CARMA processes as solutions of integral equations

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Abstract

A CARMA(p,q) process is defined by suitable interpretation of the formal p^{th} order differential equation $a(D)Y_t = b(D)DL_t$, where L is a two-sided Lévy process, $a(z)$ and $b(z)$ are polynomials of degrees p and q, respectively, with $p > q$, and D denotes the differentiation operator. Since derivatives of Lévy processes do not exist in the usual sense, the rigorous definition of a CARMA process is based on a corresponding state space equation. In this note, we show that the state space definition is also equivalent to the integral equation $a(D)J^pY_t = b(D)J^{p-1}L_t + r_t$, where J, defined by $Jf_t := \int_0^t f_s \, ds$, denotes the integration operator and r_t is a suitable polynomial of degree at most $p-1$. This equation has well defined solutions and provides a natural interpretation of the formal equation $a(D)Y_t = b(D)DL_t$.

Keywords: CARMA process, continuous time autoregressive moving average process, differential equation, integral equation

1. Introduction

Just as ARMA processes play a central role in the representation of time series with discrete time parameter, $(Y_n)_{n\in\mathbb{Z}}$, CARMA (continuous-time autoregressive moving average) processes play an analogous role in the representation of time series with continuous time parameter, $(Y_t)_{t\in\mathbb{R}}$. Lévy-driven CARMA processes permit the modelling of heavy-tailed and asymmetric time series, allowing for a wide variety of sample-path and distributional properties. In recent years there has been a resurgence of interest in these processes and in continuous-time processes more generally, partly as a result of the very successful application of stochastic differential equation models to problems in finance, particularly to the pricing of options. In this context the celebrated Ornstein-Uhlenbeck stochastic volatility model of Barndorff-Nielsen and Shephard uses a Lévy-driven CAR(1) process as a continuous-time model for spot volatility and Brockwell and Lindner (2013) have extended this to a more general CARMA representation. Gaussian CARMA models were used very successfully by Jones (1981) for the modelling of irregularly-spaced data. More recent applications include the application of stable CARMA processes to futures pricing in electricity markets (Garcia et al. (2011)), the CARMA interest rate model (Andresen et al. (2014)), and applications to signal extraction (McElroy (2013)). They have also been used to approximate high-frequency discrete-time data encountered in both financial and turbulence studies. In this note we give a new and direct characterization of CARMA processes.

Let $L = (L_t)_{t\in\mathbb{R}}$ be a two-sided Lévy process, i.e. a process with homogeneous independent increments, continuous in probability, with càdlàg sample paths (i.e. right-continuous with finite left-limits) and $L_0 = 0$. For integers p and q such that $p > q$, let $a_1, \ldots, a_p, b_0, \ldots, b_{p-1}$ be complex valued coefficients such that $b_j = 0$ for $j > q$, $b_q \neq 0$, let $a_0 := 1$, and define the polynomials $a(z)$ and $b(z)$ by

$$a(z) = z^p + a_1 z^{p-1} + \ldots + a_p, \quad \text{and} \quad b(z) = b_0 + b_1 z + \ldots + b_{p-1} z^{p-1}.$$

Denote by D the differentiation operator with respect to t. It is natural, by analogy with the definition of an ARMA process, to attempt to define a CARMA(p, q) process as a process Y satisfying the formal p^{th}-order stochastic differential equation

$$a(D)Y_t = b(D)DL_t. \tag{1}$$

However, since Lévy processes are not differentiable, DL_t does not exist in the usual sense, so that CARMA processes are rigorously defined by a corresponding state-space representation. More precisely, let

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_p & -a_{p-1} & -a_{p-2} & \cdots & -a_1 \end{bmatrix}, \; \mathbf{e}_p = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \; \mathbf{b} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{p-2} \\ b_{p-1} \end{bmatrix} \in \mathbb{C}^p$$

with $A \in \mathbb{C}^{p\times p}$. For $p = 1$ the matrix A is to be understood as $A = (-a_1)$. A (complex valued) *CARMA*(p, q)*-process* $Y = (Y_t)_{t\in\mathbb{R}}$ driven by L and with characteristic polynomials $a(z)$ and $b(z)$ is then defined as

$$Y_t = \mathbf{b}'\mathbf{X}_t, \quad t \in \mathbb{R}, \tag{2}$$

(a prime will be used throughout to denote the transpose of a matrix or vector), where $\mathbf{X} = (\mathbf{X}_t)_{t\in\mathbb{R}}$ is a $\mathbb{C}^p$-valued state-vector process satisfying the stochastic differential equation,

$$\mathrm{d}\mathbf{X}_t = A\mathbf{X}_t \,\mathrm{d}t + \mathbf{e}_p \,\mathrm{d}L_t,$$

i.e.

$$\mathbf{X}_t = \mathbf{X}_s + \int_s^t A\mathbf{X}_u \,\mathrm{d}u + \mathbf{e}_p(L_t - L_s), \quad \forall\, s \le t \in \mathbb{R}, \tag{3}$$

see Brockwell (2001). The solution of Equation (3) is unique for any given $\mathbf{X}_0$ and satisfies

$$\mathbf{X}_t = \mathrm{e}^{A(t-s)}\mathbf{X}_s + \int_{(s,t]} \mathrm{e}^{A(t-u)}\mathbf{e}_p \, dL_u, \ \forall s \le t \in \mathbb{R}. \tag{4}$$

The state vector process $\mathbf{X}$ is not necessarily uniquely determined by Y, L, $a(z)$ and $b(z)$, but it is if $a(z)$ and $b(z)$ have no common zeroes, see (Brockwell and Lindner, 2015, Proposition 2.1 (ii)). Observe that CARMA processes and state vector processes are necessarily càdlàg processes.

The aim of the present note is to bypass the state-space representation, giving a rigorous interpretation of the formal differential equation (1) itself in such a way that the solution for Y coincides with the CARMA process as defined by (2) and (3). In so doing we shall clarify the precise nature of the sample-paths of Y and indeed give a rigorous interpretation of (1) when L is assumed only to be a càdlàg process with $L_0 = 0$.

The idea is simply to formally integrate the differential equation (1) p times. This gives the equation,

$$a(D)J^pY_t = b(D)J^{p-1}L_t + (\text{polynomial of degree} \le p-1),$$

both sides of which are obviously well-defined if Y is a càdlàg process. Here J denotes the integration operator, defined by $Jf_t := \int_0^t f_s \,\mathrm{d}s$ for any càdlàg process $(f_t)_{t\in\mathbb{R}}$. We shall show that the resulting specification of Y coincides indeed with the definition via (2) and (3), providing a natural and rigorous interpretation of (1).

2. Results

Let J be the integration operator, which associates with any càdlàg function $f = (f_t)_{t\in\mathbb{R}} : \mathbb{R} \to \mathbb{C}, t \mapsto f_t$, the function Jf, defined by

$$Jf_t := \int_0^t f_s \,\mathrm{d}s,$$

(where $\int_0^t$ is interpreted as $-\int_t^0$ if $t < 0$). We say that a function $g : \mathbb{R} \to \mathbb{C}$ is *differentiable with càdlàg derivative* Dg, if g is continuous and there exists a càdlàg function Dg such g is right-differentiable and left-differentiable at every point $t \in \mathbb{R}$ with right-derivative Dg_t and left-derivative $Dg_{t-} = \lim_{\varepsilon\downarrow 0, \varepsilon\neq 0} Dg_{t-\varepsilon}$, respectively. Since a càdlàg function has only countably many jumps, this implies that g is differentiable at all but countably many points, and since Dg is bounded on compacts, it follows that g must be absolutely continuous, (see e.g. Cohn (2013), Theorem 6.3.11).The *derivative* Dg is then obviously uniquely determined.

It is easy to see that Jf is differentiable with càdlàg derivative $DJ(f) = f$ for any càdlàg function f. We say that a function U is p-times differentiable with p^{th} càdlàg derivative, if it is $(p-1)$-times continuously differentiable in the usual sense, and if the $(p-1)^{\text{st}}$-derivative is differentiable with càdlàg derivative. We then write $U_t^{(j)} = D^j U_t$ for the j^{th} derivative, $j = 0, \dots, p$.

With $a_0, \dots, a_p, b_0, \dots, b_{p-1}$ and $a(z)$ and $b(z)$ as in the introduction, define the polynomials

$$\widetilde{a}(z) := 1 + a_1 z + a_2 z^2 + \ldots + a_p z^p \quad \text{and} \quad \widetilde{b}(z) := b_0 z^{p-1} + b_1 z^{p-2} + \ldots + b_{p-1}.$$

Then

$$a(D)J^p f = \widetilde{a}(J) f \quad \text{and} \quad b(D)J^{p-1} f = \widetilde{b}(J) f$$

for each càdlàg function f.

Theorem 1. *Suppose that $q < p$ and $L = (L_t)_{t\in\mathbb{R}}$ is a two-sided Lévy process. Then, with the notation already introduced, the following statements are true.*
(a) If Y is a CARMA(p, q)-process driven by L with characteristic polynomials $a(z)$ and $b(z)$, then there exists a $\mathbb{C}^p$-valued random vector $\mathbf{V}_0$ such that

$$a(D)J^p Y_t = b(D)J^{p-1} L_t + a(D)J^p(\mathbf{b}' \mathrm{e}^{At} \mathbf{V}_0), \quad t \in \mathbb{R}, \tag{5}$$

or equivalently

$$\widetilde{a}(J)Y_t = \widetilde{b}(J)L_t + \widetilde{a}(J)(\mathbf{b}'\mathrm{e}^{At}\mathbf{V}_0). \tag{6}$$

(b) Conversely, for every $\mathbb{C}^p$-valued random vector $\mathbf{V}_0$, there exists a unique càdlàg solution $Y = (Y_t)_{t\in\mathbb{R}}$ of (5) (equivalently (6)), and this solution is a CARMA(p,q)-process driven by L with characteristic polynomials $a(z)$ and $b(z)$. The state-vector process $(\mathbf{X}_t)_{t\in\mathbb{R}}$ of this CARMA process can be chosen to satisfy $\mathbf{X}_0 = \mathbf{V}_0$.
(c) Denoting by $\lambda_1, \ldots, \lambda_p$ the (not necessarily distinct) zeroes of the polynomial $a(z)$ and by J_λ the operator which maps any càdlàg function $f : \mathbb{R} \to \mathbb{C}$ into the function,

$$J_\lambda f_t = \int_0^t e^{\lambda(t-u)} f_u \,\mathrm{d}u, \;\; t \in \mathbb{R},$$

the unique càdlàg solution Y of (5) is

$$Y_t = b(D)D[J_{\lambda_1} \cdots J_{\lambda_p} L_t] + \mathbf{b}'e^{At}\mathbf{V}_0, \quad t \in \mathbb{R}. \tag{7}$$

Proof. (a) Consider first a CARMA process $(Y_t = \mathbf{b}'\mathbf{X}_t)_{t\in\mathbb{R}}$ driven by L with characteristic polynomials $a(z)$ and $b(z)$, and such that $\mathbf{X}_0 = \mathbf{0}$. Write $\mathbf{X}_t = (X_t^1, \ldots, X_t^p)'$. Since $\mathbf{X}_0 = \mathbf{0}$ and $L_0 = 0$, equation (3) is equivalent to

$$\begin{aligned} X_t^j &= \int_0^t X_s^{j+1}\,\mathrm{d}s = JX_t^{j+1} \quad \forall\, j = 1, \ldots, p-1, \quad \text{and} \\ X_t^p &= -a_p \int_0^t X_s^1\,\mathrm{d}s - \ldots - a_1 \int_0^t X_s^p\,\mathrm{d}s + L_t, \end{aligned}$$

hence

$$X_t^j = J^{p-j} X_t^p \quad \forall\, j = 1, \ldots, p \tag{8}$$

and

$$\begin{aligned} L_t &= X_t^p + a_1 J X_t^p + \ldots + a_p J X_t^1 = X_t^p + a_1 J X_t^p + a_2 J^2 X_t^p + \ldots + a_p J^p X_t^p \\ &= \widetilde{a}(J) X_t^p. \end{aligned} \tag{9}$$

From (8) and (9) we conclude that

$$\widetilde{a}(J)X_t^j = \widetilde{a}(J)J^{p-j}X_t^p = J^{p-j}\widetilde{a}(J)X_t^p = J^{p-j}L_t.$$

Hence, by (2),

$$\widetilde{a}(J)Y_t = \widetilde{a}(J)\mathbf{b}'\mathbf{X}_t = \sum_{j=1}^p b_{j-1}\widetilde{a}(J)X_t^j = \sum_{j=1}^p b_{j-1}J^{p-j}L_t = \widetilde{b}(J)L_t.$$

This shows that Y satisfies (5), or equivalently (6), with $\mathbf{V}_0 = \mathbf{0}$.

Now let $Y = (Y_t = \mathbf{b}'\mathbf{X}_t)_{t\in\mathbb{R}}$ be a CARMA-process driven by L with characteristic polynomials $a(z)$ and $b(z)$ and some state vector process $\mathbf{X} = (\mathbf{X}_t)_{t\in\mathbb{R}}$, which does not necessarily satisfy $\mathbf{X}_0 = \mathbf{0}$. Let $\widetilde{Y} = (\widetilde{Y}_t = \mathbf{b}'\widetilde{\mathbf{X}}_t)_{t\in\mathbb{R}}$ be the CARMA-process whose state vector process $\widetilde{\mathbf{X}}$ satisfies $\widetilde{\mathbf{X}}_0 = \mathbf{0}$ (and (3)). Then $\mathrm{d}(\mathbf{X}_t - \widetilde{\mathbf{X}}_t) = A(\mathbf{X}_t - \widetilde{\mathbf{X}}_t)\,\mathrm{d}t$, hence

$$\mathbf{X}_t = \widetilde{\mathbf{X}}_t + \mathrm{e}^{At}(\mathbf{X}_0 - \widetilde{\mathbf{X}}_0) = \widetilde{\mathbf{X}}_t + \mathrm{e}^{At}\mathbf{X}_0 \quad \forall\, t \in \mathbb{R}$$

(this follows easily for both $t \geq 0$ and $t < 0$ from (4)). We conclude that $Y_t = \widetilde{Y}_t + \mathbf{b}'\mathrm{e}^{At}\mathbf{X}_0$ and hence

$$a(D)J^pY_t = a(D)J^p\widetilde{Y}_t + a(D)J^p(\mathbf{b}'\mathrm{e}^{At}\mathbf{X}_0) = b(D)J^{p-1}L_t + a(D)J^p(\mathbf{b}'\mathrm{e}^{At}\mathbf{X}_0).$$

This shows that Y satisfies (5) (equivalently (6)) with $\mathbf{V}_0 := \mathbf{X}_0$.

(b) Now let $\mathbf{V}_0$ be a $\mathbb{C}^p$-valued random vector. We have already seen that the CARMA process with state vector process $\mathbf{X}$ satisfying $\mathbf{X}_0 = \mathbf{V}_0$ is a solution of (5), hence we only have to address uniqueness. Let Y and $\widehat{Y}$ be two solutions of (5) with the same $\mathbf{V}_0$ and let $U_t = J^p(Y_t - \widehat{Y}_t)$. Then $a(D)U_t = 0$. Writing $\mathbf{W}_t = (U_t, U_t^{(1)}, \ldots, U_t^{(p-1)})'$, this is equivalent to $D\mathbf{W}_t = A\mathbf{W}_t$, hence $\mathbf{W}_t = \mathrm{e}^{At}\mathbf{W}_0$ from (4). But $\mathbf{W}_0 = \mathbf{0}$ by definition of U. This shows $U = 0$ and hence $Y = \widehat{Y}$.

(c) Let Y be the unique càdlàg solution of (5). Observe that by partial integration (see e.g. (Cohn, 2013, Corollary 6.3.9)),

$$J_\lambda(D-\lambda)f_t = e^{\lambda t}\int_0^t \left(e^{-\lambda u}Df_u - \lambda e^{-\lambda u}f_u\right)\,\mathrm{d}u = e^{\lambda t}(e^{-\lambda t}f_t - f_0) = f_t \quad (10)$$

for any differentiable function f with càdlàg derivative such that $f_0 = 0$ and any $\lambda \in \mathbb{C}$. Since $a(D) = (D-\lambda_p)\cdots(D-\lambda_1)$, applying $J_{\lambda_p}, \ldots, J_{\lambda_1}$ and D^p successively to (5) gives

$$Y_t = D^pJ_{\lambda_1}\cdots J_{\lambda_p}[b(D)J^{p-1}L_t] + \mathbf{b}'e^{At}\mathbf{V}_0. \quad (11)$$

Since $DJ_\lambda f_t = \lambda J_\lambda f_t + f_t = J_\lambda Df_t$ by (10) for any differentiable function f with càdlàg derivative such that $f_0 = 0$, and since $J^{p-1}L_t$ is $(p-1)$-times differentiable with $(p-1)^{\mathrm{st}}$ càdlàg derivative with $D^jJ^{p-1}L_t|_{t=0} = 0$ for all $j \in \{0, \ldots, p-1\}$, it follows that $J_{\lambda_1}\cdots J_{\lambda_p}$ and $b(D)$ in (11) commute. Since also $J_\lambda J_\mu f_t = J_\mu J_\lambda f_t$ for any $\lambda, \mu \in \mathbb{C}$ and càdlàg function f (as a consequence of $(D-\lambda)(D-\mu) = (D-\mu)(D-\lambda)$ and the facts that $(D-\lambda)J_\lambda f_t = f_t$ and $J_\lambda J_\mu f_t|_{t=0} = J_\mu f_t|_{t=0} = 0$), we may also interchange $J_{\lambda_1}\cdots J_{\lambda_p}$ with J^{p-1}, so that $Y_t = D^p b(D) J^{p-1} J_{\lambda_1}\cdots J_{\lambda_p} L_t + \mathbf{b}' e^{At}\mathbf{V}_0$, giving (7). □

Remark 1. Equations (5) and (7) give rigorous and intuitive interpretations of (1). Equation (7) can be understood as applying formally the operator $J_{\lambda_1}\cdots J_{\lambda_p}$ to (1), which can be regarded as the inverse operator of $a(D)$ as seen in the proof of part (c).

Corollary 1. *For any $\mathbb{C}^p$-valued random vector $\mathbf{V}_0$ let $U = U(\mathbf{V}_0)$ be the unique p-times differentiable solution with p^{th} càdlàg derivative of the differential equation*

$$a(D)U_t = b(D)J^{p-1}L_t + a(D)J^p(\mathbf{b}' e^{At}\mathbf{V}_0) \tag{12}$$

with initial conditions $U_0 = U_0^{(1)} = \ldots = U_0^{(p-1)} = 0$. Let Y be a stochastic process. Then Y is a CARMA(p,q)-process driven by L with characteristic polynomials $a(z)$ and $b(z)$ if and only if $Y = U^{(p)}(\mathbf{V}_0)$ for some random variable $\mathbf{V}_0$.

Proof. This follows as in the proof of Theorem 1. □

As expected, $a(D)J^p(\mathbf{b}'\mathrm{e}^{At}\mathbf{V}_0) = \widetilde{a}(J)(\mathbf{b}'\mathrm{e}^{At}\mathbf{V}_0)$ turns out to be a polynomial in t of degree at most $p-1$, but it is interesting to observe that when varying $\mathbf{V}_0$ over all p-variate random vectors, we do not necessarily get all random polynomials of degree at most $p-1$. In fact this happens only in the case when $a(z)$ and $b(z)$ have no common zeroes. More precisely, we have the following result.

Proposition 1. *If $q < p$ and $\mathbf{V}_0$ is a $\mathbb{C}^p$-valued random vector then:*

(a)

$$a(D)J^p(\mathbf{b}'\mathrm{e}^{At}\mathbf{V}_0) = \widetilde{a}(J)(\mathbf{b}'\mathrm{e}^{At}\mathbf{V}_0) = \sum_{m=0}^{p-1}\frac{1}{m!}\left(\sum_{k=0}^{m} a_{m-k}\mathbf{b}'A^k\mathbf{V}_0\right)t^m, \tag{13}$$

a random polynomial in t of degree at most $p-1$. Hence there is a $\mathbb{C}^p$-valued random vector $\mathbf{W} = (W^0,\ldots,W^{p-1})'$ such that

$$a(D)J^p(\mathbf{b}'\mathrm{e}^{At}\mathbf{V}_0) = \widetilde{a}(J)(\mathbf{b}'\mathrm{e}^{At}\mathbf{V}_0) = \sum_{m=0}^{p-1} W^m t^m.$$

(b) If R_{p-1} is the set of all random polynomials of degree at most $p-1$, i.e. if $R_{p-1} := \{\sum_{m=0}^{p-1} W^m t^m : (W^0, \ldots, W^{p-1})'$ a $\mathbb{C}^p$-valued random vector$\}$, and if Q_{p-1} is the set of all random polynomials that are expressible in the form $\widetilde{a}(J)(\mathbf{b}' \mathrm{e}^{At} \mathbf{V}_0)$ for some $\mathbb{C}^p$-valued random vector $\mathbf{V}_0$, then $Q_{p-1} \subset R_{p-1}$ by (a), and $Q_{p-1} = R_{p-1}$ if and only if $a(z)$ and $b(z)$ have no common zeroes.

Proof. (a) From $\mathrm{e}^{At} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$ we obtain $J^l \mathrm{e}^{At} = \sum_{k=0}^{\infty} \frac{t^{k+l}}{(k+l)!} A^k$, hence

$$\begin{aligned}\widetilde{a}(J)\mathrm{e}^{At} &= \sum_{l=0}^{p} a_l \sum_{k=0}^{\infty} \frac{t^{k+l}}{(k+l)!} A^k = \sum_{m=0}^{p-1} \sum_{k=0}^{m} a_{m-k} A^k \frac{t^m}{m!} + \sum_{m=p}^{\infty} \sum_{l=0}^{p} a_l A^{m-l} \frac{t^m}{m!} \\ &= \sum_{m=0}^{p-1} \sum_{k=0}^{m} a_{m-k} A^k \frac{t^m}{m!},\end{aligned}$$

since $a(z)$ is the characteristic polynomial of A and

$$\sum_{l=0}^{p} a_l A^{m-l} = \left(\sum_{l=0}^{p} a_l A^{p-l} \right) A^{m-p} = a(A)\, A^{m-p} = 0$$

since $a(A) = 0$ by the Cayley-Hamilton theorem. This gives (13).

(b) That $Q_{p-1} \subset R_{p-1}$ follows from (a). Define $\mathbf{c}'_m := \frac{1}{m!} \sum_{k=0}^{m} a_{m-k} \mathbf{b}' A^k$ for $m \in \{0, \ldots, p-1\}$. Then $\mathbf{c}_m \in \mathbb{C}^p$ and $a(D)J^p(\mathbf{b}' \mathrm{e}^{At} \mathbf{V}_0) = \sum_{m=0}^{p-1} \mathbf{c}'_m \mathbf{V}_0 t^m$. It follows that $Q_{p-1} = R_{p-1}$ if and only if $(\mathbf{c}_0, \ldots, \mathbf{c}_{p-1}) \in \mathbb{C}^{p \times p}$ has rank p, i.e. if and only if $\mathbf{c}_0, \ldots, \mathbf{c}_{p-1}$ are linearly independent. But by definition of the vectors $\mathbf{c}_m$ and the fact that $a_0 = 1$, this is equivalent to the linear independence of $\mathbf{b}', \mathbf{b}'A, \mathbf{b}'A^2, \ldots, \mathbf{b}'A^{p-1}$, which in turn is equivalent to the matrix $(\mathbf{b}, A'\mathbf{b}, (A')^2\mathbf{b}, \ldots, (A')^{p-1}\mathbf{b}) \in \mathbb{C}^{p \times p}$ having rank p. In the language of control theory this is expressed as *observability* of the pair $(A, \mathbf{b}')$. As shown in Proposition 2.1(i) of Brockwell and Lindner (2015), this follows if $a(z)$ and $b(z)$ have no common zeroes and it is easy to see from the proof that the reverse implication also holds. □

Remark 2. When $a(z)$ and $b(z)$ have no common zeroes, Proposition 1 allows an obvious reformulation of Corollary 1 with $a(D)J^p(\mathbf{b}' \mathrm{e}^{At} \mathbf{V}_0)$ replaced by any random polynomial $P(t)$ of degree at most $p-1$ and $U(\mathbf{V}_0)$ replaced by $U(P)$. This is not possible however if $a(z)$ and $b(z)$ have common zeroes.

When $a(z)$ and $b(z)$ have common zeroes, then by Theorem 4.1 in Brockwell and Lindner (2009), the common zeroes of $a(z)$ and $b(z)$ can be factored out to give polynomials $\bar{a}(z)$ and $\bar{b}(z)$, and every CARMA process with polynomials $a(z)$ and $b(z)$ is also a CARMA process with polynomials $\bar{a}(z)$ and $\bar{b}(z)$ and the same driving noise. Conversely, it is easy to see by inverting the arguments in the proof of Theorem 4.1 in Brockwell and Lindner (2009) that every CARMA process with polynomials $\bar{a}(z)$ and $\bar{b}(z)$ is a CARMA process with polynomials $a(z)$ and $b(z)$ and the same driving noise.

Equation (5) gives sense to the formal equation $a(D)Y_t = b(D)DL_t$ by integrating it p-times, which ensures that J^pY_t is p-times differentiable with càdlàg p^{th} derivative and $J^pDL_t = J^{p-1}L_t$ is $(p-1)$-times differentiable with càdlàg $(p-1)^{\text{st}}$ derivative. If the order q of the CARMA-process is less than $p-1$, then it seems natural to integrate the formal equation (1) only $(q+1)$-times in order to give sense to it. That this is true in a rigorous way is the content of the next result, which together with Theorem 1 gives another characterization of CARMA processes.

Proposition 2. *Suppose that $q < p$, $a(z)$, $b(z)$ and L are as in Theorem 1, $\mathbf{V}_0$ is a $\mathbb{C}^p$-valued random vector and Y is a càdlàg stochastic process. Then Y is the (unique) solution of* (5) *if and only if, for some (equivalently, all) $k \in \{q+1, \dots, p\}$, Y is $(p-k)$-times differentiable with $(p-k)^{\text{th}}$ càdlàg derivative and satisfies the equations*

$$a(D)J^kY_t = b(D)J^{k-1}L_t + a(D)J^k(\mathbf{b}'\mathrm{e}^{At}\mathbf{V}_0), \tag{14}$$

$$Y_0 = a(D)J^p(\mathbf{b}'\mathrm{e}^{At}\mathbf{V}_0)\big|_{t=0}, \quad \textit{and} \tag{15}$$

$$Y_0^{(j)} = a(D)J^{p-j}(\mathbf{b}'\mathrm{e}^{At}\mathbf{V}_0)\big|_{t=0} - \sum_{i=1}^{j} a_i Y_0^{(j-i)},\ 1 \le j < p-k. \tag{16}$$

(Note the recursive nature of (15) *and* (16) *in $Y_0^{(j)}$.)*

Proof. Suppose that $k \in \{q+1, \dots, p\}$ and let $g = (g_t)_{t\in\mathbb{R}}$ and $f = (f_t)_{t\in\mathbb{R}}$ be two stochastic processes such that f is $(p-k)$-times differentiable with $(p-k)^{\text{th}}$ càdlàg-derivative. By the fundamental theorem of calculus for the Lebesgue-integral (see e.g. Cohn (2013), Theorem 6.3.11) we have $g = f$ if and only if g is $(p-k)$-times differentiable with $(p-k)^{th}$ càdlàg derivative such that

$$g^{(p-k)} = f^{(p-k)} \quad \text{and} \quad g_0^{(j)} = f_0^{(j)} \quad \forall\, j \in \{0, \dots, p-k-1\}. \tag{17}$$

Now let $f_t = b(D)J^{p-1}L_t + a(D)J^p(\mathbf{b}'\mathrm{e}^{At}\mathbf{V}_0)$ and $g_t = a(D)J^pY_t$. Then f is $(p-k)$-times differentiable with càdlàg derivative $f^{(p-k)}$ and $f_t^{(j)} =$

$b(D)J^{p-1-j}L_t + a(D)J^{p-j}(\mathbf{b}'\mathrm{e}^{At}\mathbf{V}_0)$ for all $j \in \{0, \ldots, p-k\}$, in particular $f_0^{(j)} = a(D)J^{p-j}(\mathbf{b}'\mathrm{e}^{At}\mathbf{V}_0)\big|_{t=0}$. From (17) we then see that Y satisfies (5) if and only if Y is $(p-k)$-times differentiable with càdlàg derivative $Y^{(p-k)}$ and such that Y satisfies (14) along with

$$a(D)J^{p-j}Y_t\big|_{t=0} = f_0^{(j)} = a(D)J^{p-j}(\mathbf{b}'\mathrm{e}^{At}\mathbf{V}_0), \quad j = 0, \ldots, p-k-1. \tag{18}$$

But (18) is equivalent to $Y_0 = f_0^{(0)}$ for $j = 0$ and to

$$(D^p + a_1 D^{p-1} + \ldots + a_p D^0)J^{p-j}Y_t = Y_0^{(j)} + a_1 Y_0^{(j-1)} + \ldots + a_j Y_0^{(0)} = f_0^{(j)}$$

for $j = 1, \ldots, p-k-1$, which gives the desired equations (15) and (16). □

We have seen what the CARMA equations mean when integrating k times with $k \in \{q+1, \ldots, p\}$. It is also possible to integrate to an order $k > p$:

Proposition 3. *Suppose that $q < p$, $a(z)$, $b(z)$, $\widetilde{a}(z)$, $\widetilde{b}(z)$ and L are as in Theorem 1, $\mathbf{V}_0$ is a $\mathbb{C}^p$-valued random vector and Y is a càdlàg stochastic process. Then Y is the (unique) solution of* (5) *(equivalently of* (6)*) if and only if, for some (equivalently, all) $k \in \{p+1, p+2, \ldots\}$,*

$$\widetilde{a}(J)J^{k-p}Y_t = \widetilde{b}(J)J^{k-p-1}L_t + \widetilde{a}(J)J^{p-k}(\mathbf{b}'\mathrm{e}^{At}\mathbf{V}_0).$$

Proof. This is clear since two càdlàg process f and g satisfy $f = g$ if and only if $J^{k-p}f = J^{k-p}g$. □

Remark 3. Stationarity of Lévy driven CARMA processes was investigated in detail in Brockwell and Lindner (2009). Applying these results, it follows that the solution (7) is strictly stationary if the zeroes of $a(z)$ all have non-zero real parts, if $E\log(\max\{|L_1|, 1\}) < \infty$ and if $\mathbf{V}_0 = \int_{-\infty}^{\infty} \mathbf{f}(-u)\,\mathrm{d}L_u$, where $\mathbf{f}(t) = \mathbf{f}_1(t)\mathbf{1}_{[0,\infty)}(t) - \mathbf{f}_2(t)\mathbf{1}_{(-\infty,0)}(t)$ and $\mathbf{f}_1(t)$ and $\mathbf{f}_2(t)$ are the residues of the mapping $z \mapsto [1\ z\ \ldots\ z^{p-1}]'e^{zt}/a(z)$ in the left and right halves of the complex plane respectively. To see this, observe that by Theorem 1(b), Y has a state-space representation with $\mathbf{X}_0 = \mathbf{V}_0$. But if $\mathbf{X}_0 = \int_{-\infty}^{\infty} \mathbf{f}(-u)\,\mathrm{d}L_u$ (observe that this converges almost surely as a consequence of the assumptions on $a(z)$ and L) then, since the solution of (3) is uniquely determined by $\mathbf{X}_0$ and since $\mathbf{X}_t = \int_{-\infty}^{\infty} \mathbf{f}(t-u)\,\mathrm{d}L_u,\ t \in \mathbb{R}$, is known to be a solution which is also strictly stationary (Brockwell and Lindner (2009)), it follows that Y satisfies $Y_t = \int_{-\infty}^{\infty} \mathbf{b}'\mathbf{f}(t-u)\,\mathrm{d}L_u,\ t \in \mathbb{R}$, and is therefore strictly stationary. By similar arguments, if all the zeroes of $a(z)$ all have strictly negative real parts, if $E\log(\max\{|L_1|, 1\}) < \infty$ and if $\mathbf{V}_0$ is independent of L with the distribution of $\int_0^{\infty} e^{At}\mathbf{e}_p\,\mathrm{d}L_t$, then $(Y_t)_{t\geq 0}$ given by (7) is strictly stationary.

Remark 4. Theorem 1 also shows why it is in general impossible to define CARMA(p, q) processes with $q \geq p$ unless allowing generalized stochastic processes as done in Brockwell and Hannig (2010). To see this, suppose that there is a stochastic process Y_t that satisfies the formal equation $a(D)Y_t = b(D)DL_t$ with $q \geq p$, where L is a non-degenerate Lévy process. Let us interpret this by saying that there should exist some càdlàg process $Y = (Y_t)_{t \in \mathbb{R}}$ which satisfies the equation

$$a(D)J^{q+1}Y_t = b(D)J^q L_t + r_t,$$

where r_t is some random polynomial and $b_q \neq 0$. Since $q \geq p$, $a(D)J^{q+1}Y_t - r_t$ must be differentiable (with càdlàg derivative). On the other hand, $b(D)J^q L_t$ is not differentiable, as $b_q D^q J^q L_t = b_q L_t$ is not differentiable, but $b_j D^j J^q L_t$ is differentiable with càdlàg derivative for $j < q$. This indicates that solutions to CARMA equations when $q \geq p$ are only possible in terms of generalized random processes.

Remark 5. It should be noted that none of our results from Theorem 1 to Proposition 3 depend on the fact that L is a Lévy process, but require only that L be a càdlàg process with $L_0 = 0$. Observe also that for (2) and (3) to make sense it is not necessary for L to be an (increment) semimartingale. The càdlàg property suffices since integration is with respect to t only.

Remark 6. Let L be a stochastic process with $L_0 = 0$ whose sample paths are p-times continuously differentiable (when L is additionally a Lévy process, this means that $L_t = \rho t$ for some $\rho \in \mathbb{R}$). For each $\mathbb{C}^p$-valued random vector $\mathbf{V}_0$, let U be the unique solution of (12) with $U_0 = U_0^{(1)} = \ldots = U_0^{(p-1)} = 0$ and let $Y = U^{(p)}$ as in Corollary 1. Since, by Proposition 1, $a(D)J^p(\mathbf{b}' e^{At}\mathbf{V}_0)$ is a polynomial of degree at most $p - 1$, we obtain

$$a(D)Y_t = D^p a(D)U_t = D^p b(D)J^{p-1}L_t + D^p a(D)J^p(\mathbf{b}' e^{At}\mathbf{V}_0) = b(D)DL_t,$$

which is (1). Observe that (1) makes sense here in the usual way since L is sufficiently smooth. Observe further that for each fixed ω in the underlying sample space Ω, the set of (pathwise) solutions of (1) is a p-dimensional affine function space. On the other hand, as seen in the proof of Theorem 1 and by the argument just given, the set of (pathwise) solutions of (5) with varying $\mathbf{V}_0(\omega) \in \mathbb{C}^p$ is a k-dimensional affine subspace of this with $k \leq p$, where $k = p$ if and only if $a(z)$ and $b(z)$ have no common zeroes by Proposition

1. This shows that the seemingly obvious equivalence of (1) and (5) is not as straightforward as it appears at first sight, and actually is an equivalence only if $a(z)$ and $b(z)$ have no common zeroes.

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