Integration of CARMA Processes and Spot Volatility Modelling

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Abstract

Continuous-time autoregressive moving average (CARMA) processes with a non-negative kernel and driven by a non-decreasing Lévy process constitute a useful and very general class of stationary, non-negative continuous-time processes which have been used, in particular, for the modelling of stochastic volatility. Brockwell, Davis and Yang (2011) considered the fitting of CARMA models to closely and uniformly spaced data, illustrating their results by fitting a CARMA(2,1) model to daily realized volatility of the Deutsche Mark/US dollar (DM/US$) exchange rate from December 1986 through June, 1999. A more fundamental quantity in financial modelling is the (unobserved) spot, or instantaneous, volatility process. In the celebrated stochastic volatility model of Barndorff-Nielsen and Shephard (2001), the spot volatility is represented by a stationary Lévy-driven Ornstein-Uhlenbeck process. This has the shortcoming that its autocorrelation function is necessarily a decreasing exponential function, which limits its ability to generate integrated volatility series with autocorrelation functions of the forms encountered in practice. (A realized volatility series is a sequence of estimated integrals of spot volatility over successive intervals of fixed length, typically one day.) If instead of the stationary Ornstein-Uhlenbeck process, we use a CARMA process to represent spot volatility, we can overcome the restriction to exponentially decaying autocorrelation function and obtain a more realistic model for the dependence observed in realized volatility. In this paper we show how to use realized volatility data to estimate parameters of a CARMA model for spot volatility and apply the analysis to the DM/US$ exchange rate.

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1 Introduction

In financial econometrics a Lévy-driven CAR(1) (or stationary Ornstein-Uhlenbeck) process was used by Barndorff-Nielsen and Shephard (2001) to represent the spot volatility $V(t)$ in their celebrated model,

$$dX^*(t) = (\mu + \beta V(t))dt + \sqrt{V(t)}dW(t), \tag{1.1}$$

for the logarithm, $X^*(t)$, of the price of an asset at time t. In this model μ and β are constants, W is standard Brownian motion and the volatility process V is a stationary causal non-negative Lévy-driven Ornstein-Uhlenbeck process, independent of W, satisfying

$$dV(t) + aV(t)dt = dL(t), \quad a > 0,$$

i.e.

$$V(t) = \int_{-\infty}^{t} \exp(-a(t-u))dL(u).$$

Since V is necessarily non-negative, the Lévy process L must have non-decreasing sample-paths. Lévy processes with this property are known as subordinators. A subordinator-driven CARMA(2,1) process was used by Todorov and Tauchen (2006) and Todorov (2011) to represent realized daily volatility in the Deutsche Mark/US Dollar (DM/US$) daily exchange rate. Brockwell, Davis and Yang (2011) considered the problem of estimation for subordinator-driven non-negative CARMA(p, q) processes based on uniformly and closely-spaced observations. For non-negativity of the CARMA process it is also necessary that the kernel of the process ($g(t) = e^{-at}\mathbf{1}_{(0,\infty)}(t)$ for the Ornstein-Uhlenbeck process) be non-negative. Conditions under which this holds were given by Tsai and Chan (2004) and, in the special case of the CARMA(2,1) process, by Brockwell and Davis (2001). Brockwell, Davis and Yang (2011) also considered the problem of recovering the increments of the driving subordinator from closely-spaced observations of the CARMA process and found that for the DM/US$ exchange rate a gamma-driven CARMA(2,1) process fitted the daily realized volatility series reasonably well.

In this paper we take a different point of view. Starting from a strictly stationary causal subordinator-driven CARMA(p, q) model for the *spot* volatility V in (1.1), we show that for any fixed $\Delta > 0$, the Δ-integrated volatility sequence,

$$I_n^{\Delta} := \int_{(n-1)\Delta}^{n\Delta} V(t)dt, \ n \in \mathbb{Z}, \tag{1.2}$$

is a strictly stationary solution of the difference equations,

$$\phi(B)I_n^{\Delta} = U_n, \ n \in \mathbb{Z}, \tag{1.3}$$

where B denotes the backward shift operator ($B^j Y_n := Y_{n-j}$), $\phi(z)$ is a polynomial of the form,

$$\phi(z) = \prod_{i=1}^{p}(1 - e^{\lambda_i \Delta} z),$$

where $\Re(\lambda_i) < 0$, $i = 1, \ldots, p$, and $(U_n)_{n\in\mathbb{Z}}$ is a p-dependent sequence. In the case when the driving subordinator has the property $EL(1)^2 < \infty$, then EU_n^2 is also finite and we know (see e.g. Brockwell and Davis (1991), Proposition 3.2.1) that $(U_n)_{n\in\mathbb{Z}}$ can then be expressed as a moving average of order p,

$$U_n = \theta(B)Z_n,$$

where $\theta(z)$ is a polynomial of the form,

$$\theta(z) = 1 + \theta_1 z + \cdots + \theta_p z^p,$$

and $(Z_n)_{n\in\mathbb{Z}}$ is an uncorrelated (but not necessarily independent) white noise sequence. This implies that $(I_n^{\Delta})_{n\in\mathbb{Z}}$ is a weak ARMA(p, q) process with $q \leq p$. In the case $p = 1$ this is already well-known (Barndorff-Nielsen and Shephard (2001)), however the autocorrelation function of the ARMA(1,1) model is restricted for lags h greater than zero to functions of the form $c\phi_1^h$, $c, \phi_1 > 0$. The purpose of introducing the finite variance CARMA(p, q) model for spot volatility is to escape from this restriction in order to obtain a more realistic representation of integrated volatility as estimated in practice by the so-called realized volatility, denoted henceforth by V^{Δ}.

In order to find a spot volatility model for which the Δ-integrated volatility I^{Δ} provides a good representation of V^{Δ} we derive, under the additional assumption that $EL(1)^2 < \infty$, expressions for the autocovariance functions of I^{Δ} and U. The former can be used to obtain preliminary parameter estimates of the CARMA(p, q) process for spot volatility V by choosing them in such a way that the autocorrelation function of I^{Δ} matches, in some ad hoc sense, the sample autocorrelation function of the realized volatility V^{Δ}. A more systematic approach is to search for a causal and invertible CARMA model for V which minimizes the sum of squares of the linear one-step prediction errors, when the corresponding weak ARMA model for I^{Δ} is applied to the data V^{Δ}. The sum of squares of the one-step linear prediction errors can be calculated directly as a function of the CARMA parameters using the state-space representation of the CARMA process. We show, under the assumptions of Theorem 3.6, that numerical minimization of this sum of squares gives strongly consistent estimators of the CARMA coefficients.

Using these estimates, we can simulate the spot volatility process V and the corresponding Δ-integrated volatility I^{Δ} using a variety of driving subordinators. Choosing the mean and variance of the driving subordinators so as to match the sample mean and variance of the realized volatility process V^{Δ}, we can then compare the empirical marginal distribution of the simulated integrated volatility series with that of V^{Δ}. Applying this technique to the DM/US$ daily realized volatility series, we obtain a remarkably good fit using the least squares CARMA(2,1) spot volatility model driven by a gamma subordinator with appropriately chosen $\mathrm{E}L(1)$ and $\mathrm{Var}L(1)$.

In Section 2 we shall review necessary and sufficient conditions under which a strictly stationary causal solution of the equations defining a Lévy-driven CARMA process exists. In Section 3 we shall impose the additional conditions that the driving Lévy process has non-decreasing sample paths (i.e. is a subordinator) and that $EL(1)^2 < \infty$. Under these conditions we determine the integrated process I^{Δ} and its second-order properties. Section 4 is concerned with estimation of the parameters of the process I based on observations of V^{Δ}, again assuming that $EL(1)^2 < \infty$. We illustrate the results by applying them to the DM/US$ daily realized volatility series considered earlier by Todorov (2011) and Brockwell, Davis and Yang (2011).

2 Lévy-driven CARMA processes

If L is a Lévy process with index set $\mathbb{R}$ (i.e. a process with homogeneous independent increments, càdlàg sample paths and $L(0) = 0$) and $p > q$, then an L-driven CARMA(p, q) process with real coefficients $\{a_1, \ldots, a_p;\ b_1, \ldots, b_q\}$ is defined (see Brockwell (2001)) as a strictly stationary solution of the state-space representation of the formal stochastic differential equation

$$a(D)V(t) = b(D)DL(t), \tag{2.1}$$

where D denotes differentiation with respect to t,

$$\begin{aligned} a(z) &:= z^p + a_1 z^{p-1} + \cdots + a_p\,, \\ b(z) &:= b_0 + b_1 z + \cdots + b_{p-1} z^{p-1}, \end{aligned}$$

and the coefficients b_j satisfy $b_q = 1$ and $b_j = 0$ for $q < j < p$. By Theorem 4.1 of Brockwell and Lindner (2009), there is no loss of generality in assuming that $a(z)$ and $b(z)$ have no common factors. Since $DL(t)$ does not exist in the usual sense, we interpret the differential equation (2.1) by means of its state-space representation, consisting of the *observation* and *state* equations,

$$V(t) = \mathbf{b}'\mathbf{X}(t), \tag{2.2}$$

and

$$d\mathbf{X}(t) - \mathbf{A}\mathbf{X}(t)dt = \mathbf{e}\, dL(t), \tag{2.3}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_p & -a_{p-1} & -a_{p-2} & \cdots & -a_1 \end{bmatrix}, \ \mathbf{e} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{p-2} \\ b_{p-1} \end{bmatrix}.$$

Every solution of (2.3) satisfies the relations

$$\mathbf{X}(t) = e^{\mathbf{A}(t-s)}\mathbf{X}(s) + \int_s^t e^{\mathbf{A}(t-u)}\mathbf{e}\, dL(u), \text{ for all } t > s, s, t \in \mathbb{R}. \tag{2.4}$$

Brockwell and Lindner (2009), Theorem 4.2, show that if L is not deterministic and $a(z)$ and $b(z)$ have no common factors, then necessary and sufficient conditions for (2.2) and (2.3) to have a strictly stationary solution V are that $E\max(0, \log|L_1|) < \infty$ and $a(z)$ is non-zero on the imaginary axis. In this case the strictly stationary solution is unique and is given by

$$V(t) = \int_{-\infty}^{\infty} g(t-u)\ dL(u), \tag{2.5}$$

with

$$g(t) = \left(\sum_{\lambda:\Re\lambda<0} \sum_{k=0}^{\nu(\lambda)-1} c_{\lambda k} t^k e^{\lambda t} \mathbf{1}_{(0,\infty)}(t) - \sum_{\lambda:\Re\lambda>0} \sum_{k=0}^{\nu(\lambda)-1} c_{\lambda k} t^k e^{\lambda t} \mathbf{1}_{(-\infty,0)}(t) \right), \tag{2.6}$$

where the sums are over the distinct zeroes λ of the polynomial $a(z)$ and $\nu(\lambda)$ denotes the multiplicity of λ. The sum $\sum_{k=0}^{\nu(\lambda)-1} c_{\lambda k} t^k e^{\lambda t}$ is the residue of $z \mapsto e^{zt} b(z)/a(z)$ at λ, i.e.

$$\sum_{k=0}^{\nu(\lambda)-1} c_{\lambda k} t^k e^{\lambda t} = \frac{1}{(\nu(\lambda)-1)!} \left[D_z^{\nu(\lambda)-1} \left((z-\lambda)^{\nu(\lambda)} e^{zt} b(z)/a(z) \right) \right]_{z=\lambda},$$

and D_z denotes differentiation with respect to z. (For a zero λ with $\nu(\lambda) = 1$ the last sum reduces to $b(\lambda)e^{\lambda t}/a'(\lambda)$.) The unique strictly stationary solution is causal if and only if $a(z)$ has no zeroes with positive real part, in which case the second sum in (2.6) disappears.

Now define the p^{th}-degree polynomial $\phi(z)$ and the coefficents $d_0, d_1, \ldots, d_p$ by

$$\phi(z) := \prod_{\lambda} (1 - e^{\lambda\Delta} z)^{\nu(\lambda)} =: \sum_{j=0}^{p} d_j z^j, \tag{2.7}$$

where the product is over the zeroes λ of $a(z)$. It follows at once from Lemma 2.1 of Brockwell and Lindner (2009) that for any fixed $s \in [0, \Delta]$ and for all $n \in \mathbb{Z}$, every solution V of (2.2) and (2.3) satisfies the *difference* equations,

$$\phi(B)V(n\Delta + s) = W_n(s), \; n \in \mathbb{Z}, \tag{2.8}$$

where

$$W_n(s) = Z_n^1(s) + Z_{n-1}^2(s) + \cdots + Z_{n-p+1}^p(s) \tag{2.9}$$

and

$$Z_n^r(s) := \int_{(n-1)\Delta+s}^{n\Delta+s} \boldsymbol{b}' \left(e^{(r-1)A\Delta} + \sum_{j=1}^{r-1} d_j e^{(r-1-j)A\Delta} \right) e^{A(n\Delta+s-u)} \boldsymbol{e} \, dL(u), \quad r = 1, \ldots, p. \tag{2.10}$$

Remark 2.1. Equation (2.8) is the starting point for the study of the integrated process I^Δ in Section 3. It also shows clearly, because of the independence of the increments of L, that if we sample any CARMA(p,q) process at uniformly spaced intervals, we obtain an autoregression driven by the $(p-1)$-dependent sequence $(W_n(s))_{n\in\mathbb{Z}}$. If $EL(1)^2 < \infty$ then this $(p-1)$-dependent sequence can be written as a weak moving average process (see Brockwell and Davis (1991), Proposition 3.2.1),

$$W_n(s) = EW_0(s) + Z_n(s) + \psi_1 Z_{n-1}(s) + \cdots + \psi_{p-1} Z_{n-p+1}(s),$$

for some zero-mean, constant variance, uncorrelated sequence $(Z_n(s))_{n\in\mathbb{Z}}$. This means that the sampled process is a weak ARMA(p,q) process with $q = p-1$ (or $q < p-1$ if ψ_{p-1} happens to be zero).

Remark 2.2. From now on we shall restrict attention to CARMA processes for which the zeroes of $a(z)$ and $b(z)$ all have strictly negative real parts. These conditions are the continuous-time analogues of causality and invertibility for discrete-time processes. Causality implies, by (2.5) and (2.6), that V has the representation,

$$V(t) = \int_{-\infty}^{t} g(t-u) dL(u), \tag{2.11}$$

with the kernel g specified by (2.6). By Proposition 3.2 of Brockwell and Lindner (2009), V also has the representation,

$$V(t) = \int_{-\infty}^{t} \boldsymbol{b}' e^{A(t-u)} \boldsymbol{e} dL(u). \tag{2.12}$$

Remark 2.3. In the special case when the zeroes of $a(z)$ all have multiplicity one and we assume causality, we obtain the extremely simple and useful characterization of V as a linear combination of dependent and possibly complex-valued CAR(1) processes all driven by L, namely

$$V(t) = \sum_{r=1}^{p} V^{(r)}(t), \tag{2.13}$$

where

$$V^{(r)}(t) = \alpha_r \int_{-\infty}^{t} e^{\lambda_r(t-u)} dL(u), \tag{2.14}$$

and

$$\alpha_r = \frac{b(\lambda_r)}{a'(\lambda_r)}, \ r = 1, \ldots, p, \tag{2.15}$$

where a' is the derivative of the function a.

Example 2.4 (The CARMA(2,1) Process). Consider the Lévy-driven CARMA(2,1) process specified by the equation

$$(D^2 + a_1 D + a_2)V(t) = (b_0 + D)DL(t).$$

In this case $b(z) = b_0 + z$ and $a(z) = z^2 + a_1 z + a_2 = (z - \lambda_1)(z - \lambda_2)$. Assuming that $\lambda_1 \neq \lambda_2$ and that both λ_1 and λ_2 have strictly negative real parts, then from (2.6) the kernel of the process V is

$$g(h) = (\alpha_1 e^{\lambda_1 h} + \alpha_2 e^{\lambda_2 h}) I_{[0,\infty)}(h),$$

where $\alpha_r = (b_0 + \lambda_r)/(\lambda_r - \lambda_{3-r})$, $r = 1, 2$. Consequently V has the representation,

$$V(t) = \alpha_1 \int_{-\infty}^{t} e^{\lambda_1(t-u)} dL(u) + \alpha_2 \int_{-\infty}^{t} e^{\lambda_2(t-u)} dL(u).$$

Remark 2.4. The representation (2.5) shows that if the kernel g is non-negative and L has non-decreasing sample-paths, i.e. is a subordinator, then V will be non-negative, as required if it is to represent spot volatility. Tsai and Chan (2005) showed that the kernel is non-negative if and only if the ratio $b(\cdot)/a(\cdot)$ is completely monotone. For the CARMA(2,1) process, which is of particular interest because of its financial applications, the condition is equivalent to the statement that the roots of $a(z) = 0$ are both real and that $b_0 \geq \min(|\lambda_i|)$ (see Brockwell and Davis (2001)).

3 The integrated process, $\{I_n^\Delta = \int_{(n-1)\Delta}^{n\Delta} V(t)dt,\ n \in \mathbb{Z}\}$

In this section we specify the integrated volatility process and then, under the assumption that $EL(1)^2 < \infty$, we determine its first and second-order moments.

Theorem 3.1. *If V is any solution of the equations (2.2) and (2.3) then the integrated sequence, $I_n^\Delta := \int_{(n-1)\Delta}^{n\Delta} V(t)dt,\ n \in \mathbb{Z}$, is an autoregression driven by a p-dependent sequence U. It satisfies the difference equations,*

$$\phi(B)I_n^\Delta = U_n := \sum_{r=1}^{p} \int_{-\Delta}^{0} Z_{n-r+1}^r(s)ds,\ n \in \mathbb{Z}, \tag{3.1}$$

where $\phi(z)$ was defined in (2.7) and $Z_n^r(s)$ in (2.10).

Proof. The validity of (3.1) follows at once from (2.8) by observing that $I_n^\Delta = \int_{-\Delta}^{0} V(n\Delta + s)\, ds$. The summands on the right of (3.1) are not independent, but they depend only on increments of L over the interval $[(n-p-1)\Delta, n\Delta]$ and hence the sequence $(U_n)_{n\in\mathbb{Z}}$ is p-dependent. □

Corollary 3.2. *If $EL(1)^2 < \infty$ then, by the stationarity and independence of the increments of L, the sequence $(U_n)_{n\in\mathbb{Z}}$ is a p-dependent finite variance identically distributed sequence (with not-necessarily zero mean). It follows that U_n can be expressed uniquely as a moving average with real-valued coefficients,*

$$U_n = EU_0 + \epsilon_n + \theta_1\epsilon_{n-1} + \cdots + \theta_p\epsilon_{n-p}, \tag{3.2}$$

where $(\epsilon_n)_{n\in\mathbb{Z}}$ is a weak white noise sequence with zero mean and the polynomial, $\theta(z) := 1 + \theta_1 z + \cdots + \theta_p z^p$, has no zeros in the interior of the unit disc. In other words $(I_n^\Delta)_{n\in\mathbb{Z}}$ is a weak ARMA(p, q) process with $q = p$ or $q < p$ if θ_p happens to be zero. The term weak here refers to the fact that $(\epsilon_n)_{n\in\mathbb{Z}}$ is uncorrelated but not necessarily independent.

Proof. By Proposition 3.2.1 in Brockwell and Davis (1991), there exist real-valued coefficients $\theta_1, \ldots, \theta_p$ and a white-noise sequence $(\epsilon_n)_{n\in\mathbb{Z}}$ such that (3.2) holds. A spectral density argument shows that the coefficients can be chosen in such a way that $\theta(z)$ has no zeroes in the interior of the unit disc and that $\theta(z)$ is uniquely determined by this constraint. The uniqueness of the sequence $(\epsilon_n)_{n\in\mathbb{Z}}$ then follows from Proposition 4.4.1 of Brockwell and Davis (1991) and the uniqueness of the Wold decomposition. □

The mean of the process V is readily found from (2.12) to be

$$EV(t) = m\boldsymbol{b}'A^{-1}\boldsymbol{e} = mb_0/a_p,$$

where $m := EL(1)$. From this it follows at once that

$$E(I_n^\Delta) = mb_0\Delta/a_p.$$

From now on we shall assume that $EL(1)^2 < \infty$. In order to compute the autocovariance functions of the processes I^Δ and U we need the autocovariance function of V which is given (see Brockwell (2001)) by,

$$\gamma_V(h) = \sigma^2 S(h), \ h \geq 0, \tag{3.3}$$

where $\sigma^2 = \mathrm{Var}(L(1))$ and $S(h)$ is the sum of residues of the mapping $z \mapsto e^{zh}\frac{b(z)b(-z)}{a(z)a(-z)}$ at the zeroes of $a(z)$. Thus, if $\nu(\lambda)$ denotes the multiplicity of the root λ of $a(z) = 0$,

$$\gamma_V(h) = \sum_{\lambda: a(\lambda)=0} \frac{\sigma^2}{(\nu(\lambda)-1)!}\left[\frac{d^{\nu(\lambda)-1}}{dz^{\nu(\lambda)-1}}\frac{(z-\lambda)^{\nu(\lambda)}e^{z|h|}b(z)b(-z)}{a(z)a(-z)}\right]_{z=\lambda}, \ h \in \mathbb{R}, \tag{3.4}$$

or equivalently

$$\gamma_V(h) = \sum_{\lambda: a(\lambda)=0}\sum_{j=0}^{\nu(\lambda)-1} \beta_{\lambda j}|h|^j e^{\lambda|h|}, \ h \in \mathbb{R}, \tag{3.5}$$

where the coefficients $\beta_{\lambda j}$ are easily found from (3.4). In the case when the roots are distinct, (3.5) simplifies to

$$\gamma_V(h) = \sigma^2 \sum_{\lambda: a(\lambda)=0} \frac{e^{\lambda|h|}b(\lambda)b(-\lambda)}{a'(\lambda)a(-\lambda)}, \ h \in \mathbb{R}.$$

From these results we obtain the following expression for the autocovariance function γ_{I^Δ} of the sequence I^Δ.

Proposition 3.3.

$$\gamma_{I^\Delta}(0) = 2\sum_\lambda \sum_{j=0}^{\nu(\lambda)-1} \frac{j!\beta_{\lambda j}}{(-\lambda)^{j+2}}\left(-\lambda\Delta - \sum_{s=0}^{j}\left(1 - \frac{j+1-s}{s!}(-\lambda\Delta)^s e^{\lambda\Delta}\right)\right),$$

where $\sum_\lambda$ *denotes summation over the zeroes,* λ*, of* $a(z)$ *and* $\nu(\lambda)$ *is the multiplicity of* λ*. For* $h \in \mathbb{N}$*,*

$$\gamma_{I^\Delta}(h) = \sum_\lambda \sum_{j=0}^{\nu(\lambda)-1} \frac{j!\beta_{\lambda j}e^{\lambda h\Delta}}{(-\lambda)^{j+2}}\left(\sum_{s=0}^{j}\frac{j+1-s}{s!}(-\lambda\Delta)^s\left((h+1)^s e^{\lambda\Delta} - 2h^s + (h-1)^s e^{-\lambda\Delta}\right)\right).$$

Proof. The result follows by straightforward integration starting from the relation, for $h \in \mathbb{N}_0$,

$$\begin{aligned}\gamma_{I^\Delta}(h) &= \int_{h\Delta}^{(h+1)\Delta}\int_0^\Delta \mathrm{Cov}(V(u), V(y))du\,dy \\ &= \int_{h\Delta}^{(h+1)\Delta}\int_0^\Delta \sum_\lambda\sum_{j=0}^{\nu(\lambda)-1}\beta_{\lambda j}|y-u|^j e^{\lambda|y-u|}du\,dy.\end{aligned}$$

□

Corollary 3.4. *If $\nu(\lambda) = 1$ for each λ such that $a(\lambda) = 0$ then*

$$\gamma_{I^\Delta}(h) = \begin{cases} \sum_\lambda 2\beta_{\lambda 0}\lambda^{-2}(e^{\lambda\Delta} - 1 - \lambda\Delta), & \text{if } h = 0, \\ \\ \sum_\lambda \beta_{\lambda 0}\lambda^{-2}(e^{\lambda\Delta} - 1)^2 e^{\lambda(|h|-1)\Delta}, & \text{if } h \in \mathbb{Z}\backslash\{0\}. \end{cases} \tag{3.6}$$

Corollary 3.5. *Defining U_n as in Theorem 3.1 and $d_0, \ldots, d_p$ as in (2.7), the autocovariance function γ_U of U is given by*

$$\gamma_U(h) = \begin{cases} \sum_{i=0}^p \sum_{j=0}^p d_i d_j \gamma_{I^\Delta}(|h| - j + i), & \text{if } |h| \in \{0, 1, \ldots, p\}, \\ \\ 0, & \text{otherwise.} \end{cases} \tag{3.7}$$

Proof. The result follows immediately from (3.1). □

The following theorem plays a crucial role in establishing the consistency of the least squares estimation procedure used in Section 4.

Theorem 3.6. *(i) Let $(V_t)_{t\geq 0}$ be a causal and invertible CARMA(p, q) process (see Remark 2.2) such that $a(z)$ and $b(z)$ have no common zeroes and such that all zeroes of $a(z)$, denoted by $\lambda_1, \ldots, \lambda_p$, have multiplicity 1. Suppose that $\Delta > 0$ and that $\Im(\lambda_i) \in (-\frac{\pi}{\Delta}, \frac{\pi}{\Delta})$ for all $i = 1 \ldots, p$. Then there is a unique pair $(\widetilde{\phi}, \widetilde{\theta})$ of polynomials of degree at most p with $\widetilde{\phi}(0) = \widetilde{\theta}(0) = 1$, such that $\widetilde{\phi}$ has no zeroes inside or on the unit circle, $\widetilde{\theta}$ has no zeroes inside the unit circle, and such that $(I_n^\Delta)_{n\in\mathbb{Z}}$ is a weak ARMA process with autoregressive polynomial $\widetilde{\phi}$ and moving average polynomial $\widetilde{\theta}$. These uniquely determined polynomials $\widetilde{\phi}$ and $\widetilde{\theta}$ have no common zeroes, and $\widetilde{\phi} = \phi$ with ϕ as defined in (2.7). In particular, $(I_n^\Delta)_{n\in\mathbb{Z}}$ cannot be represented as a weak ARMA(p', p) process with $p' < p$.*

(ii) For CARMA$(p, p-1)$ processes which satisfy the conditions in part (i), the mapping of the characteristic polynomials with p fixed into the characteristic polynomials of the corresponding ARMA processes I^Δ is one-to-one with a continuous inverse.

Proof. (i) The existence of $(\widetilde{\phi}, \widetilde{\theta})$ with $\widetilde{\phi} = \phi$ follows from Corollary 3.2. Hence we only have to show uniqueness and that $\widetilde{\phi}$ and $\widetilde{\theta}$ have no common zeroes.

It follows from Corollary 3.4 that if

$$c_i := \sigma^2 \frac{b(\lambda_i)b(-\lambda_i)}{a'(\lambda_i)a(-\lambda_i)} \lambda_i^{-2}(e^{\lambda_i\Delta} - 1)^2, \quad i = 1, \ldots, p, \tag{3.8}$$

then $\gamma_{I^\Delta}(h) = \sum_{i=1}^p c_i e^{\lambda_i\Delta(h-1)}$ for $h \in \mathbb{N}$. Since $|\Im\lambda_i| < \pi/\Delta$, $e^{\lambda_i\Delta} \neq 1$ and $e^{\lambda_1\Delta}, \ldots, e^{\lambda_p\Delta}$ are all distinct. Since $a(z)$ and $b(z)$ have no common zeroes, it follows that $c_i \neq 0$ for

$i = 1, \ldots, p$. Now if I^Δ were to be represented as a causal ARMA(p,p) process with autoregressive polynomial $\tilde{\phi}$ different from ϕ, then we could write

$$\gamma_{I^\Delta}(h) = \sum_{i=1}^{k} \sum_{j=0}^{r_i-1} \alpha_{ij} h^j z_i^h, \quad h \geq p+1,$$

for some complex numbers z_i (the reciprocals of the roots of $\widetilde{\phi}$) and α_{ij}, where $\{z_1, \ldots, z_k\} \neq \{e^{\lambda_1 \Delta}, \ldots, e^{\lambda_p \Delta}\}$. By the linear independence of solutions of difference equations (see, e.g., Brockwell and Davis (1991), Theorem 3.6.2), this is not possible, so that $\widetilde{\phi} = \phi$. The uniqueness of $\widetilde{\theta}$ then follows from Corollary 3.2. The polynomials ϕ and $\widetilde{\theta}$ have no common zeroes since otherwise I^Δ could be represented as an ARMA process with lower autoregressive order.

(ii) The one-to one property was established in (i). To show the continuity of the inverse mapping let log denote the principal branch of the natural logarithm (whose values have imaginary part in $(-\pi, \pi]$). Let $\phi(z)$ and $\theta(z)$ be the characteristic polynomials of the causal ARMA representation of I^Δ. Then the autoregressive roots $w_1, \ldots, w_p$ all have absolute values greater than 1 and they can be labelled in such a way that they are continuous functions of the coefficients of the autoregressive polynomial (see, e.g., Theorem 3.9.1 of Tyrtyshnikov (1997)).

Let $c_1, \ldots, c_p$ be the uniquely determined complex numbers such that

$$\gamma_{I^\Delta}(h) = \sum_{i=1}^{p} c_i w_i^{h-1}, h \in \mathbb{N}.$$

Then $\prod_{i=1}^p c_i \neq 0$ by (i) and $c_1, \ldots, c_p$ depend continuously on the ARMA coefficients.

Define $\lambda_1, \ldots, \lambda_p$, (all of which necessarily satisfy $\Im(\lambda_i) \in (-\pi/\Delta, \pi/\Delta)$) by

$$\lambda_i := -\frac{\log w_i}{\Delta}, \ i = 1, \ldots, p.$$

Then $a(z) = \prod_{i=1}^p (z - \lambda_i)$.

Now define

$$f_i := c_i a'(\lambda_i) a(-\lambda_i) \lambda_i^2 (e^{\lambda_i \Delta} - 1)^{-2} = -2 c_i \lambda_i^3 (e^{\lambda_i \Delta} - 1)^{-2} \prod_{j \in \{1,\ldots,p\} \setminus \{i\}} (\lambda_j^2 - \lambda_i^2).$$

Then $(\lambda_1, \ldots, \lambda_p, f_1, \ldots, f_p)$ depends continuously on $(w_1, \ldots, w_p, c_1, \ldots, c_p)$.

Define

$$P(z) := \sum_{i=1}^{p} f_i \prod_{k \in \{1,\ldots,p\} \setminus \{i\}} \frac{z - \lambda_k^2}{\lambda_i^2 - \lambda_k^2}, \ z \in \mathbb{C}$$

to be the (unique) Lagrange polynomial of degree less than or equal to $p-1$ such that $P(\lambda_i^2) = f_i$, $i = 1, \ldots, p$, and denote its zeroes by $\tilde{\mu}_1^2, \ldots, \tilde{\mu}_{p'}^2$ with $p' \leq p-1$. Then the coefficients of P depend continuously on $(\lambda_1, \ldots, \lambda_p, f_1, \ldots, f_p)$.

Writing $b(z) = \prod_{j=1}^{p-1}(z - \mu_i)$, it follows that

$$\sigma^2 b(z)b(-z) = \sigma^2(-1)^{p-1}\prod_{j=1}^{p-1}(z^2 - \mu_j^2)$$

is a polynomial in z^2 satisfying

$$\sigma^2 b(\lambda_i)b(-\lambda_i) = f_i,\ \ i = 1,\dots,p,$$

by (3.8). The uniqueness of polynomial interpolation (see, e.g., Tyrtyshnikov (1997), Section 12.2) then gives

$$\sigma^2(-1)^{p-1}\prod_{j=1}^{p-1}(z^2 - \mu_j^2) = \sigma^2 b(z)b(-z) = P(z^2).$$

Hence $\mu_1^2,\dots,\mu_{p-1}^2$ are the roots of $P(z)$, in particular $p' = p - 1$ and the roots of $b(z)$ can be labelled such that $\mu_1^2 = \tilde{\mu}_1^2,\dots,\mu_{p-1}^2 = \tilde{\mu}_{p-1}^2$, and hence depend continuously on the ARMA parameters. □

Remark 3.7. (Strict stationarity and ergodicity) The sequence I^Δ is strictly stationary by the strict stationarity of the process V and ergodic by the β-mixing property of V established in Proposition 3.34 of Marquardt and Stelzer (2007). (Recall that we assume that V is causal and that $EL(1)^2 < \infty$.)

Example 3.8. The CAR(1) process with defining stochastic differential equation,

$$(D - \lambda)V_t = DL_t,\quad \lambda < 0,$$

has strictly positive causal kernel, $g(t) = e^{\lambda t}I_{[0,\infty)}(t)$, and, if L is a subordinator, has non-negative sample-paths, making it a useful model for stochastic volatility as in Barndorff-Nielsen and Shephard (2001). From (3.5) the autocovariance function of V is

$$\gamma_V(h) = \frac{\sigma^2}{2|\lambda|}e^{\lambda|h|},\quad h \in \mathbb{R}.$$

Equations (3.1) and (3.2) for the integrated sequence $I = I^\Delta$ with $\Delta = 1$ take the form,

$$(1 - \phi B)I_n = (1 + \theta B)\epsilon_n + \frac{m}{|\lambda|}(1 - \phi),$$

where $\phi = e^\lambda$ and θ is found by evaluating the autocorrelation function of the right-hand side from (3.6) and (3.7) and calculating the corresponding value of θ. This value is readily found to be

$$\theta(\lambda) = -r - \sqrt{r^2 - 1}, \tag{3.9}$$

where

$$r = \frac{1 + \lambda - e^{2\lambda}(1-\lambda)}{1 + 2\lambda e^{\lambda} - e^{2\lambda}}.$$

The mapping $\lambda \mapsto (\phi, \theta)$ is one-to-one from $(-\infty, 0)$ onto the one-dimensional submanifold of the Cartesian product, $(-1,1) \times (-1,1)$, which is shown as the curved line in Figure 1. The limits of θ as $\lambda \to -\infty$ and $\lambda \to 0$ are 0 and $2 - \sqrt{3}$ respectively.

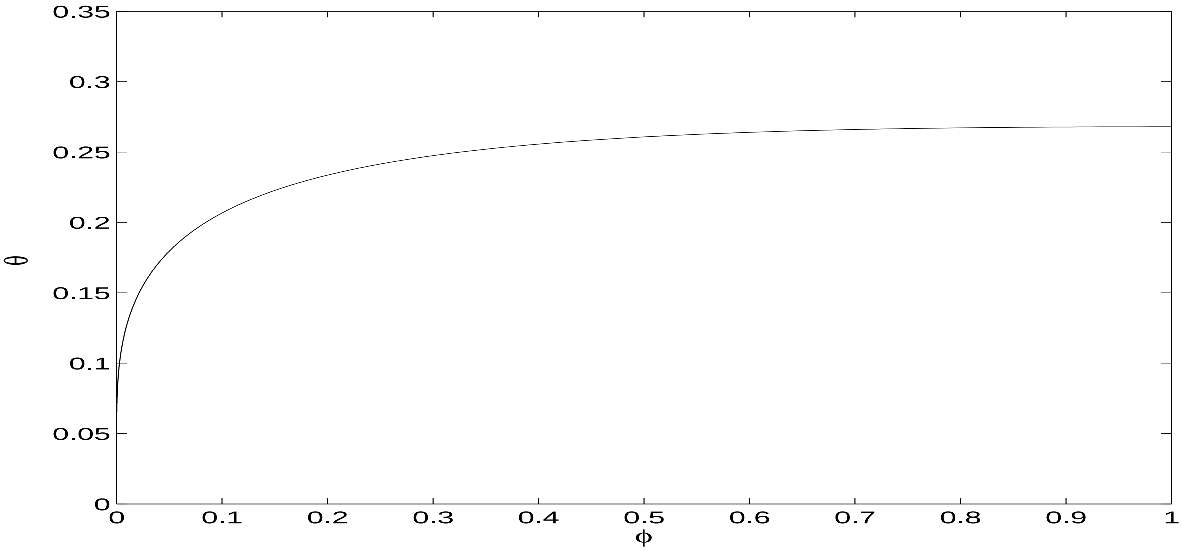

Figure 1: The parameter set for the ARMA(1,1) coefficients of the integrated CAR(1) process in Example 3.8.

Example 3.9. The CARMA(2,1) process with defining stochastic differential equation,

$$(D - \lambda_1)(D - \lambda_2)Y_t = (D - \mu_1)DL_t,$$

where $\lambda_1 \neq \lambda_2$, is invertible with strictly positive causal kernel if and only if the parameters satisfy the conditions,

$$0 > \lambda_1 > \lambda_2 \text{ and } 0 > \lambda_1 > \mu_1. \tag{3.10}$$

We assume also that $\mu_1 \neq \lambda_2$ since otherwise Y would be a CAR(1) process. The integrated sequence $I = I^{\Delta}$ with $\Delta = 1$ satisfies the ARMA(2,2) difference equations (3.1) and (3.2), i.e.

$$(1 - \eta_1 B)(1 - \eta_2 B)I_n = (1 - \eta_1)(1 - \eta_2)\frac{m|\mu_1|}{\lambda_1 \lambda_2} + (1 - \xi_1 B)(1 - \xi_2 B)\epsilon_n,$$

where $\eta_1 = e^{\lambda_1}$, $\eta_2 = e^{\lambda_2}$ and ξ_1 and ξ_2 (with $|\xi_1| \geq |\xi_2|$) can be found from the autocorrelation function, γ_U, calculated from Corollaries 3.4 and 3.5 as in Example 3.8. Explicit expressions for the mapping $(b_0, \lambda_1, \lambda_2) \mapsto (\eta_1, \eta_2, \xi_1, \xi_2)$ are extremely unwieldy but we know from Theorem 3.6 that it is injective with continuous inverse.

4 Estimation

Given a sequence of daily realized volatilities we aim to find a CARMA model for spot volatility V such that the corresponding integrated volatility process provides a good representation of the realized volatility sequence. A complete solution requires the estimation of the defining polynomials $a(z)$ and $b(z)$ and specification of L.

The daily realized volatility of the DM/US$ exchange rate from December 1, 1986, through June 30, 1999 (kindly provided to us by Viktor Todorov), is shown in Figure 2 and its sample autocorrelation function is shown as the bar-graph in Figure 3. For details on the determination of the realized volatility see Andersen et al. (2001). It is clear that a good match between the sample autocorrelation function in Figure 3 for lags greater than zero and a single exponential function (as would be derived from the Ornstein-Uhlenbeck (CAR(1)) model for spot volatility) is not possible. The line graph in Figure 3 is based on a CARMA(2,1) model for spot volatility which leads to a linear combination of two exponential functions for the autocorrelations of integrated volatility at lags greater than zero. Since a linear combination of two exponential functions appears to give a reasonably good fit to the sample autocorrelation of realized volatility we shall focus attention on modelling this data with a CARMA(2,1) model for the spot volatility.

Our goal then is to estimate the parameters a_1, a_2, b_1 (or equivalently the parameters λ_1, λ_2 and μ_1 in the model (2.1) for spot volatility with

$$a(z) = (z - \lambda_1)(z - \lambda_2)$$

and

$$b(z) = (z - \mu_1),$$

where λ_1, λ_2 and μ_1 satisfy the constraints (3.10). We also need to estimate $EL(1)$ and $\mathrm{Var}L(1)$ and to recover whatever additional information we can about the process L.

Method 1. An ad hoc method for choosing suitable polynomials $a(z)$ and $b(z)$ is to compute, for given $a(z)$ and $b(z)$, the autocorrelation function of I^Δ using Proposition 3.3 and to minimize numerically the sum of squares of deviations of the autocorrelation function from the sample autocorrelation function of the realized volatility at selected lags. This will give preliminary estimates.

Method 2. An alternative approach, which gives strongly consistent estimators of the CARMA coefficients a_1, a_2 and b_1, is to find numerically the values of a_1, a_2 and b_1 which minimize the sum of squares, S, of the one-step linear prediction errors of I^Δ based on the implied weak ARMA$(2,2)$ model. The sum of squares can be computed directly in terms of a_1, a_2 and b_1 by first computing the zeroes λ_1 and λ_2 of $a(z)$. These determine the polynomial $\phi(z)$ in (3.1). The moving average coefficients in (3.2) are then determined by computing the autocovariance function (3.7) and hence the corresponding coefficients

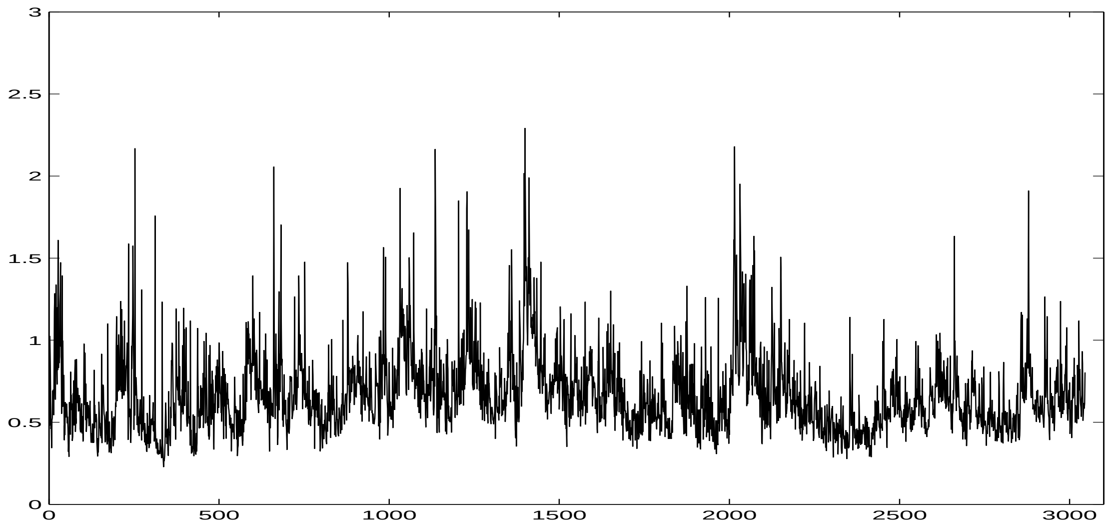

Figure 2: The realized daily volatility of the DM/US$ exchange rate, December 1, 1986, through June 30, 1999 .

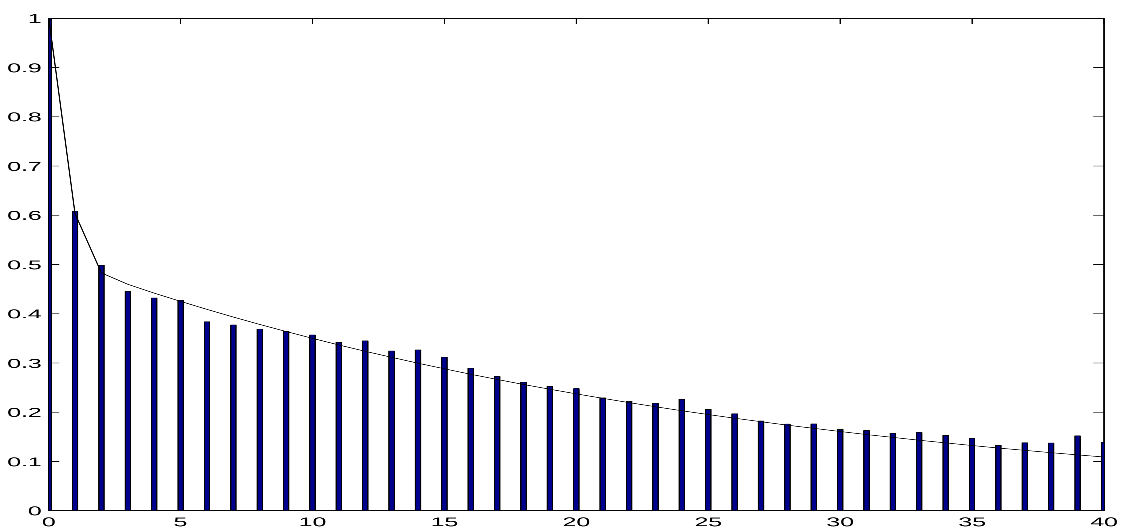

Figure 3: The vertical bars represent the sample autocorrelation function of the daily realized volatility of the DM/US$ exchange rate. The line graph is the autocorrelation function of the integrated volatility corresponding to a CARMA(2,1) model for spot volatility.

θ_1 and θ_2. (These coefficients can be found analytically from (3.7) when $p = 2$ and numerically using Wilson's algorithm (Wilson (1969)) for larger values of p.) Once the autoregressive and moving average polynomials have been determined from $a(z)$ and $b(z)$, the sum of squares S of the one-step prediction errors for I^{Δ} can be calculated by standard time-series methods. Minimization of S with respect to a_1, a_2 and b_1 is equivalent to minimization with respect to the corresponding ARMA(2,2) coefficients, restricted to a three-dimensional submanifold of $\mathbb{R}^4$. By the argument of Theorem 10.8.1 of Brockwell and Davis (1991) the estimators of the ARMA coefficients are strongly consistent and by continuity of the inverse of the injective mapping of the CARMA parameters into the ARMA parameters (a special case of Theorem 3.6) the estimators of the CARMA coefficients are also strongly consistent.

Remark 4.1. The procedure outlined above can clearly be extended to the estimation of parameters of any CARMA$(p, p-1)$ process satisfying the assumptions of Theorem 3.6 and yields consistent estimators based on the observations of I^Δ.

Example 4.2. We now illustrate the procedure with the DM/US$ daily realized volatility series shown in Figure 1. Measuring the spot volatility in units of volatility per day, the realized volatility series corresponds to volatility integrated over time intervals of length 1, i.e I^Δ with $\Delta = 1$.

Using Proposition 3.3 and minimizing the sum of squares of the deviations of the autocorrelation functions of I^Δ and V^Δ at lags 1, 2, 10, 20 and 40 gives the preliminary spot-volatility model,

$$(D^2 + 3.09054D + .10983)V(t) = (.23302 + D)DL(t),$$

with corresponding $\lambda_1 = -.035956$ and $\lambda_2 = -3.05458$.

Using these coefficients as initial values and minimizing the sum of squares of the one-step linear prediction errors, computed as described above, we find the least-squares model to be,

$$(D^2 + 3.07141D + .11793)V(t) = (.23938 + D)DL(t), \tag{4.1}$$

with corresponding $\lambda_1 = -.038890$ and $\lambda_2 = -3.02152$. The autocorrelation function of the daily integrated volatility corresponding to this model is plotted as the line graph in Figure 3.

It remains to identify a subordinator L which yields daily integrated volatilities compatible with the realized daily volatility series shown in Figure 2. This was done by trying out potential subordinators, with parameters chosen to match the mean and variance of the realized volatility series, and then simulating sample paths of the corresponding CARMA(2,1) process defined by (4.1), integrating the sample-paths over successive days and comparing the empirical cumulative distribution functions and kernel density estimates of the realized volatility series with those of the integrated volatilities calculated from the models. The results are shown in Figure 4 for three different subordinators.

The top graphs were generated by simulating the CARMA(2,1) process (4.1) driven by a compound Poisson subordinator with exponentially distributed jumps. The mean

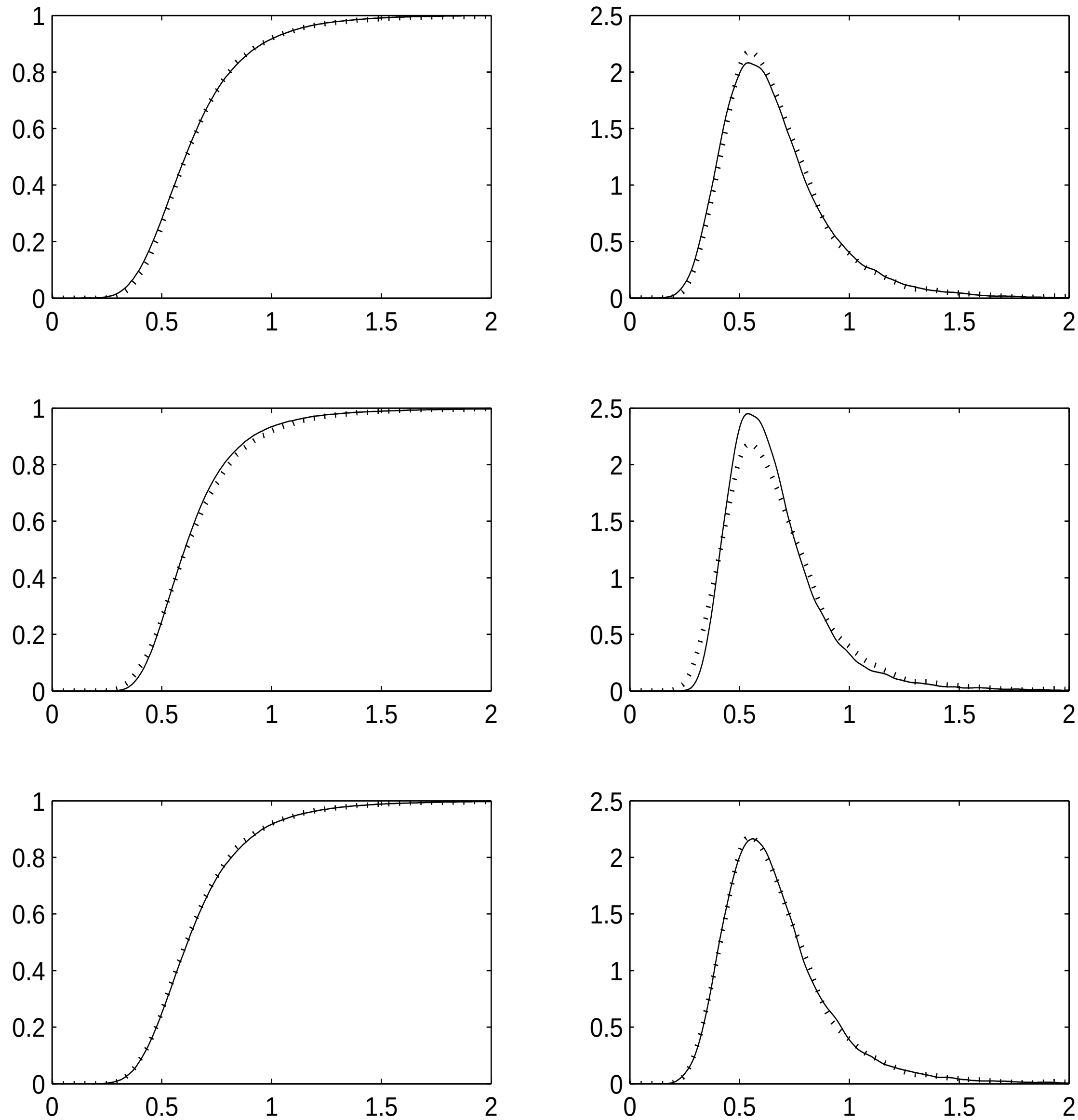

Figure 4: The empirical cdf (left) and kernel density estimate (right) of the daily realized volatility of the DM/US$ exchange rate are shown as dotted lines. The solid lines are the corresponding graphs for daily integrated volatility of three subordinator-driven CARMA(2,1) spot volatility processes with subordinator moments chosen to match the mean and variance of the integrated volatility with those of the realized volatility. For the top graphs the driving subordinator is a compound Poisson process with exponential jumps, for the middle graphs an inverse Gaussian subordinator and for the bottom graphs a gamma subordinator.

jump rate of the process was .5478 and the mean jump size was .6008. The simulation of the CARMA process is greatly simplified by the decomposition (2.14) which reduces the simulation to that of two Ornstein-Uhlenbeck processes with the same driving subordinator. In fact from the simulated jump-times and jump-sizes the complete sample-path can be constructed and the daily integrals easily computed. The same is true for compound-Poisson-driven CARMA processes of any order as long as the zeroes of $a(z)$ are distinct.

The middle graphs are derived from the spot volatility process (4.1) with inverse Gaussian subordinator having $EL(1) = .3291$ and $\mathrm{Var}L(1) = .3954$. Simulation in this case was carried out by using an Euler approximation to generate values of the spot volatility at intervals of .01 days and integrating numerically to get 40,000 daily integrated volatility values.

The bottom graphs were derived in the same way, using a gamma subordinator with $EL(1) = .3291$ and $\mathrm{Var}L(1) = .3954$. The empirical cdf and kernel density estimates were again based on 40,000 daily integrated volatility values.

The results show that all three models gave reasonably good fits to the marginal distribution of the realized volatility series. The discrepancies are shown more clearly in the kernel density estimates, but in the case of the gamma subordinator the empirical and simulated distributions are virtually indistinguishable.

5 Conclusions.

By making use of the properties of integrated CARMA(p, q) processes, we have developed a technique for estimating the parameters of a CARMA(p, q) model for spot volatility which is compatible with realized (integrated) volatility. The procedure was illustrated by application to a daily realized volatility sequence for the DM/US$ exchange rate. All three candidate subordinators for driving the spot volatility process gave reasonably good matches with the empirical marginal distribution of daily realized volatility, indicating the difficulty of discriminating between them on this basis. The goodness of fit of the gamma-driven model however was remarkable.

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