Density results for frames of exponentials

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Dedicated to Professor John Benedetto on the occasion of his 65' birthday.

Summary. For a separated sequence $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ of real numbers there is a close link between the lower and upper densities $D^-(\Lambda), D^+(\Lambda)$ and the frame properties of the exponentials $\{e^{i\lambda_k x}\}_{k\in\mathbb{Z}}$: in fact, $\{e^{i\lambda_k x}\}_{k\in\mathbb{Z}}$ is a frame for its closed linear span in $L^2(-\gamma,\gamma)$ for any $\gamma \in]0, \pi D^-(\Lambda)[\cup]\pi D^+(\Lambda), \infty[$. We consider a classical example presented already by Levinson [10] with $D^-(\Lambda) = D^+(\Lambda) = 1$; in this case, the frame property is guaranteed for all $\gamma \in]0,\infty[\setminus\{\pi\}$. We prove that the frame property actually breaks down for $\gamma = \pi$. Motivated by this example, it is natural to ask whether the frame property can break down on an interval if $D^-(\Lambda) \neq D^+(\Lambda)$. The answer is yes: We present an example of a family Λ with $D^-(\Lambda) \neq D^+(\Lambda)$ for which $\{e^{i\lambda_k x}\}_{k\in\mathbb{Z}}$ has no frame property in $L^2(-\gamma,\gamma)$ for any $\gamma \in]\pi D^-(\Lambda), \pi D^+(\Lambda)[$.

1 Introduction

While frames nowadays are a recognized tool in many branches of harmonic analysis and signal processing, it is interesting to remember that Duffin and Schaeffer [6] actually introduced the concept in the context of systems of complex exponentials. The starting point was the possibility of nonharmonic Fourier expansions, as discovered by Paley and Wiener. Much of the study is also rooted in the study of sampling theories tracing back to Paley-Wiener, Levinson, Plancherel-Polya and Boas; a complete treatment is given by John Benedetto in *Irregular Sampling and Frames* [2]. That paper also contains original work on irregular sampling using Fourier frames, as well as new properties of Fourier frames. In particular, density issues for Fourier frames and distinctions as well as interconnections among *uniform density*, *natural density* and the *lower and upper Beurling densities* $D^-(\Lambda)$ and $D^+(\Lambda)$ of a sequence $\Lambda \equiv \{\lambda_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ are discussed in detail in *Sections 7* and *9* of [2].

The frame properties for systems of exponentials $\{e^{i\lambda_k x}\}_{k\in\mathbb{Z}}$ are closely related to density issues concerning the sequence $\{\lambda_k\}_{k\in\mathbb{Z}}$, as revealed by, e.g., [8], [9], [12]. A combination of well known results shows that for a separated sequence $\{\lambda_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$, the exponentials $\{e^{i\lambda_k x}\}_{k\in\mathbb{Z}}$ form a frame sequence in $L^2(-\gamma,\gamma)$ for all $\gamma \in]0, \pi D^-(\Lambda)[\, \cup \,]\pi D^+(\Lambda), \infty[$. On the other hand, it is known that there might be no frame property for the limit cases $\gamma = \pi D^-(\Lambda), \gamma = \pi D^+(\Lambda)$. This appears, e.g., in a classical example presented by Levinson [10] which we consider in Example 2: here $D^-(\Lambda) = D^+(\Lambda) = 1$, and the exponentials $\{e^{i\lambda_k x}\}_{k\in\mathbb{Z}}$ form a frame sequence in $L^2(-\gamma,\gamma)$ exactly for $\gamma \in]0,\infty[\, \setminus \{\pi\}$. The above considerations leave an interesting gap on the interval $]\pi D^-(\Lambda), \pi D^+(\Lambda)[$. In particular, it is natural to ask whether there are exponentials $\{e^{i\lambda_k x}\}_{k\in\mathbb{Z}}$ with $D^-(\Lambda) \neq D^+(\Lambda)$ and having no frame property in the gap $]\pi D^-(\Lambda), \pi D^+(\Lambda)[$. The answer turns out to be yes; and we provide a concrete example.

The paper is organized as follows. *Section 2* concerns the general frame terminology and definitions. *Section 3* summarizes known results of Seip and Beurling, in which it is shown that exponentials $\{e^{i\lambda_k x}\}_{k\in\mathbb{Z}}$ form a frame sequence in $L^2(-\gamma,\gamma)$ for all $\gamma \notin [\pi D^-(\Lambda), \pi D^+(\Lambda)]$. A necessary condition for $\{e^{i\lambda_k x}\}_{k\in\mathbb{Z}}$ being a frame for $L^2(-\gamma,\gamma)$, due to Landau [9], is phrased as a no-go theorem as a preparation for applications in later sections. In *Section 4*, a lemma by Jaffard is presented and discussed; this is the foundation for the construction of the aforementioned example in *Section 6*. *Section 5* analyzes the limit case of the frame version of the classical Kadec's 1/4-Theorem and discuss the example presented by Levinson. Then in *Section 6*, we give an explicit example with no frame property for $\gamma \in]\pi D^-(\Lambda), \pi D^+(\Lambda)[$.

2 General frames

We will formulate the basic concepts in somewhat larger generality than needed in the present paper. Thus, in this section we consider a separable Hilbert space $\mathcal{H}$ with the inner product $\langle \cdot, \cdot \rangle$ linear in the first entry. In later sections we will mainly consider $\mathcal{H} = L^2(-\gamma,\gamma)$ for some $\gamma \in]0,\infty[$.

We begin with some definitions.

Definition 1. *Let $\{f_k\}_{k=1}^\infty$ be a sequence in $\mathcal{H}$. We say that*

(i) $\{f_k\}_{k=1}^\infty$ is a frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\ ||f||^2 \leq \sum_{k=1}^\infty |\langle f, f_k\rangle|^2 \leq B\ ||f||^2,\ \forall f \in \mathcal{H}. \tag{1}$$

(ii) $\{f_k\}_{k=1}^\infty$ is a frame sequence if there exist constants $A, B > 0$ such that

$$A \,||f||^2 \le \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \le B \,||f||^2, \ \forall f \in \overline{span}\{f_k\}_{k=1}^{\infty}.$$

(iii) $\{f_k\}_{k=1}^{\infty}$ *is a Riesz basis for* $\mathcal{H}$ *if* $\overline{span}\{f_k\}_{k=1}^{\infty} = \mathcal{H}$ *and there exist constants* $A, B > 0$ *such that*

$$A \sum |c_k|^2 \le \left|\left|\sum c_k f_k\right|\right|^2 \le B \sum |c_k|^2 \tag{2}$$

for all finite scalar sequences $\{c_k\}$*.*

(iv) $\{f_k\}_{k=1}^{\infty}$ *is a Riesz sequence if there exist constants* $A, B > 0$ *such that*

$$A \sum |c_k|^2 \le \left|\left|\sum c_k f_k\right|\right|^2 \le B \sum |c_k|^2$$

for all finite scalar sequences $\{c_k\}$*.*

The frame definition goes back to the paper [6] by Duffin and Schaeffer. More recent treatments can be found in the books [13], [5], or [3].

Any numbers $A, B > 0$ which can be used in (1) will be called *frame bounds.* If $\{f_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$, there exists a dual frame $\{g_k\}_{i=1}^{\infty}$ such that

$$f = \sum_{i=1}^{\infty} \langle f, g_k\rangle f_k = \sum_{i=1}^{\infty} \langle f, f_k\rangle g_k, \ \forall f \in \mathcal{H}. \tag{3}$$

The series in (3) converges unconditionally; for this reason, we can index the frame elements in an arbitrary way. In particular, we can apply the general frame results discussed in this section to frames of exponentials, which are usually indexed by $\mathbb{Z}$.

A Riesz basis is a frame, so a representation of the type (3) is also available if $\{f_k\}_{k=1}^{\infty}$ is a Riesz basis. Furthermore, the possible values of A, B in (2) coincide with the frame bounds. Riesz bases are characterized as the class of frames which are ω-independent, cf. [7]: that is, a frame $\{f_k\}_{k=1}^{\infty}$ is a Riesz basis if and only if $\sum_{k=1}^{\infty} c_k f_k = 0$ implies that $c_k = 0$ for all $k \in \mathbb{N}$. Thus, Riesz bases are the frames which are at the same time bases. This means that a frame which is not a Riesz basis is *redundant:* it is possible to remove elements without destroying the frame property. However, in general, not *arbitrary* elements can be removed:

Example 1. If $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$, then

$$\{e_1, e_2, e_2, e_3, e_3, e_4, e_4, ...\}$$

is a frame for $\mathcal{H}$. If any $e_k, k \ge 2$, is removed, the remaining family is still a frame for $\mathcal{H}$. However, if e_1 is removed, the remaining family is merely a frame sequence.

We note in passing that the above example is typical: the removal of an element from a frame might leave an incomplete family, which can't be a frame for the same space. However, the remaining family will always be a frame sequence. These considerations of course generalize to the removal of a *finite* number of elements; but *not* to removal of arbitrary collections.

Note that a frame sequence also leads to representations of the type (3) – but only for $f \in \overline{span}\{f_k\}_{k=1}^{\infty}$.

3 Frames of exponentials

In this section we consider the frame properties for a sequence of complex exponentials $\{e^{i\lambda_k x}\}_{k\in\mathbb{Z}}$, where $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ is a sequence of real numbers. Before we discuss the frame properties, we introduce some central concepts related to the sequence Λ.

We say that Λ is *separated* if there exists $\delta > 0$ such that $|\lambda_k - \lambda_l| \geq \delta$ for all $k \neq l$. If Λ is a finite union of separated sets, we say that Λ is *relatively separated*. It can be proved that $\{e^{i\lambda_k x}\}_{k\in\mathbb{Z}}$ satisfies the upper frame condition if and only if Λ is relatively separated.

In this paper we concentrate on separated sequences Λ. Given $r > 0$, let $n^-(r)$ (resp. $n^+(r)$) denote the minimal (resp. maximal) number of elements from Λ to be found in an interval of length r. The lower (resp. upper) density of Λ is defined by

$$D^-(\Lambda) = \lim_{r\to\infty} \frac{n^-(r)}{r} \text{ resp. } D^+(\Lambda) = \lim_{r\to\infty} \frac{n^+(r)}{r}.$$

The sufficiency parts of Theorems 2.1 and 2.2 in [12] can be formulated as follows:

Theorem 1. *Let $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ be a separated sequence of real numbers. Then the following holds:*

(a) $\{e^{i\lambda_k x}\}_{k\in\mathbb{Z}}$ forms a frame for $L^2(-\gamma,\gamma)$ for any $\gamma < \pi D^-(\Lambda)$.
(b) $\{e^{i\lambda_k x}\}_{k\in\mathbb{Z}}$ forms a Riesz sequence in $L^2(-\gamma,\gamma)$ for any $\gamma > \pi D^+(\Lambda)$.

As illustration of this result, we encourage the reader to consider the family $\{e^{ikx}\}_{k\in\mathbb{Z}}$: it is an orthonormal basis for $L^2(-\pi,\pi)$, a frame for $L^2(-\gamma,\gamma)$ for any $\gamma \in]0,\pi[$, and a (non-complete) Riesz sequence in $L^2(-\gamma,\gamma)$ for $\gamma > \pi$. This corresponds to the fact that $D^-(\mathbb{Z}) = D^+(\mathbb{Z}) = 1$. This example is considered in detail in the paper [7] in this volume.

In terms of frame sequences Theorem 1 gives the following:

Corollary 1. *Let Λ be a separated sequence of real numbers. Then $\{e^{i\lambda_k x}\}_{k\in\mathbb{Z}}$ is a frame sequence in $L^2(-\gamma,\gamma)$ whenever*

$$\gamma \in]0, \pi D^-(\Lambda)[\cup]\pi D^+(\Lambda), \infty[.$$

Corollary 1 serves as motivation for our new results in Section 5 and Section 6. In Section 5 we consider an example with $D^-(\Lambda) = D^+(\Lambda) = 1$, where it turns out that the frame property holds in $L^2(-\gamma,\gamma)$ for any $\gamma \neq \pi$. In Section 6 we prove that if $D^-(\Lambda) \neq D^+(\Lambda)$, then $\{e^{i\lambda_k x}\}_{k\in\mathbb{Z}}$ might not have any frame property when considered in $L^2(-\gamma,\gamma), \gamma \in]\pi D^-(\Lambda), \pi D^+(\Lambda)[$.

We need a deep result due to Landau [9] (see Ortega-Cerda and Seip [11], p. 791-792 for a discussion of this result).

Theorem 2. *A separated family of complex exponentials $\{e^{i\lambda x}\}_{\lambda\in\Lambda}$ is not a frame for $L^2(-\gamma,\gamma)$ if $\pi D^-(\Lambda) < \gamma$.*

In [12], Seip considers removal of elements from a frame of exponentials. In particular, he proves that it always is possible to remove elements in a way such that the remaining family is still a frame, but now corresponding to a separated family:

Lemma 1. *Assume that Λ is relatively separated and that $\{e^{i\lambda_k x}\}_{k\in\mathbb{Z}}$ is a frame for $L^2(-\gamma,\gamma)$ for some $\gamma > 0$. Then there exists a separated subfamily $\Lambda' \subseteq \Lambda$ such that $\{e^{i\lambda x}\}_{k\in\Lambda'}$ is a frame for $L^2(-\gamma,\gamma)$.*

4 Jaffard's lemma

We need a lemma by Jaffard, [8] (Lemma 4, p. 344). Jaffard states the lemma as follows: *Suppose a sequence of functions $\{e_k\}_{k\in\mathbb{Z}}$ is a frame for $L^2(I)$ for an interval I. Then $\{e_k\}_{k\neq 0}$ is a frame on each interval $I' \subset I$ such that $|I'| < |I|$.*

However, the lemma is false in that generality. Before we illustrate this with an example, let us examine the mistake in the proof which appears in the first two lines. Here the following is stated: *The $\{e_k\}_{k\in\mathbb{Z}}$ are a frame of $L^2(I')$. Then, either $\{e_k\}_{k\neq 0}$ is a frame of $L^2(I')$, and we have nothing to prove, or the $\{e_k\}_{k\in\mathbb{Z}}$ are a Riesz basis of $L^2(I')$.* This, of course, is not true. If $\{e_k\}_{k\in\mathbb{Z}}$ is a redundant frame for $L^2(I)$ and if e_0 is not in the closed linear span of $\{e_k\}_{k\neq 0}$ in $L^2(I')$ then $\{e_k\}_{k\neq 0}$ is not a frame for $L^2(I')$ but $\{e_k\}_{k\in\mathbb{Z}}$ is still a redundant frame for its span and not a Riesz basis. For example, let $\{e_k\}_{k\in\mathbb{Z}}$ be an orthonormal basis for $L^2(0,1)$ with supp $e_0 \subset [0,1/2]$. Since $e_0 \perp e_k$ for all $k \neq 0$ on $L^2(0,1)$ and $e_0(t) = 0$, for all $1/2 \leq t \leq 1$, it follows that $e_0 \perp e_k$ for all $k \neq 0$ on $L^2(0,1/2)$. Hence, $\{e_k\}_{k\neq 0}$ is not a frame for $L^2(0,1/2)$.

We need, as did Jaffard, a very special case of this lemma which is true.

Lemma 2. *Let $\{e^{i\lambda x}\}_{\lambda\in\Lambda}$ be a frame for $L^2(-R,R)$ and let $J \subset \Lambda$ be a finite set. Then, $\{e^{i\lambda x}\}_{\lambda\in\Lambda-J}$ is a frame for $L^2(-A,A)$, for all $0 < A < R$.*

Jaffard's proof works to prove Lemma 2 because of a result of Seip [12] (p. 142, Lemma 3.15) which makes the first two lines of Jaffard's proof correct in this special case. We state the Lemma below.

Lemma 3. *Let $\{e^{i\lambda x}\}_{\lambda \in \Lambda}$ be a frame for $L^2(-R,R)$. Then either $\{e^{i\lambda x}\}_{\lambda \in \Lambda}$ is a Riesz basis for $L^2(-R,R)$ or $\{e^{i\lambda x}\}_{\lambda \in \Lambda - \{\lambda_0\}}$ is a frame for $L^2(-R,R)$ for every $\lambda_0 \in \Lambda$.*

However, for completeness we give an independent proof of Lemma 2. It was communicated to us by D. Speegle.

Proof of Lemma 2: We will rely on Theorem 2. First, we assume that Λ is separated. By Theorem 2 we know that $D^-(\Lambda) \geq \frac{R}{\pi}$. It is clear that D^- is unchanged if we delete a finite number of elements. So for all $0 < A < R$,

$$D^-(\Lambda - J) = D^-(\Lambda) \geq \frac{R}{\pi} > \frac{A}{\pi}.$$

So by Theorem 1 we have that $\{e^{i\lambda x}\}_{\lambda \in \Lambda - J}$ is a frame for $L^2(-A,A)$.

If Λ is not separated, Lemma 1 shows that there is a $\Lambda^{'} \subset \Lambda$, with $\Lambda^{'}$ separated so that $\{e^{i\lambda x}\}_{\lambda \in \Lambda'}$ is also a frame for $L^2(-R,R)$. Now, $\{e^{i\lambda x}\}_{\lambda \in \Lambda' - J}$ is a frame for $L^2(-A,A)$ for all $A < R$ and hence so is $\{e^{i\lambda x}\}_{\lambda \in \Lambda - J}$. $\square$

5 The limit case of Kadec's 1/4-Theorem

The classical Kadec's 1/4-theorem concerns perturbations of the orthonormal basis $\{e^{ikx}\}_{k \in \mathbb{Z}}$ for $L^2(-\pi,\pi)$: it states that if $\{\lambda_k\}_{k \in \mathbb{Z}}$ is a sequence of real numbers and $\sup_{k \in \mathbb{Z}} |\lambda_k - k| < 1/4$, then $\{e^{i\lambda_k x}\}_{k \in \mathbb{Z}}$ is a Riesz basis for $L^2(-\pi,\pi)$. It is well known that the result is sharp, in the sense that $\{e^{i\lambda_k x}\}_{k \in \mathbb{Z}}$ might not be a Riesz basis for $L^2(-\pi,\pi)$ if $\sup_{k \in \mathbb{Z}} |\lambda_k - k| = 1/4$ (see Example 2 below). In Proposition 1 we shall sharpen this result, using the following extension of Kadec's theorem to frames, which was proved independently by Balan [1] and Christensen [4]:

Theorem 3. *Let $\{\lambda_k\}_{k \in \mathbb{Z}}$ and $\{\mu_k\}_{k \in \mathbb{Z}}$ be real sequences. Assume that $\{e^{i\lambda_k x}\}_{k \in \mathbb{Z}}$ is a frame for $L^2(-\pi,\pi)$ with bounds A, B, and that there exists a constant $L < 1/4$ such that*

$$|\mu_k - \lambda_k| \leq L \text{ and } 1 - \cos \pi L + \sin \pi L < \frac{A}{B}.$$

Then $\{e^{i\mu_k x}\}_{k \in \mathbb{Z}}$ is a frame for $L^2(-\pi,\pi)$ with bounds

$$A\left(1 - \frac{B}{A}(1 - \cos \pi L + \sin \pi L)\right)^2, \quad B(2 - \cos \pi L + \sin \pi L)^2.$$

With the help of Theorem 3 we can now prove the following:

Proposition 1. *Let $\{\lambda_k\}_{k \in \mathbb{Z}}$ be a sequence of real numbers such that*

$$\sup_{k \in \mathbb{Z}} |\lambda_k - k| = 1/4.$$

Then either $\{e^{i\lambda_k x}\}_{k \in \mathbb{Z}}$ is a Riesz basis for $L^2(-\pi,\pi)$ or it is not a frame for $L^2(-\pi,\pi)$.

Proof: For $t \in [0,1]$, define

$$\lambda_k(t) := k + t(\lambda_k - k).$$

Then

$$\sup_{k\in\mathbb{Z}} |\lambda_k(t) - k| = t \cdot \sup_{k\in\mathbb{Z}} |\lambda_k - k| = \frac{t}{4}.$$

Furthermore, $\lambda_k(1) = \lambda_k$ for all $k \in \mathbb{Z}$. Now, suppose that $\{e^{i\lambda_k x}\}_{k\in\mathbb{Z}}$ is a frame for $L^2(-\pi,\pi)$, with lower bound A, say. By Theorem 3, there exists a $t_0 \in [0,1[$ such that $\{e^{i\lambda_k(t)x}\}_{k\in\mathbb{Z}}$ is a frame for $L^2(-\pi,\pi)$ with lower bound $A/2$ for any $t \in [t_0,1[$. But, by Kadec's 1/4-Theorem, $\{e^{i\lambda_k(t)x}\}_{k\in\mathbb{Z}}$ is a Riesz basis for $L^2(-\pi,\pi)$ for all $t \in [t_0,1[$, too; since Riesz bounds and frame bounds coincide, it is thus a Riesz basis with lower bound $A/2$. Thus we have for $t \in [t_0,1[$:

$$\frac{A}{2}\sum_{k=-N}^{N} |c_k|^2 \leq \left\| \sum_{k=-N}^{N} c_k e^{i\lambda_k(t)(\cdot)} \right\|^2_{L^2(-\pi,\pi)} \qquad \forall\, N \in \mathbb{N}, c_{-N}, \ldots, c_N \in \mathbb{C}.$$

Taking the limit $t \to 1$, we obtain

$$\frac{A}{2}\sum_{k=-N}^{N} |c_k|^2 \leq \left\| \sum_{k=-N}^{N} c_k e^{i\lambda_k(\cdot)} \right\|^2_{L^2(-\pi,\pi)} \qquad \forall\, N \in \mathbb{N}, c_{-N}, \ldots, c_N \in \mathbb{C}.$$

Thus $\{e^{i\lambda_k x}\}_{k\in\mathbb{Z}}$ satisfies the lower Riesz sequence condition in $L^2(-\pi,\pi)$. Since it is also a frame by assumption, it is complete and satisfies the upper condition, too. Thus it is a Riesz basis for $L^2(-\pi,\pi)$. □

We will now reconsider the classical example presented by Levinson [10].

Example 2. Consider the sequence

$$\lambda_k := \begin{cases} k - 1/4 & \text{if } k > 0 \\ k + 1/4 & \text{if } k < 0 \\ 0 & \text{if } k = 0. \end{cases}$$

It is clear that $D^-(\Lambda) = D^+(\Lambda) = 1$, thus, by Corollary 1 $\{e^{i\lambda_k x}\}_{k\in\mathbb{Z}}$ is a frame sequence in $L^2(-\gamma,\gamma)$ when $\gamma \in]0,\pi[\cup]\pi,\infty[$. It is also known [13] that $\{e^{i\lambda_k x}\}_{k\in\mathbb{Z}}$ is complete in $L^2(-\pi,\pi)$ but not a Riesz basis for $L^2(-\pi,\pi)$. Thus, by Lemma 1 we conclude that $\{e^{i\lambda_k x}\}_{k\in\mathbb{Z}}$ is not a frame sequence in $L^2(-\pi,\pi)$. □

6 Exponentials $\{e^{i\lambda_k x}\}_{k\in\mathbb{Z}}$ with no frame property in $L^2(-\gamma,\gamma)$ for $\gamma \in]\pi D^-(\Lambda), \pi D^+(\Lambda)[$

Corollary 1 does not provide us with any conclusion for $\gamma \in [\pi D^-(\Lambda), \pi D^+(\Lambda)]$; and for $\gamma = \pi D^-(\Lambda)$ and $\gamma = \pi D^+(\Lambda)$, Example 2 shows that no frame property might be available. If $D^-(\Lambda) < D^+(\Lambda)$ we are thus missing information on a whole interval. Our purpose is now to show that Corollary 1 is optimal in the sense that for any $a, b \in]0, \infty[,\ a < b$, we can construct sequences $\Lambda \subset \mathbb{R}$ with $a = D^-(\Lambda),\ b = D^+(\Lambda)$, for which $\{e^{i\lambda_k x}\}_{k\in\mathbb{Z}}$ has no frame property for *any* $\gamma \in]\pi a, \pi b[$. This is stronger than Example 2, where we had $D^-(\Lambda) = D^+(\Lambda)$.

Theorem 4. *For any* $0 < a < b$*, there are real numbers* $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ *satisfying:*

(1) $D^-(\Lambda) = a$.
(2) $D^+(\Lambda) = b$.
(3) $\{\frac{1}{\sqrt{2\pi b}} e^{i\lambda_k x}\}_{k\in\mathbb{Z}}$ *is a subsequence of an orthonormal basis for* $L^2(-\pi b, \pi b)$.
(4) $\{e^{i\lambda_k x}\}_{k\in\mathbb{Z}}$ *spans* $L^2(-\gamma, \gamma)$ *for every* $0 < \gamma < \pi b$.
It follows that:
(5) $\{e^{i\lambda_k x}\}_{k\in\mathbb{Z}}$ *is a frame for* $L^2(-\gamma, \gamma)$ *for all* $0 < \gamma \le \pi a$.
(6) $\{e^{i\lambda_k x}\}_{k\in\mathbb{Z}}$ *is not a frame sequence in* $L^2(-\gamma, \gamma)$ *for* $\pi a < \gamma < \pi b$.
(7) $\{e^{i\lambda_k x}\}_{k\in\mathbb{Z}}$ *is a Riesz sequence in* $L^2(-\gamma, \gamma)$ *for all* $\pi b \le \gamma$.

Proof. To simplify the notation, we will do the case $a = 1,\ b = 2$. The general case follows immediately from here by a change of variables. We let

$$\{f_k\}_{k\in J} = \{e^{ikx}, e^{i(k+1/2)x}\}_{k\in\mathbb{Z}}.$$

Our purpose is to exhibit a subfamily $\{f_k\}_{k\in\Lambda},\ \Lambda \subset J$, which has the required properties.

Now, $\{f_k\}_{k\in J}$ is an orthogonal basis for $L^2(-2\pi, 2\pi)$. The idea of the construction is to carefully delete (by induction) a family of subsets of J, $J_1 \subset J_2 \subset \cdots$, so that $\Lambda = J - \cup_{n=1}^{\infty} J_n$ has the required properties. The difficult part is to maintain property (4). First, let $J_1 = \{1 + 1/2\}$ and $\alpha = 1 + 1/2$. By Lemma 2, there is a sequence $\{a_k^{\alpha,1}\} \in \ell^2$ so that

$$e^{i\alpha x} = \sum_{k\in J-J_1} a_k^{\alpha,1} f_k \in L^2(-2\pi + \pi, 2\pi - \pi).$$

Choose a natural number $N_1 > 1 + \frac{1}{2}$ so that

$$\| \sum_{|k|\ge N_1} a_k^{\alpha,1} f_k \|_{L^2(-2\pi+\pi, 2\pi-\pi)} \le 1.$$

Let

$$J_2 = \{1 + 1/2, N_1 + 1/2, N_1 + 1 + 1/2\} = J_1 \cup \{N_1 + 1/2, N_1 + 1 + 1/2\}.$$

Since $\{f_k\}_{k\in J-J_2}$ is a frame for $L^2(-2\pi + \frac{\pi}{2}, 2\pi - \frac{\pi}{2})$ by Lemma 2, for each $\alpha \in J_2$ there is a sequence of scalars $\{a_k^{\alpha,2}\} \in \ell^2$ so that

$$e^{i\alpha x} = \sum_{k\in J-J_2} a_k^{\alpha,2} f_k \in L^2(-2\pi + \frac{\pi}{2}, 2\pi - \frac{\pi}{2}).$$

Choose a natural number $N_2 > N_1 + 1 + \frac{1}{2}$ so that for each $\alpha \in J_2$ we have

$$\| \sum_{|k| \geq N_2} a_k^{\alpha,2} f_k \|_{L^2(-2\pi + \frac{\pi}{2}, 2\pi - \frac{\pi}{2})} \leq 1/2.$$

Now by induction, we can for each n choose natural numbers $\{N_j\}_{j=1}^n$ with $N_{j-1} + j - 1 + \frac{1}{2} < N_j$ and sets $J_1 \subset J_2 \subset \cdots \subset J_n$ with

$$J_n = \cup_{j=1}^{n-1} J_j \cup \{N_{n-1} + 1/2, N_{n-1} + 1 + 1/2, \cdots, N_{n-1} + n - 1 + 1/2\}$$

satisfying:

(i) $\{f_k\}_{k \in J - J_n}$ is a frame for $L^2(-2\pi + \pi/n, 2\pi - \pi/n)$.

(ii) For every $\alpha \in J_n$ there is a sequence $\{a_k^{\alpha,n}\} \in \ell^2$ so that

$$e^{i\alpha x} = \sum_{k \in J - J_n} a_k^{\alpha,n} f_k \in L^2(-2\pi + \pi/n, 2\pi - \pi/n),$$

and

(iii)

$$\| \sum_{|k| \geq N_n} a_k^{\alpha,n} f_k \|_{L^2(-2\pi + \pi/n, 2\pi - \pi/n)} \leq \frac{1}{n}.$$

Now, let $\Lambda = J - \cup_{n=1}^{\infty} J_n$; we claim that $\{f_k\}_{k \in \Lambda}$ has the required properties. For (1), by the definition of the sets J_n,

$$\frac{|\Lambda \cap [N_{n-1} + \frac{1}{2}, N_{n-1} + n - 1 + \frac{1}{2}]|}{(N_{n-1} + n - 1 + \frac{1}{2}) - (N_{n-1} + \frac{1}{2})} =$$

$$\frac{|\{N_{n-1} + \frac{1}{2}, N_{n-1} + 1 + \frac{1}{2}, \ldots, N_{n-1} + n - 1 + \frac{1}{2}\}|}{(N_{n-1} + n - 1 + \frac{1}{2}) - (N_{n-1} + \frac{1}{2})} = \frac{n}{n-1}.$$

So $D^-(\Lambda) \leq 1$. But, since Λ contains $\mathbb{Z}$ it follows that $D^-(\Lambda) \geq 1$.

For (2), for any $N \in \mathbb{N}$ we have

$$\frac{|\Lambda \cap [-2N, -N]|}{-N - (-2N)} = \frac{|\{-2N, -2N + \frac{1}{2}, -2N + 1, \ldots, -N\}|}{N} = \frac{2N + 1}{N}.$$

So $2 \leq D^+(\Lambda) \leq D^+(J) = 2$.

(3) This is obvious.

(4) Fix $0 < \gamma < 2\pi$ and choose $M > 0$ such that $\gamma < 2\pi - \frac{\pi}{M}$. Fix $j \in \mathbb{N}$ and choose any $\alpha \in J_j$. For all $n \geq \max\{j, M\}$, we have

$$\|e^{i\alpha x} - \sum_{k \in J - J_n, |k| \leq N_n} a_k^{\alpha,n} f_k \|_{L^2(-\gamma,\gamma)} \leq$$

$$\|e^{i\alpha x} - \sum_{k \in J - J_n, |k| \leq N_n} a_k^{\alpha,n} f_k \|_{L^2(-2\pi + \pi/n, 2\pi - \pi/n)} \leq \frac{1}{n}.$$

In the rest of the argument for (4) we consequently consider the vectors f_k as elements in the vector space $L^2(-\gamma,\gamma)$. Since $J_1 \subset J_2 \subset \cdots$ and max $J_n \le N_n < \min(J_{n+1}-J_n)$, it follows that

$$\sum_{k\in J-J_n,|k|\le N_n} a_k^{\alpha,n} f_k \in \text{span } \{f_k\}_{k\in\Lambda}.$$

Hence,

$$e^{i\alpha x} \in \overline{span}\ \{f_k\}_{k\in\Lambda}.$$

That is, for all j,

$$f_j \in \overline{span}\ \{f_k\}_{k\in\Lambda}.$$

Since $\{f_k\}_{k\in J}$ is a frame for $L^2(-\gamma,\gamma)$, we have (4).

(5) For $0<\gamma<\pi a$, this follows from Theorem 1(a) and the fact that $D^-(\Lambda)=a=1$. For $\gamma=\pi a$, we note that $\{f_k\}_{k\in\Lambda}$ contains $\{e^{ikx}\}_{k\in\mathbb{Z}}$, which is an orthonormal basis for $L^2(-\pi,\pi)$. Therefore $\{f_k\}_{k\in\Lambda}$ is a frame for $L^2(-\pi,\pi)$.

(6) Since the closed linear span of $\{f_k\}_{k\in\Lambda}$ equals $L^2(-\gamma,\gamma)$ by (4), if it was a frame sequence then it would be a frame, contradicting Theorem 2.

(7). For $\gamma>\pi b$, this is a consequence of Theorem 1(b). For $\gamma=\pi b$, we note that $\{f_k\}_{k\in\Lambda}$ is a subset of $\{f_k\}_{k\in J}$, which is an orthogonal basis for $L^2(-2\pi,2\pi)$; this implies that $\{f_k\}_{k\in\Lambda}$ is a Riesz sequence in $L^2(-2\pi,2\pi)$.

This completes the proof of the theorem. □

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