CHAPTER 6

Ornstein–Uhlenbeck related models driven by Lévy processes

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6.1 Introduction

Recently, there has been increasing interest in continuous-time stochastic models with jumps, a class of models which has applications in the fields of finance, insurance mathematics and storage theory, to name just a few. In this chapter we shall collect known results about a prominent class of these continuous-time models with jumps, namely the class of Lévy-driven Ornstein–Uhlenbeck processes, and their generalisations. In Section 6.2, basic facts about Lévy processes, needed in the sequel, are reviewed. Then, in Section 6.3.1 the Lévy-driven Ornstein–Uhlenbeck process, defined as a solution of the stochastic differential equation

$$dV_t = -\lambda V_t dt + dL_t,$$

where L is a driving Lévy process, is introduced. An application to storage theory is mentioned, followed by the volatility model of Barndorff-Nielsen and Shephard (2001a, 2001b). Then, in Sections 6.3.2 and 6.3.3, two generalisations of Ornstein–Uhlenbeck processes are considered, both of which are based on the fact that an Ornstein–Uhlenbeck process can be seen as a continuous-time analogue of an AR(1) process with i.i.d. noise. In Section 6.3.2 we consider CARMA processes, which are continuous-time analogues of discrete time ARMA processes, and in Section 6.3.3 we consider generalised Ornstein–Uhlenbeck processes, which are continuous time analogues of AR(1) processes with i.i.d. random coefficients. Special emphasis is given to conditions for stationarity of these processes and existence of moments. Then, in Section 6.3.4, we introduce the COGARCH(1,1) process, which is a continuous-time analogue of the ARCH process of Engle (1982) and the GARCH(1,1) process of Bollerslev (1986). It is an example of a generalised Ornstein–Uhlenbeck process. An extension to COGARCH(q, p) processes is also given.

Finally, in Section 6.4 we present some estimation methods for these processes, mostly based on high frequency data. In Section 6.4.1, Jongbloed, van der Meulen and van der Vaart's (2005) non-parametric estimator for the underlying Lévy measure of a subordinator driven Ornstein–Uhlenbeck process is presented, together with an estimator for the underlying parameter λ and a method for reconstruction of the sample paths of the driving Lévy process. In Section 6.4.2, a least squares estimator for the parameters of a CARMA process is considered, and Section 6.4.3 presents a generalised method of moments estimator for the COGARCH(1,1) process.

Throughout the paper, $\mathbb{N} = \{1, 2, \ldots\}$ is the set of strictly positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\Re(z)$ is the real part of the complex number z, and $\mathbf{b}'$ is the transpose of a vector $\mathbf{b} \in \mathbb{C}^n$, where $\mathbb{C}^n$ is understood to consist of column vectors. For a real number x, $\lfloor x \rfloor$ denotes the largest integer not exceeding x. The symbol "$\stackrel{d}{=}$" will be used to denote equality in distribution, and "a.s." as well as "i.i.d." are abbreviations for "almost surely" and "independent and identically distributed".

6.2 Lévy processes

In this section we collect some basic aspects of Lévy processes which will be needed in the following. We refer to the books by Applebaum (2004), Bertoin (1996), Kyprianou (2006) and Sato (1999) for the cited results and for further information about Lévy processes.

Definition 6.1 *A* Lévy process *with values in* $\mathbb{R}^d$ $(d \in \mathbb{N})$ *defined on a probability space* $(\Omega, \mathcal{F}, P)$ *is a stochastic process* $M = (M_t)_{t\geq 0}$*,* $M_t : \Omega \to \mathbb{R}^d$*, which satisfies the following properties:*

(i) it has independent increments, i.e. the random variables M_{t_0}*,* $M_{t_1} - M_{t_0}$*,* $M_{t_2} - M_{t_1}, \ldots, M_{t_n} - M_{t_{n-1}}$ *are independent for every* $n \in \mathbb{N}$ *and* $0 \leq t_0 < t_1 < \ldots < t_n$*,*

(ii) it has stationary increments, i.e. $M_{s+t} - M_s \stackrel{d}{=} M_t$ *for every* $s, t \geq 0$*,*

(iii) it starts almost surely at 0, i.e. $M_0 = 0$ *a.s.,*

(iv) its paths are almost surely càdlàg functions, i.e. there is a set $\Omega_0 \in \mathcal{F}$ *such that* $P(\Omega_0) = 1$ *and for every* $\omega \in \Omega_0$ *the path* $[0, \infty) \to \mathbb{R}^d$*,* $t \mapsto M_t(\omega)$ *is right-continuous in* $t \geq 0$ *and has finite left limits in* $t > 0$*.*