Exchangeability and infinite divisibility

Martin Drapatz and Alexander Lindner

Abstract We characterize exchangeability of infinitely divisible distributions in terms of the characteristic triplet. This is applied to stable distributions and self-decomposable distributions, and a connection to Lévy copulas is made. We further study general mappings between classes of measures that preserve exchangeability and give various examples which arise from discrete time settings, such as stationary distributions of AR(1) processes, or from continuous time settings, such as Ornstein–Uhlenbeck processes or Upsilon-transforms.

Key words: Exchangeability, exchangeability preserving transformation, infinitely divisible distribution, Lévy copula, Ornstein-Uhlenbeck process, random recurrence equation, Upsilon-transform.
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1 Introduction

Throughout, let $d \in \{2,3,4,\ldots\}$ be an integer. The set of all *permutations* π of $\{1,\ldots,d\}$, i.e. of all bijections $\pi : \{1,\ldots,d\} \to \{1,\ldots,d\}$, is denoted by $[d]$. For fixed d, exactly $d!$ permutations of $\{1,\ldots,d\}$ exist. Corresponding to a permutation $\pi \in [d]$ we define the *permutation matrix of* π by

$$P_\pi = \left(e_{\pi(1)}, e_{\pi(2)}, \ldots, e_{\pi(d)}\right) \in \mathbb{R}^{d\times d}$$

where e_i is the i-th unit (column) vector in $\mathbb{R}^d$. A permutation matrix P_π is an orthogonal matrix, i.e. $P_\pi P_\pi^T = \mathrm{Id}_d$, where Id_d is the identity matrix in $\mathbb{R}^{d\times d}$. Thus, $P_\pi^T = P_\pi^{-1} = P_{\pi^{-1}}$. A distribution $\mu = \mathscr{L}(X)$ of a random vector $X = (X_1,\ldots,X_d)^T$ is *exchangeable*, if

$$\mathscr{L}(X) = \mathscr{L}((X_{\pi(1)},\ldots,X_{\pi(d)})^T) \quad \forall \pi \in [d],$$

and we shall also say that the *random vector X is exchangeable*. Since

$$P_\pi(X_1,\ldots,X_d)^T = (X_{\pi^{-1}(1)},\ldots,X_{\pi^{-1}(d)})^T$$

and since the distribution of $P_\pi(X_1,\ldots,X_d)^T$ is the image measure $P_\pi\mu$, defined by

$$(P_\pi\mu)(B) = \mu(P_\pi^{-1}(B)) \quad \forall B \in \mathscr{B}_d,$$

where $\mathscr{B}_d$ denotes the σ-algebra of Borel sets in $\mathbb{R}^d$, this is equivalent to saying that $P_\pi\mu = \mu$ for all $\pi \in [d]$. This can be extended to general (positive) measures on $(\mathbb{R}^d, \mathscr{B}_d)$:

Definition 1. A measure μ on $(\mathbb{R}^d, \mathscr{B}_d)$ is *exchangeable*, if $P_\pi\mu = \mu$ for all permutations $\pi \in [d]$, where $P_\pi\mu$ denotes the image measure of μ under P_π.

Exchangeable probability distributions have various applications, e.g. permutation tests ([13]). The dependence structure of exchangeable random vectors is limited to some extend. This can be used for deriving inequalities for tail probabilities of sums of exchangeable random variables, see e.g. [27] for large deviation results or [11] for a concentration of measure result. Furthermore, an important class of copulas are exchangeable Archimedian copulas, see e.g. Nelsen [19], Chapter 4.

In this paper, we are interested in infinitely divisible distributions which are exchangeable and in transformations of (infinitely divisible) distributions and Lévy measures that preserve exchangeability. Recall that a distribution μ on $(\mathbb{R}^d, \mathscr{B}_d)$ is *infinitely divisible*, if for every $n \in \mathbb{N}$ there exists a distribution μ_n on $(\mathbb{R}^d, \mathscr{B}_d)$, such that the n-fold convolution of μ_n with itself is equal to μ. By the Lévy-Khintchine formula (e.g. Sato [22, Thm. 8.1]), a distribution μ on $(\mathbb{R}^d, \mathscr{B}_d)$ is infinitely divisible if and only if its characteristic function $\mathbb{R}^d \ni z \mapsto \widehat{\mu}(z) = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}x^T z} \mu(dx)$ can be represented in the form

$$\begin{aligned} \widehat{\mu}(z) &:= \exp\{\Psi_\mu(z)\}, \quad \text{with} \\ \Psi_\mu(z) &= -\frac{1}{2} z^T A z + \mathrm{i}\gamma^T z + \int_{\mathbb{R}^d} (\mathrm{e}^{\mathrm{i}x^T z} - 1 - \mathrm{i}x^T z \mathbb{1}_{\mathbb{D}}(x))\, \nu(\mathrm{d}x) \quad \forall\, z \in \mathbb{R}^d, \end{aligned}$$

where $\gamma = (\gamma_1, \ldots, \gamma_d) \in \mathbb{R}^d$, $A \in \mathbb{R}^{d \times d}$ is symmetric and nonnegative definite and ν is a measure on $(\mathbb{R}^d, \mathscr{B}_d)$ such that $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d} (|x|^2 \wedge 1)\, \nu(dx) < \infty$. Here,

$$\mathbb{D} := \{x \in \mathbb{R}^d : |x|^2 \le 1\}$$

denotes the unit ball in $\mathbb{R}^d$ and $|x| = \sqrt{x^T x}$ the Euclidean norm, where x^T denotes the transpose of $x = (x_1, \ldots, x_d)^T \in \mathbb{R}^d$ which we will throughout understand as column vectors. The quantities Ψ_μ and (A, ν, γ) are unique and called the *characteristic exponent* and *characteristic triplet* of μ, respectively. The matrix A is called the *Gaussian covariance matrix* of μ and ν the *Lévy measure* of μ. Conversely, to every triplet (A, ν, γ) with these properties there exists a unique infinitely divisible distribution having this characteristic triplet. Infinitely divisible distributions are closely connected to *Lévy processes*, i.e. to $\mathbb{R}^d$-valued stochastic processes $L = (L_t)_{t \ge 0}$ that have stationary and independent increments, start in 0 and have almost surely right-continuous sample paths with finite left-limits: for every Lévy process L, the distribution $\mathscr{L}(L_t)$ is infinitely divisible for every $t \ge 0$, and conversely, to every infinitely divisible distribution μ on $(\mathbb{R}^d, \mathscr{B}_d)$, there exists a Lévy process L, unique in law, such that $\mathscr{L}(L_1) = \mu$. See Sato [22] for further information regarding Lévy processes and infinitely divisible distributions.

Exchangeability of multivariate Poisson distributions has been considered by Griffiths and Milne [17], but to our knowledge no systematic study of exchangeability of infinitely divisible distributions has been carried out. In the next section, we shall characterize exchangeability of infinitely divisible distributions and apply it to stable distributions. We also give a general construction criterion for exchangeable measures and give relations between exchangeable Lévy measures and their Lévy copula. Then, in Section 3, we summarise some results about exchangeability preserving linear transformations due to Dean and Verducci [15] and Commenges [13]. In Section 4, we study general exchangeability preserving transformations and give a result when the inverse of such a transformation preserves exchangeability. This is then applied in Section 5 to various transformations based on time series, such as the mapping which transforms a (noise) distribution to the stationary distribution of an associated autoregressive process of order 1, moving average processes with random coefficients, or random recurrence equations. Finally, in Section 6 we study mappings of the form $\mu \mapsto \mathscr{L}(\int_0^\infty f(t)\, \mathrm{d}L_t^\mu)$, where f is some suitable (mostly deterministic) function and $(L_t^\mu)_{t\geq 0}$ a Lévy process with distribution μ at time 1. This is in particular applied to self-decomposable distributions and their background driving Lévy process, or to Upsilon-transforms.

Throughout, we use the following notation: the characteristic function of a random vector X will be denoted by φ_X and the Fourier transform of a measure μ by $\widehat{\mu}$, so that $\varphi_X(z) = \mathbb{E}(\mathrm{e}^{\mathrm{i}X^T z})$ and $\widehat{\mu}(z) = \int e^{\mathrm{i}x^T z} \mu(\mathrm{d}x)$ for $z \in \mathbb{R}^d$. When we speak of a measure μ on $\mathbb{R}^d$, we always mean a measure on $\mathbb{R}^d$ with the corresponding Borel-σ-algebra which we denoted by $\mathscr{B}_d$. Sometimes we will consider a Borel-σ-algebra on a subset F of $\mathbb{R}^d$ which we will denote by $\mathscr{B}(F)$. Equality in distribution of random vectors will be denoted by "$\stackrel{d}{=}$", and convergence in distribution by "$\stackrel{d}{\to}$". We write $\log^+(x) = \log(\max\{x,1\})$ for $x \in \mathbb{R}$.

2 Infinitely divisible exchangeable distributions

In this section we shall characterise exchangeability of infinitely divisible distributions in terms of their characteristic triplet. For that, we need the following definition:

Definition 2. A matrix $A \in \mathbb{R}^{d\times d}$ *commutes with permutations* if

$$P_\pi A = A P_\pi \quad \forall\, \pi \in [d].$$

Commenges [13] calls matrices that commute with permutations *exchangeable matrices*, but we shall stick to our notation. In the next section in Theorem 5 we shall summarise some known results about matrices that commute with permutations, in particular give an explicit description for them, but for the moment we shall confine ourselves with the fact that a feasible characterization of these matrices exists.

Remark 1. Let X be a normal random vector in $\mathbb{R}^d$ with mean m and covariance matrix Σ. Then $P_\pi X$ is $N(P_\pi m, P_\pi \Sigma P_\pi^T)$ distributed, and from that it is easy to see that X is exchangeable if and only if $m = (m_1,\dots,m_1)^T$ for some $m_1 \in \mathbb{R}$ and Σ commutes with permutations.

More generally, we can characterise when an infinitely divisible distribution is exchangeable:

Theorem 1. [Exchangeable infinitely divisible distributions]
Let μ be an infinitely divisible distribution on $\mathbb{R}^d$ with characteristic exponent Ψ_μ and characteristic triplet (A,ν,γ). Then the following are equivalent:
(i) μ is exchangeable.
(ii) $\Psi_\mu(P_\pi z) = \Psi_\mu(z)$ for all $z \in \mathbb{R}^d$ and $\pi \in [d]$.
(iii) The Gaussian covariance matrix A commutes with permutations, the Lévy measure ν is exchangeable and $\gamma_i = \gamma_j$ for all $i,j \in \{1,\dots,d\}$, where γ_i denotes the i'th component of γ.

Proof. Let $X = (X_1,\dots,X_d)^T \in \mathbb{R}^d$ be a random vector with infinitely divisible distribution $\mu = \mathscr{L}(X)$. Since $\mathbb{E}e^{i(P_\pi X)^T z} = \widehat{\mu}(P_\pi^{-1} z) = e^{\Psi_\mu(P_{\pi^{-1}} z)}$ for all $\pi \in [d]$ and since $z \mapsto \Psi_\mu(P_{\pi^{-1}} z)$ defines indeed the characteristic exponent of the infinitely divisible distribution $\mathscr{L}(P_\pi X)$ (as a consequence of Lemma 7.6 and Prop. 11.10 in [22]), the equivalence of (i) and (ii) follows.

To see the equivalence of (i) and (iii), observe that since μ is uniquely described by its characteristic triplet, it is exchangeable if and only if the characteristic triplets of $\mathscr{L}(X)$ and $\mathscr{L}(P_\pi X)$ coincide for all permutations π on $[d]$. Denote the characteristic triplet of $\mathscr{L}(P_\pi X)$ by $(A_\pi, \nu_\pi, \gamma_\pi)$. Then by Prop. 11.10 of [22] the characteristic triplet of $\mathscr{L}(P_\pi X)$ is given by

$$A_\pi = P_\pi A P_\pi^T = P_\pi A P_{\pi^{-1}}, \quad \nu_\pi = P_\pi \nu, \quad \gamma_\pi = P_\pi \gamma, \tag{1}$$

where we used that P_π is orthogonal and hence $(P_\pi \nu)(\{0\}) = 0$, and $P_\pi x \in \mathbb{D}$ if and only if $x \in \mathbb{D}$. The equivalence of (i) and (iii) then follows from (1).

Corollary 1. *Let $(L_t)_{t\geq 0}$ be a Lévy process in $\mathbb{R}^d$ and $s > 0$. Then $\mathscr{L}(L_s)$ is exchangeable if and only if $\mathscr{L}(L_1)$ is exchangeable.*

Proof. This is immediate from Theorem 1, since $\Psi_{\mathscr{L}(L_s)}(z) = s\Psi_{\mathscr{L}(L_1)}(z)$.

As an application of Theorem 1, let us characterize all exchangeable stable distributions. Recall that a distribution μ on $\mathbb{R}^d$ is *stable* if and only if it is Gaussian (2-stable case), or if it is infinitely divisible with Gaussian covariance matrix A being 0 and such that there are $\alpha \in (0,2)$ and a finite measure λ on $\mathbb{S} := \{x \in \mathbb{R}^d : |x| = 1\}$ such that the Lévy measure ν of μ can be represented in the form

$$\nu(B) = \int_{\mathbb{S}} \lambda(d\xi) \int_0^\infty \mathbb{1}_B(r\xi) \frac{dr}{r^{1+\alpha}}, \quad B \in \mathscr{B}(\mathbb{R}^d), \tag{2}$$

see Sato [22], Theorems 13.15, 14.1 and 14.3. We speak of α-*stable* distributions in this case. The measure λ is unique and called *spectral measure* or *spherical part* of ν.

Exchangeability of 2-stable, i.e. normal distributions, has been already settled in Remark 1, so we shall restrict to $\alpha \in (0,2)$. By extending the spherical measure to a measure on $\mathbb{R}^d$ by setting it 0 outside $\mathbb{S}$, we can speak of exchangeability of a spherical measure.

Theorem 2. [Exchangeable stable distributions]
Let μ be an α-stable distribution with characteristic triplet $(0, \nu, (\gamma_1, \dots, \gamma_d)^T)$, where $\alpha \in (0,2)$. Then μ is exchangeable if and only if the spherical part λ of ν is exchangeable and if $\gamma_i = \gamma_j$ for all $i, j \in \{1, \dots, d\}$.

Proof. By Theorem 1, it is enough to show that ν is exchangeable if and only if λ is exchangeable. Observe that each P_π maps $\mathbb{S}$ bijectively onto $\mathbb{S}$.

Suppose ν is exchangeable, i.e. for all permutations π on $[d]$ and $B \in \mathscr{B}_d$ we have $\nu(B) = \nu(P_\pi(B))$. Now consider the system of subsets $\mathscr{A} := \{(b,\infty)C : b > 0, C \in \mathscr{B}(\mathbb{S})\}$ of $\mathbb{R}^d$, where $(b,\infty)C := \{x \in \mathbb{R}^d : |x| \in (b,\infty), \frac{x}{|x|} \in C\}$. By (2) the Lévy measure of $(b,\infty)C \in \mathscr{A}$ is

$$\nu((b,\infty)C) = \lambda(C)\alpha^{-1}b^{-\alpha}. \tag{3}$$

Combining exchangeability of ν with equation (3) yields $\lambda(C) = \lambda(P_\pi(C))$ for all permutations π on $[d]$ and all $C \in \mathscr{B}(\mathbb{S})$. Thus, λ is exchangeable.

For the converse, assume now that λ is exchangeable. Using Equation (3) we get for all $b > 0$ and $C \in \mathscr{B}(\mathbb{S})$

$$\nu((b,\infty)C) = \lambda(C)\alpha^{-1}b^{-\alpha} = \lambda(P_\pi(C))\alpha^{-1}b^{-\alpha} = \nu((b,\infty)P_\pi(C)) = \nu(P_\pi((b,\infty)C)).$$

The system $\mathscr{A}$ is a generator of $\mathscr{B}(\mathbb{R}^d \setminus \{0\})$ and a π-system. Furthermore $\mathscr{D} = \{B \in \mathscr{B}(\mathbb{R}^d \setminus \{0\}) : \nu(B) = (P_\pi \nu)(B)\}$ is a λ-system containing $\mathscr{A}$ and this implies by Theorem 1.3.2 in [12] that $\mathscr{D} = \mathscr{B}(\mathbb{R}^d \setminus \{0\})$.

Let us mention another result regarding exchangeable stable distributions of Nguyen [20], who showed that a random vector $(X_1, \dots, X_d)^T$ of identically distributed random variables and with α-stable conditional margin $X_1 | X_2 = x_2, \dots, X_d = x_d$, is α-multivariate stable distributed, if $(X_1, \dots, X_d)^T$ is exchangeable.

In view of Theorems 1 and 2 it is interesting to know how to construct exchangeable measures, in particular exchangeable Lévy measures or exchangeable spherical measures, which is the contents of the next result:

Theorem 3. [Construction of exchangeable measures]
Let

$$F := \{(x_1,\ldots,x_d) \in \mathbb{R}^d : x_1 \geq x_2 \geq \ldots \geq x_d\}$$

and $F_\pi := P_\pi(F)$ *for* $\pi \in [d]$. *Then* $\cup_{\pi\in[d]} F_\pi = \mathbb{R}^d$ *and the following two statements hold:*
(i) A measure ν *on* $(\mathbb{R}^d, \mathscr{B}_d)$ *is exchangeable if and only if for all* $\pi \in [d]$ *and all* $C \in \mathscr{B}_d$

$$\nu(C \cap F_\pi) = \nu(P_\pi^{-1}(C) \cap F). \tag{4}$$

In particular, an exchangeable measure is uniquely determined by its values on $\mathscr{B}(F)$.
(ii) Let $\tilde{\nu}$ *be a measure on* $(F, \mathscr{B}(F))$. *Then the measure* ν *on* $(\mathbb{R}^d, \mathscr{B}_d)$*, defined by*

$$\nu(C \cap F_\pi) := \tilde{\nu}(P_\pi^{-1}(C) \cap F), \quad C \in \mathscr{B}_d, \quad \pi \in [d],$$

is well-defined, exchangeable and satisfies $\nu|_{\mathscr{B}(F)} = \tilde{\nu}$. *Further,* ν *is finite if and only if* $\tilde{\nu}$ *is finite, and* ν *is a Lévy measure on* $\mathbb{R}^d$ *if and only if* $\tilde{\nu}$ *is a Lévy measure on F, the latter meaning that* $\tilde{\nu}(\{0\}) = 0$ *and* $\int_F (|x|^2 \wedge 1)\, \tilde{\nu}(\mathrm{d}x) < \infty$.

Proof. That $\cup_{\pi\in[d]} F_\pi = \mathbb{R}^d$ is clear.
(i) If ν is exchangeable, i.e. $\nu = P_\pi \nu$ for all $\pi \in [d]$, then for all $\pi \in [d]$ and all $C \in \mathscr{B}_d$

$$\nu(C \cap F_\pi) = (P_\pi \nu)(C \cap F_\pi) = \nu(P_\pi^{-1}(C \cap F_\pi)) = \nu(P_\pi^{-1}(C) \cap F),$$

i.e. Equation (4) holds. For the converse suppose Equation (4) is satisfied. For $\pi, \sigma \in [d]$ we define $\tau = \pi^{-1} \circ \sigma$. Then for every $C \in \mathscr{B}_d$ we have

$$\begin{aligned} (P_\pi \nu)(C \cap F_\sigma) &= \nu(P_\pi^{-1}(C) \cap P_\pi^{-1}(F_\sigma)) \\ &\overset{(4)}{=} \nu(P_\tau^{-1}(P_\pi^{-1}(C)) \cap F) \\ &= \nu(P_\sigma^{-1}(C) \cap F) \overset{(4)}{=} \nu(C \cap F_\sigma), \end{aligned}$$

i.e. $\nu|_{\mathscr{B}(F_\sigma)} = (P_\pi \nu)|_{\mathscr{B}(F_\sigma)}$, and $\nu = P_\pi \nu$ follows since $\cup_{\sigma\in[d]} F_\sigma = \mathbb{R}^d$.
(ii) If ν is well-defined, it is clear that ν is a finite, respectively a Lévy measure, if and only if $\tilde{\nu}$ is, and by (i) ν will be exchangeable. Further, $\nu|_{\mathscr{B}(F)} = \tilde{\nu}$. Hence we only have to show that ν is well-defined. Let $C \in \mathscr{B}_d$ with $C \subset F_{\pi_1} \cap F_{\pi_2}$ for some $\pi_1, \pi_2 \in [d]$. We have to show that

$$\tilde{\nu}(P_{\pi_1}^{-1}(C)) = \tilde{\nu}(P_{\pi_2}^{-1}(C)). \tag{5}$$

Define $C_1 = P_{\pi_1}^{-1}(C)$ and $C_2 = P_{\pi_2}^{-1}(C)$ and notice that $C_1, C_2 \subset F$. Furthermore, let $\pi_3 = \pi_2^{-1} \circ \pi_1$ such that $P_{\pi_3}(C_1) = C_2$. If now $x = (x_1, \ldots, x_d)^T \in C_1$, then $x_1 \geq x_2 \geq \ldots \geq x_d$ since $C_1 \subset F$. Since $P_{\pi_3} x = (x_{\pi_3^{-1}(1)}, \ldots, x_{\pi_3^{-1}(d)})^T \in C_2 \subset F$ we conclude $x_{\pi_3^{-1}(1)} \geq x_{\pi_3^{-1}(2)} \geq \ldots \geq x_{\pi_3^{-1}(d)}$. Hence $x_i = x_{\pi_3^{-1}(i)}$ for all $i = 1, \ldots, d$ and $P_{\pi_3} x = x$. Thus, $P_{\pi_3}(C_1) = C_1$ and $C_1 = C_2$, which gives Equation (5).

As an example, define for $d = 2$ a measure λ on $(\mathbb{S} \cap F, \mathscr{B}(\mathbb{S} \cap F))$ by

$$\lambda(\{e_1\}) = w_1, \quad \lambda(\{\frac{e_1 - e_2}{\sqrt{2}}\}) = w_2, \quad \lambda((\mathbb{S} \cap F) \setminus \{e_1, \frac{e_1 - e_2}{\sqrt{2}}\}) = 0, \tag{6}$$

where $e_1 = (1,0)^T$ and $e_2 = (0,1)^T$ denote the unit vectors in $\mathbb{R}^2$ and $w_1, w_2 > 0$. Then

$$\tilde{\nu}(B) = \int_{\mathbb{S}} \lambda(d\xi) \int_0^\infty \mathbb{1}_B(r\xi) \frac{\mathrm{e}^{-r^2 \lambda(\{\xi\})} dr}{r}, \quad B \in \mathscr{B}(F),$$

defines a Lévy measure on $(F, \mathscr{B}(F))$. By equation (6),

$$\tilde{\nu}(B) = w_1 \int_0^\infty \mathbb{1}_B(re_1) \frac{\mathrm{e}^{-r^2 w_1} dr}{r} + w_2 \int_0^\infty \mathbb{1}_B(r \frac{e_1 - e_2}{\sqrt{2}}) \frac{\mathrm{e}^{-r^2 w_2} dr}{r}, \quad B \in \mathscr{B}(F).$$

Denote by σ the permutation such that $\sigma(1) = 2$ and $\sigma(2) = 1$. By Theorem 3,

$$\nu(C \cap F_\pi) := \tilde{\nu}(P_\pi^{-1}(C) \cap F), \quad C \in \mathscr{B}_2, \quad \pi \in [2],$$

defines an exchangeable measure on $(\mathbb{R}^2, \mathscr{B}_2)$. One easily concludes for $C \in \mathscr{B}_2$

$$\nu(C \cap F_\sigma) = w_1 \int_0^\infty \mathbb{1}_C(rP_\sigma e_1) \frac{\mathrm{e}^{-r^2 w_1} dr}{r} + w_2 \int_0^\infty \mathbb{1}_C(rP_\sigma \frac{e_1 - e_2}{\sqrt{2}}) \frac{\mathrm{e}^{-r^2 w_2} dr}{r},$$

and hence with $f_A(rx) := \mathbb{1}_A(rx) + \mathbb{1}_A(rP_\sigma x)$ for $A \in \mathscr{B}_2$, $r \geq 0$ and $x \in \mathbb{R}^2$,

$$\begin{aligned} \nu(A) &= \nu(A \cap F_\sigma) + \nu(A \cap F) \\ &= w_1 \int_0^\infty f_A(re_1) \frac{\mathrm{e}^{-r^2 w_1} dr}{r} + w_2 \int_0^\infty f_A(r \frac{e_1 - e_2}{\sqrt{2}}) \frac{\mathrm{e}^{-r^2 w_2} dr}{r}, \quad A \in \mathscr{B}_2, \end{aligned}$$

is an exchangeable measure on $(\mathbb{R}^2, \mathscr{B}_2)$.

It is well known that a distribution $\mu = \mathscr{L}(X)$ is exchangeable if and only if its one-dimensional margins are equal and if it admits a copula which is exchangeable, see e.g. Nelsen [19, Theorem 2.7.4]. It is natural to ask if the same result holds for the relation between Lévy measures and Lévy copulas. Lévy copulas have been introduced by Tankov [26] and Cont and Tankov [14] for Lévy measures concentrated on $[0,\infty)^d$ and by Kallsen and Tankov [18] for Lévy measures on $\mathbb{R}^d$. The concept for measures on $\mathbb{R}^d$ is more complicated and we shall restrict ourselves to Lévy measures on $[0,\infty)^d$ and call the corresponding Lévy copula a positive Lévy copula. In order to define it, let ν be a Lévy measure concentrated on $[0,\infty)^d$. Then its *tail integral* U_ν is defined as the function $U_\nu : [0,\infty]^d \to [0,\infty]$ given by

$$U_\nu(x_1,\ldots,x_d) := \begin{cases} \nu([x_1,\infty)\times\ldots\times[x_d,\infty)), & (x_1,\ldots,x_d) \neq (0,\ldots,0), \\ \infty, & (x_1,\ldots,x_d) = (0,\ldots,0). \end{cases}$$

Its *marginal tail integrals* U_{ν_i} are defined as the tail integrals of the (one-dimensional) marginal Lévy measures $\nu_1,\ldots,\nu_d$, i.e.

$$U_{\nu_i}(x_i) = U_\nu(0,\ldots,0,x_i,0,\ldots,0) = \begin{cases} \nu_i([x_i,\infty)), & x_i \in (0,\infty], \\ \infty, & x_i = 0. \end{cases}$$

It is clear that a Lévy measure on $[0,\infty)^d$ is uniquely determined by its tail integral. Now a *positive Lévy copula* is a function $C : [0,\infty]^d \to [0,\infty]$ such that $C(x_1,\ldots,x_d) = 0$ if at least one of the x_i is zero, such that

$$C(\infty,\ldots,\infty,x_i,\infty,\ldots,\infty) = x_i \quad \forall\, x_i \in [0,\infty], \quad i = 1,\ldots,d,$$

such that $C(x_1,\ldots,x_d) \neq \infty$ unless $x_1 = \ldots, x_d = \infty$, and such that C is a d-increasing function (cf. Cont and Tankov [14, Def. 5.11]). Similar to copulas, Lévy copulas allow to separate the margins and the dependence structure of Lévy measures. More precisely, for every Lévy measure ν on $[0,\infty)^d$ there exists a positive Lévy copula C such that

$$U_\nu(x_1,\ldots,x_d) = C(U_{\nu_1}(x_1),\ldots,U_{\nu_d}(x_d)) \quad \forall\, x_1,\ldots,x_d \in [0,\infty]. \tag{7}$$

The Lévy copula is uniquely determined on $U_{\nu_1}([0,\infty])\times\ldots\times U_{\nu_d}([0,\infty])$ and we shall call every positive Lévy copula C satisfying (7) a positive Lévy copula *associated with* ν. Conversely, if $\nu_1,\ldots,\nu_d$ are one-dimensional Lévy measures on $[0,\infty)$ and if C is a positive Lévy copula, then the right-hand side of (7) defines the tail-integral of a Lévy measure ν on $[0,\infty)^d$ with margins $\nu_1,\ldots,\nu_d$ and associated Lévy copula C; see Cont and Tankov [14, Thm. 5.6].

Barndorff-Nielsen and Lindner [1, Thm. 6] showed that there is a one-to-one correspondence between (d-dimensional) positive Lévy copulas and Lévy measures on $[0,\infty)^d$ with “unit” 1-stable margins, i.e. Lévy measures with marginal tail integrals $[0,\infty] \to [0,\infty]$, $x_k \mapsto x_k^{-1}$ (the tail integral of a one-dimensional stable Lévy measure is $x_k \mapsto a x_k^{-1}$ for some $a > 0$, and by "unit" we mean that $a = 1$). More precisely, if C is a positive Lévy copula, then there exists a unique Lévy measure ν_C on $[0,\infty)^d$ such that

$$\nu_C([x_1^{-1},\infty) \times \ldots \times [x_d^{-1},\infty)) = C(x_1,\ldots,x_d) \quad \forall\, x_1,\ldots,x_d \in [0,\infty] \tag{8}$$

which then has unit 1-stable margins (more precisely, is the Lévy measure of an infinitely divisible distribution with unit 1-stable margins). Conversely, to any Lévy measure ν_C on $[0,\infty)^d$ with unit 1-stable margins, the left-hand side of (8) defines a positive Lévy copula.

Definition 3. A positive Lévy copula $C : [0,\infty]^d \to [0,\infty]$ is *exchangeable*, if

$$C(x_1,\ldots,x_d) = C(x_{\pi(1)},\ldots,x_{\pi(d)}) \quad \forall\, x_1,\ldots,x_d \in [0,\infty],\ \pi \in [d].$$

We now give a connection between exchangeable Lévy copulas and exchangeable Lévy measures:

Theorem 4. [Exchangeability and Lévy copulas]
(i) A positive Lévy copula C is exchangeable if and only if the Lévy measure ν_C with unit 1-stable margins defined by (8) *is exchangeable.*
(ii) Let ν be a Lévy measure on $[0,\infty)^d$ with marginal Lévy measures $\nu_1,\ldots,\nu_d$. If $\nu_1 = \ldots = \nu_d$ and if an associated positive Lévy copula C exists which is exchangeable, then ν is exchangeable. Conversely, if ν is exchangeable and $U_{\nu_1}([0,\infty]) = [0,\infty]$ (i.e. ν_1 has no atoms and is infinite), then $\nu_1 = \ldots = \nu_d$ and the unique associated positive Lévy copula C is exchangeable.

It seems very likely that the result in (ii) can be extended to the case when $U_{\nu_1}([0,\infty]) \neq [0,\infty]$, to a general statement of the form that ν is exchangeable if and only if $\nu_1 = \ldots = \nu_d$ and if it admits an exchangeable associated positive Lévy copula C (observe that then the positive Lévy copula is not necessarily unique any more). For simplicity, we have not pursued this issue further.

Proof. First observe that as in the proof of Theorem 2, a Lévy measure ν on $[0,\infty)^d$ is exchangeable if and only if

$$(P_\pi \nu)\big([x_1,\infty) \times \ldots \times [x_d,\infty)\big) = \nu\big([x_1,\infty) \times \ldots \times [x_d,\infty)\big) \quad \forall\, x_1,\ldots,x_d \in [0,\infty],\ \pi \in [d],$$

i.e. if and only if

$$U_{P_\pi \nu} = U_\nu \quad \forall\, \pi \in [d]. \tag{9}$$

(i) For $(x_1,\ldots,x_d) \neq (0,\ldots,0)$ we have

$$U_{P_\pi \nu_C}(x_1,\ldots,x_d) = \nu_C([x_{\pi(1)},\infty)\times\ldots\times[x_{\pi(d)},\infty)) = C(x^{-1}_{\pi(1)},\ldots,x^{-1}_{\pi(d)})$$

by (8), hence we conclude from (9) that ν_C is exchangeable if and only if C is exchangeable.
(ii) Suppose first that $\nu_1 = \ldots = \nu_d$ and that ν admits an exchangeable associated Lévy copula C. Then $U_{\nu_1} = \ldots = U_{\nu_d}$ and it follows from (7) that

$$\begin{aligned}
&(P_\pi \nu)\big([x_1,\infty)\times\ldots\times[x_d,\infty)\big)\\
&= U_\nu(x_{\pi(1)},\ldots,x_{\pi(d)})\\
&= C(U_{\nu_1}(x_{\pi(1)}),\ldots,U_{\nu_d}(x_{\pi(d)}))\\
&= C(U_{\nu_1}(x_1),\ldots,U_{\nu_1}(x_d))\\
&= U_\nu(x_1,\ldots,x_d) = \nu\big([x_1,\infty)\times\ldots\times[x_d,\infty)\big) \quad \forall\, x_1,\ldots,x_d \in [0,\infty],\ \pi \in [d].
\end{aligned}$$

Hence, ν is exchangeable.
Conversely, suppose that ν is an exchangeable Lévy measure. Then it is easy to see that $\nu_1 = \ldots = \nu_d$, hence $U_{\nu_1} = \ldots = U_{\nu_d}$. If now additionally $U_{\nu_1}([0,\infty]) = [0,\infty]$, then for every $y \in [0,\infty]$ there exists some $U^{\leftarrow}_{\nu_1}(y) \in [0,\infty]$ such that $U_{\nu_1}(U^{\leftarrow}_{\nu_1}(y)) = y$. For $y_1,\ldots,y_d \in [0,\infty]$ we then have necessarily by (7)

$$C(y_1,\ldots,y_d) = U_\nu(U^{\leftarrow}_{\nu_1}(y_1),\ldots,U^{\leftarrow}_{\nu_1}(y_d)),$$

and exchangeability of C follows from (9).

Many positive Lévy copulas used in practice are *Archimedian Lévy copulas*. They are of the form

$$C(x_1,\ldots,x_d) = \phi^{-1}(\phi(x_1)+\ldots+\phi(x_d))$$

for some strictly decreasing function $\phi : [0,\infty] \to [0,\infty]$ (the *generator*) such that $\phi(0) = \infty$, $\phi(\infty) = 0$, and ϕ^{-1} has derivatives up to order d on $(0,\infty)$ with alternating signs, see Cont and Tankov [14, Prop. 5.7]. It is evident that Archimedian Lévy copulas are exchangeable. We hence have

Corollary 2. *Let ν be a Lévy measure on $[0,\infty)^d$ which admits an Archimedian Lévy copula. Then ν is exchangeable if and only if $\nu_1 = \ldots = \nu_d$.*

Proof. This is immediate from Theorem 4 and its proof.

3 Matrices that preserve exchangeability

In this section we summarise some results of Dean and Verducci [15] and Commenges [13] about matrices that preserve (second order) exchangeability, which will be needed later. Denote by $J_d \in \mathbb{R}^{d\times d}$ the matrix with all entries equal to 1 and recall that $\mathrm{Id}_d \in \mathbb{R}^{d\times d}$ denotes the identity matrix.

Theorem 5. (Commenges [13])
(i) A matrix $A \in \mathbb{R}^{d\times d}$ commutes with permutations, if and only if there are $a,b \in \mathbb{R}$ such that

$$A = a\,\mathrm{Id}_d + bJ_d.$$

(ii) Suppose $A,B \in \mathbb{R}^{d\times d}$ commute with permutations. Then $A+B$ and AB commute with permutations.
(iii) If $A \in \mathbb{R}^{d\times d}$ commutes with permutations and $\det A \neq 0$, then A^{-1} commutes with permutations.
(iv) Let $C \in \mathbb{R}^{d\times d}$. Then $\mathscr{L}(CX)$ is exchangeable for every exchangeable normal distribution $\mathscr{L}(X)$ on $\mathbb{R}^d$ if and only if C can be represented in the form $C = AQ$, where $A \in \mathbb{R}^{d\times d}$ commutes with permutations and $Q = (q_{ij})_{i,j=1,\dots,d} \in \mathbb{R}^{d\times d}$ is an orthogonal matrix that satisfies $\sum_{j=1}^d q_{ij} = \sum_{j=1}^d q_{1j}$ for all $i \in \{1,\dots,d\}$.

Commenges [13] calls property (iv) above the *preservation of second moment exchangeability*. For later use, we mention the following consequence.

Corollary 3. *Let $A \in \mathbb{R}^{d\times d}$ commute with permutations. Then also e^A commutes with permutations. If A is additionally positive definite, then the converse also holds.*

Proof. Since $\mathrm{e}^A = \sum_{k=0}^\infty \frac{1}{k!}A^k$, the first statement is immediate from Theorem 5 (ii). For the converse suppose that $\mathrm{e}^A \in \mathbb{R}^{d\times d}$ commutes with permutations. Then for every $\pi \in [d]$

$$\exp(A) = P_\pi \exp(A) P_{\pi^{-1}} = \sum_{k=0}^\infty \frac{P_\pi A^k P_{\pi^{-1}}}{k!} = \sum_{k=0}^\infty \frac{(P_\pi A P_{\pi^{-1}})^k}{k!} = \exp(P_\pi A P_{\pi^{-1}}).$$

Thus, we proved $\exp(A) = \exp(P_\pi A P_{\pi^{-1}})$. By positive definiteness of A, Proposition 11.2.9 of [7] implies $A = P_\pi A P_{\pi^{-1}}$.

While matrices that preserve exchangeability of *normal* random vectors have been characterized in Theorem 5, the corresponding question which linear transformations preserve exchangeability of any random vector has been solved already earlier by Dean and Verducci [15].

Definition 4. Let $A \in \mathbb{R}^{d\times d}$. Then A is said to be *exchangeability preserving*, if $\mathscr{L}(AX)$ is exchangeable for every exchangeable random vector X in $\mathbb{R}^d$. Denote

$$\mathscr{E}_d = \{A \in \mathbb{R}^{d\times d} : A \text{ is exchangeability preserving}\},$$
$$\mathscr{E}_d^0 = \{A \in \mathbb{R}^{d\times d} : A \text{ is exchangeability preserving and } \det(A) \neq 0\}.$$

Dean and Verducci [15, Thm. 4, Cor. 1.2, Cor. 1.3] gave the following characterization of exchangeability preserving matrices:

Theorem 6. [Dean and Verducci]
A matrix $A \in \mathbb{R}^{d\times d}$ is exchangeability preserving if and only if for every $\pi \in [d]$ there exists $\pi' \in [d]$ such that

$$P_\pi A = AP_{\pi'}.$$

Further, $\mathscr{E}_d^0$ can be characterized as

$$\mathscr{E}_d^0 = \{A \in \mathbb{R}^{d\times d} : \exists a, b \in \mathbb{R}, a \neq 0, a \neq -db, \pi \in [d] \text{ such that } A = aP_\pi + bJ_d\}.$$

Notice that for $d = 2$ the set $\mathscr{E}_2^0$ of invertible exchangeability preserving matrices is precisely the set of invertible matrices which commute with permutations. Dean and Verducci [15, Cor. 1.1] also gave an explicit expression for $\mathscr{E}_d$, which is however more complicated to formulate. However, it is easy to see that

$$\{A \in \mathbb{R}^d : \exists\, a, b \in \mathbb{R}, \pi \in [d] : A = aP_\pi + bJ_d\} \subset \mathscr{E}_d.$$

4 Exchangeability preserving transformations

In this section we study more general transformations that preserve exchangeability, not necessarily linear ones. We define:

Definition 5. Let $\mathscr{M}_1$ and $\mathscr{M}_2$ be two classes of measures on $\mathbb{R}^d$ and $G : \mathscr{M}_1 \to \mathscr{M}_2$ a mapping. We say that G
(i) is *exchangeability preserving* if $G(\mu)$ is exchangeable whenever μ is exchangeable,
(ii) *commutes weakly with permutations*, if for every $\mu \in \mathscr{M}_1$ and $\pi \in [d]$ there exists $\pi' \in [d]$ such that $P_{\pi'}\mu \in \mathscr{M}_1$ and

$$P_\pi G(\mu) = G(P_{\pi'}\mu),$$

(iii) *commutes with permutations* if $P_\pi\mu \in \mathscr{M}_1$ for all $\mu \in \mathscr{M}_1$ and $\pi \in [d]$, and

$$P_\pi G(\mu) = G(P_\pi \mu) \quad \forall\, \mu \in \mathscr{M}_1, \quad \pi \in [d].$$

When $\mathscr{M}_1 = \mathscr{M}_2$ is the class of all probability distributions on $\mathbb{R}^d$, and G is induced by a linear mapping $A : \mathbb{R}^d \to \mathbb{R}^d$, i.e. $G(\mu) := A\mu$, the image measure of μ under A, then G commutes with permutations if and only if A does in the sense of Definition 2, and G is exchangeability preserving if and only if A is in the sense of Definition 4. Further, by Theorem 6, in this case G is exchangeability preserving if and only if it is commutes weakly with permutations. For general mappings that are not necessarily linear this is no longer true, as the following simple example shows:

Example 1. (i) If the set $\mathscr{M}_1$ does not contain any exchangeable measure, then any $G : \mathscr{M}_1 \to \mathscr{M}_2$ will be exchangeability preserving, but it will commute weakly with permutations only in special cases.
(ii) Let $\mathscr{M}_1 = \mathscr{M}_2$ be the class of all probability distributions on $\mathbb{R}^d$, and define

$$G(\mu) := \begin{cases} \delta_{(0,\dots,0)^T}, & \text{if } \mu \text{ is exchangeable,} \\ \delta_{(1,0,\dots,0)^T}, & \text{if } \mu \text{ is not exchangeable,} \end{cases}$$

where δ_x denotes the Dirac measure at x. Then G is exchangeability preserving, but does not commute weakly with permutations, since $P_\pi \delta_{(1,0,\dots,0)^T}$ is not in the range of G for π a permutation different from the identity.
(iii) If $\mathscr{M}_2$ contains only exchangeable measures, then any $G : \mathscr{M}_1 \to \mathscr{M}_2$ is exchangeability preserving and it even commutes weakly with permutations, since for all $\pi \in [d]$ and $\mu \in \mathscr{M}_1$ we can choose the identity for π' and obtain $P_\pi G(\mu) = G(\mu) = G(P_{\pi'}\mu)$.

So we have seen that there are mappings that are exchangeability preserving but do not commute weakly with permutations. On the other hand, it is easy to see that any transformation that commutes weakly with permutations is exchangeability preserving. More precisely, we have:

Proposition 1. *Let $\mathscr{M}_1$ and $\mathscr{M}_2$ be two classes of measures on $\mathbb{R}^d$. Then every mapping $G : \mathscr{M}_1 \to \mathscr{M}_2$ that commutes with permutations also commutes weakly with permutations, and every mapping that commutes weakly with permutations is exchangeability preserving.*

Proof. That every mapping which commutes with permutations commutes weakly with permutations is clear. Now let $G : \mathscr{M}_1 \to \mathscr{M}_2$ commute weakly with permutations and let $\mu \in \mathscr{M}_1$ be exchangeable. Let $\pi \in [d]$. Then there exists $\pi' \in [d]$ such that

$$P_\pi G(\mu) = G(P_{\pi'}\mu) = G(\mu),$$

which shows that $G(\mu)$ is exchangeable.

Dean and Verducci [15, Cor. 1.3, Thm. 4] showed that an invertible matrix $A \in \mathbb{R}^{d\times d}$ is exchangeability preserving if and only if its inverse A^{-1} is exchangeability preserving. This then trivially transfers to the mappings G and G^{-1} induced by A and A^{-1}. For general transformations G, such a result is not true:

Example 2. Let $G : \mathscr{M}_1 \to \mathscr{M}_2$ be bijective and assume that $\mathscr{M}_2$ consists only of exchangeable measures, while $\mathscr{M}_1$ contains at least one non-exchangeable measure. Then G commutes weakly with permutations by Example 1 (iii), but the inverse G^{-1} is not exchangeability preserving, in particular it cannot commute weakly with permutations.

When G commutes with permutations and G is bijective, then however also the inverse commutes with permutations:

Theorem 7. *Let $\mathscr{M}_1$ and $\mathscr{M}_2$ be two classes of measures on $\mathbb{R}^d$ and $G : \mathscr{M}_1 \to \mathscr{M}_2$ an injective mapping that commutes with permutations. Then its inverse G^{-1} : $G(\mathscr{M}_1) \to \mathscr{M}_1$ also commutes with permutations, in particular G^{-1} is exchangeability preserving.*

Proof. Let G be injective. Let $\pi \in [d]$ and $\nu \in G(\mathscr{M}_1)$. Let $\mu := G^{-1}(\nu)$. Then $P_\pi\mu \in \mathscr{M}_1$ and

$$P_\pi\nu = P_\pi G(\mu) = G(P_\pi\mu),$$

showing that $P_\pi\nu \in G(\mathscr{M}_1)$. Applying G^{-1} to the equation above gives

$$G^{-1}(P_\pi\nu) = P_\pi\mu = P_\pi G^{-1}(\nu).$$

Hence G^{-1} commutes with permutations.

In the next two sections we give some examples of exchangeability preserving mappings.

5 Exchangeability preserving transformations based on discrete time

In this section we discuss some transformations related to time series analysis and their exchangeability preserving property.

Example 3. [Convolution with an exchangeable distribution]
Let $\rho = \mathscr{L}(X)$ be an exchangeable distribution on $\mathbb{R}^d$ and let $\mathscr{M}_1 = \mathscr{M}_2$ be the class of all probability distributions on $(\mathbb{R}^d, \mathscr{B}_d)$. Then the mapping

$$G_\rho : \mathscr{M}_1 \to \mathscr{M}_2, \quad \mu \mapsto \mu * \rho$$

commutes with permutations, in particular is exchangeability preserving, since for $\pi \in [d]$ and $B \in \mathscr{B}_d$ we have

$$\begin{aligned}(P_\pi(\mu * \rho))(B) &= \int_{\mathbb{R}^d} \mu(P_\pi^{-1}(B) - x)\, \rho(\mathrm{d}x) \\ &= \int_{\mathbb{R}^d} (P_\pi \mu)(B - P_\pi x)\, \rho(\mathrm{d}x) \\ &= \int_{\mathbb{R}^d} (P_\pi \mu)(B - y)(P_\pi \rho)(\mathrm{d}y) = ((P_\pi \mu) * \rho)(B).\end{aligned}$$

(i) Now assume that the characteristic function $z \mapsto \widehat{\rho}(z)$ is different from 0 for z from a dense subset of $\mathbb{R}^d$, e.g. if ρ is infinitely divisible (cf. [22, Lem. 7.5]). Since $\widehat{G_\rho(\mu)}(z) = \widehat{\rho}(z)\widehat{\mu}(z)$, it follows that G_ρ is injective and hence the inverse $G_\rho^{-1} : G_\rho(\mathscr{M}_1) \to \mathscr{M}_2$ commutes with permutations by Theorem 7. Hence $\rho * \mu$ is exchangeable if and only if μ is exchangeable, provided ρ is exchangeable and $\widehat{\rho}$ does not vanish on a dense set. Similarly, using the Laplace transform to establish injectivity, if ρ and μ are both concentrated on $[0,\infty)^d$ and ρ is exchangeable, then $\rho * \mu$ is exchangeable if and only if μ is exchangeable.
(ii) Let X_1, X_2, X_3, Y_1 be four independent one-dimensional random variables such that

$$\mathscr{L}(X_1) = \mathscr{L}(X_2) = \mathscr{L}(X_3) \neq \mathscr{L}(Y_1),$$

but

$$\mathscr{L}(X_1) * \mathscr{L}(Y_1) = \mathscr{L}(X_1) * \mathscr{L}(X_1).$$

Examples of such distributions can be found in Feller [16, p. 506]. Now let $d = 2$ and consider the two-dimensional distributions

$$\rho := \mathscr{L}((X_1, X_2)^T), \quad \text{and} \quad \mu := \mathscr{L}((Y_1, X_3)^T).$$

Then ρ and

$$\rho * \mu = \mathscr{L}((X_1 + Y_1, X_2 + X_3)^T) = \rho * \rho$$

are exchangeable, but μ is not exchangeable. Hence without extra assumptions on the exchangeable ρ (as done e.g. in (i)), it is not true that $\rho * \mu$ is exchangeable if and only if μ is exchangeable.

(iii) It is worth noting that the convolution of two non-exchangeable distributions can be exchangeable. To see this, let $d=2$ and let X_1,X_2,Y_1,Y_2 be independent one-dimensional random variables such that $\mathscr{L}(X_1)=\mathscr{L}(Y_2)\neq\mathscr{L}(Y_1)=\mathscr{L}(X_2)$. Define the distributions $\mu=\mathscr{L}((X_1,X_2)^T)$ and $\rho=\mathscr{L}((Y_1,Y_2)^T)$. Then $\mu*\rho=\mathscr{L}((X_1+Y_1,X_2+Y_2)^T)$ is exchangeable, although neither μ nor ρ are exchangeable.

Next we shall consider distributions that arise from infinite moving average processes. We first note that exchangeability is closed under weak convergence:

Lemma 1. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of exchangeable random vectors in $\mathbb{R}^d$ that converges in distribution to a random vector Y as $n\to\infty$. Then Y is exchangeable.*

Proof. The convergence $X_n\xrightarrow{d}Y$ for $n\to\infty$ implies $P_\pi X_n\xrightarrow{d}P_\pi Y$. Since $P_\pi X_n\stackrel{d}{=}X_n$ by assumption the two limits coincide.

Corollary 4. [Two sided moving average processes]
Let $(Z_t)_{t\in\mathbb{Z}}$ be an i.i.d. sequence of exchangeable random vectors in $\mathbb{R}^d$ and $(A_t)_{t\in\mathbb{Z}}$ be an i.i.d. sequence, independent of $(Z_t)_{t\in\mathbb{Z}}$, taking values in $\mathscr{E}_d$, the space of exchangeability preserving $(d\times d)$-matrices. Assume that $\sum_{j=-\infty}^{\infty}A_jZ_{t-j}$ converges almost surely (equivalently, in distribution as a sum with independent increments). Then $\sum_{j=-\infty}^{\infty}A_jZ_{t-j}$ is exchangeable.

Proof. Let ρ be the distribution of A_j. Conditioning on $A_j=M$ for $M\in\mathscr{E}_d$, we have for the characteristic function

$$\varphi_{P_\pi A_jZ_{t-j}}(z)=\int_{\mathscr{E}_d}\varphi_{P_\pi MZ_{t-j}}(z)\,\rho(\mathrm{d}M)=\int_{\mathscr{E}_d}\varphi_{MZ_{t-j}}(z)\,\rho(\mathrm{d}M)=\varphi_{A_jZ_{t-j}}(z)$$

for all $z\in\mathbb{R}^d$ and $\pi\in[d]$, hence A_jZ_{t-j} is exchangeable. Since $(A_jZ_{t-j})_{j\in\mathbb{Z}}$ is independent, Lemma 1 together with Example 3 shows that $\sum_{j=-\infty}^{\infty}A_jZ_{t-j}$ is exchangeable.

Under conditions which guarantee the existence of mean and covariance, we give a necessary condition for the stationary distribution of an infinite moving average process to be exchangeable:

Proposition 2. *Let $(Z_t)_{t\in\mathbb{Z}}$ be an i.i.d. sequence of $\mathbb{R}^d$-valued random vectors with finite variance and $(A_k)_{k\in\mathbb{Z}}$ be an $\mathbb{R}^{d\times d}$-valued deterministic sequence such that $(A_k^{ij})_{k\in\mathbb{Z}}$ is absolutely summable for each $i,j\in\{1,\dots,d\}$, where A_k^{ij} denotes the (i,j)-component of A_k. Then $\sum_{j=-\infty}^{\infty}A_jZ_{t-j}$ converges almost surely absolutely and in L^2, and necessary conditions for $\sum_{j=-\infty}^{\infty}A_jZ_{t-j}$ to be exchangeable are that $(\sum_{j=-\infty}^{\infty}A_j)\mathbb{E}Z_0=(\gamma_1,\dots,\gamma_1)^T$ for some $\gamma_1\in\mathbb{R}$ and that $\sum_{k=-\infty}^{\infty}A_kCov(Z_0)A_k^T$ commutes with permutations, where $Cov(Z_0)=\mathbb{E}(Z_0Z_0^T)-\mathbb{E}(Z_0)\mathbb{E}(Z_0^T)$ denotes the covariance matrix of Z_0.*

Proof. Almost sure and L^2-convergence under the stated conditions is well known, and $X := \sum_{j=-\infty}^{\infty} A_j Z_{t-j}$ has mean $(\sum_{j=-\infty}^{\infty} A_j)\mathbb{E}Z_0$ and covariance matrix

$$\sum_{k=-\infty}^{\infty} A_k \mathrm{Cov}(Z_0) A_k^T.$$

If X is exchangeable, then $P_\pi X$ and X must share the same mean and covariance matrix for all $\pi \in [d]$, from which follows that $\mathbb{E}X = (\gamma_1, \ldots, \gamma_1)^T$ for some $\gamma_1 \in \mathbb{R}$ and $P_\pi \mathrm{Cov}(X) P_\pi^T = \mathrm{Cov}(X)$, which is the claim.

Now consider the multivariate AR model of first order

$$Y_t - \Phi Y_{t-1} = Z_t, \quad t \in \mathbb{Z}, \tag{10}$$

where $\Phi \in \mathbb{R}^{d \times d}$ and $(Z_t)_{t \in \mathbb{Z}}$ is an i.i.d. sequence of d-dimensional random vectors. Necessary and sufficient conditions for the existence of a strictly stationary solution to this equation have been derived by Brockwell et al. [10, Thm. 1]. For simplicity, we shall assume that all eigenvalues of Φ lie in the open unit ball $\{z \in \mathbb{C} : |z| < 1\}$. Then a sufficient condition for the existence of a strictly stationary solution of (10) is $\mathbb{E}\log^+ |Z_0| < \infty$, in which case the stationary solution is unique and given by

$$Y_t = \sum_{k=0}^{\infty} \Phi^k Z_{t-k}, \quad t \in \mathbb{Z}, \tag{11}$$

where the right-hand side converges almost surely absolutely. If Φ is additionally invertible, then the condition $\mathbb{E}\log^+ |Z_0| < \infty$ is also necessary for the existence of a strictly stationary solution, see [10, Cor. 1].

Theorem 8. [Stationary solution of AR(1) equation]
Let $\Phi \in \mathbb{R}^{d \times d}$ such that all eigenvalues of Φ lie in $\{z \in \mathbb{C} : |z| < 1\}$. Let $\mathscr{M}_1$ be the set of all probability distributions $\mathscr{L}(X)$ on $\mathbb{R}^d$ with $\mathbb{E}\log^+ |X| < \infty$ (i.e. with finite log-moment) and $\mathscr{M}_2$ be the set of all probability distributions on $\mathbb{R}^d$. Consider the mapping

$$G_\Phi : \mathscr{M}_1 \to \mathscr{M}_2, \quad \mathscr{L}(Z_0) \mapsto \mathscr{L}\left(\sum_{k=0}^{\infty} \Phi^k Z_{-k}\right), \quad t \in \mathbb{Z},$$

where $(Z_{-k})_{k \in \mathbb{N}_0}$ is an i.i.d. sequence with distribution $\mathscr{L}(Z_0)$, so that G_Φ associates to each $\mathscr{L}(Z_0)$ the distribution of the corresponding stationary solution of the AR(1) equation (10)*. We then have:*
(i) If Φ is exchangeability preserving (commutes with permutations), then G_Φ is exchangeability preserving (commutes with permutations), so that Y_t given by (11) *is*

exchangeable whenever $\mathscr{L}(Z_0)$ is exchangeable.
(ii) Let $\mathscr{M}_1'$ be the subset of all infinitely divisible $\mu \in \mathscr{M}_1$, and denote by G'_Φ the restriction of G_Φ to $\mathscr{M}_1'$. Assume that Φ commutes with permutations. Then G'_Φ commutes with permutations, G'_Φ is injective, and the inverse $(G'_\Phi)^{-1} : G_\Phi(\mathscr{M}_1') \to \mathscr{M}_1'$ commutes with permutations. In particular, for $\mathscr{L}(Z_0) \in \mathscr{M}_1'$, $\mathscr{L}\left(\sum_{k=0}^\infty \Phi^k Z_{-k}\right)$ is exchangeable if and only if $\mathscr{L}(Z_0)$ is exchangeable.
(iii) Suppose that $(Z_t)_{t\in\mathbb{Z}}$ is i.i.d. $N(0,\Sigma)$-distributed. Then a necessary and sufficient condition for exchangeability of $\mathscr{L}\left(\sum_{k=0}^\infty \Phi^k Z_{-k}\right)$ is that $\sum_{k=0}^\infty \Phi^k \Sigma (\Phi^T)^k$ commutes with permutations.

Proof. (i) If Φ is exchangeability preserving this is an immediate consequence of Corollary 4, since with Φ obviously also Φ^k is exchangeability preserving for each $k \in \mathbb{N}_0$. In the case when $P_\pi \Phi = \Phi P_\pi$ for all $\pi \in [d]$, we have also $P_\pi \Phi^k = \Phi^k P_\pi$ for all $k \in \mathbb{N}_0$, hence

$$P_\pi \sum_{k=0}^\infty \Phi^k Z_{-k} = \sum_{k=0}^\infty \Phi^k P_\pi Z_{-k},$$

which shows that G_Φ commutes with permutations.
(ii) That G'_Φ commutes with permutations is easy to see from (i). Let $Y_t := \sum_{k=0}^\infty \Phi^k Z_{t-k}$. Then Y_t is infinitely divisible and hence its characteristic function φ_Y has no zeros. By (10),

$$\varphi_Y(z) = \varphi_Y(\Phi^T z)\varphi_Z(z) \quad \forall\, z \in \mathbb{R}^d,$$

where φ_Z is the characteristic function of Z_t. Hence G'_Φ is injective. The rest follows from Theorem 7.
(iii) By (11), we have $\mathscr{L}(Y_t) = N(0, \sum_{k=0}^\infty \Phi^k \Sigma (\Phi^T)^k)$. The result then follows from Remark 1.

The following example shows that some conditions on $\mathscr{L}(Z_0)$ are needed in order for the assertion in Theorem 8 (ii) to hold:

Example 4. Let $(U_t)_{t\in\mathbb{Z}}$, $(V_t)_{t\in\mathbb{Z}}$ and $(W_t)_{t\in\mathbb{Z}}$ be one-dimensional independent i.i.d. sequences such that U_0 and V_0 have characteristic function

$$\varphi_{U_0}(z) = \varphi_{V_0}(z) = \begin{cases} 1 - |z|, & |z| \le 1, \\ 0, & |z| > 1, \end{cases}$$

and $\varphi_{W_0}(z) = \varphi_{U_0}(z)$ for $|z| \le 1$ and otherwise φ_{W_0} being periodic with period 4. These functions are indeed characteristic functions of random variables, see Feller [16, p. 506]. Observe that U_0 has density $\mathbb{R} \to \mathbb{R}$, $x \mapsto \pi^{-1}(1-\cos(x))x^{-2}$ ([16, p.503]), hence $\mathbb{E}\log^+|U_0| < \infty$ and it follows that $\sum_{k=0}^\infty 2^{-k}U_{-k}$ converges almost surely. Hence $\prod_{k=0}^\infty \varphi_{U_0}(2^{-k}z)$ converges pointwise to some characteristic function, and it is easy to see that $\prod_{k=0}^\infty \varphi_{U_0}(2^{-k}z) = \prod_{k=0}^\infty \varphi_{W_0}(2^{-k}z)$ for $z \in \mathbb{R}$. By Lévy's continuity theorem, $\sum_{k=0}^\infty 2^{-k}W_{-k}$ converges in distribution, hence almost surely. In particular, $\mathbb{E}\log^+|W_0| < \infty$ and $\mathscr{L}(\sum_{k=0}^\infty 2^{-k}U_{-k}) = \mathscr{L}(\sum_{k=0}^\infty 2^{-k}W_{-k})$. Now let $d=2$,

$$\Phi = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}, \quad Z_t = (U_t, W_t)^T, \quad \text{and} \quad X_t = (U_t, V_t)^T.$$

Then $\mathscr{L}\left(\sum_{k=0}^\infty \Phi^k Z_{-k}\right) = \mathscr{L}\left(\sum_{k=0}^\infty \Phi^k X_{-k}\right)$ is exchangeable and Φ is a diagonal matrix, but $\mathscr{L}(Z_0)$ is not exchangeable.

Observe that exchangeability of $\mathscr{L}(Y_t)$ and $\mathscr{L}(Z_t)$ in (11) does not imply that Φ is exchangeability preserving, as can be seen in the next example.

Example 5. Let $(Z_t)_{t\in\mathbb{Z}}$ be an i.i.d sequence of two-dimensional $N(0,\mathrm{Id}_2)$-distributed random vectors and let $\Phi \in \mathbb{R}^{2\times 2}$ be of the form $\Phi = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ with $0 < a < c < 1$, and let Y_t be defined by (11). Then easy calculations show that

$$\Phi^k = \begin{pmatrix} a^k & b\frac{a^k-c^k}{a-c} \\ 0 & c^k \end{pmatrix} \quad \text{and} \quad \sum_{k=0}^\infty \Phi^k(\Phi^T)^k = \sum_{k=0}^\infty \begin{pmatrix} a^{2k} + b^2\left(\frac{a^k-c^k}{a-c}\right)^2 & c^k b\frac{a^k-c^k}{a-c} \\ c^k b\frac{a^k-c^k}{a-c} & c^{2k} \end{pmatrix}.$$

By Theorem 8 (iii) and Theorem 5 (i) we conclude that $\mathscr{L}(Y_t)$ is exchangeable if and only if $\sum_{k=0}^\infty \left(a^{2k} + b^2\left(\frac{a^k-c^k}{a-c}\right)^2\right) = \sum_{k=0}^\infty c^{2k}$. Therefore, setting

$$b^2 := (a-c)^2 \frac{\sum_{k=0}^\infty c^{2k} - \sum_{k=0}^\infty a^{2k}}{\sum_{k=0}^\infty (a^k - c^k)^2} = \frac{1-ac}{1+ac}(c^2 - a^2)$$

gives exchangeability of $\mathscr{L}(Y_t)$. However, from Theorem 6 it is easy to see that Φ is not exchangeability preserving (observe that Φ is invertible).

Now we consider random recurrence equations. Let $(A_t, Z_t)_{t\in\mathbb{Z}}$ be an $\mathbb{R}^{d\times d}\times\mathbb{R}^d$-valued i.i.d. sequence. Suppose that

$$E\log^+|Z_1| < \infty,\ E\log^+\|A_1\| < \infty, \text{ and } \gamma := \inf_{n\in\mathbb{N}}\left\{\frac{1}{n}\mathbb{E}\log\|A_1A_2\cdots A_n\|\right\} < 0, \tag{12}$$

where $\|A\| := \sup_{|x|=1}|Ax|$ for $A \in \mathbb{R}^{d\times d}$ denotes the matrix norm induced by the Euclidean vector norm $|x| = \sqrt{x^T x}, x \in \mathbb{R}^d$. The quantity γ is called the *top Lyapunov exponent* of the sequence $(A_t)_{t\in\mathbb{Z}}$. Under these conditions, there exists a unique strictly stationary solution $(X_t)_{t\in\mathbb{Z}}$ of the random recurrence equation

$$X_t = A_t X_{t-1} + Z_t, \quad t \in \mathbb{Z}, \tag{13}$$

and it is given by

$$X_t = \sum_{k=0}^{\infty} A_t A_{t-1} \cdots A_{t-k+1} Z_{t-k}, \tag{14}$$

where the sum converges almost surely absolutely. In particular, the unique strictly stationary solution is non-anticipative in the sense that $(X_s)_{s\le t}$ is independent of $(A_s, Z_s)_{s\ge t+1}$. Further, for any $\mathbb{R}^d$-valued random variable V_0 on the same probability space, let $(V_n)_{n\in\mathbb{N}}$ be defined recursively by

$$V_n = A_n V_{n-1} + Z_n, \quad n \in \mathbb{N}. \tag{15}$$

Then V_n converges in distribution to $\mathscr{L}(X_0)$ as $n \to \infty$, where X_0 is given by (14), see Brandt [9, Thm. 1] for the one-dimensional case and Bougerol and Picard [8, Thm. 1.1] and Stelzer [25, Thm. 4.1] for the multivariate case. In [8, Thm. 2.5], under finite log-moment conditions and certain irreducibility conditions, a characterization for the existence of strictly stationary non-anticipative solutions of (13) in terms of negativity of the top Lyapunov exponent is achieved.

Under the above conditions, we obtain a mapping $G : \mathscr{L}(Z_0) \mapsto \mathscr{L}(X_0)$ as described below, where X_0 is given by (14).

Theorem 9. *Let $(A_t)_{t\in\mathbb{Z}}$ be an i.i.d. sequence in $\mathbb{R}^{d\times d}$ such that* (12) *is satisfied (i.e. $E\log^+\|A_1\| < \infty$ and $\gamma < 0$). Let $\mathscr{M}_1$ be the set of all probability distributions $\mathscr{L}(Z_0)$ with $\mathbb{E}\log^+|Z_0| < \infty$, $\mathscr{M}_2$ be the set of all probability distributions, and let $G : \mathscr{M}_1 \to \mathscr{M}_2$ be defined by*

$$\mathscr{L}(Z_0) \mapsto \mathscr{L}(X_0)$$

where $(Z_t)_{t\in\mathbb{Z}}$ is an i.i.d. sequence with distribution $\mathscr{L}(Z_0)$, independent of $(A_t)_{t\in\mathbb{Z}}$, and X_0 is given by (14).
(i) If $(A_t)_{t\in\mathbb{Z}}$ takes only values in the set $\mathscr{E}_d$ of exchangeability preserving matrices, then G is exchangeability preserving, i.e. $\mathscr{L}(X_0)$ is exchangeable provided that $\mathscr{L}(Z_0)$ is exchangeable.
(ii) If $(A_t)_{t\in\mathbb{Z}}$ takes only values in the set of all matrices that commute with permutations, then G commutes with permutations.
(iii) Suppose that $(A_t)_{t\in\mathbb{Z}}$ takes only values in the set of matrices that commute with permutations and that have only non-negative entries. Let $\mathscr{M}_1'$ be the set of all $\mathscr{L}(Z_0) \in \mathscr{M}_1$ with distribution concentrated on $[0,\infty)^d$, and denote by G' the restriction of G to $\mathscr{M}_1'$. Then G' is injective, and its inverse $(G')^{-1} : G'(\mathscr{M}_1') \to \mathscr{M}_1$

commutes with permutations. In particular, in this case, $\mathscr{L}(X_0)$ is exchangeable if and only if $\mathscr{L}(Z_0)$ is exchangeable.

Proof. (i) Suppose that $\mathscr{L}(Z_0) \in \mathscr{M}_1$ is exchangeable. Let $V_0 := Z_0$ and define V_n recursively by (15). Since $\mathscr{L}(V_n)$ converges in distribution to $\mathscr{L}(X_0)$, by Lemma 1 it is enough to show that each V_n is exchangeable. We do that by induction on n. Observe that V_0 is exchangeable and independent of $(A_t, Z_t)_{t\geq 1}$. Suppose that V_{n-1} is proved to be exchangeable. Observe that V_{n-1} is independent of (A_n, Z_n). Since A_n takes values in the space of exchangeability preserving matrices, by conditioning on A_n we see that $A_n V_{n-1}$ is exchangeable, similar to the proof of Corollary 4. Since Z_n is independent of $A_n V_{n-1}$, we conclude that $V_n = A_n V_{n-1} + Z_n$ is exchangeable by Example 3. This gives the claim.
(ii) By (14), for every $\pi \in [d]$ we have

$$P_\pi X_t = \sum_{k=0}^{\infty} P_\pi A_t \cdots A_{t-k+1} Z_{t-k} = \sum_{k=0}^{\infty} A_t \cdots A_{t-k+1} P_\pi Z_{t-k},$$

and $(P_\pi Z_t)_{t\in\mathbb{Z}}$ is i.i.d. with $\mathscr{L}(P_\pi Z_1) \in \mathscr{M}_1$. Hence G commutes with permutations.
(iii) Under the given conditions, it follows from (14) that also X_t is concentrated on $[0,\infty)^d$. Then also $A_t X_{t-1}$ is concentrated on $[0,\infty)^d$ and is independent of Z_t by (14). Hence we may take Laplace transforms in (13) and obtain

$$\mathbb{L}_{X_1}(u) = \mathbb{L}_{A_1 X_0}(u)\, \mathbb{L}_{Z_1}(u) \quad \forall\, u \in [0,\infty)^d,$$

where $\mathbb{L}_{X_1}(u) = \mathbb{E}\mathrm{e}^{-X_1^T u}$ denotes the Laplace transform of X_1 for $u \in [0,\infty)^d$. From this it follows that G' is injective, and it obviously commutes with permutations by (ii). The claim then follows from Theorem 7.

6 Exchangeability preserving transformations based on continuous time

In this section we consider stochastic integrals of the form

$$\int_0^T f(t)\mathrm{d}L_t \quad \text{and} \quad \int_0^\infty f(t)\,\mathrm{d}L_t := \mathrm{dlim}_{T\to\infty} \int_0^T f(t)\,\mathrm{d}L_t,$$

where $L = (L_t)_{t\geq 0}$ is a Lévy process in $\mathbb{R}^d$, f is an $\mathbb{R}^{d\times d}$-valued stochastic process with càglàd paths, and dlim denotes the limit in distribution (provided it exists). Here, we assume implicitly that there is an underlying filtration $\mathbb{F} = (\mathscr{F}_t)_{t\geq 0}$ satisfying the usual conditions such that L is a semimartingale with respect to $\mathbb{F}$ and that $(f_t)_{t\geq 0}$ is adapted with respect to $\mathbb{F}$; if the processes f and L are independent, such a filtration always exists, see Protter [21, Thm. VI.2], so that $\int_0^T f(t)\,\mathrm{d}L_t$ always exists in this case. See [21] also for the definition and facts about stochastic integration. When f is deterministic, then $T \mapsto \int_0^T f(t)\,\mathrm{d}L_t$ has independent increments, and it follows that it converges in distribution as $T \to \infty$ if and only if it converges in probability (or even almost surely). For deterministic f, the integral coincides with integrals with respect to the induced independently scattered random measure as defined in Sato [23, Prop. 2.11, Ex. 2.12, Def. 2.16, Def. 2.20], and $V := \int_0^\infty f(t)\mathrm{d}L_t$ is infinitely divisible (provided it exists); denoting by Ψ_L and Ψ_V the characteristic exponents of L_1 and V, we then have

$$\Psi_V(z) = \lim_{T\to\infty} \int_{[0,T]} \Psi_L(f(s)^T z)\,\mathrm{d}s \quad \forall\, z \in \mathbb{R}^d, \tag{16}$$

see [23, Prop. 2.17]. A relation between the characteristic triplets of L and V can also be established in this case, as well as a characterization when $\int_0^\infty f(t)\,\mathrm{d}L_t$ actually exists as a limit in probability; see [23, Prop. 2.17, Cor. 2.19]. If f is not deterministic but independent of L, then $\int_0^\infty f(t)\,\mathrm{d}L_t$ needs not to be infinitely divisible, but its characteristic function can be calculated from (16) by conditioning on f.

Theorem 10. *Let $f = (f_t)_{t\geq 0}$ be an $\mathbb{R}^{d\times d}$-valued stochastic process with càglàd paths. Let $\mathscr{M}_1$ be a set of infinitely divisible distributions μ, such that $\int_0^\infty f(t)\,\mathrm{d}L_t^\mu$ is definable as a limit in distribution for each $\mu \in \mathscr{M}_1$, where $(L_t^\mu)_{t\geq 0}$ is a d-dimensional Lévy process with $\mathscr{L}(L_1^\mu) = \mu$, independent of f (possibly defined on a suitably enlarged probability space). Denote by $\mathscr{M}_2$ the set of all probability distributions on $\mathbb{R}^d$. This gives a mapping*

$$G : \mathscr{M}_1 \to \mathscr{M}_2, \quad \mu \mapsto \mathscr{L}\left(\int_0^\infty f(t)\,\mathrm{d}L_t^\mu\right).$$

(i) Suppose that f takes only values in the set $\mathscr{E}_d$ of exchangeability preserving matrices. Then G is exchangeability preserving, i.e. exchangeability of μ implies exchangeability of $G(\mu)$.
(ii) Suppose that f takes only values in the set of matrices that commute with permutations. Then $\int_0^\infty f(t)\,\mathrm{d}L_t^\mu$ exists as a limit in distribution if and only $\int_0^\infty f(t)\,\mathrm{d}(P_\pi L_t^\mu)$ exists as a limit in distribution for all $\pi \in [d]$, so that (by possibly enlarging $\mathscr{M}_1$) we can assume that $P_\pi\mu \in \mathscr{M}_1$ for all $\pi \in [d]$ and $\mu \in \mathscr{M}_1$. Then G commutes with permutations. If additionally G is injective, then the inverse $G^{-1} : G(\mathscr{M}_1) \to \mathscr{M}_1$ also commutes with permutations, so that in this case, $\int_0^\infty f(t)\,\mathrm{d}L_t^\mu$ is exchangeable if and only if μ is exchangeable.

Proof. (i) Let $V = \int_0^\infty f(t)\,\mathrm{d}L_t^\mu$. Suppose first that f is deterministic and has compact support in $[0,T]$. Since $f(s)$ takes values in $\mathscr{E}_d$, for every $\pi \in [d]$ and $s \geq 0$ there exists $\pi'(s) \in [d]$ such that $f(s)^T P_\pi = P_{\pi'(s)} f(s)^T$. Hence

$$\Psi_V(P_\pi z) = \int_0^T \Psi_{L^\mu}(f(s)^T P_\pi z)\,\mathrm{d}s = \int_0^T \Psi_{L^\mu}(P_{\pi'(s)} f(s)^T z)\,\mathrm{d}s \quad \forall\, z \in \mathbb{R}^d.$$

Now if μ is exchangeable, then $\Psi_{L^\mu}(P_{\pi'(s)} z) = \Psi_{L^\mu}(z)$, and we conclude

$$\Psi_V(P_\pi z) = \int_0^T \Psi_{L^\mu}(f(s)^T z) = \Psi_V(z) \quad \forall\, z \in \mathbb{R}^d,$$

so that G is exchangeability preserving.
Next assume that f is not deterministic but has support in $[0,T]$ and its distribution is ρ. Then conditioning on the paths of f, and since f and L^μ are independent, we obtain

$$\begin{aligned}\varphi_{P_\pi \int_0^T f(t)\,\mathrm{d}L_t^\mu}(z) &= \int_{\mathscr{E}_d} \varphi_{P_\pi \int_0^T M(t)\,\mathrm{d}L_t^\mu}(z)\rho(\mathrm{d}M) \\ &= \int_{\mathscr{E}_d} \varphi_{\int_0^T M(t)\,\mathrm{d}L_t^\mu}(z)\rho(\mathrm{d}M) = \varphi_{\int_0^T f(t)\,\mathrm{d}L_t^\mu}(z), \quad z \in \mathbb{R},\end{aligned}$$

where the second equality follows from the previous case, so that G must also be exchangeability preserving in this case.
Finally, the case when f is random (but independent) and does not have compact support follows from Lemma 1 since

$$\begin{aligned}\varphi_{P_\pi \int_0^\infty f(t)\,\mathrm{d}L_t^\mu} &= \lim_{T\to\infty} \varphi_{P_\pi \int_0^T f(t)\,\mathrm{d}L_t^\mu}(z) \\ &= \lim_{T\to\infty} \varphi_{\int_0^T f(t)\,\mathrm{d}L_t^\mu}(z) = \varphi_{\int_0^\infty f(t)\,\mathrm{d}L_t^\mu}(z), \quad z \in \mathbb{R}.\end{aligned}$$

(ii) Suppose that $f(t)$ commutes with permutations for each t. Consider first the case that f is deterministic and has support in $[0,T]$. For each $\pi \in [d]$, we then have

$$\begin{aligned}\Psi_{\int_0^T f(t)\,\mathrm{d}(P_\pi L_t^\mu)}(z) &= \int_0^T \Psi_{P_\pi L^\mu}(f(s)^T z)\,\mathrm{d}s = \int_0^T \Psi_{L^\mu}(P_\pi^T f(s)^T z)\,\mathrm{d}s \\ &= \int_0^T \Psi_{L^\mu}(f(s)^T P_\pi^T z)\,\mathrm{d}s = \Psi_{\int_0^T f(t)\,\mathrm{d}L_t^\mu}(P_\pi^T z) \\ &= \Psi_{P_\pi \int_0^T f(t)\,\mathrm{d}L_t^\mu}(z), \quad z \in \mathbb{R}^d.\end{aligned}$$

Now if f is stochastic with support in $[0,T]$ but independent of L^μ, then by conditioning on f we see that

$$\varphi_{\int_0^T f(t)\,\mathrm{d}(P_\pi L_t^\mu)}(z) = \varphi_{\int_0^T f(t)\,\mathrm{d}L_t^\mu}(P_\pi^T z) = \varphi_{P_\pi \int_0^T f(t)\,\mathrm{d}L_t^\mu}(z), \quad z \in \mathbb{R}^d. \tag{17}$$

Since $\int_0^T f(t)\,\mathrm{d}(P_\pi L_t^\mu)$ converges in distribution if and only if $\varphi_{\int_0^T f(t)\,\mathrm{d}(P_\pi L_t^\mu)}$ converges pointwise to a characteristic function as $T \to \infty$, it follows that $\int_0^T f(t)\,\mathrm{d}(P_\pi L_t^\mu)$ converges in distribution if and only if $\int_0^T f(t)\,\mathrm{d}L_t^\mu$ does. Hence we can assume that $P_\pi \mu \in \mathscr{M}_1$ for all $\pi \in [d]$ and $\mu \in \mathscr{M}_1$. Taking the limit as $T \to \infty$ in (17) then shows that G commutes with permutations. The rest follows from Theorem 7.

Remark 2. (i) When restricting to deterministic f, we can use integration theory with respect to independently scattered random measures as in Sato [23], and the proof above carries easily over to deterministic, but not necessarily càglàd f, as long it is integrable with respect to L^μ.
(ii) Another approach to prove Theorem 10 (ii) is to use that

$$P_\pi \int_0^T f(t)\,\mathrm{d}L_t^\mu = \int_0^T f(t)\,\mathrm{d}(P_\pi L_t^\mu),$$

as a consequence of approximating the stochastic integrals by Riemann sums (e.g. [21, Thm. II.21]) and the fact that $f(t)$ commutes with permutations.

Theorem 10 will be mostly used in the case when $f(t) = g(t)\mathrm{Id}_d$ for a deterministic scalar valued càglàd function g. Observe that in that case, $f(t)$ obviously commutes with permutations. As a first application, we consider self-decomposable distributions.

Recall that a distribution μ on $\mathbb{R}^d$ is *self-decomposable* if and only if for each $b > 1$ there exists a probability measure ρ_b on $\mathbb{R}^d$ such that

$$\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\rho_b}(z) \quad \forall\, z \in \mathbb{R}^d$$

(e.g. [22, Def. 15.1]). It is well known that self-decomposable distributions constitute exactly the class of stationary distributions of Ornstein–Uhlenbeck processes. More precisely, given $c > 0$, a distribution σ is self-decomposable if and only if there exists a Lévy process $L^\mu = (L_t^\mu)_{t\ge 0}$ with $\mu = \mathscr{L}(L_1^\mu)$ such that $\mathbb{E}\log^+|L_1^\mu| < \infty$ (i.e. $\int_{\mathbb{R}} \log^+|x|\,\mu(\mathrm{d}x) < \infty$) and

$$\sigma = \mathscr{L}\left(\int_0^\infty \mathrm{e}^{-ct}\,\mathrm{d}L_t^\mu\right),$$

where the integral converges almost surely (equivalently, in distribution). It is known that $\mathbb{E}\log^+|L_1^\mu| < \infty$ is a necessary and sufficient condition for convergence of $\int_0^\infty \mathrm{e}^{-ct}\,\mathrm{d}L_t^\mu$, and that the mapping

$$G : \mathscr{M}_1 \to \mathscr{M}_2, \quad \mu \mapsto \mathscr{L}\left(\int_0^\infty \mathrm{e}^{-ct}\,\mathrm{d}L_t^\mu\right), \tag{18}$$

where $\mathscr{M}_1$ is the class of all infinitely distributions μ on $\mathbb{R}^d$ with $\mathbb{E}\log^+|L_1^\mu| < \infty$, and $\mathscr{M}_2$ is the class of all self-decomposable distributions, is a bijection; cf. Sato [22, Thm. 17.5]. For self-decomposable σ, the unique (in distribution) Lévy process L^μ such that $G(\mathscr{L}(L_1^\mu)) = \sigma$ is then called the *background driving Lévy process* of σ, or μ the *background driving infinitely divisible distribution* of σ.

The following is now immediate:

Corollary 5. [Exchangeable self-decomposable distributions]
With the notations above, the mapping $G : \mathscr{M}_1 \to \mathscr{M}_2$ defined in (18) *as well as its inverse commute with permutations. In particular, a self-decomposable distribution is exchangeable if and only if its background driving infinitely distribution is exchangeable (for fixed $c > 0$).*

Let us give yet another proof that a self-decomposable distribution can only be exchangeable if the background driving infinitely divisible distribution is exchangeable, without referring to the general result Theorem 7. For that, we need the following Lemma, which is interesting in its own right. It is well known in one dimension (e.g. Barndorff-Nielsen and Shephard [4, Eq. (4.15)]) and proved similarly in higher dimensions, but since we were not able to find a ready reference we provide a proof:

Lemma 2. *Let $c > 0$, $(L_t)_{t\geq 0}$ a Lévy process with $\mathbb{E}\log^+|L_1| < \infty$ and $V = \int_0^\infty \mathrm{e}^{-cs}\mathrm{d}L_s$. Denote by Ψ_V and Ψ_L the characteristic exponents of V respectively of L_1. Then for $\xi := \frac{z}{|z|}, z \in \mathbb{R}^d \setminus \{0\}$*

$$\frac{\partial \Psi_V(z)}{\partial \xi} = \frac{\Psi_L(z)}{c|z|}.$$

Proof. We have $\Psi_V(z) = \int_0^\infty \Psi_L(\mathrm{e}^{-cs}z)\mathrm{d}s$ for $z \in \mathbb{R}^d$ as a limit by (16), but it even holds $\int_0^\infty |\Psi_L(\mathrm{e}^{-cs}z)|\,\mathrm{d}s < \infty$ (cf. [22], proof of Theorem 17.5). Letting $\xi := z/|z|$ for $z \neq 0$ and substituting $x = \mathrm{e}^{-cs}|z|$, we obtain

$$\Psi_V(z) = \int_0^\infty \Psi_L(e^{-cs}|z|\xi)\mathrm{d}s = \int_0^{|z|} \Psi_L(x\xi)\frac{\mathrm{d}x}{cx}.$$

Differentiation with respect to ξ gives

$$\frac{\partial \Psi_V(z)}{\partial \xi} = \lim_{h\to 0} h^{-1}\int_{|z|}^{|z|+h} \Psi_L(x\xi)\frac{\mathrm{d}x}{cx} = \frac{\Psi_L(z)}{c|z|},$$

which finishes the proof of Lemma 2.

Let us now show again that $V := \int_0^\infty e^{-ct}\, dL_t^\mu$ can only be exchangeable if μ is exchangeable. So suppose that V is exchangeable. Then $\Psi_V(z) = \Psi_V(P_\pi z)$ for all $\pi \in [d]$ and $z \in \mathbb{R}^d \setminus \{0\}$. Therefore for $\xi_\pi := P_\pi \xi = \frac{P_\pi z}{|z|}$,

$$\frac{\partial \Psi_V(z)}{\partial \xi} = \lim_{h\to 0} \frac{\Psi_V(z+\xi h) - \Psi_V(z)}{h} = \lim_{h\to 0} \frac{\Psi_V(P_\pi z + \xi_\pi h) - \Psi_V(P_\pi z)}{h} = \frac{\partial \Psi_V(P_\pi z)}{\partial \xi_\pi},$$

which implies by the previous lemma $\Psi_{L_\mu}(z) = \Psi_{L_\mu}(P_\pi z)$ for all $z \in \mathbb{R}^d$ and $\pi \in [d]$, showing that μ is exchangeable.

Next we generalise our results on self-decomposable distributions to A-decomposable distributions. Let $A \in \mathbb{R}^{d\times d}$ such that all eigenvalues of A have strictly positive real part. Then $\int_0^\infty e^{-As}\, dL_s^\mu$ is definable for a Lévy process L^μ if and only if $\mathbb{E}\log^+ |L_1^\mu| < \infty$. By Theorem 4.1 in Sato and Yamazato [24], the mapping

$$G: \mathscr{M}_1 \to \mathscr{M}_2, \quad \mu \mapsto \mathscr{L}\left(\int_0^\infty e^{-As}\, dL_s^\mu\right) \tag{19}$$

defines a bijection from the class $\mathscr{M}_1$ of all infinitely divisible distributions on $\mathbb{R}^d$ with finite log-moment to the class $\mathscr{M}_2$ of all A-decomposable distributions; here, a distribution σ on $\mathbb{R}^d$ is A-decomposable, if for every $t > 0$ there exists a probability measure ρ_t on $\mathbb{R}^d$ such that

$$\widehat{\sigma}(z) = \widehat{\sigma}(e^{-tA^T} z)\widehat{\rho}_t(z), \quad z \in \mathbb{R}^d.$$

The distribution $G(\mu)$ is then the unique stationary distribution of the Ornstein–Uhlenbeck process

$$dX_t(\omega) = dL_t^\mu(\omega) - AX_{t-}(\omega)dt, \quad t \geq 0.$$

All this can be found in [24]. If A commutes with permutations, then also e^{-As} commutes with permutations for all $s \geq 0$ by Corollary 3. Hence, the following result is immediate from Theorem 10:

Corollary 6. [Exchangeable A-decomposable distributions]
Let $A \in \mathbb{R}^{d\times d}$ such that all eigenvalues of A have strictly positive real part and that A commutes with permutations. With the notations above, the mapping $G: \mathscr{M}_1 \to \mathscr{M}_2$ defined in (19) *as well as its inverse commute with permutations. In particular, $\int_0^\infty e^{-As}\, dL_s^\mu$ is exchangeable if and only if μ is exchangeable.*

The distribution $\int_0^\infty \mathrm{e}^{-As}\,\mathrm{d}L_s$ can be exchangeable without A being exchangeability preserving, as follows from the following result applied to $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

Proposition 3. *Suppose $(L_t)_{t\geq 0}$ is a standard Brownian motion, and A a normal matrix (i.e. $AA^T = A^TA$) such that all eigenvalues of A have only strictly positive real parts. Then $\int_0^\infty \mathrm{e}^{-As}\,\mathrm{d}L_s$ is exchangeable if and only if $A+A^T$ commutes with permutations.*

Proof. Since A is normal, it follows that $\mathrm{e}^{-As}\mathrm{e}^{-A^Ts} = \mathrm{e}^{-A^Ts}\mathrm{e}^{-As} = \mathrm{e}^{-(A+A^T)s}$, see [7, Prop. 11.2.8]. Therefore, the infinitely divisible distribution $\mu = \mathscr{L}(\int_0^\infty e^{-As} dL_s)$ is normal distributed with mean 0 and variance

$$B = \int_0^\infty e^{-sA}e^{-sA^T}\,ds = \int_0^\infty e^{-s(A+A^T)}\,ds = -(A+A^T)^{-1}.$$

By Remark 1, μ is exchangeable if and only if B commutes with permutations, which by Theorem 5 (iii) is equivalent to the fact that $A+A^T$ commutes with permutations.

Let us consider some other mappings. Barndorff-Nielsen and Thorbjørnsen ([5] and others) have introduced the Upsilon-transform in dimension one, and this has been generalised by Barndorff-Nielsen et al. [2] to a multivariate setting. For an infinitely divisible distribution μ on $\mathbb{R}^d$ with associated Lévy process L^μ, the Upsilon-transform is defined by

$$\Upsilon(\mu) = \mathscr{L}\left(\int_0^1 \log\frac{1}{t}\,\mathrm{d}L_t^\mu\right).$$

Observe that the integrand $t \mapsto \log t^{-1}$ has a singularity at 0 and hence is not càglàd, but Theorem 10 carries over by Remark 2. As shown in [2, Thm. A], Υ defines a bijection from the class of all infinitely divisible distributions to the Goldie-Steuel-Bondesson class $B(\mathbb{R}^d)$. By Theorem 10, we then have:

Theorem 11. *The Υ-transform as defined above commutes with permutations, as does its inverse. In particular, $\Upsilon(\mu)$ is exchangeable if and only if μ is exchangeable.*

By restricting Υ to the class of self-decomposable distributions, one obtains the Thorin class [2, Thm. B], and by composing Υ with the mapping G of (18), one obtains a bijection from the class of infinitely divisible distributions with finite log-moment to the Thorin class [2, Thm. C]. Results similar to Theorem 11 can then be stated for this composition.

We note that the Upsilon-transform has been generalised in various directions, sometimes acting on infinitely divisible distributions, sometimes acting directly on Lévy measures. We just mention the general Upsilon-transforms of Barndorff–Nielsen et al. [3]: let ρ be a σ-finite measure on $(0,\infty)$ and for each σ-finite measure ν on $\mathbb{R}^d$ with $\nu(\{0\}) = 0$ define Υ_ρ as the positive measure on $\mathbb{R}^d$ given by

$$[\Upsilon_\rho(\nu)](B) = \int_0^\infty \nu(x^{-1}B)\,\rho(\mathrm{d}x), \quad B \in \mathscr{B}_d;$$

Υ_ρ is called the *Upsilon transformation with dilation measure* ρ. Restricting the domain of Υ_ρ to the set of all σ-finite measures ν such that $\Upsilon_\rho(\nu)$ is a Lévy measure, it is easy to see that Υ_ρ commutes with permutations. The domain has been derived in [3, Sect. 3] in various cases. In [3, Sect. 6], the injectivity property of Υ_ρ has been further studied, and shown to be equivalent to the *cancellation property* of the multiplicative convolution; see [3, Eq. (6.1)] for details. It is clear that Theorem 10 applies for injective Υ_ρ.
Finally, we mention that a natural continuous time analogue of random recurrence equations with iid coefficients is the multivariate generalized Ornstein-Uhlenbeck process, as introduced in Behme and Lindner [6]. It would be interesting to investigate conditions under which an analogue of Theorem 9 holds for these processes, but we leave this for future research.

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