. all common zeroes with multiplicity on the unit circle, is a consequence of the non-uniqueness associated with an autoregressive root on the unit circle. For example, if $\Theta(z) = \Phi(z) = (1+z)^2$ and $(Z_t)_{t\in\mathbb{Z}}$ is non-deterministic and i.i.d., then Theorem 2 shows that every strictly stationary solution Y of $\Phi(B)Y_t = \Theta(B)Z_t$ is also a strictly stationary solution of $Y_t + Y_{t-1} = Z_t + Z_{t-1}$ and vice versa, but is not necessarily a solution of $Y_t = Z_t$, as shown in Lemma 2.

If $(Z_t)_{t\in\mathbb{Z}}$ is deterministic the conditions for existence of a strictly stationary solution of (1.1) are a little different from those which apply when $(Z_t)_{t\in\mathbb{Z}}$ is i.i.d. and non-deterministic as has been assumed until now. The deterministic case is covered by the following theorem.

Theorem 3. *Suppose that $(Z_t)_{t\in\mathbb{Z}}$ is deterministic, i.e. there is a constant $c \in \mathbb{C}$ such that $Z_t = c$ with probability 1 for every t. Then if $(Y_t)_{t\in\mathbb{Z}}$ is a strictly stationary solution of (1.1) it must be expressible as*

$$Y_t = K + \sum_{\lambda:|\lambda|=1} A_{\lambda,0}\lambda^{-t}, \tag{3.2}$$

where the sum is over the distinct zeroes of $\Phi(z)$ on the unit circle, the coefficients $A_{\lambda,0}$ are complex-valued random variables and

$$K = \begin{cases} \Theta(1)c/\Phi(1) & \textit{if } m_\Phi(1) = 0, \\ 0 & \textit{if } m_\Phi(1) > 0, \end{cases} \tag{3.3}$$

where $m_\Phi(1)$ denotes the multiplicity of 1 as a zero of $\Phi(z)$. A strictly stationary solution of (1.1) exists in this case if and only if

$$m_\Phi(1)\Theta(1)c = 0. \tag{3.4}$$

Proof. If $Z_t = c$ with probability 1 for every t, the defining equation (1.1) becomes

$$\Phi(B)Y_t = \Theta(1)c, \tag{3.5}$$

and every random sequence satisfying (3.5) can be written as

$$Y_t = f_0(t) + \sum_\lambda \left(\sum_{i=0}^{m_\Phi(\lambda)-1} A_{\lambda,i}t^i \right) \lambda^{-t}, \tag{3.6}$$

where $\sum_\lambda$ denotes the sum over distinct zeroes of $\Phi(z)$, $m_\Phi(\lambda)$ denotes the multiplicity of the zero λ, the coefficients $A_{\lambda,i}$ are random variables and the deterministic complex-valued function f_0 is the particular solution of (3.5) defined by

$$f_0(t) = \frac{\Theta(1)c}{\widetilde{\Phi}(1)m_\Phi(1)!}\, t^{m_\Phi(1)}, \tag{3.7}$$

where

$$\widetilde{\Phi}(z) = \frac{\Phi(z)}{(1-z)^{m_\Phi(1)}}.$$

(The validity of (3.6) for each fixed ω in the underlying probability space $(\Omega, \mathcal{F}, P)$ is a standard result from the theory of linear difference equations (see, e.g., Brockwell and Davis [1], pp. 105–110). The required measurability of the functions $A_{\lambda,i}$ follows from that of the random variables Y_t.)

In order for the sequence $(Y_t)_{t\in\mathbb{Z}}$ specified by (3.6) to be strictly stationary, it is clear (on considering the behaviour of the solution as $t \to -\infty$) that the terms corresponding to the zero or zeroes with largest absolute value greater than 1 must have zero coefficients. Having set these coefficients to zero, it is then clear that coefficients of terms corresponding to the zero or zeroes with next largest absolute value greater than 1 must also be zero. Continuing with this argument we find that all terms in the sum (3.5) corresponding to zeroes with absolute values greater than 1 must vanish. In the same way, by letting $t \to \infty$ we conclude that all the terms corresponding to zeroes with absolute values less than 1 must vanish. The same argument can also be applied to eliminate all terms of the form $A_{\lambda,i}t^i\lambda^{-t}$ with $|\lambda| = 1$ and $i > 0$, and to see that $\Theta(1)c = 0$ is necessary for a strictly stationary solution to exist if $m_\Phi(1) > 0$. Thus $f_0(t)$ reduces to K as defined in (3.3), Y has representation (3.2), and it is apparent that the condition (3.4) is necessary for the existence of a strictly stationary solution. On the other hand if this condition is satisfied then $Y_t = K$ is in fact a strictly stationary solution. □

As in the non-deterministic case, the strictly stationary solution is unique if $\Phi(z)$ has no zeroes on the unit circle, while the argument of Lemma 2 shows that in general $Y_t = K$ is not the only strictly stationary solution if $\Phi(z)$ has zeroes on the unit circle.

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