Frames containing a Riesz basis and approximation of the inverse frame operator.

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Abstract

A frame in a Hilbert space $\mathcal{H}$ allows every element in $\mathcal{H}$ to be written as a linear combination of the frame elements, with coefficients called frame coefficients. Calculations of those coefficients and many other situations where frames occur, require knowledge of the inverse frame operator. But usually it is hard to invert the frame operator if the underlying Hilbert space is infinite dimensional. We introduce a method for approximation of the inverse frame operator using finite subsets of the frame. In particular this allows to approximate the frame coefficients (even in ℓ^2-sense) using finite-dimensional linear algebra. We show that the general method simplifies when the frame contains a Riesz basis.

1 Introduction

Let $\mathcal{H}$ be a separable Hilbert space with the inner product $< \cdot, \cdot >$ linear in the first entry.

Definition 1.1. *$\{f_i\}_{i\in I} \subseteq \mathcal{H}$ is a frame if there exist constants $A, B > 0$ such that*

$$A||f||^2 \leq \sum_{i\in I} | < f, f_i > |^2 \leq B||f||^2$$

for all $f \in \mathcal{H}$. A, B are called frame bounds.

Given a frame $\{f_i\}_{i\in I}$, the *frame operator* is defined by

$$S : \mathcal{H} \to \mathcal{H}, \quad Sf = \sum_{i\in I} < f, f_i > f_i.$$

S is bounded, invertible, and self-adjoint; this leads to the important *frame decomposition*:

$$f = SS^{-1}f = \sum_{i\in I} < f, S^{-1}f_i > f_i, \ \forall f \in \mathcal{H}. \tag{1.1}$$

For practical purposes, it is a problem that calculation of the frame coefficients $< f, S^{-1}f_i >$ requires inversion of S. In the following we develope methods for approximation of S^{-1} using finite subsets of $\{f_i\}_{i\in I}$.

For convenience, we describe the theory for a frame indexed by the natural numbers. Given a frame $\{f_i\}_{i=1}^{\infty}$, let $n \in \mathbb{N}$ and consider $\{f_i\}_{i=1}^{n}$, which is a frame for $\mathcal{H}_n := \text{span}\{f_i\}_{i=1}^{n}$. One can prove that the orthogonal projection P_n of $\mathcal{H}$ onto $\mathcal{H}_n$ is given by

$$P_n f = \sum_{i=1}^{n} < f, {S_n}^{-1} f_i > f_i, \quad f \in \mathcal{H},$$

where

$$S_n : \mathcal{H}_n \to \mathcal{H}_n, \quad S_n f = \sum_{i=1}^{n} < f, f_i > f_i.$$

Observe that ${S_n}^{-1}$ (and hence P_n) can be found using finite-dimensional linear algebra.

Our starting point is the theorem below, which is proved in [3].

Theorem 1.2. *Let $\{f_i\}_{i=1}^{\infty}$ be a frame. Given $n \in \mathbb{N}$, let A_n denote a lower frame bound for $\{f_i\}_{i=1}^{n}$ (as frame for span$\{f_i\}_{i=1}^{n}$) and choose $m(n)$ such that*

$$\sum_{i=n+m(n)+1}^{\infty} | < f_j, f_i > |^2 \leq \frac{A_n}{n^2} \text{ for } j = 1, ..., n.$$

Let $V_n : \mathcal{H}_n \to \mathcal{H}_n$ denote the frame operator for the finite family $\{P_n f_i\}_{i=1}^{n+m(n)}$. Then

$$V_n^{-1} P_n f \to S^{-1} f, \ \forall f \in \mathcal{H}.$$

Theorem 1.2 demonstrates that S^{-1} can be approximated arbitrary well in the strong operator topology using finite-dimensional linear algebra. However, for practical calculation of V_n^{-1} it is desirable that $m(n)$ is not too large. But for most frames of practical interest (see section 3 and 4) one can prove that

$$A_n \to 0 \text{ as } n \to \infty,$$

which forces $m(n)$ to be large. In the next section we show that Theorem 1.2 can be improved under an extra assumption.

2 Frames containing a Riesz basis

Recall that $\{f_i\}_{i\in I} \subseteq \mathcal{H}$ is a *Riesz basis for* $\mathcal{H}$ if $\overline{span}\{f_i\}_{i\in I} = \mathcal{H}$ and there exist constants $A, B > 0$ such that

$$A\sum |c_i|^2 \leq ||\sum c_i f_i||^2 \leq B\sum |c_i|^2$$

for all finite sequences $\{c_i\}$.

Given a Riesz basis $\{f_i\}_{i\in I}$, it is well known [8] that there exists a dual Riesz basis $\{g_i\}_{i\in I}$ such that

$$f = \sum_{i\in I} < f, g_i > f_i, \ \forall f \in \mathcal{H} \tag{2.1}$$

Observe the similarity between the equations (1.1) and (2.1)! In both cases the convergence is unconditional, i.e., independent of the order of summation. Intuitively, it is natural to think about a frame as an "overcomplete basis", but it turns out that this does not hold in the strict sense: There exist frames $\{f_i\}_{i\in I}$ for which no subfamily is a basis for $\mathcal{H}$. This is the reason for the following definition.

Definition 2.1. *A frame $\{f_i\}_{i\in I}$ contains a Riesz basis if there is an index set $J \subseteq I$ for which $\{f_i\}_{i\in J}$ is a Riesz basis for $\mathcal{H}$.*

General information about frames containing a Riesz basis can be found in [1] and [2].

For a frame containing a Riesz basis, Theorem 1.2 can be improved:

Theorem 2.2. *Let $\{f_i\}_{i=1}^{\infty}$ be a frame, containing a Riesz basis $\{f_i\}_{i\in J}$ with lower (Riesz)bound A. Choose finite index sets I_n for which*

$$I_1 \subseteq I_2 \subseteq ... \subseteq I_n \uparrow J.$$

and let P_n be the orthogonal projection onto $span\{f_i\}_{i\in I_n}$. Given $n \in \mathbb{N}$, choose a finite set J_n containing I_n such that

$$\sum_{i\notin J_n} | < f_j, f_i > |^2 \leq \frac{A}{n\cdot |I_n|}, \ \forall j \in I_n.$$

Let $V_n : \mathcal{H}_n \to \mathcal{H}_n$ denote the frame operator for the finite family $\{P_n f_i\}_{i\in J_n}$. Then

$$V_n{}^{-1} P_n f \to S^{-1} f \ \text{as } n \to \infty, \ \forall f \in \mathcal{H}.$$

Proof. Let $n \in \mathbb{N}$. First, it can be proved that for all $f \in \mathcal{H}_n$,

$$\sum_{i \notin J_n} | < f, f_i > |^2 \quad \leq \tfrac{1}{n} ||f||^2.$$

and

$$< (P_n S - V_n) f, f >= \sum_{i \notin J_n} | < f, f_i > |^2.$$

So $P_n S - V_n$ is a positive operator on $\mathcal{H}_n$ and $||(P_n S - V_n)_{|\mathcal{H}_n}|| \leq \frac{1}{n}$.

We leave it to the reader to prove that $A - \frac{1}{n}$ is a lower frame bound for $\{P_n f_i\}_{i \in J_n}$; this implies that $||V_n{}^{-1}|| \leq \frac{1}{A - \frac{1}{n}}$. Now, for $f \in \mathcal{H}$ we obtain that

$$\begin{aligned}
& ||S^{-1} f - V_n{}^{-1} P_n f|| \\
\leq \; & ||(I - P_n) S^{-1} f|| + ||P_n S^{-1} f - V_n{}^{-1} P_n f|| \\
\leq \; & ||(I - P_n) S^{-1} f|| + ||V_n{}^{-1}|| \cdot ||V_n P_n S^{-1} f - P_n f|| \\
\leq \; & ||(I - P_n) S^{-1} f|| + \frac{1}{A - \frac{1}{n}} (||V_n P_n S^{-1} f - P_n S P_n S^{-1} f|| + ||P_n S P_n S^{-1} f - P_n f||) \\
\leq \; & ||(I - P_n) S^{-1} f|| + \frac{1}{A - \frac{1}{n}} (||(V_n - P_n S) P_n S^{-1} f|| + ||S P_n S^{-1} f - f||) \\
\leq \; & ||(I - P_n) S^{-1} f|| + \frac{1}{A - \frac{1}{n}} (\frac{1}{n} \cdot ||P_n S^{-1} f|| + ||S|| \cdot ||P_n S^{-1} f - S^{-1} f||) \\
\leq \; & \frac{1}{nA(A - \frac{1}{n})} \cdot ||f|| + (\frac{B}{A - \frac{1}{n}} + 1) ||(I - P_n) S^{-1} f||.
\end{aligned}$$

□

The importance of Theorem 2.2 lies in the fact that the lower bound A_n appearing in Theorem 1.2 is replaced by A - independently of n. Furthermore, a typical value for A is ~ 1, while A_n is usually much smaller.

3 Frames of exponentials

Let $\{\lambda_n\}_{n \in \mathbb{Z}} \subseteq \mathbb{R}$. A frame for $L^2(-\pi, \pi)$ of the form $\{e^{i \lambda_n x}\}_{n \in \mathbb{Z}}$ is called a *frame of exponentials.* This is actually the context in which frames were introduced in the original paper [6] by Duffin/Schaeffer in 1952!

Whether $\{e^{i\lambda_n x}\}_{n\in\mathbb{Z}}$ is a frame or not depends on the *density* of $\{\lambda_n\}_{n\in\mathbb{Z}}$. Given $r > 0$, let $n^-(r)$ denote the minimal number of points from $\{\lambda_n\}_{n\in\mathbb{Z}}$ to be found in an interval of length r and let

$$D^-(\{\lambda_n\}) := \lim_{r\to\infty} \frac{n^-(r)}{r}.$$

$D^-(\{\lambda_n\})$ is called the *lower Beurling density* of $\{\lambda_n\}_{n\in\mathbb{Z}}$.

Recall that $\{\lambda_n\}_{n\in\mathbb{Z}}$ is *separated* (with separation constant δ) if

$$|\lambda_n - \lambda_m| \geq \delta > 0 \text{ whenever } n \neq m.$$

$\{\lambda_n\}_{n\in\mathbb{Z}}$ is *relatively separated* if $\{\lambda_n\}_{n\in\mathbb{Z}}$ is a finite union of separated sets.

Seip [7] proved the following:

Theorem 3.1. *If $\{\lambda_n\}_{n\in\mathbb{Z}}$ is separated and has lower density strictly larger than one, then $\{e^{i\lambda_n x}\}_{n\in\mathbb{Z}}$ is a frame for $L^2(-\pi,\pi)$ containing a Riesz basis.*

It can be proved that if $\{e^{i\lambda_n x}\}_{n\in\mathbb{Z}}$ is a Riesz basis, then $D^-(\{\lambda_n\}) = 1$. For a regular distribution of points, i.e., $\lambda_n = nb$ for some $b > 0$, the lower density is $D^-(\{\lambda_n\}) = \frac{1}{b}$. When $b < 1$, it follows by Theorem 3.1 that $\{e^{inbx}\}_{n\in\mathbb{Z}}$ is a frame for $L^2(-\pi,\pi)$ containing a Riesz basis $\{e^{inbx}\}_{n\in J}$ for some $J \subseteq \mathbb{Z}$. It is interesting to observe that if b is irrational, no subfamily of the form $\{nNb\}_{n\in\mathbb{Z}}$ has density one; thus the Riesz basis $\{e^{inbx}\}_{n\in J}$ necessarily corresponds to points $\{nb\}_{n\in J}$ which are irregular distributed.

Seip also proved that for $\{\lambda_n\} = \{n(1-|n|^{-1/2})\}_{|n|>1}$, $\{e^{i\lambda_n x}\}$ is a frame for $L^2(-\pi,\pi)$ which does not contain a Riesz basis.

We now return to the question about approximation of the inverse frame operator. First, it turns out to be very difficult to get good estimates for the lower frame bounds for a finite set of exponentials $\{e^{i\lambda_n x}\}_{n=1}^N$ in $L^2(-\pi,\pi)$. Assuming that $\{\lambda_n\}_{n=1}^N$ is separated with separation constant δ, it has been proved in [5] that

$$A_N := 1.6 \cdot 10^{-14} \cdot (\delta/2)^{2N+1} \cdot ((N+1)!)^{-8}$$

is a lower bound, but this bound is clearly too small for practical purposes. We would like to pose it as an open problem:

How can one obtain good estimates for the lower frame bound for a finite set of exponentials?

Without good estimates for the lower frame bound, Theorem 1.2 is not very useful for general frames of exponentials. The situation improves if $\{e^{i\lambda_n x}\}_{n\in\mathbb{Z}}$ contains a Riesz basis $\{e^{i\lambda_k x}\}_{k\in J}$. For convenience, assume that $\{\lambda_k\}_{k\in\mathbb{Z}}$ is separated and ordered such that

$$\cdots \lambda_{-1} \leq \lambda_0 \leq \lambda_1 \leq \cdots$$

Choose $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \uparrow J$. For $n \in \mathbb{N}$, let $\tilde{n} := \max_{k\in I_n}|k|$ and let

$$J_n = \{k\}_{|k|\leq m(n)+\tilde{n}};$$

As before, P_n denotes the orthogonal projection onto $\text{span}\{e^{i\lambda_k x}\}_{k\in I_n}$.

Theorem 3.2. *Suppose that the frame $\{e^{i\lambda_k x}\}_{k\in\mathbb{Z}}$ contains a Riesz basis $\{e^{i\lambda_k x}\}_{k\in J}$ with lower bound A. Given $n \in \mathbb{N}$, choose*

$$m(n) \geq \frac{8\cdot n\cdot |I_n|}{\delta^2 A}$$

and let $V_n : \mathcal{H}_n \to \mathcal{H}_n$ denote the frame operator for the finite family $\{P_n e^{i\lambda_k x}\}_{k\in J_n}$. Then for all $f \in L^2(-\pi,\pi)$,

$$V_n{}^{-1}P_n f \to S^{-1}f, \text{ as } n \to \infty.$$

The proof can be found in [5].

4 Gabor frames

Definition 4.1. *Let $g \in L^2(\mathbb{R}),\ a, b > 0$. A frame for $L^2(\mathbb{R})$ of the form*

$$\{e^{2\pi imbx}g(x-na)\}_{m,n\in\mathbb{Z}},$$

is called a Gabor frame.

By introducing the operators on $L^2(\mathbb{R})$

$$\text{Translation by } a\in\mathbb{R}:\ (T_a f)(x) = f(x-a),\ x\in\mathbb{R}$$

and

$$\text{Modulation by } \ b\in\mathbb{R}:\ (E_b g)(x) = e^{2\pi ibx}f(x),\ x\in\mathbb{R},$$

we can use $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ as short notation for a Gabor frame.

Usually, one thinks about a Gabor frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ as the set of time-frequency shifts of g along the *lattice* $\{(na, mb)\}_{m,n\in\mathbb{Z}} \subseteq \mathbb{R}^2$. In order for $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ to be a frame, the lattice has to be dense enough, in the sense that $ab \le 1$. Also, if $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame, then

$$\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}} \text{ is a Riesz basis } \Leftrightarrow ab = 1 \tag{4.1}$$

It is easy to construct Gabor frames containing a Riesz basis. For example, by choosing g, a, b such that $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a Riesz basis and letting $N \in \mathbb{N}$, the family $\{e^{i2\pi mbx}g(x - n\frac{a}{N})\}_{m,n\in\mathbb{Z}}$ is a frame containing a Riesz basis.

For another example, let $\{\lambda_m\}_{m\in I} \subseteq \mathbb{R}$ be a separated set for which $D^-(\{\lambda_m\}) > 1$. By Theorem 3.1, $\{e^{i\lambda_m x}\}_{m\in I}$ is a frame for $L^2(-\pi,\pi)$ containing a Riesz basis. Thus, for every function g for which

$$\text{supp}\,(g) \subseteq [-\pi,\pi], \quad A \le |g(x)| \le B, a.e\ x \in [-\pi,\pi],$$

the family $\{e^{i\lambda_m x}g(x - n2\pi)\}_{m\in I, n\in\mathbb{Z}}$ is a Gabor frame for $L^2(\mathbb{R})$ which contains a Riesz basis. The remark after Theorem 3.1 implies that even in the lattice case $\lambda_m = 2\pi mb$, only for very special values of b, the Riesz basis contained in $\{e^{i2\pi mbx}g(x - n2\pi)\}_{m,n\in\mathbb{Z}}$ will correspond to a sublattice of the form $\{e^{i2\pi mb'x}g(x - n2\pi)\}_{m,n\in\mathbb{Z}}$, where $b' > 0$.

However, besides such constructions, it is very difficult to decide when a given Gabor frame contains a Riesz basis. We put it as an open problem:

When does a Gabor frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ contains a Riesz basis?

We now want to apply the general approximation theory to Gabor frames $\{E_{kb}T_{la}g\}_{k,l\in\mathbb{Z}}$ containing a Riesz basis $\{E_{kb}T_{la}g\}_{(k,l)\in J}$. First, choose finite sets $\{I_n\}$ such that

$$I_1 \subseteq I_2 \subseteq ... \subseteq I_n \uparrow J.$$

Given $n \in \mathbb{N}$, let $\tilde{n} = \max_{(k,l)\in I_n}\{|k|, |l|\}$ and let

$$J_n = \{(k,l) : |k|, |l| \le \tilde{n} + m(n)\}.$$

As before, the question is how to choose $m(n) \in \mathbb{N}$ such that Theorem 2.2 can be applied. We need the Lemma below.

Lemma 4.2. *If $|k'|, |l'| \leq \tilde{n}$, then for all choices of $m(n) \in \mathbb{N}$,*

$$\sum_{(k,l)\notin J_n} |< E_{kb}T_{la}g, E_{k'b}T_{l'a}g >|^2 \leq \sum_{\{(k,l):|k|,|l|\leq m(n)\}^c} |< E_{kb}T_{la}g, g >|^2.$$

In particular, the estimate holds when $(k', l') \in I_n$.

Proof. Suppose that $|k'|, |l'| \leq \tilde{n}$. Then

$$\begin{aligned} &\textstyle\sum_{(k,l)\notin J_n} |< E_{kb}T_{la}g, E_{k'b}T_{l'a}g >|^2 \\ &= \textstyle\sum_{(k,l)\notin J_n} |< E_{(k-k')b}T_{(l-l')a}g, g >|^2 \\ &= \textstyle\sum_{k,l\in \mathbb{Z}} |< E_{(k-k')b}T_{(l-l')a}g, g >|^2 \\ &- \textstyle\sum_{\{(k,l):|k|,|l|\leq \tilde{n}+m(n)\}} |< E_{(k-k')b}T_{(l-l')a}g, g >|^2. \end{aligned}$$

Now, since $|k'|, |l'| \leq \tilde{n}$,

$$\begin{aligned} &\textstyle\sum_{\{(k,l):|k|,|l|\leq \tilde{n}+m(n)\}} |< E_{(k-k')b}T_{(l-l')a}g, g >|^2 \\ &\geq \textstyle\sum_{\{(k,l):|k|,|l|\leq m(n)\}} |< E_{kb}T_{la}g, g >|^2, \end{aligned}$$

from which the Lemma follows. □

Now, with the above definition of J_n and with P_n beeing the orthogonal projection onto $\text{span}\{E_{kb}T_{la}g\}_{(k,l)\in I_n}$ we get the following consequence of Theorem 2.2:

Theorem 4.3. *Suppose that $\{E_{kb}T_{la}g\}_{k,l\in \mathbb{Z}}$ contains a Riesz basis $\{E_{kb}T_{la}g\}_{(k,l)\in J}$ with lower bound A. For $n \in \mathbb{N}$, choose a number $m(n)$ such that*

$$\textstyle\sum_{\{(k,l):|k|,|l|\leq m(n)\}^c} |< E_{kb}T_{la}g, g >|^2 \leq \dfrac{A}{n|I_n|}.$$

Let $V_n : \mathcal{H}_n \to \mathcal{H}_n$ be the frame operator for $\{P_n E_{kb}T_{la}g\}_{(k,l)\in J_n}$. Then, as $n \to \infty$,

$$V_n{}^{-1}P_n f \to S^{-1}f, \ \forall f \in L^2(\mathbb{R}).$$

If a Gabor frame does not contain a Riesz basis, one has to use Theorem 1.2 instead of Theorem 4.3. However, then we need some lower frame bounds for finite Gabor systems. The following theorem gives some lower bounds for certain functions g which have one-sided bounded support. Since it does not present any extra difficulty to handle *irregular* systems $\{e^{2\pi i\lambda_m x}g(x-a_n)\}_{m,n=1}^{M,N}$, i.e. where λ_m and a_n are not necessarily of the form $\lambda_m = bm$ and $a_n = an$ for some $a, b > 0$, we will immediately do it for these systems.

Theorem 4.4. *Let $a_1,\dots,a_N$ and $\lambda_1,\dots,\lambda_M$ be two finite separated sequences of real numbers, the latter separated by $\varepsilon>0$. Let $g\in L^2(\mathbb{R})$ be such that* $\operatorname{supp} g\subset(-\infty,c]$ *for some $c\in\mathbb{R}$, and suppose there is a non-degenerate interval $I\subset[c-\varepsilon,c]$ and a positive number d such that*

$$|g(x)|\geq d \quad \forall\, x\in I.$$

Denote a lower bound for $\{e^{2\pi i\lambda_m x}\}_{m=1}^M$ in $L^2(I)$ by A, and an upper bound for $\{e^{2\pi i\lambda_m x}g(x)\}_{m=1}^M$ in $L^2(\mathbb{R})$ by B'. Then $\{e^{2\pi i\lambda_m x}g(x-a_n)\}_{m=1,n=1}^{M,N}$ is linearly independent with lower frame bound

$$A_N=d^2A\left(\frac{d^2A}{16B'}\right)^{N-1}.$$

Proof. W.l.o.g. we suppose $a_1<\ldots<a_N$. Since a finite sequence is a Riesz basis for its linear span if and only if it is linearly independent, and since in that case the frame bounds and the Riesz bounds coincide, it suffices to show that

$$\left\|\sum_{n=1}^{k}\sum_{m=1}^{M}c_{mn}e^{2\pi i\lambda_m(\cdot)}g(\cdot-a_n)\right\|_{L^2(\mathbb{R})}\geq\sqrt{A_k\sum_{n=1}^{k}\sum_{m=1}^{M}|c_{mn}|^2} \tag{4.2}$$

holds for all $k\in\{1,\dots,N\}$ and all sequences $\{c_{mn}\}_{m=1,n=1}^{M,N}$ of complex scalars. We do this by induction on k:
For $k=1$, we have

$$\left\|\sum_{m=1}^{M}c_{m1}e^{2\pi i\lambda_m(\cdot)}g(\cdot-a_1)\right\|_{L^2(\mathbb{R})}\geq\left\|\sum_{m=1}^{M}c_{m1}e^{2\pi i\lambda_m(\cdot)}g(\cdot-a_1)\right\|_{L^2(I+a_1)}=$$

$$\left\|\sum_{m=1}^{M}c_{m1}e^{2\pi i\lambda_m(\cdot+a_1)}g\right\|_{L^2(I)}\geq d\sqrt{A\sum_{m=1}^{M}|c_{m1}|^2}.$$

Now suppose that $k\geq 2$ and that (4.2) holds for $k-1$. We distinguish between two cases:

Case 1:

$$\frac{1}{2}\sqrt{A_{k-1}\sum_{n=1}^{k-1}\sum_{m=1}^{M}|c_{mn}|^2}\geq\sqrt{B'\sum_{m=1}^{M}|c_{mk}|^2} \tag{4.3}$$

We then have

$$\left\|\sum_{n=1}^{k}\sum_{m=1}^{M}c_{mn}e^{2\pi i\lambda_m(\cdot)}g(\cdot-a_n)\right\|_{L^2(\mathbb{R})}\geq$$

$$\left\| \sum_{n=1}^{k-1} \sum_{m=1}^{M} c_{mn} e^{2\pi i \lambda_m (\cdot)} g(\cdot - a_n) \right\|_{L^2(\mathbb{R})} - \left\| \sum_{m=1}^{M} c_{mk} e^{2\pi i \lambda_m (\cdot)} g(\cdot - a_k) \right\|_{L^2(\mathbb{R})} \geq$$

$$\sqrt{A_{k-1} \sum_{n=1}^{k-1} \sum_{m=1}^{M} |c_{mn}|^2} - \sqrt{B' \sum_{m=1}^{M} |c_{mk}|^2} \overset{(4.3)}{\geq}$$

$$\frac{1}{2} \sqrt{A_{k-1} \sum_{n=1}^{k-1} \sum_{m=1}^{M} |c_{mn}|^2} \overset{(4.3)}{\geq}$$

$$\frac{1}{4} \sqrt{A_{k-1} \sum_{n=1}^{k-1} \sum_{m=1}^{M} |c_{mn}|^2} + \frac{1}{2} \sqrt{B' \sum_{m=1}^{M} |c_{mk}|^2} \geq$$

$$\frac{1}{4} \sqrt{A_{k-1}} \sqrt{\sum_{n=1}^{k} \sum_{m=1}^{M} |c_{mn}|^2} \geq \sqrt{A_k \sum_{n=1}^{k} \sum_{m=1}^{M} |c_{mn}|^2}.$$

Case 2:

$$\frac{1}{2} \sqrt{A_{k-1} \sum_{n=1}^{k-1} \sum_{m=1}^{M} |c_{mn}|^2} \leq \sqrt{B' \sum_{m=1}^{M} |c_{mk}|^2} \tag{4.4}$$

Then we have

$$\left\| \sum_{n=1}^{k} \sum_{m=1}^{M} c_{mn} e^{2\pi i \lambda_m (\cdot)} g(\cdot - a_n) \right\|_{L^2(\mathbb{R})} \geq \left\| \sum_{n=1}^{k} \sum_{m=1}^{M} c_{mn} e^{2\pi i \lambda_m (\cdot)} g(\cdot - a_n) \right\|_{L^2(I+a_k)} \geq$$

$$\left\| \sum_{m=1}^{M} c_{mk} e^{2\pi i \lambda_m (\cdot)} g(\cdot - a_k) \right\|_{L^2(I+a_k)} - \sum_{n=1}^{k-1} \left\| \sum_{m=1}^{M} c_{mn} e^{2\pi i \lambda_m (\cdot)} g(\cdot - a_n) \right\|_{L^2(I+a_k)} \geq$$

$$d \sqrt{A \sum_{m=1}^{M} |c_{mk}|^2} \overset{(4.4)}{\geq} d\sqrt{A} \left(\frac{1}{2} \sqrt{\sum_{m=1}^{M} |c_{mk}|^2} + \frac{1}{4} \sqrt{\frac{A_{k-1}}{B'} \sum_{n=1}^{k-1} \sum_{m=1}^{M} |c_{mn}|^2} \right) \geq$$

$$\frac{d\sqrt{AA_{k-1}}}{4\sqrt{B'}} \sqrt{\sum_{n=1}^{k} \sum_{m=1}^{M} |c_{mn}|^2} = \sqrt{A_k \sum_{n=1}^{k} \sum_{m=1}^{M} |c_{mn}|^2},$$

thus completing the induction step. The proof is over. □

With the help of Theorems 1.2, 4.4 and Lemma 4.2, one can now state an approximation result similar to Theorem 4.3, for certain Gabor frames which do not necessarily contain a Riesz basis. We leave the details to the reader.

Remark 4.5.

a) Note that an explicit value for the occuring lower bound A of $\{e^{2\pi i\lambda_m x}\}_{m=1}^{M}$ in $L^2(I)$ has been given in section 3. Also, without further assumptions we can use $B' = N \cdot \|g\|^2$. In case $\{e^{2\pi i\lambda_m x}g(x-a_n)\}_{m=1,n=1}^{M,N}$ is a subset of a frame $\{e^{2\pi i\lambda_m x}g(x-a_n)\}_{m=1,n=1}^{\infty}$ with upper bound B we can use $B = B'$ independently of N.

b) Since the Fourier Transform of the function $e^{2\pi i\lambda_m x}g(x-a_n)$ is given by $e^{2\pi i\lambda_m a_n}e^{-2\pi i a_n y}\hat{g}(y-\lambda_m)$ and since $|e^{2\pi i\lambda_m a_n}| = 1$, $\{e^{2\pi i\lambda_m x}g(x-a_n)\}_{m=1,n=1}^{M,N}$ is linearly independent if and only if $\{e^{-2\pi i a_n y}\hat{g}(y-\lambda_m)\}_{m=1,n=1}^{M,N}$ is, and the lower frame bounds are the same. Thus it is clear that an analogue statement to Theorem 4.4 holds if suitable conditions are posed on $\hat{g}$ instead on g.

c) Theorem 4.4 does only cover functions g with one-sided bounded support. In [4] there have been obtained lower bounds for a more general class of funtions g, which e.g. also includes the gaussian.

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